Assignment 3

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1 Amortization

We need to calculate the amortized cost of this function.

$$c_i = \begin{cases} i + c & ; i \neq k^2, k \in \mathbb{N} \\ i^2 & ; i = k^2, k \in \mathbb{N} \end{cases}$$

I did this using the aggregate method. Between 1 and N (both inclusive) there is $\sqrt{N}$ numbers that are perfect squares, that's what I use to sum cost when the second case is true in the function above. I also subtracted cost of the first case in the second sum as we count it $\sqrt{N}$ too many times otherwise

$$\frac{1}{n}\sum_{i=1}^{n} c_i = \frac{1}{n}(\sum_{i=1}^{n}(i + c) + \sum_{i=1}^{\lfloor\sqrt{n}\rfloor}(i^4 - (i^2 + c))) =$$

$$\frac{1}{n}(\sum_{i=1}^{n} i + \sum_{i=1}^{n} c) + \frac{1}{n}(\sum_{i=1}^{\lfloor\sqrt{n}\rfloor} i^4 - \sum_{i=1}^{\lfloor\sqrt{n}\rfloor}(i^2 + c))$$

I also used the following formulas

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\frac{1}{n}(\sum_{i=1}^{n} i + \sum_{i=1}^{n} c) + \frac{1}{n}(\sum_{i=1}^{\lfloor\sqrt{n}\rfloor} i^4 - \sum_{i=1}^{\lfloor\sqrt{n}\rfloor} (i^2 + c)) =$$

$$\frac{n(n+1)}{2n} + \frac{nc}{n} + \frac{\sqrt{n}(\sqrt{n}+1)(6\sqrt{n}^3 + 9\sqrt{n}^2 + \sqrt{n} - 1)}{30n} - \frac{1}{n}\sum_{i=1}^{\lfloor\sqrt{n}\rfloor} (i^2 + c) =$$

$$\frac{n+1}{2} + c + \frac{(n+\sqrt{n})(6\sqrt{n}^3 + 9\sqrt{n}^2 + \sqrt{n} - 1)}{30n} - \frac{\sqrt{n}(\sqrt{n}+1)(2\sqrt{n}+1)}{6n} - \frac{\sqrt{n}c}{n} =$$

$$\frac{n}{2} + \frac{1}{2} + c + \frac{\sqrt{n}^3}{5} + \frac{n}{2} + \frac{\sqrt{n}}{3} - \frac{1}{30\sqrt{n}} - \frac{(n+\sqrt{n})(2\sqrt{n}+1)}{6n} - \frac{c}{\sqrt{n}} =$$

$$n + \frac{1}{2} + c + \frac{\sqrt{n}^3}{5} + \frac{\sqrt{n}}{3} - \frac{1}{30\sqrt{n}} - \frac{\sqrt{n}}{3} - \frac{1}{6\sqrt{n}} - \frac{1}{2} - \frac{c}{\sqrt{n}} =$$

$$n + c + \frac{\sqrt{n}^3}{5} - \frac{1}{30\sqrt{n}} - \frac{1}{6\sqrt{n}} - \frac{c}{\sqrt{n}}$$

Want to know the amortized cost as n approaches infinity, so we use limit to get the final result

$$\lim_{n\to\infty}(n + c + \frac{\sqrt{n}^3}{5} - \frac{1}{30\sqrt{n}} - \frac{1}{6\sqrt{n}} - \frac{c}{\sqrt{n}}) = \frac{\sqrt{n}^3}{5} = \sqrt{n}^3 = c_i'$$

The amortized cost for this function is $c_i' = \sqrt{n}^3$ where n is the number of calls of the function.

2 Amortization

In order to prove that even when we only expand the table by 10% amortized cost is still constant, I presumed it true, and worked my way from the end to the start while taking lectures as an example.

$$c_i' = c_i + \Phi(D_i) - \Phi(D_{i-1})$$
$$\Phi(D_i) = 11 \cdot num_i - 10 \cdot size_i$$

11 and 10 were selected as coefficients as they nicely resolve the equation (when I worked from my way from the end to start) and are basically just multiplied by 10 from the original equation $\Phi(D_i) = 1,1 \cdot num_i - size_i$ from lectures. They also satisfy the condition that $\Phi(D_i) \geq 0$

a) Expansion on i-th operation
   When expanding the following holds

$$size_i = 1.1 \cdot size_{i-1}$$
$$size_{i-1} = num_{i-1} = num_i - 1$$

   Now we prove that amortized cost is still constant

$$\begin{gathered} c_i' = c_i + \Phi(D_i) - \Phi(D_{i-1}) = \\ num_i + (11 \cdot num_i - 10 \cdot size_i) - (11 \cdot num_{i-1} - 10 \cdot size_{i-1}) = \\ 12num_i - 10(1.1 \cdot size_{i-1}) - 11(num_i - 1) + 10(num_i - 1) = \\ 12num_i - 11 \cdot size_{i-1} - 11num_i + 11 + 10num_i - 10 = \\ 11num_i - 11(num_i - 1) + 1 = 11num_i - 11num_i + 11 + 1 = 12 \end{gathered}$$

b) NO expansion on i-th operation
   When expanding the following holds

$$size_i = size_{i-1}$$
$$num_{i-1} = num_i - 1$$

   Now we prove that amortized cost is still constant

$$\begin{gathered} c_i' = c_i + \Phi(D_i) - \Phi(D_{i-1}) = \\ 1 + (11 \cdot num_i - 10 \cdot size_i) - (11 \cdot num_{i-1} - 10 \cdot size_{i-1}) = \\ 1 + 11 \cdot num_i - 10 \cdot size_i - 11(num_i - 1) + 10 \cdot size_i = \\ 1 + 11 \cdot num_i - 11 \cdot num_i + 11 = 12 \end{gathered}$$

3 Approximation

First we need to calculate the probability of satisfying one pair of symmetric clauses

$$P(clause_1 \wedge clause_2 = 1) = 1 - P(clause_1 \wedge clause_2 = 0)$$

There are 16 possible inputs to each for each clause, but only one where the output is 0 since each clause is in conjunctive normal form

$$P(\textit{clause is not satisfied}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$$

For $clause_1 \wedge clause_2$ to be 0, either has to be 0 and since they cannot be 0 at the same time (their terms are all negated), this means 2 of 16 possible inputs result in output being 0.

$$P(clause_1 \wedge clause_2 = 0) = \frac{2}{16}$$
$$P(clause_1 \wedge clause_2 = 1) = 1 - \frac{2}{16} = \frac{14}{16} = \frac{7}{8} \Rightarrow E(Y_i) = \frac{7}{8}$$

Now we can start finding the approximation factor for this algorithm

$$Y = \textit{number of satisfied pairs of clauses} = Y_1 + Y_2 + \cdots + Y_n$$
$$E(Y) = E(Y_1 + Y_2 + \cdots + Y_n) = E(\sum_{i=1}^{n} Y_i) = \sum_{i=1}^{n} E(Y_i) = \sum_{i=1}^{n} \frac{7}{8}$$
$$E(Y) = \frac{7}{8}n = C; C^* = n \Rightarrow \frac{C^*}{C} = \frac{n}{\frac{7}{8}n} = \frac{8}{7}$$