Patrolling Games

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Introduction to Patrolling Games

A Patrolling game, $G = G(Q, T, m)$ is made of 3 major components

- A Graph, $Q = (N, E)$, made of nodes, N ($|N| = n$), and a set of edges, E.
- A time horizon parameter, T (with set $\mathcal{T} = \{0, 1, ..., T-1\}$).
- An attack time parameter, m.

The game involves two players, the patroller and the attacker.

- The patroller's strategy is a walk (with waiting) on the graph, $W : \mathcal{T} \to N$.
- The attacker's strategy is a node, i and starting time, τ .

The strategies are collected into the sets, $\mathcal{W}$ and $\mathcal{A}$, for the patroller and attacker respectively, with some arbitrary labelling inside the set to form strategies W_i and A_j.

Payoffs

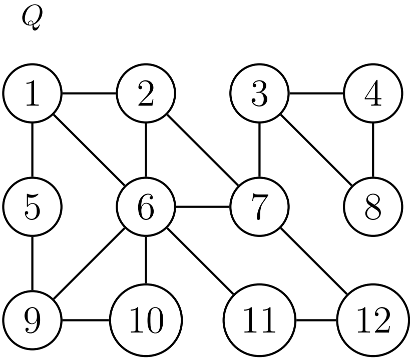
The game is formulated as win-lose (a zero-sum game) with a payoff for the patroller of

$$P(W,(i,\tau)) = \begin{cases} 1 \text{ if } i \in \{W(\tau), W(\tau+1), ..., W(\tau+m-1)\}, \\ 0 \text{ if } i \notin \{W(\tau), W(\tau+1), ..., W(\tau+m-1)\}. \end{cases}$$

With a pure payoff matrix $\mathcal{P} = (P(W_i, A_j))_{i \in \{1,...,|\mathcal{W}|\}, j \in \{1,...,|\mathcal{A}|\}}$

Example of a game

The game played on Q as below with $m = 3$ and $T = 7$

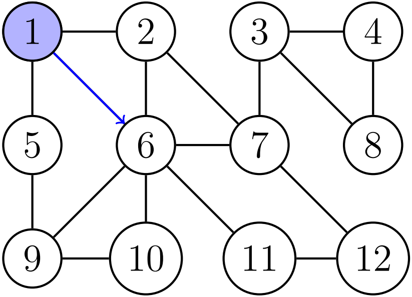


Patroller: $W(0) = 1$, $W(1) = 6$, $W(2) = 7$, $W(3) = 3$, $W(4) = 3$, $W(5) = 4$, $W(8) = 8$

Attacker: $(8, 2)$

The game played on Q as below with $m = 3$ and $T = 7$

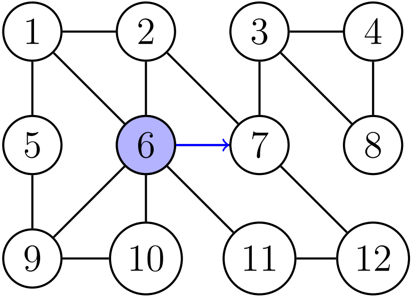
$Q \quad t = 0$



Example of a game

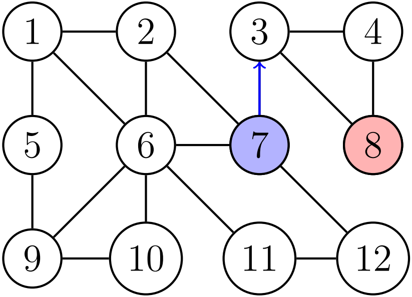
The game played on Q as below with $m = 3$ and $T = 7$

$Q \quad t = 1$



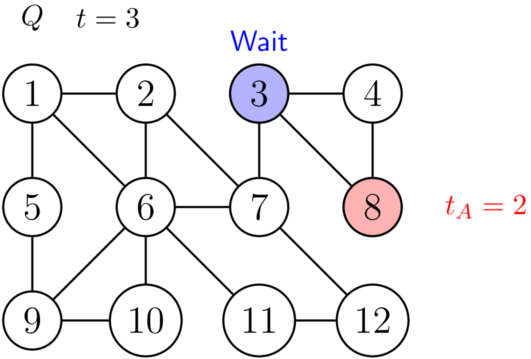
The game played on Q as below with $m = 3$ and $T = 7$

$Q \quad t = 2$



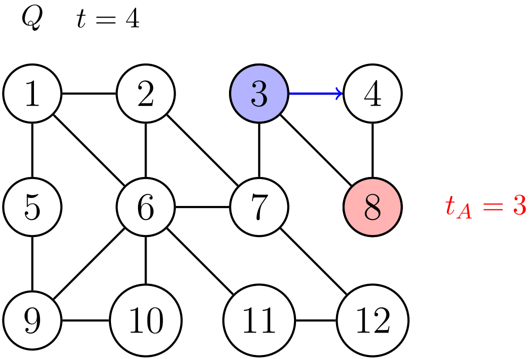
Attack Starting, $t_A = 1$

The game played on Q as below with $m = 3$ and $T = 7$



Example of a game

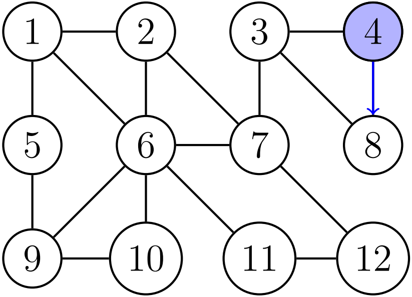
The game played on Q as below with $m = 3$ and $T = 7$



Example of a game

The game played on Q as below with $m = 3$ and $T = 7$

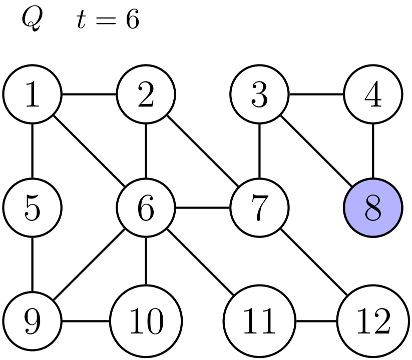
$Q \quad t = 5$



Attacker Done

Example of a game

The game played on Q as below with $m = 3$ and $T = 7$



The attacker fails to catch the patroller, therefore the patroller loses (and the attacker wins) meaning a payoff of 0 for the patroller (and -1 for the attacker).

Mixed games

Both the patroller and attacker will play their pure(realised) strategies with certain probabilities, let $\boldsymbol{\pi}$ be a mixed strategy for the patroller and let $\boldsymbol{\phi}$ be a mixed strategy for the attacker. We collect these into the sets Π and Φ for the patroller and attacker respectively.
Then the payoff for the patroller of this mixed game becomes

$$P(\boldsymbol{\pi}, \boldsymbol{\phi}) = \sum_{i=1}^{|\mathcal{W}|} \sum_{j=1}^{|\mathcal{I}|} \mathcal{P}_{i,j} \boldsymbol{\pi}_i \boldsymbol{\phi}_j = \boldsymbol{\pi} \mathcal{P} \boldsymbol{\phi}$$

By using the pure payoff as 1 when capture occurs and 0 otherwise, the mixed payoff is equivalent to the probability of capture.

Equilibrium and Value

Mixed Nash equilibrium

A choice of $\boldsymbol{\pi}^*$ and $\boldsymbol{\phi}^*$ is said to be in *Nash equilibrium* if

$$P(\boldsymbol{\pi}^*, \boldsymbol{\phi}^*) \geq P(\boldsymbol{\pi}, \boldsymbol{\phi}^*) \quad \forall \boldsymbol{\pi} \in \Pi,$$
$$P(\boldsymbol{\pi}^*, \boldsymbol{\phi}^*) \geq P(\boldsymbol{\pi}^*, \boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in \Phi.$$

There will only be one Nash equilibrium, unless the patroller can guarantee capture.
We do this by searching for the games value,

$$V(G) \equiv \max_{\boldsymbol{\pi} \in \Pi} \min_{\boldsymbol{\phi} \in \Phi} P(\boldsymbol{\pi}, \boldsymbol{\phi}) = \min_{\boldsymbol{\phi} \in \Phi} \max_{\boldsymbol{\pi} \in \Pi} P(\boldsymbol{\pi}, \boldsymbol{\phi})$$

This is done by achieving both upper and lower bounds on the value of the game.

Solved graphs:Hamiltonian, Complete Bipartite and Star

Hamiltonian graphs

A Hamiltonian graph has the value $V = \frac{m}{n}$

Bipartite graph

A complete bipartite graph, $K_{a,b}$ as value

$$V = \frac{m}{2\max(a,b)}$$

Star graph

The star $S_n \equiv K_{1,n}$ so has the value $V = \frac{m}{2n}$

Line graph

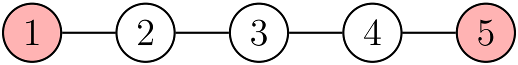
The line graph, L_n made of n nodes has a value dependent on (n, m)

1. If $m > 2(n-1)$ then $V = 1$.
2. If $n-1 \leq m \leq 2(n-1)$ then $V = \frac{m}{2(n-1)}$
3. If $m = 2, n \geq 3$ then $V = \frac{1}{\lceil \frac{n}{2} \rceil}$
4. If $m = n-1$ or $m = n-2$ and $m = 2k$ for some $k \geq 2$ then $V = \frac{1}{2}$
5. If $m \leq n-3$ or $m = n-2$ and $m = 2k+1$ for some $k \geq 1$ then $V = \frac{m}{m+n-1}$

Correction to Line graph

It was initially state that in the region two $n-1 \leq m \leq 2(n-1)$, the value $\frac{m}{2(n-1)}$ is achieved by the attacker attacking using the diametric strategy, which is to attack at opposite ends of the line with equal probability for all possible starting times. This is supposed to guarantee that $V \leq \frac{m}{2(n-1)}$.

Example. Consider L_5 with $T = m = 5$, then the patroller only needs to walk between the end nodes to win.



The walk $\{1, 2, 3, 4, 5\}$ guarantee's the capture of all attacks made.

Example. Consider L_{31} with $m = 45$

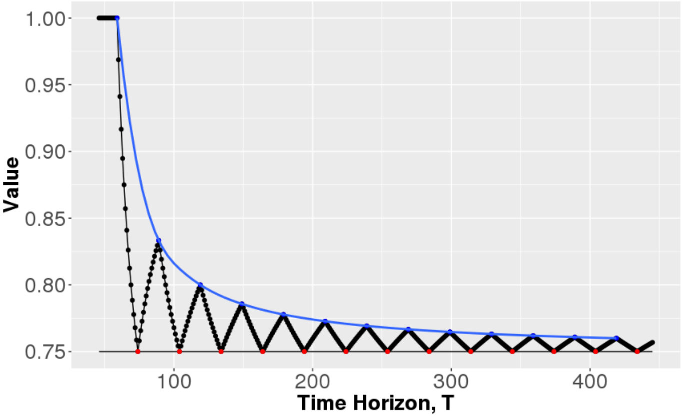


Figure: Best Upper Bound achievable under the diametric strategy

Limiting result

Lemma (Condition on T for bound to hold)

When $T = m - 1 + (k+1)(n-1)$ for some $k \in \mathbb{N}_0$ then the diametric bound holds. Otherwise as $T \to \infty$ then the diametric bound ($V \leq \frac{m}{2(n-1)}$) holds.

Solution to diametric bound line problem

We propose a solution to the problem, by limiting the time

Definition (Time limited diametric attack)

When $T \geq m + n - 2$ we have the *time limited diametric attack* (on the line) strategy is for the attacker to attack at both ends of the line with equal probability for starting times $0, 1, ..., n - 2$

This restriction to the attacking time guarantees to get the sought upper bound of $V \leq \frac{m}{2(n-1)}$.
This works due to the spacing of the attack times.

General time limited diametric attack

Let $\bar{d}$ be the diameter of the graph, i.e the distance between the furthest apart nodes then using

Definition (Time limited diametric attack)

When $T \geq m - 1 + \bar{d}$ we have the *time limited diametric attack* strategy is for the attacker to attack at both ends of the line with equal probability for starting times $\tau, \tau + 1, ..., \tau + \bar{d} - 1$ (for a chosen initial τ).

Lemma (Time limited diametric bound)

When $T \geq m - 1 + \bar{d}$ the attacker can get the bound

$$V \leq \frac{m}{2\bar{d}}.$$

Polygonal attack

Using $d(i, j)$ as the distance between nodes i and j we can get a more general type of attack.

Definition (Polygonal attack)

A d-*polygonal attack* is an attack at a set of nodes $D = \{i \in N \mid d(i, i') = d\}$ at the time intervals $\tau, \tau + 1, ..., \tau + d - 1$ (for a chosen initial τ) all equally probable.

Lemma (Polygonal bound)

When $T \geq m + d - 1$ *and a set* D *as in the* d*-polygonal attack, the bound* $V \leq \max\{\frac{1}{|D|}, \frac{m}{|D|d}\}$ *is valid.*

Mixing Lines and stars

We now wish to integrate features of a star into a line. We will form the elongated star S_n^k. We will use the labelling as below

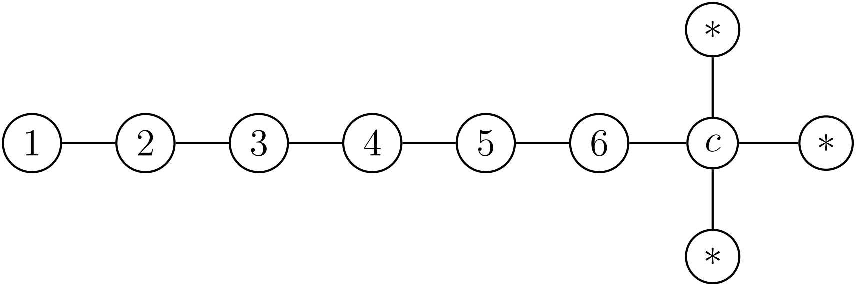


Figure: Labeling on the graph S_4^5.