Patrolling games on graphs

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Outline

- Literature review
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    - Pure game
    - Mixed game
  - Solved Graphs
    - Hamiltonian graphs
    - Line graph
- Problem with line graph strategy
- Correction of line graph strategy
- Extension of correction strategy
- Future work

Pure game

A Patrolling game, $G = G(Q, T, m)$ is made of 3 major components

- A Graph, $Q = (N, E)$, made of nodes, N ($|N| = n$), and a set of edges, E.
- A time horizon parameter, T (with set $\mathcal{T} = \{0, 1, ..., T-1\}$).
- An attack length parameter, m.

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The game involves two players, the patroller and the attacker.

- The patroller's strategy is a walk (with waiting) on the graph, $W : \mathcal{T} \to N$.
- The attacker's strategy is a node, i and starting time, τ.

The strategies are collected into the sets, $\mathcal{W}$ and $\mathcal{A}$, for the patroller and attacker respectively, with some arbitrary labelling inside the set to form strategies W_i and A_j.

Payoff

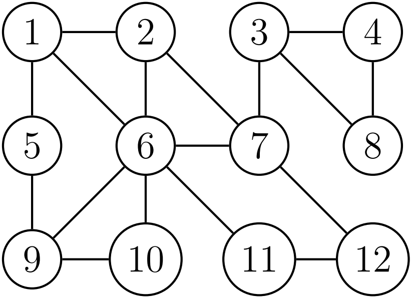
The game is formulated as win-lose (a zero-sum) game with a payoff for the patroller of

$$P(W,(i,\tau)) = \begin{cases} 1 \text{ if } i \in \{W(\tau), W(\tau+1), ..., W(\tau+m-1)\}, \\ 0 \text{ if } i \notin \{W(\tau), W(\tau+1), ..., W(\tau+m-1)\}. \end{cases}$$

With a pure payoff matrix $\mathcal{P} = (P(W_i, A_j))_{i\in\{1,...,|\mathcal{W}|\}, j\in\{1,...,|\mathcal{A}|\}}$
Note. The attackers payoff is simply negative the patroller payoff.

Example of a pure game

The game played on Q as below with $m = 3$ and $T = 7$

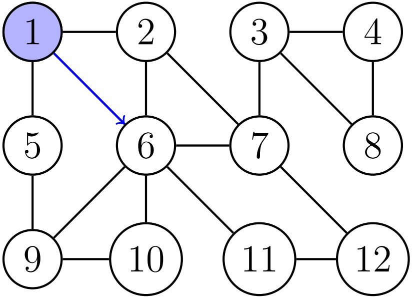


Patroller: $W(0) = 1$, $W(1) = 6$, $W(2) = 7$, $W(3) = 3$, $W(4) = 3$, $W(5) = 4$, $W(8) = 8$

Attacker: $(8, 2)$

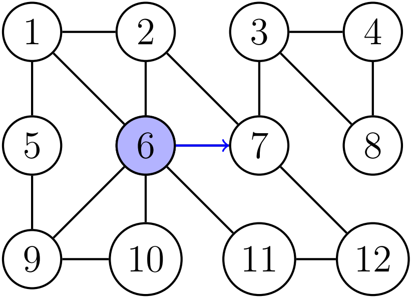
The game played on Q as below with $m = 3$ and $T = 7$

$$t = 0$$



The game played on Q as below with $m = 3$ and $T = 7$

$$t = 1$$



Example of a pure game

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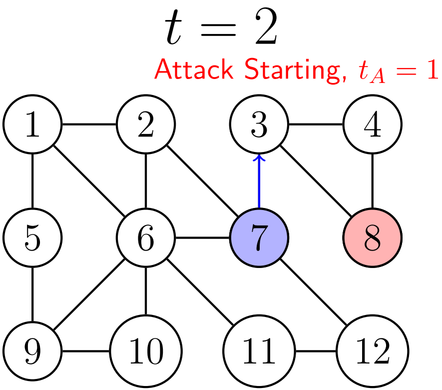
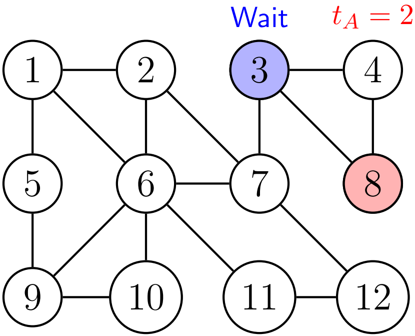


Example of a pure game

The game played on Q as below with $m = 3$ and $T = 7$

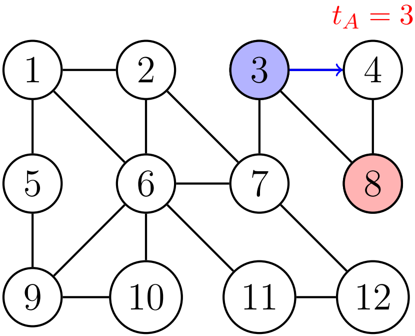
$$t = 3$$



Example of a pure game

The game played on Q as below with $m = 3$ and $T = 7$

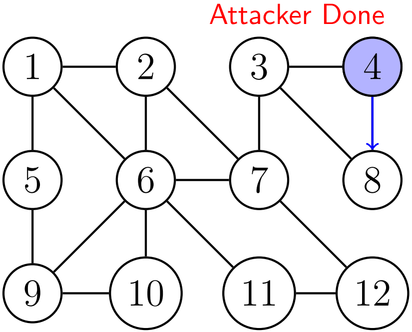
$$t = 4$$



Example of a pure game

The game played on Q as below with $m = 3$ and $T = 7$

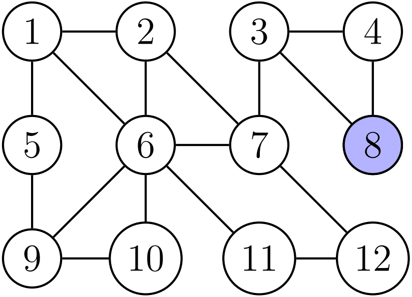
$$t = 5$$



Example of a pure game

The game played on Q as below with $m = 3$ and $T = 7$

$$t = 6$$



The attacker fails to catch the patroller, therefore the patroller loses (and the attacker wins) meaning a payoff of 0 for the patroller (and -1 for the attacker).

Mixed game formulation

Both the patroller and attacker will play their pure strategies with certain probabilities, let π be a mixed strategy for the patroller and let ϕ be a mixed strategy for the attacker. We collect these into the sets Π and Φ for the patroller and attacker respectively.

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Then the payoff for the patroller of this mixed game becomes

$$P(\boldsymbol{\pi}, \boldsymbol{\phi}) = \sum_{i=1}^{|\mathcal{W}|} \sum_{j=1}^{|\mathcal{I}|} \mathcal{P}_{i,j} \boldsymbol{\pi}_i \boldsymbol{\phi}_j = \boldsymbol{\pi} \mathcal{P} \boldsymbol{\phi}$$

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By using the pure payoff as 1 when capture occurs and 0 otherwise, the mixed payoff is equivalent to the probability of capture.
By standard game theory results, we define the value of the game to be

$$V(G) \equiv \max_{\boldsymbol{\pi} \in \Pi} \min_{\boldsymbol{\phi} \in \Phi} P(\boldsymbol{\pi}, \boldsymbol{\phi}) = \min_{\boldsymbol{\phi} \in \Phi} \max_{\boldsymbol{\pi} \in \Pi} P(\boldsymbol{\pi}, \boldsymbol{\phi})$$

and we will seek its value by getting upper and lower bounds.

A graph is Hamiltonian if it is possible to find a cycle which visits every node exactly once (apart from the start/finish).

Hamiltonian graphs

A Hamiltonian graph has the value $V = \frac{m}{n}$

Two common Hamiltonian graphs are the Cyclic graph (of n nodes C_n) and the Complete graph (of n nodes K_n).

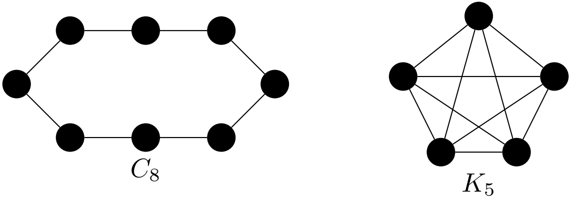

Figure: Examples of Cyclic and Complete graphs

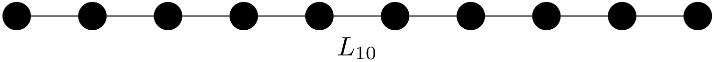

Figure: Example of a line graph

The regions are:

$m > 18$

$9 < m \leq 18$

$m = 2$

$m = 9, 8$

$3 \leq m < 8$, $m = 1$

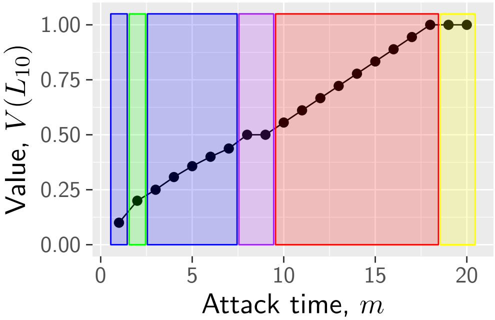

Figure: Value of the line graph, L_{10}

Strategies used in the second region

Focusing in on $n-1 < m \leq 2(n-1)$. We will look at the strategies used to get the bounds $V \leq \frac{m}{2(n-1)}$ and $V \geq \frac{m}{2(n-1)}$.

- Patroller Strategy, $\boldsymbol{\pi}_H$, the embedded random Hamiltonian patrol.
- Attacker Strategy, ϕ_D, the diametric attack.

Embedded patrols

An embedded random Hamiltonian patrol, $\boldsymbol{\pi}_H$, is made by 'expanding' the line to be Hamiltonian (meaning every non-end node becomes two nodes).

That is the patroller looks at $C_{2(n-1)}$ instead of L_n, then we get a bound of $V(C_{2(n-1)}) = \frac{m}{2(n-1)}$. Now the patroller cannot do worse in L_n than in $C_{2(n-1)}$, so a lower bound of $V \geq \frac{m}{2(n-1)}$ is achieved.

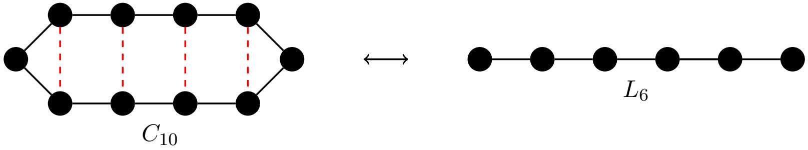

Figure: C_{10} and L_6 relation by node expanding and simplification.

Diametric attack

Let $d(i, i')$ be the distance between nodes i and i' with the distance measured by the minimum number of edges.

Definition (Graph Diameter)

The diameter of a graph Q is defined by $\bar{d} = \max_{i,i' \in N} d(i, i')$. The node pairs, (i, i'), satisfying this are called diametrical.

A diametric attack, ϕ_D is made by attacking the pair of diametric nodes 1 and n (the ends), starting with equal probability at every available start time. It is stated to give

$$V \leq \min\{\tfrac{1}{2}, \tfrac{m}{2(n-1)}\} = \begin{cases} \frac{1}{2}, \text{ if } m < n-1 \\ \frac{m}{2(n-1)}, \text{ if } n-1 \leq m \leq 2(n-1), \end{cases}$$

however....

Counter-example for diametric attack

In the region of $n-1 \leq m \leq 2(n-1)$ the proposed bound is $V \leq \frac{m}{2(n-1)}$. However a simple counter-example shows the bound to be false.

Counter-example. Consider L_5 with $T = m = 5$, then the patroller only needs to walk between the end nodes to win.

The walk $\{1, 2, 3, 4, 5\}$ guarantees the capture of all attacks made.

Problem with the diametric strategy

Example. Consider L_{31} with $m = 45$

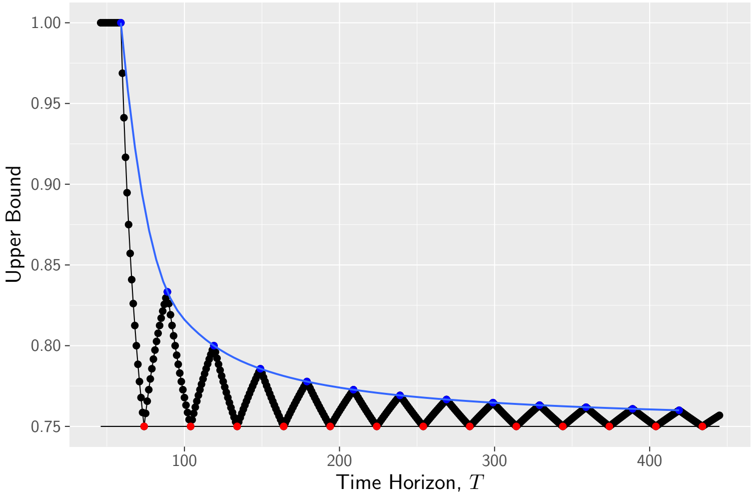

Figure: Best Upper Bound achievable under the diametric strategy

Condition for diametric bound to hold

The problem is that under the diametric attack, the upper bound is provided by the maximum number of attacks the patroller can catch. However unless this is a particular collection we are able to catch more than the bound we want to use.

Lemma (Condition on T for bound to hold)

When $T = m - 1 + (k+1)(n-1)$ for some $k \in \mathbb{N}_0$ then the diametric bound, $V \leq \frac{m}{2(n-1)}$ holds. Otherwise as $T \to \infty$ then the diametric bound, $V \leq \frac{m}{2(n-1)}$, holds.

We now propose a solution to the problem, by limiting the time.

Definition (Time limited diametric attack)

When $T \geq m + n - 2$ the *time limited diametric attack* (on the line) strategy is for the attacker to attack at both ends of the line with equal probability for starting times $0, 1, ..., n-2$.

This restriction on the attacking time guarantees to get the upper bound of $V \leq \frac{m}{2(n-1)}$.

Extension to time-limited diametric strategy

Definition (Time limited diametric attack)

When $T \geq m - 1 + \bar{d}$ the *time limited diametric attack* strategy is for the attacker to attack at both nodes of the diametric pair with equal probability for starting times $\tau, \tau + 1, ..., \tau + \bar{d} - 1$ (for a chosen initial $\tau \leq T + 1 - \bar{d} - m$).

Lemma (Time limited diametric bound)

When $T \geq m - 1 + \bar{d}$ the attacker can get the upper bound

$$V \leq \frac{m}{2\bar{d}}.$$

Polygonal attack

Definition (Polygonal attack)

A d-*polygonal attack* is an attack at a set of nodes $D = \{i \in N \mid d(i,j) = d, \forall j \in D\}$ at the starting times $\tau, \tau+1, ..., \tau+d-1$ (for a chosen initial $\tau \leq T+1-d-m$) all equally probably.

Lemma (Polygonal bound)

When $T \geq m+d-1$ and a set D (as in the d-polygonal attack) exists, the value has an upper bound of $V \leq \max\{\frac{1}{|D|}, \frac{m}{|D|d}\}$.

Example.

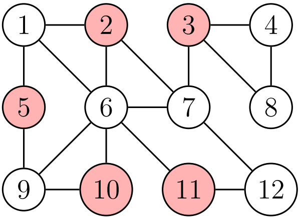

Figure: 2-polygonal attack on $D = \{2, 3, 5, 10, 11\}$

Giving $V \leq \max\{\frac{1}{5}, \frac{m}{10}\}$.

Future Work

- Look at analysing different types of Polygonal attacks, i.e the best choice of d and how to select the set D.
- Look at analysing the elongated star graph, that is the star graph with one of the end points further away from the centre.
- Alter the problem to have 'semi-random' attackers arriving on predetermined sections of the graph.

Nash equilibria and optimal solutions

Definition (Mixed Nash equilibrium)

A choice of $\boldsymbol{\pi}^*$ and ϕ^* is said to be in *Nash equilibrium* if

$$P(\boldsymbol{\pi}^*, \boldsymbol{\phi}^*) \geq P(\boldsymbol{\pi}, \boldsymbol{\phi}^*) \quad \forall \boldsymbol{\pi} \in \Pi,$$
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Note. It is possible that no optimal solution exists
Luckily in zero-sum games are guaranteed to to have at least one optimal solution and any nash-equilbrium is optimal.

Game Value

Theorem (Minimax Theorem (John Neumann, 1928))

Let $X \subset \mathbb{R}^n$ *and* $Y \subset \mathbb{R}^m$ *be compact convex sets and let* $f : X \times Y \to \mathbb{R}$ *be a continuous convex-concave function then*

$$\min_{x \in X} \max_{y \in Y} f(x, y) = \max_{y \in Y} \min_{x \in X} f(x, y)$$

Note. Convex-concave means convex for a fixed y and concave for a fixed x.

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Now Π and Φ are both compact convex sets and as our game is zero-sum our payoff function $P(\boldsymbol{\pi}, \boldsymbol{\phi})$ is convex-concave. Solving the minimax problem is equivalent to Nash-equilibrium and are hence optimal solutions.

This gives rise to the game's value, as given by

$$V(G) \equiv \max_{\boldsymbol{\pi} \in \Pi} \min_{\boldsymbol{\phi} \in \Phi} P(\boldsymbol{\pi}, \boldsymbol{\phi}) = \min_{\boldsymbol{\phi} \in \Phi} \max_{\boldsymbol{\pi} \in \Pi} P(\boldsymbol{\pi}, \boldsymbol{\phi})$$