Patrolling Games

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A Patrolling game, $G = G(Q, T, m)$ is made of 3 major components

- A Graph, $Q = (N, E)$, made of nodes, N ($|N| = n$), and a set of edges, E.
- A time horizon parameter, T (with set $\mathcal{T} = \{0, 1, ..., T-1\}$).
- An attack time parameter, m.

The game involves two players, the patroller and the attacker.

- The patroller's strategy is a walk (with waiting) on the graph, $W : \mathcal{T} \rightarrow N$.
- The attacker's strategy is a node, i and starting time, τ .

The strategies are collected into the sets, $\mathcal{W}$ and $\mathcal{A}$, for the patroller and attacker respectively, with some arbitrary labelling inside the set to form strategies W_i and A_j.

Payoffs

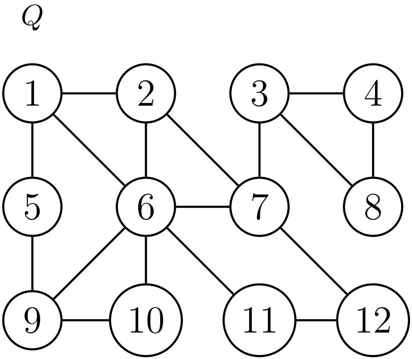
The game is formulated as win-lose (a zero-sum game) with a payoff for the patroller of

$$P(W,(i,\tau)) = \begin{cases} 1 \text{ if } i \in \{W(\tau), W(\tau+1), ..., W(\tau+m-1)\}, \\ 0 \text{ if } i \notin \{W(\tau), W(\tau+1), ..., W(\tau+m-1)\}. \end{cases}$$

With a pure payoff matrix $\mathcal{P} = (P(W_i, A_j))_{i \in \{1,...,|\mathcal{W}|\}, j \in \{1,...,|\mathcal{A}|\}}$

Example of a game

The game played on Q as below with $m = 3$ and $T = 7$



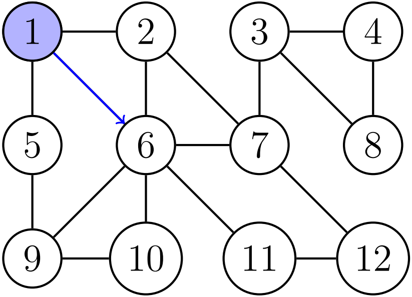
Patroller: $W(0) = 1$, $W(1) = 6$, $W(2) = 7$, $W(3) = 3$, $W(4) = 3$, $W(5) = 4$, $W(8) = 8$

Attacker: $(8, 2)$

Example of a game

The game played on Q as below with $m = 3$ and $T = 7$

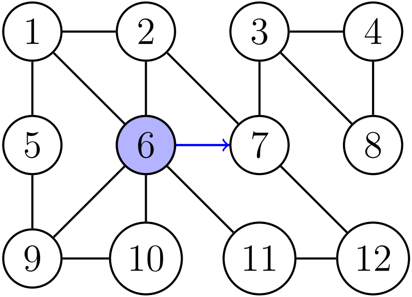
$Q \quad t = 0$



Example of a game

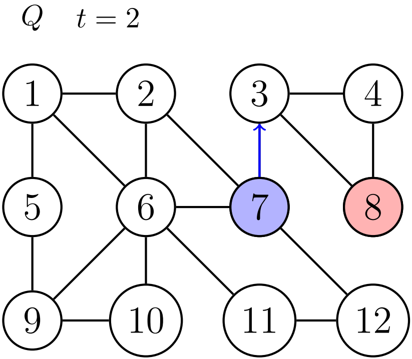
The game played on Q as below with $m = 3$ and $T = 7$

$Q \quad t = 1$



Example of a game

The game played on Q as below with $m = 3$ and $T = 7$

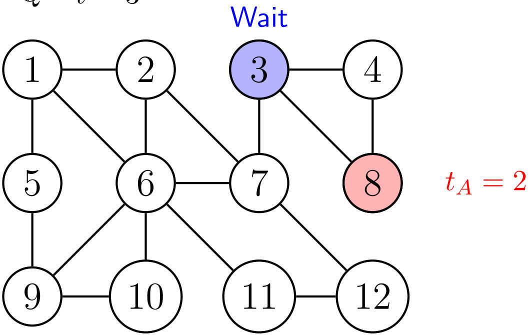


Attack Starting, $t_A = 1$

Example of a game

The game played on Q as below with $m = 3$ and $T = 7$

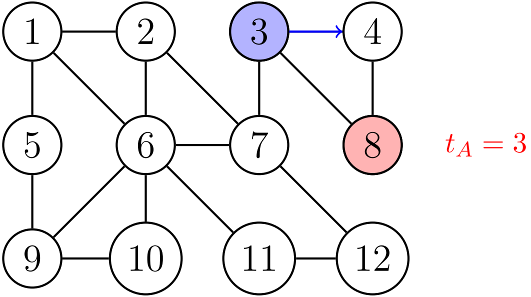
$Q \quad t = 3$



Example of a game

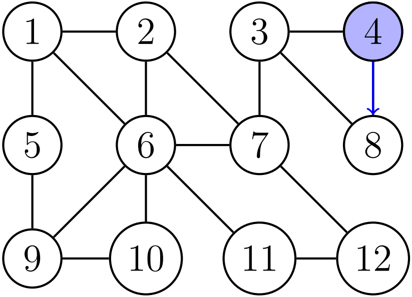
The game played on Q as below with $m = 3$ and $T = 7$

$Q \quad t = 4$



The game played on Q as below with $m = 3$ and $T = 7$

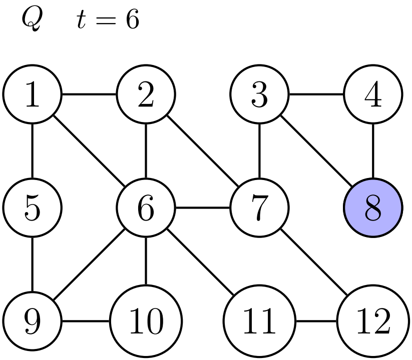
$Q \quad t = 5$



Attacker Done

Example of a game

The game played on Q as below with $m = 3$ and $T = 7$



The attacker fails to catch the patroller, therefore the patroller loses (and the attacker wins) meaning a payoff of 0 for the patroller (and -1 for the attacker).

Mixed games

Both the patroller and attacker will play their pure(realised) strategies with certain probabilities, let $\boldsymbol{\pi}$ be a mixed strategy for the patroller and let $\boldsymbol{\phi}$ be a mixed strategy for the attacker. We collect these into the sets Π and Φ for the patroller and attacker respectively.
Then the payoff for the patroller of this mixed game becomes

$$P(\boldsymbol{\pi}, \boldsymbol{\phi}) = \sum_{i=1}^{|\mathcal{W}|} \sum_{j=1}^{|\mathcal{I}|} \mathcal{P}_{i,j} \boldsymbol{\pi}_i \boldsymbol{\phi}_j = \boldsymbol{\pi} \mathcal{P} \boldsymbol{\phi}$$

By using the pure payoff as 1 when capture occurs and 0 otherwise, the mixed payoff is equivalent to the probability of capture.

Equilibrium and Value

Mixed Nash equilibrium

A choice of $\boldsymbol{\pi}^*$ and ϕ^* is said to be in *Nash equilibrium* if

$$P(\boldsymbol{\pi}^*, \boldsymbol{\phi}^*) \geq P(\boldsymbol{\pi}, \boldsymbol{\phi}^*) \quad \forall \boldsymbol{\pi} \in \Pi,$$
$$P(\boldsymbol{\pi}^*, \boldsymbol{\phi}^*) \geq P(\boldsymbol{\pi}^*, \boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in \Phi.$$

There will only be one Nash equilibrium, unless the patroller can guarantee capture.
We do this by searching for the games value,

$$V(G) \equiv \max_{\boldsymbol{\pi} \in \Pi} \min_{\boldsymbol{\phi} \in \Phi} P(\boldsymbol{\pi}, \boldsymbol{\phi}) = \min_{\boldsymbol{\phi} \in \Phi} \max_{\boldsymbol{\pi} \in \Pi} P(\boldsymbol{\pi}, \boldsymbol{\phi})$$

This is done by achieving both upper and lower bounds on the value of the game.

Solved graphs:Hamiltonian

Hamiltonian graphs

A Hamiltonian graph has the value $V = \frac{m}{n}$

Solved graphs:Bipartite

Bipartite graph

A bipartite graph with no-adjacency partition into sets A and B has the value $V = \frac{m}{2(\max(|A|,|B|))}$

Solved graphs:Star

Star graph

The star $S_n \equiv K_{1,n}$ so has the value $V = \frac{m}{2n}$

Solved graphs:Line

Line graph

The line graph, L_n made of n nodes has a value dependent on (n, m)

1. If $m > 2(n-1)$ then $V = 1$.
2. If $n - 1 \leq m \leq 2(n-1)$ then $V = \frac{m}{2(n-1)}$
3. If $m = 2, n \geq 3$ then $V = \frac{1}{\lceil \frac{n}{2} \rceil}$
4. If $m = n - 1$ or $m = n - 2$ and $m = 2k$ for some $k \geq 2$ then $V = \frac{1}{2}$
5. If $m \leq n - 3$ or $m = n - 2$ and $m = 2k + 1$ for some $k \geq 1$ then $V = \frac{m}{m+n-1}$