Patrolling Games

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December 13, 2017

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- Problem with line graph strategy
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Introduction to Game: Pure game

A Patrolling game, $G = G(Q, T, m)$ is made of 3 major components

- A Graph, $Q = (N, E)$, made of nodes, N ($|N| = n$), and a set of edges, E.
- A time horizon parameter, T (with set $\mathcal{T} = \{0, 1, ..., T-1\}$).
- An attack time parameter, m.

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The game involves two players, the patroller and the attacker.

- The patroller's strategy is a walk (with waiting) on the graph, $W : \mathcal{T} \to N$.
- The attacker's strategy is a node, i and starting time, τ .

The strategies are collected into the sets, $\mathcal{W}$ and $\mathcal{A}$, for the patroller and attacker respectively, with some arbitrary labelling inside the set to form strategies W_i and A_j.

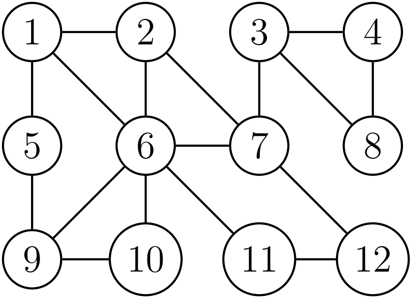
Introduction to Game: Pure game

The game is formulated as win-lose (a zero-sum game) with a payoff for the patroller of

$$P(W,(i,\tau)) = \begin{cases} 1 \text{ if } i \in \{W(\tau), W(\tau+1), ..., W(\tau+m-1)\}, \\ 0 \text{ if } i \notin \{W(\tau), W(\tau+1), ..., W(\tau+m-1)\}. \end{cases}$$

With a pure payoff matrix $\mathcal{P} = (P(W_i, A_j))_{i \in \{1,...,|\mathcal{W}|\}, j \in \{1,...,|\mathcal{A}|\}}$

The game played on Q as below with $m = 3$ and $T = 7$

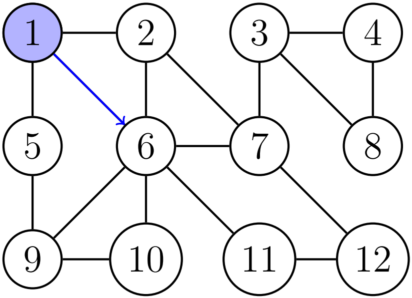


Patroller: $W(0) = 1$, $W(1) = 6$, $W(2) = 7$, $W(3) = 3$, $W(4) = 3$, $W(5) = 4$, $W(8) = 8$

Attacker: $(8, 2)$

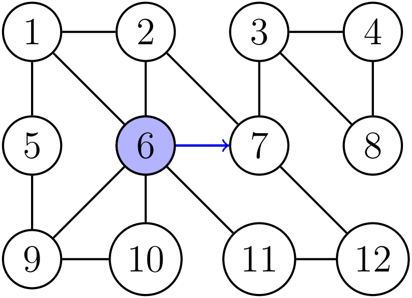
The game played on Q as below with $m = 3$ and $T = 7$

$$t = 0$$



The game played on Q as below with $m = 3$ and $T = 7$

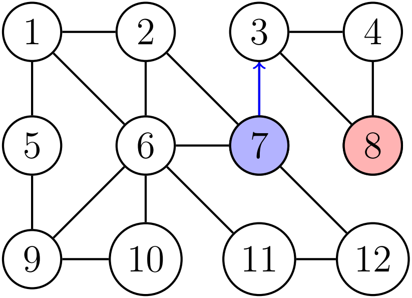
$$t = 1$$



The game played on Q as below with $m = 3$ and $T = 7$

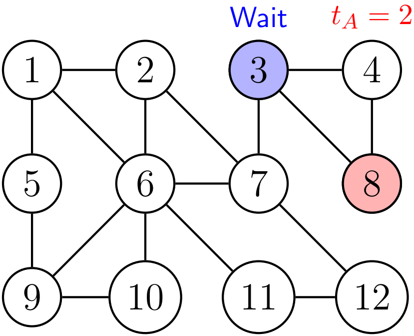
$$t = 2$$

Attack Starting, $t_A = 1$



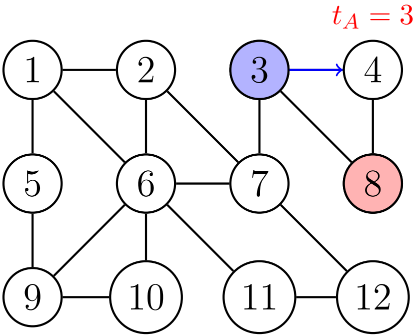
The game played on Q as below with $m = 3$ and $T = 7$

$$t = 3$$



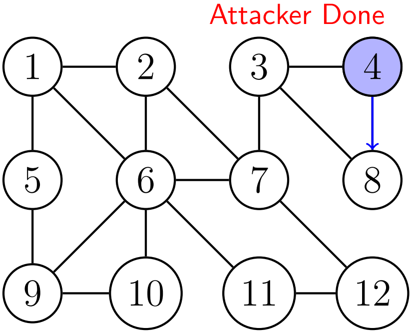
The game played on Q as below with $m = 3$ and $T = 7$

$$t = 4$$



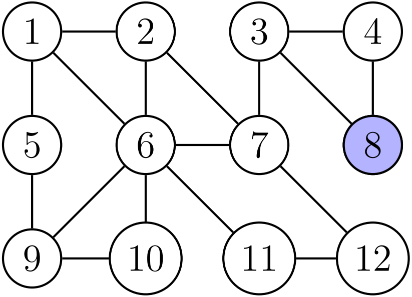
The game played on Q as below with $m = 3$ and $T = 7$

$$t = 5$$



The game played on Q as below with $m = 3$ and $T = 7$

$$t = 6$$



The attacker fails to catch the patroller, therefore the patroller loses (and the attacker wins) meaning a payoff of 0 for the patroller (and -1 for the attacker).

Both the patroller and attacker will play their pure(realised) strategies with certain probabilities, let $\boldsymbol{\pi}$ be a mixed strategy for the patroller and let ϕ be a mixed strategy for the attacker. We collect these into the sets Π and Φ for the patroller and attacker respectively.
Then the payoff for the patroller of this mixed game becomes

$$P(\boldsymbol{\pi}, \phi) = \sum_{i=1}^{|\mathcal{W}|} \sum_{j=1}^{|\mathcal{I}|} \mathcal{P}_{i,j} \boldsymbol{\pi}_i \phi_j = \boldsymbol{\pi} \mathcal{P} \boldsymbol{\phi}$$

By using the pure payoff as 1 when capture occurs and 0 otherwise, the mixed payoff is equivalent to the probability of capture.

Mixed Nash equilibrium

A choice of $\boldsymbol{\pi}^*$ and ϕ^* is said to be in *Nash equilibrium* if

$$P(\boldsymbol{\pi}^*, \boldsymbol{\phi}^*) \geq P(\boldsymbol{\pi}, \boldsymbol{\phi}^*) \quad \forall \boldsymbol{\pi} \in \Pi,$$
$$P(\boldsymbol{\pi}^*, \boldsymbol{\phi}^*) \geq P(\boldsymbol{\pi}^*, \boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in \Phi.$$

There will only be one Nash equilibrium, unless the patroller can guarantee capture.
We do this by searching for the games value,

$$V(G) \equiv \max_{\boldsymbol{\pi} \in \Pi} \min_{\boldsymbol{\phi} \in \Phi} P(\boldsymbol{\pi}, \boldsymbol{\phi}) = \min_{\boldsymbol{\phi} \in \Phi} \max_{\boldsymbol{\pi} \in \Pi} P(\boldsymbol{\pi}, \boldsymbol{\phi})$$

This is done by achieving both upper and lower bounds on the value of the game.

Solved graphs: Hamiltonian graphs

A graph is Hamiltonian if it is possible to find a cycle which visits every node exactly one (apart from the start/finish).

Hamiltonian graphs

A Hamiltonian graph has the value $V = \frac{m}{n}$

Two common Hamiltonian graphs are the Cyclic graph (of n nodes C_n) and the Complete graph (of n nodes K_n).

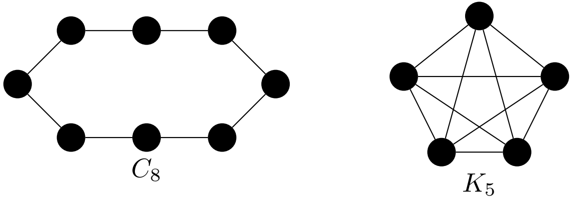
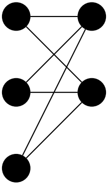


Figure: Examples of Cyclic and Complete graphs

A bipartite graph is a graph made of two non-adjacent sets, the complete version has all connections.

Complete bipartite graph

A complete bipartite graph, $K_{a,b}$ as value $V = \frac{m}{2\max(a,b)}$



$K_{3,2}$

Figure: Example of a complete bipartite graph

The star graph, S_n, is n nodes adjacent only to the centre.

Star graph

The star $S_n \equiv K_{1,n}$ so has the value $V = \frac{m}{2n}$

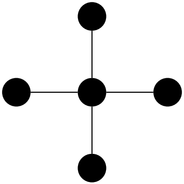


Figure: Example of a star graph

The line graph, L_n, made of n nodes each adjacent to two other nodes (apart from the ends)

Line graph

The line graph, L_n has a value dependent on (n, m)

1. If $m > 2(n-1)$ then $V = 1$.
2. If $n-1 < m \leq 2(n-1)$ then $V = \frac{m}{2(n-1)}$
3. If $m = 2, n \geq 3$ then $V = \frac{1}{\lceil \frac{n}{2} \rceil}$
4. If $m = n-1$ or $m = n-2$ and $m = 2k$ for some $k \geq 2$ then $V = \frac{1}{2}$
5. If $3 \leq m \leq n-3$ or $m = n-2$ and $m = 2k+1$ for some $k \geq 1$ then $V = \frac{m}{m+n-1}$

Note. The solution for $m = 1$ is know for every graph as $V = \frac{1}{|N|} = \frac{1}{n}$, and if $m = n = 2$ then we know $V = 1$.

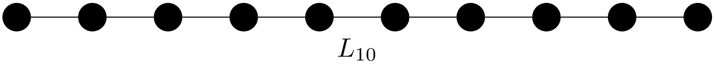


Figure: Example of a line graph

Regions are:

1. $m > 18$
2. $9 < m \leq 18$
3. $m = 2$
4. $m = 9, 8$
5. $3 \leq m < 8$, $m = 1$

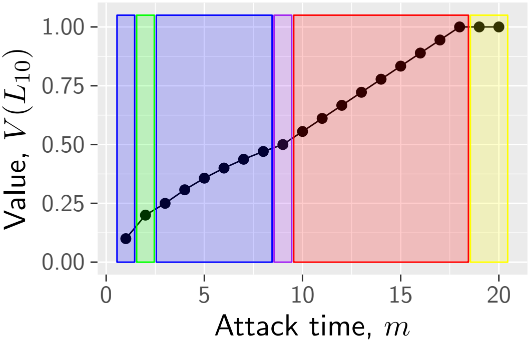


Figure: Value of the line graph, L_{10}

Solved graphs: Line graph

Focusing in on $n-1 < m \leq 2(n-1)$. We will look at the strategies used to get the bounds $V \leq \frac{m}{2(n-1)}$ and $V \geq \frac{m}{2(n-1)}$.

- Patroller Strategy, $\boldsymbol{\pi}_H$, the embedded random Hamiltonian patrol.
- Attacker Strategy, ϕ_D, the diametric attack.

An embedded random Hamiltonian patrol, $\boldsymbol{\pi}_H$, is made by 'expanding' the line to be Hamiltonian (meaning every non-end node becomes two nodes). That is the patroller looks at $C_{2(n-1)}$ instead of L_n, then we get a bound of $V(C_{2(n-1)}) = \frac{m}{2(n-1)}$. Now the patroller cannot do worse in L_n than in $C_{2(n-1)}$, so a lower bound of $V \geq \frac{m}{2(n-1)}$ is achieved. In the line graph this is also known as oscillation.

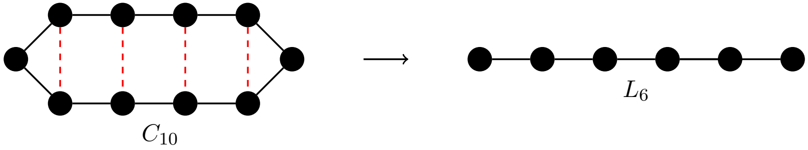


Figure: C_{10} can be simplified to L_6 by node identifying.

Solved graphs: Line graph

Let $d(i, i')$ is the distance between nodes i and i' with the distance measured by the minimum number of edges.

Definition (Graph Diameter)

The diameter of a graph Q is definded by $\bar{d} = \max_{i,i' \in N} d(i, i')$. The node pairs satisfying this are called diametrical.

A diametric attack, ϕ_D is made by attacking the pair of diametric nodes 1 and n (the ends), starting with equal probability at every available star time. It is stated to give a bound of

$$V \leq \min\{\tfrac{1}{2}, \tfrac{m}{2(n-1)}\} = \begin{cases} \frac{1}{2}, \text{ if } m < n-1 \\ \frac{m}{2(n-1)}, \text{ if } n-1 \leq m \leq 2(n-1), \end{cases}$$

however....

Problem with the diametric strategy

In the region of $n-1 \leq m \leq 2(n-1)$ the proposed bound is $V \leq \frac{m}{2(n-1)}$. However a simple counter shows this to be false.

Counter-example. Consider L_5 with $T = m = 5$, then the patroller only needs to walk between the end nodes to win.

The walk $\{1, 2, 3, 4, 5\}$ guarantee's the capture of all attacks made.

Problem with the diametric strategy

Example. Consider L_{31} with $m = 45$

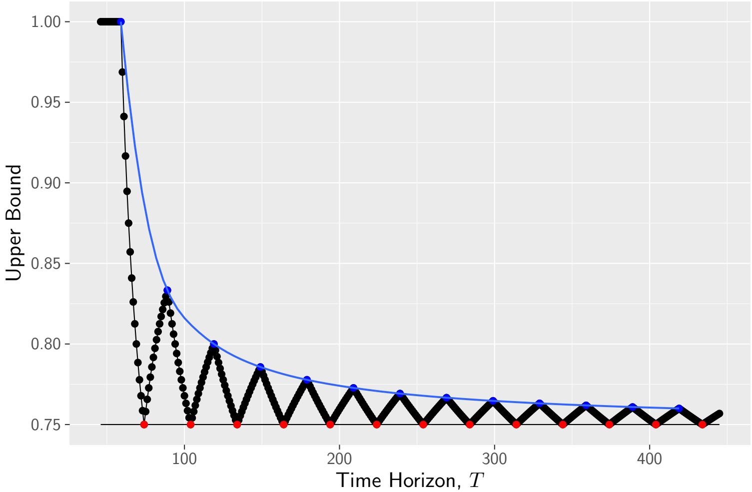


Figure: Best Upper Bound achievable under the diametric strategy

Problem with the diametric strategy

The problem is under the diametric attack, a patroller can catch

$$m-\bar{d}+\left(m\times(\left\lfloor\frac{T-2m+1}{\bar{d}}\right\rfloor+1)\right)_{+}+\left(T-(m-1+(\left\lfloor\frac{T-2m+1}{\bar{d}}\right\rfloor+1)\bar{d})\right)_{+}+ \left(T-(m-1+(\left\lfloor\frac{T-2m+1}{\bar{d}}\right\rfloor+2)\bar{d})\right)_{+} \quad (1)$$

attacks. From this we can get

Lemma (Condition on T for bound to hold)

When $T=m-1+(k+1)(n-1)$ for some $k\in\mathbb{N}_0$ then the diametric bound holds. Otherwise as $T\to\infty$ then the diametric bound, $V\leq\frac{m}{2(n-1)}$, holds.

Correction to diametric line strategy

We propose a solution to the problem, by limiting the time

Definition (Time limited diametric attack)

When $T \geq m + n - 2$ we have the *time limited diametric attack* (on the line) strategy is for the attacker to attack at both ends of the line with equal probability for starting times $0, 1, ..., n-2$

This restriction to the attacking time guarantees to get the upper bound of $V \leq \frac{m}{2(n-1)}$.

Extension to time limited diametric strategy

Definition (Time limited diametric attack)

When $T \geq m - 1 + \bar{d}$ we have the *time limited diametric attack* strategy is for the attacker to attack at both ends of the line with equal probability for starting times $\tau, \tau + 1, ..., \tau + \bar{d} - 1$ (for a chosen initial τ).

Lemma (Time limited diametric bound)

When $T \geq m - 1 + \bar{d}$ the attacker can get the bound

$$V \leq \frac{m}{2\bar{d}}.$$

Definition (Polygonal attack)

A d-*polygonal attack* is an attack at a set of nodes $D = \{i \in N \mid d(i,j) = d, \forall j \in D\}$ at the time intervals $\tau, \tau+1, ..., \tau+d-1$ (for a chosen initial τ) all equally probable.

Lemma (Polygonal bound)

When $T \geq m+d-1$ and a set D as in the d-polygonal attack exists, the value has an upper bound of $V \leq \max\{\frac{1}{|D|}, \frac{m}{|D|d}\}$.

Example.

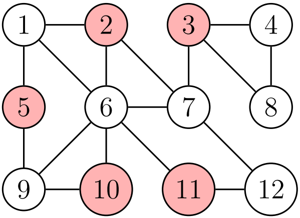


Figure: 2-polygonal attack on $D = \{2, 3, 5, 10, 11\}$

Giving $V \leq \max\{\frac{1}{5}, \frac{m}{10}\}$.

Introduction to the elongated star

We now wish to integrate features of a star into a line. We will form the elongated star S_n^k. We will use the labelling as below

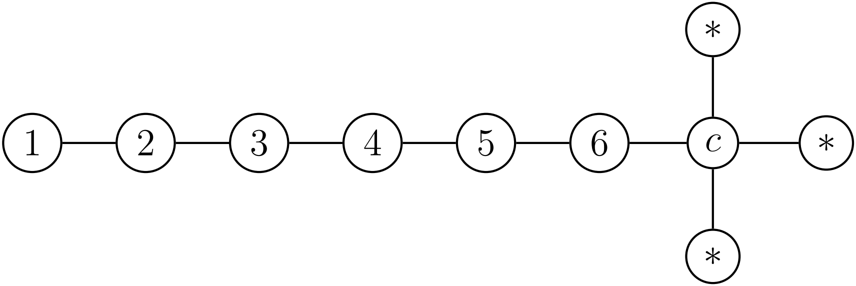


Figure: Labeling on the graph S_4^5.

Definition (Random Oscillation)

The *Oscillation* on S_n^k is any embedded Hamiltonian Patrol on $C_{2(n+k)}$.
The *Random Oscillation* on S_n^k is the embedded Random Hamiltonian Patrol on $C_{2(n+k)}$.

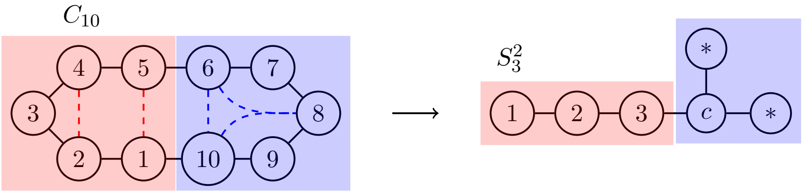


Figure: C_{10} can be simplified to S_3^2 by node identifying.

Lemma

For $m < 2(n+k)$ following the Random Oscillation,

$$V(S_n^k) \geq V(C_{2(n+k)}) = \frac{m}{2(n+k)}$$

and if $m \geq 2(n+k)$ then $V(S_n^k) = 1$, achieved by any Oscillation.

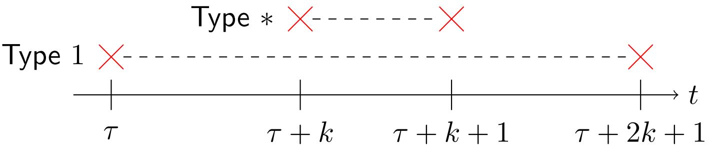
This is again because the patroller can do no worse following the embedded path from $C_{2(n+k)}$ in S_n^k

Introduction to the elongated star

We adapt the time limited diametric attack to the time-delayed attack.

Definition (Time-delayed attack)

Let the *time-delayed attack*, be the attack that attacks at the extended node labelled 1 with probability $\frac{k+1}{n+k}$ and a particular normal node labelled $*$ with probability $\frac{1}{n+k}$.
If node 1 is chosen have the attack choose probability intervals with equal probability starting attacks at $\tau, \tau+1, ..., \tau+2k+1$. If a $*$ node is chosen start the attacks at the times $\tau+k, \tau+k+1$ with equal probability.



Introduction to the elongated star

Lemma

When $T \geq m + 2k$*, the upper bound* $V \leq \max\left\{\frac{k+1}{n+k}, \frac{m}{2(n+k)}\right\}$

Hence we have a partial solution, analogous to region 1 and region 2 for the line.

Theorem

If $T \geq m + 2k$ *and* $m \geq 2(k+1)$ *then we have the value of the game is*

$$V = \min\left\{1, \frac{m}{2(n+k)}\right\} \quad (2)$$

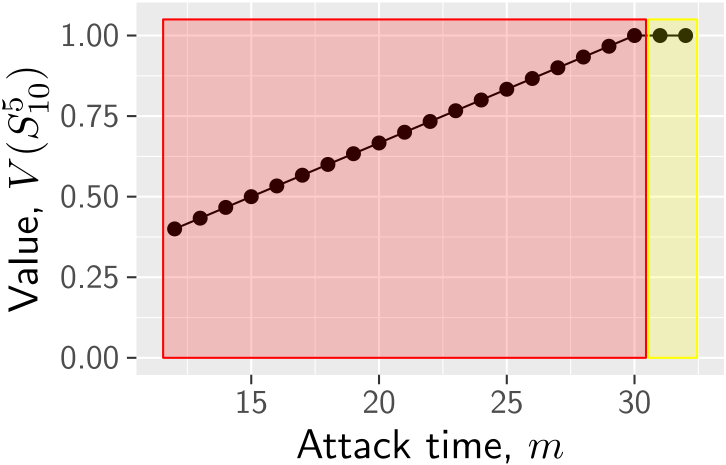


Figure: Value of the Star Graph, S^5_{10}

Future Work

- Look at analysing different types of Polygonal attacks, i.e the best choice of d and how to select the set D.
- Look at finding solutions analogous to region 3,4 and 5 for the line, which may need to be split into more regions, for (m, n, k) instead of (n, m).
- Expand the idea to a more generalised star, $S_n^{k_1,..,k_h}$, where elongation of more than one star node may occur.