# Patrolling Games

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# Introduction to Game: Pure Game

A Patrolling game, $G = G(Q, T, m)$ is made of 3 major components

- A Graph, $Q = (N, E)$, made of nodes, $N$ ($|N| = n$), and a set of edges, $E$.
- A time horizon parameter, $T$ (with set $\mathcal{T} = \{0, 1, ..., T-1\}$).
- An attack time parameter, $m$.

The game involves two players, the patroller and the attacker.

- The patroller's strategy is a walk (with waiting) on the graph, $W : \mathcal{T} \to N$ .
- The attacker's strategy is a node, $i$ and starting time, $\tau$ .

The strategies are collected into the sets, $\mathcal{W}$ and $\mathcal{A}$ , for the patroller and attacker respectively, with some arbitrary labelling inside the set to form strategies $W_i$ and $A_j$.

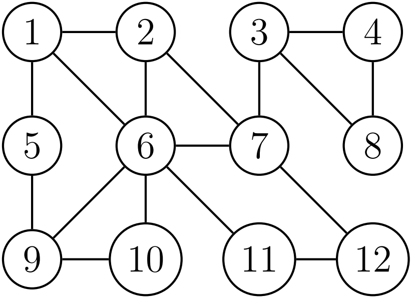
The game is formulated as win-lose (a zero-sum game) with a payoff for the patroller of

$$P(W,(i,\tau)) = \begin{cases} 1 \text{ if } i \in \{W(\tau), W(\tau+1), ..., W(\tau+m-1)\}, \\ 0 \text{ if } i \notin \{W(\tau), W(\tau+1), ..., W(\tau+m-1)\}. \end{cases}$$

With a pure payoff matrix $\mathcal{P} = (P(W_i, A_j))_{i\in\{1,...,|\mathcal{W}|\}, j\in\{1,...,|\mathcal{A}|\}}$

# Introduction to Game: Pure Game

The game played on $Q$ as below with $m = 3$ and $T = 7$

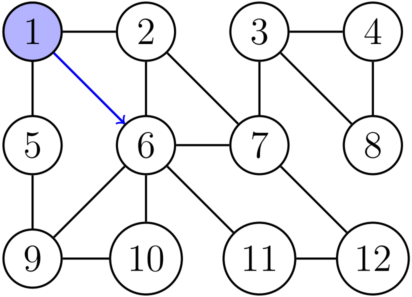



Patroller: $W(0) = 1$ , $W(1) = 6$ , $W(2) = 7$ , $W(3) = 3$ , $W(4) = 3$ , $W(5) = 4$ , $W(8) = 8$

Attacker: $(8, 2)$

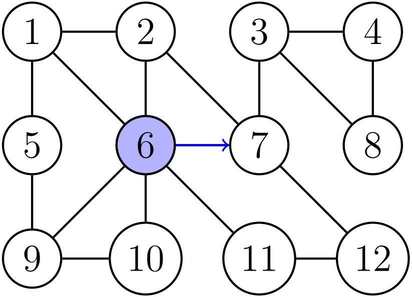
The game played on $Q$ as below with $m = 3$ and $T = 7$

$$t = 0$$



The game played on $Q$ as below with $m = 3$ and $T = 7$

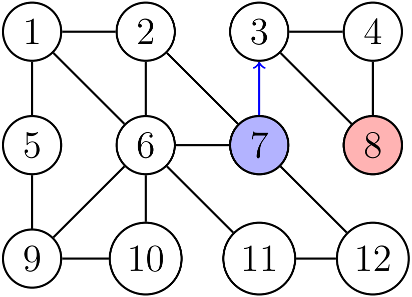
$$t = 1$$



The game played on $Q$ as below with $m = 3$ and $T = 7$

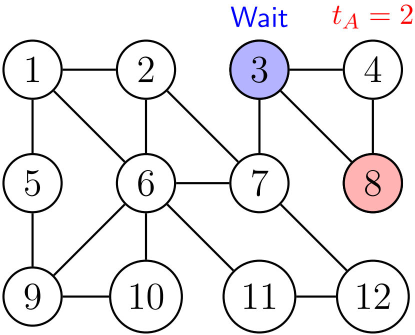
$$t = 2$$

Attack Starting, $t_A = 1$



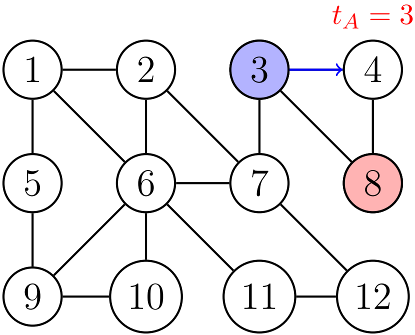
The game played on $Q$ as below with $m = 3$ and $T = 7$

$$t = 3$$



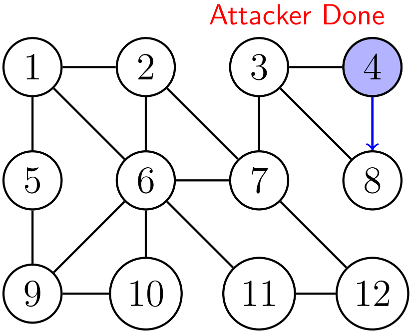
The game played on $Q$ as below with $m = 3$ and $T = 7$

$$t = 4$$



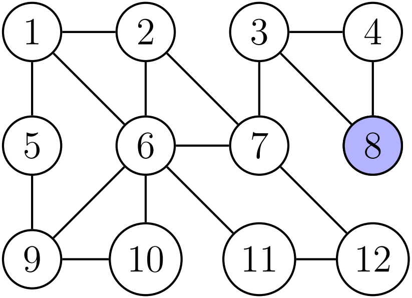
The game played on $Q$ as below with $m = 3$ and $T = 7$

$$t = 5$$



The game played on $Q$ as below with $m = 3$ and $T = 7$

$$t = 6$$




The attacker fails to catch the patroller, therefore the patroller loses (and the attacker wins) meaning a payoff of $0$ for the patroller (and $-1$ for the attacker).

# Introduction to Game: Mixed Game

Both the patroller and attacker will play their pure(realised) strategies with certain probabilities, let $\boldsymbol{\pi}$ be a mixed strategy for the patroller and let $\phi$ be a mixed strategy for the attacker. We collect these into the sets $\Pi$ and $\Phi$ for the patroller and attacker respectively.

Then the payoff for the patroller of this mixed game becomes

$$P(\boldsymbol{\pi}, \boldsymbol{\phi}) = \sum_{i=1}^{|\mathcal{W}|} \sum_{j=1}^{|\mathcal{I}|} \mathcal{P}_{i,j} \boldsymbol{\pi}_i \boldsymbol{\phi}_j = \boldsymbol{\pi} \mathcal{P} \boldsymbol{\phi}$$

By using the pure payoff as $1$ when capture occurs and $0$ otherwise, the mixed payoff is equivalent to the probability of capture.

# Introduction to Game: Mixed Game

## Mixed Nash equilibrium

A choice of $\boldsymbol{\pi}^*$ and $\phi^*$ is said to be in *Nash equilibrium* if

$$P(\boldsymbol{\pi}^*, \boldsymbol{\phi}^*) \geq P(\boldsymbol{\pi}, \boldsymbol{\phi}^*) \quad \forall \boldsymbol{\pi} \in \Pi,$$
$$P(\boldsymbol{\pi}^*, \boldsymbol{\phi}^*) \geq P(\boldsymbol{\pi}^*, \boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in \Phi.$$

There will only be one Nash equilibrium, unless the patroller can guarantee capture.
We do this by searching for the games value,

$$V(G) \equiv \max_{\boldsymbol{\pi} \in \Pi} \min_{\boldsymbol{\phi} \in \Phi} P(\boldsymbol{\pi}, \boldsymbol{\phi}) = \min_{\boldsymbol{\phi} \in \Phi} \max_{\boldsymbol{\pi} \in \Pi} P(\boldsymbol{\pi}, \boldsymbol{\phi})$$

This is done by achieving both upper and lower bounds on the value of the game.

# Solved graphs: Hamiltonian graphs

A graph is Hamiltonian if it is possible to find a cycle which visits every node exactly one (apart from the start/finish).

**Hamiltonian graphs**

A Hamiltonian graph has the value $V = \frac{m}{n}$

Two common Hamiltonian graphs are the Cyclic graph (of n nodes $C_n$) and the Complete graph (of n nodes $K_n$).

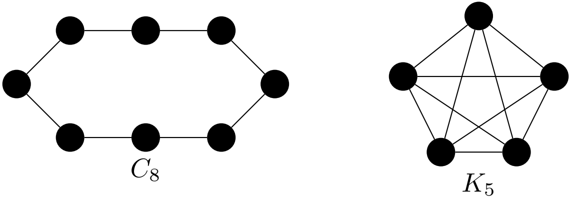


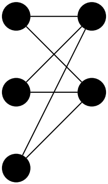

Figure: Examples of Cyclic and Complete graphs

# Solved graphs: Complete bipartite graphs

A bipartite graph is a graph made of two non-adjacent sets, the complete version has all connections.

**Complete bipartite graph**

A complete bipartite graph, $K_{a,b}$ as value $V = \frac{m}{2\max(a,b)}$



$K_{3,2}$

Figure: Example of a complete bipartite graph

# Solved graphs: Star graph

The star graph, $S_n$, is $n$ nodes adjacent only to the centre.

**Star graph**

The star $S_n \equiv K_{1,n}$ so has the value $V = \frac{m}{2n}$

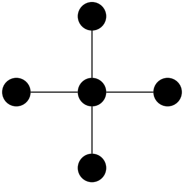



Figure: Example of a star graph

# Solved graphs: Line graph

The line graph, $L_n$, made of $n$ nodes each adjacent to two other nodes (apart from the ends)

## Line graph

The line graph, $L_n$ has a value dependent on $(n, m)$

1. If $m > 2(n-1)$ then $V = 1$.
2. If $n - 1 < m \leq 2(n-1)$ then $V = \frac{m}{2(n-1)}$
3. If $m = 2, n \geq 3$ then $V = \frac{1}{\lceil \frac{n}{2} \rceil}$
4. If $m = n - 1$ or $m = n - 2$ and $m = 2k$ for some $k \geq 2$ then $V = \frac{1}{2}$
5. If $3 \leq m \leq n - 3$ or $m = n - 2$ and $m = 2k + 1$ for some $k \geq 1$ then $V = \frac{m}{m+n-1}$

Note. The solution for $m = 1$ is know for every graph as $V = \frac{1}{|N|} = \frac{1}{n}$, and if $m = n = 2$ then we know $V = 1$.

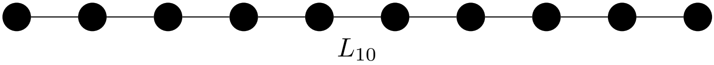



Figure: Example of a line graph

Regions are:

1. $m > 18$
2. $9 < m \leq 18$
3. $m = 2$
4. $m = 9, 8$
5. $m =$

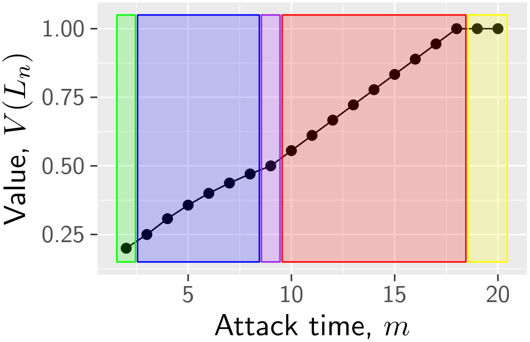



Figure: Value of the line graph for varying $m$

# Problem with line graph strategy

It was initially stated that in the region two $n-1 \leq m \leq 2(n-1)$, the value $\frac{m}{2(n-1)}$ is achieved by the attacker attacking using the diametric strategy, which is to attack at opposite ends of the line with equal probability for all possible starting times. This is supposed to guarantee that $V \leq \frac{m}{2(n-1)}$.

**Example.** Consider $L_5$ with $T = m = 5$ , then the patroller only needs to walk between the end nodes to win.


The walk $\{1, 2, 3, 4, 5\}$ guarantee's the capture of all attacks made.

# Problem with line graph strategy

**Example.** Consider $L_{31}$ with $m = 45$

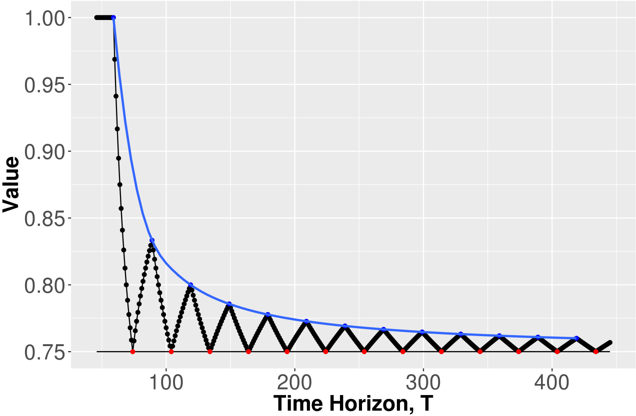



Figure: Best Upper Bound achievable under the diametric strategy

# Problem with the line graph strategy

The problem is under the diametric attack, a patroller can catch

$$m-\bar{d}+\left(m\times(\left\lfloor\frac{T-2m+1}{\bar{d}}\right\rfloor+1)\right)_{+}+\left(T-(m-1+(\left\lfloor\frac{T-2m+1}{\bar{d}}\right\rfloor+1)\bar{d})\right)_{+}+\left(T-(m-1+(\left\lfloor\frac{T-2m+1}{\bar{d}}\right\rfloor+2)\bar{d})\right)_{+}$$

attacks. From this we can get

**Lemma (Condition on $T$ for bound to hold)**

*When $T=m-1+(k+1)(n-1)$ for some $k\in\mathbb{N}_0$ then the diametric bound holds. Otherwise as $T\to\infty$ then the diametric bound, $V\leq\frac{m}{2(n-1)}$, holds.*

# Correction of line graph strategy

We propose a solution to the problem, by limiting the time

**Definition (Time limited diametric attack)**

When $T \geq m + n - 2$ we have the *time limited diametric attack* (on the line) strategy is for the attacker to attack at both ends of the line with equal probability for starting times $0, 1, ..., n-2$

This restriction to the attacking time guarantees to get the upper bound of $V \leq \frac{m}{2(n-1)}$.
This works due to the spacing of the attack times.

# Extension of correction strategy

Let $\bar{d}$ be the diameter of the graph, i.e the distance between the furthest apart nodes then using

**Definition (Time limited diametric attack)**

When $T \geq m - 1 + \bar{d}$ we have the *time limited diametric attack* strategy is for the attacker to attack at both ends of the line with equal probability for starting times $\tau, \tau + 1, ..., \tau + \bar{d} - 1$ (for a chosen initial $\tau$).

**Lemma (Time limited diametric bound)**

*When $T \geq m - 1 + \bar{d}$ the attacker can get the bound*

$$V \leq \frac{m}{2\bar{d}}.$$

# Extension of correction strategy

Using $d(i,j)$ as the distance between nodes $i$ and $j$ we can get a more general type of attack.

**Definition (Polygonal attack)**

A $d$-*polygonal attack* is an attack at a set of nodes $D = \{i \in N \mid d(i,j) = d, \forall j \in D\}$ at the time intervals $\tau, \tau+1, ..., \tau+d-1$ (for a chosen initial $\tau$) all equally probable.

**Lemma (Polygonal bound)**

*When $T \geq m+d-1$ and a set $D$ as in the $d$-polygonal attack exists, the value has an upper bound of $V \leq \max\{\frac{1}{|D|}, \frac{m}{|D|d}\}$ .*

We now wish to integrate features of a star into a line. We will form the elongated star $S_n^k$. We will use the labelling as below

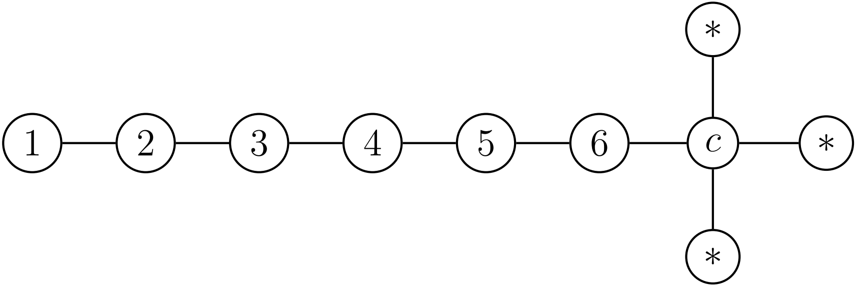



Figure: Labeling on the graph $S_4^5$.

# Future work