Patrolling games on graphs collection

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This document houses all work on patrolling games on graphs.

Contents

1 Introduction to game

1.1 Pure game

A patrolling game $G = G(Q, T, m)$ is a win-lose, zero-sum game between a maximizing patroller (often referred to as she) and a minimizing attacker (often referred to as he). The parameters of the game are:

- The graph Q which has a set of nodes, N, and of edges, E.
- The length of time over which the game takes place, the time-horizon T.
- The length of time the attack takes to complete, the attack-time m.

Two forms of the game exist: the one-off game, which is played in a finite time interval $\mathcal{T} = \{0, 1, ..., T-1\}$ denoted using G^o; and the periodic game, which is played on the time circle $\mathcal{T}^* = \{0, 1, ..., T-1\}$ (with the asterisk representing arithmetic on the time circle taking place modulo T) denoted using G^p. We will assume that $T \geq m$, otherwise all attacks will fail.

The pure strategies available to the patroller are called patrols, choosing a starting position and how to walk along the graph Q, $W : \mathcal{T} \to N$. With no restrictions in the one-off game, but the restriction that the edge $(W(T-1), W(0)) \in E$ in the periodic game (so that $W(T) = W(0)$). Let

$$\mathcal{W} = \{W \,|\, W : \mathcal{T} \to N \text{ s.t } (W(t), W(t+1)) \in E \text{ for } t = 0, ..., T-2\}$$

be the set of all pure patrols in the one-off game (and similarly $\mathcal{W}^*$ in the periodic game). Let there be some ordering to the strategies $W_k \in \mathcal{W}$ (or $W_k \in \mathcal{W}^*$) for $k = 1, ..., |\mathcal{W}|$ (or $k = 1, ..., |\mathcal{W}^*|$ in the periodic game).

The pure strategies available to the attacker are pairs, $[i, I]$ for $i \in N$, called the attack node, and $I = \{\tau, \tau+1, ..., \tau+m-1\} \subseteq \mathcal{T}$ (or $I \subseteq \mathcal{T}^*$ if periodic) called the attack interval (starting at time τ). Let $\mathcal{A} = \{[i, I] \,|\, i \in N, I \subseteq \mathcal{T}\}$ be the set of all possible pure attacks. Let there be some ordering to the strategies $A_k \in \mathcal{A}$ for $k = 1, ..., |\mathcal{A}|$.

A patrol, W, intercepts the attack, $[i, I]$, if $i \in W(I) = \{W(\tau), W(\tau+1), ..., W(\tau+m-1)\}$ and as our game is Win-Lose the pure payoff function is

$$P(W, [i, I]) = \begin{cases} 1 \text{ if } i \in W(I), \\ 0 \text{ if } i \notin W(I). \end{cases}$$

A pure payoff matrix $\mathcal{P} = (P(W_i, A_j))_{i \in \{1,...,|\mathcal{W}|\}, j \in \{1,...,|\mathcal{A}|\}}$ (with the change of $\mathcal{W}$ to $\mathcal{W}^*$ if in the periodic game) stores Win (1) or Lose (0) for each pair of pure strategies.

1.2 Mixed game

Let Π_W be the set of mixed strategies for the patroller in the one-off game and Π_W^* in the periodic game. Let Φ be the set of mixed strategies for the attacker.

In the mixed strategy game the patroller selects a strategy $\boldsymbol{\pi} \in \Pi_W$ (or $\boldsymbol{\pi} \in \Pi_W^*$ in the periodic game).

The attacker selects a strategy $\boldsymbol{\phi} \in \Phi$. Then the mixed payoff function (Probability of Capture) is

$$P(\boldsymbol{\pi}, \boldsymbol{\phi}) = \sum_{i=1}^{|\mathcal{W}|} \sum_{j=1}^{|\mathcal{I}|} \mathcal{P}_{i,j} \boldsymbol{\pi}_i \boldsymbol{\phi}_j = \boldsymbol{\pi} \mathcal{P} \boldsymbol{\phi}$$

(with the change of $\mathcal{W}$ to $\mathcal{W}^*$ if in the periodic game).

We will also use the convention that a pure strategy is in the mixed strategy set, $W_i \in \Pi_W$ (or $W_i \in \Pi_{W^*}$) and $A_j \in \Phi$, to mean $\boldsymbol{\pi}_k = \begin{cases} 1 \text{ if } k = i, \\ 0 \text{ if } k \neq i. \end{cases}$ and $\boldsymbol{\phi}_k = \begin{cases} 1 \text{ if } k = j, \\ 0 \text{ if } k \neq j. \end{cases}$ respectively

The value of the game is denoted by $V = V(Q, T, m) \equiv \max_{\boldsymbol{\pi} \in \Pi} \min_{\boldsymbol{\phi} \in \Phi} P(\boldsymbol{\pi}, \boldsymbol{\phi}) = \min_{\boldsymbol{\phi} \in \Phi} \max_{\boldsymbol{\pi} \in \Pi} P(\boldsymbol{\pi}, \boldsymbol{\phi})$ and when needed we distinguish between the one-off and period game by using the subscripts V^o and V^p respectively.

1.3 Properties

2 Current Reduction Tools

2.1 Symmetry

Symmetry of a graph's nodes allows us to reduce the number of attacks and patrols under consideration. This is because symmetric nodes must be treated identically (or some bias is formed for the other player). More formally

Definition 2.1 (Attacker Equivalence)**.** We call two nodes, n_1 and n_2, *equivalent* if there exists some automorphism, σ, of Q which takes one to the other, i.e $n_1 = \sigma(n_2)$ or $n_2 = \sigma(n_1)$. Two attack intervals, I_1 and I_2, are called *equivalent* if there exists some automorphism, γ on the time interval, $\mathcal{T}$ which is reflective, i.e $\gamma(t) = T - t$.

Lemma 2.2 (Attacker Symmetry)**.** *We only need to consider the equivalence class of nodes for the attacker, with the same attack pattern formed on the equivalent nodes. Similarly we need only consider the equivalence class of attack* *times. Furthermore the periodic game has all attack times equivalent (under some rotation of the time circle) so only the node need be considered.*

Definition 2.3 (Patroller Equivalence)**.** We call two patrols, $W_1(t)$ and $W_2(t)$, *equivalent* if there exists some automorphism, σ, of Q takes one two the other, i.e $W_1(t) = \sigma(W_2(t))$ or $W_2(t) = \sigma(W_1(t))$.

Lemma 2.4 (Patroller Symmetry)**.** *We only need to consider the equivalence class of patrols for the patroller, with the same patrol pattern formed on the equivalent patrols.*

So without affecting the value of the graph we can reduce the problem to one of searching through the equivalent class of strategies for both players.

2.2 Strategy Domination

Domination of some strategies for the patroller and attacker are given, which allows the removal of these strategies from the possible strategy space for the players.

Note. We use the term dominance to mean weak dominance.

Lemma 2.5 (Waiting Patrol Dominated)**.** *For Q connected and $T \geq 3$, $m \geq 2$, patrols staying at any node for three consecutive periods are dominated.*

Corollary 2.6 (Penultimate Attack Dominated)**.** *For $m \geq 3$, attacks at penultimate (i.e non-leaf node adjacent to a leaf node) nodes are dominated.*

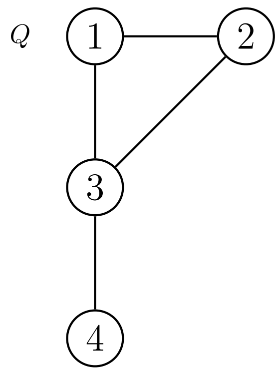


Figure 2.1: Graph Q used for examples

Example 2.7. For Q as seen Figure 2.1 We can see that nodes, 1 and 2, are equivalent nodes by using the automorphism $\sigma(1) = 2$, $\sigma(2) = 1$, $\sigma(i) = i \quad \forall i \neq 1, 2$. Similarly the patrols $W_1(t) = \begin{cases} 1 \text{ for } t \text{ even} \\ 3 \text{ for } t \text{ odd} \end{cases}$ and $W_2(t) = \begin{cases} 2 \text{ for } t \text{ even} \\ 3 \text{ for } t \text{ odd} \end{cases}$ are equivalent under the same automorphism.

The patrol $W_3(t) = \begin{cases} 1 \text{ for } t = 1, 2, 3 \\ 3 \text{ for } t \geq 4 \end{cases}$ is dominated (by $W_1(t)$)

Node 3 is penultimate and so will not be attacked.

3 Bounds

3.1 Basic Bounds(Attacker and Patroller)

We seek bounds on the value of the game V. The lower bounds are given in terms of the patroller's "good" strategy against all attacker options, similarly the upper bounds are given in terms of the attacker's "good" strategy against all the patroller options. When we reach tightness between the bound these "good" strategies become an optimal solution.

By the patroller waiting a random node they can achieve $V \geq \frac{1}{n}$ and by the attacker picking a random node with a fixed time I they can achieve $V \leq \frac{m}{n}$ (More generally $V \leq \frac{\omega}{n}$ for ω, the maximum number of nodes any patrol can cover).

Lemma 3.1 (General bounds)**.**

$$\frac{1}{n} \leq V \leq \frac{\omega}{n} \leq \frac{m}{n}$$

Where ω is the maximum number of distinct nodes that can be visited in an attack interval.

3.2 Decomposition(Patroller)

We can consider decomposing the graph so that we just operate on parts with some appropriate probability.

Lemma 3.2 (Decomposition lower bound)**.** *Consider decomposing Q into edge-preserving subgraphs Q_i for $i = 1, ..., k$ with values $V_i = V(Q_i)$ such that $Q = \bigcup_{i=1}^{k} Q_i$ then*

$$V \geq \frac{1}{\sum_{i=1}^{k} \frac{1}{V_i}}$$

Furthermore in the case of a disjoint decomposition equality is reached

More explicitly an edge-preserving subgraph is a subgraph who has all possible connection between its nodes and disjoint means both edge and vertex disjoint. This means we are really only selecting nodes for the subgraph and the edges are mandated.The above provides a solution to build disjointly decomposable graphs, so it is only worth studying connected graphs.

Example 3.3. For Q as seen in Figure 2.1. Consider when $m = 3$, the decomposition of Q into the graphs Q_1 and Q_2 (as in Example Figure 3.1).

$V_1 = V(Q_1) = 1$ as alternating between 1 and 2 can catch every attack. $V_2 = V(L_3) = \frac{3}{4}$ (as seen in [**?**]).

Then we can get the bound $V \geq \frac{1}{(\frac{1}{1}+\frac{3}{4})} = \frac{4}{7}$.

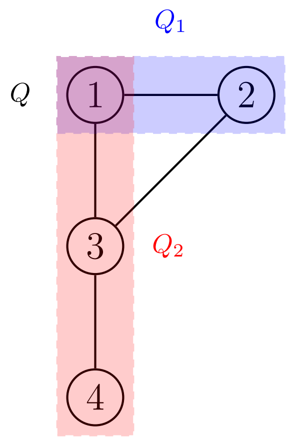


Figure 3.1: Decomposition of Q into Q_1 and Q_2.

3.3 Simplification(Patroller and Attacker)

Definition 3.4 (Node Identification)**.** The operation of Node identification on two nodes, u and v, of a graph, $G = (N, E)$ into a single node w, is a mapping $f : N \to N'$ resulting in a new graph $G' = (N', E')$ where $N' = (N \backslash \{u, v\}) \cup \{w\}$ with $E' = E \setminus \{(u, v)\}$ if $(u, v) \in E$ and under the condition that $\forall x \in N$, $f(x) \in N'$ is incident to $e' \in E'$ iff $e \in E$ is incident to $x \in N$. Furthermore if a graph, Q, undergoes repeated node identification to become Q' then we say it has been simplified.

Definition 3.5 (Embedded walk)**.** An *Embedded walk*, W', on Q' is the walk, W, done on Q under the simplification mapping of Q to Q'. i.e if $\pi : Q \to Q'$ is the simplification map, then $W' = \pi(W)$.

Lemma 3.6 (Simplification)**.** *If Q' is a simplified version of Q then* $V(Q') \geq V(Q)$

This allows us to get bounds for both the patroller and attacker.

Example 3.7. For Q as seen in Figure 2.1. Consider when $m = 3$, the Simplification of the graph by identifying $1, 2$ from Q to $Q' = L_3$ (as seen in Example Figure 3.2). Hence we can get the bound that $V(L_3) \geq V(Q)$ hence $V(Q) \leq \frac{3}{4}$

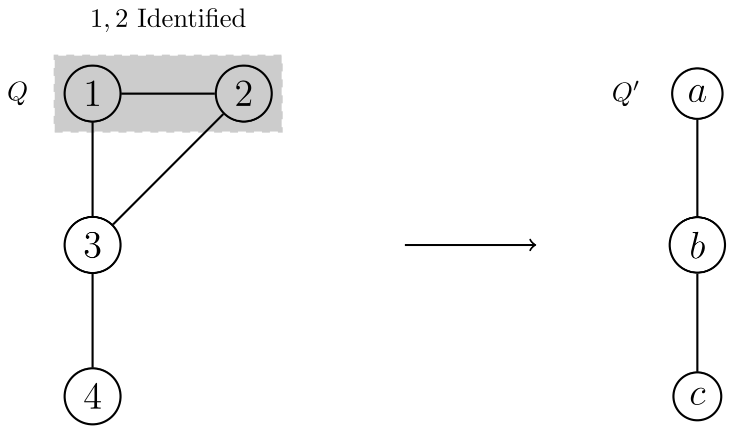


Figure 3.2: Simplifcation of Q to Q' by identification.

3.4 Old Diametric attack(Attacker)

Let $d(i, i')$ is the distance between nodes i and i' with the distance measured by the minimum number of edges.

Definition 3.8 (Graph Diameter). The diameter of a graph Q is definded by $\bar{d} = \max_{i,i' \in N} d(i, i')$. The node pairs satisfying this are called diametrical.

Lemma 3.9. *By the attacker playing equally likely at a pair of diametrical at a random time interval, called the diametric attack, gives* $V \leq \max\left\{\frac{m}{2\bar{d}}, \frac{1}{2}\right\}$

However the bound presented in [**?**] seems to indicate for large T, m the second is chosen. However a simple counter example will show the bound does not allows hold

Example 3.10 (Problem with diametric attack). Consider the graph L_5 so $\bar{d} = 4$ for $T = m = 5$, then under the diametric attack, the patroller performing the patrol $\{1, 2, 3, 4, 5\}$ allows her to catch the two attacks. Hence the bound of $V \leq \frac{5}{8}$ given by lemma does not seem to hold.

3.5 Covering(Patroller) and Independence(Attacker)

Definition 3.11 (Covering). A patrol, W, is called *intercepting* if it intercepts every possible attack at every node contained in the patrol, i.e all nodes visited by W are in any subpath of length m (i.e visits are at most m apart).

A set of intercepting patrols forms a *Covering set* if every node in Q is contained in at least one of the patrols. Furthermore the *Covering number*, $\mathcal{C}$ is the minimum cardinality of all the covering sets.

Definition 3.12 (Independence)**.** Two nodes, i and i', are called independent (under attack time, m) if any patrol intercepting an attack at i cannot also intercept an attack at i'.

For the one-off game this is equivalent to $d(i, i') \geq m$.

For the Periodic game this is equivalent to $d(i, i') \geq m$ and $2d(i, i') \leq T$(due to returning to start).

A set of independent points forms a *Independent set* if every element of the set is independent of every other element. Furthermore the independence number $\mathcal{I}$ is the maximum cardinality of all the independent sets.

Clearly $\mathcal{I} \leq \mathcal{C}$ as to cover a collection of independent nodes, at least that many covering patrols are needed (Possibly more if they also don't get every node in Q)

Lemma 3.13 (Covering and Independence)**.**

$$\frac{1}{\mathcal{C}} \leq V \leq \frac{1}{\mathcal{I}}$$

4 Simple Graph bounds and solutions

4.1 Hamiltonian

A Hamiltonian graph is a graph with a Hamiltonian cycle. Two simple examples of such graphs are cyclic graphs, C_n and the complete graph, K_n. While Hamiltonian graphs can exhibit more than one Hamiltonian cycle we shall assume that we have selected one. We shall also assume that the attack, $m < n$, as otherwise by following the Hamiltonian cycle we guarantee capture (i.e for $m \geq n$, $V = 1$).

Definition 4.1 (Random Hamiltonian Patrol)**.** A *Random Hamiltonian Patrol* is a mixed strategy starting with equal probability at all nodes and following the Hamiltonian cycle.

Theorem 4.2 (Hamiltonian)**.** *If Q is Hamiltonian, by following the Random Hamiltonian Patrol (if feasible), the patroller can achieve $V \geq \frac{m}{n}$.*

This, along with a general upper bound from [**?**], provides the solution $V = \frac{m}{n}$.

Two common Hamiltonian graphs are the Cyclic graph (of n nodes C_n) and the Complete graph (of n nodes K_n).

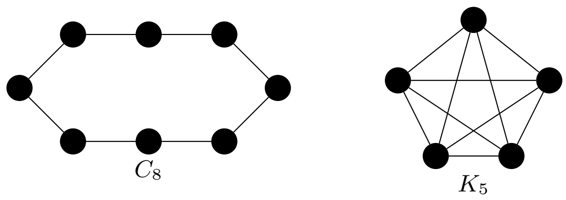


Figure 4.1: Examples of Cyclic and Complete graphs

4.2 Biparite

A bipartite graph is a graph which can be partitioned into two sets, A and B (with $|A| = a, |B| = b$, assume WLOG that $b \geq a$) were edges only exist between these two sets. A special bipartite graph is the complete bipartite graph $K_{a,b}$.

Assume that $m < 2b$, as otherwise there exists a $2b$ period patrol which covers all nodes and guarantees capture (i.e if $m \geq 2b$, $V = 1$).

Definition 4.3 (Bipartite Attack)**.** The *Bipartite Attack* selects nodes equiprobably from the larger set B for a fixed time interval, I (or for the two time intervals, I and $I + 1$ equiprobably).

Theorem 4.4 (Bipartite)**.** *If Q is bipartite with $b \geq a$, by following the Bipartite Attack, the attacker can achieve $V \leq \frac{m}{2b}$*

The reasoning behind the bound is that any patrol must alternate between $|A|$ and $|B|$, so only visits a node from B every other time step.

4.3 Complete Bipartite

Corollary 4.5 (Complete Bipartite)**.** *The value of the complete bipartite graph, $K_{a,b}$, with $b \geq a$, then $V = \frac{m}{2b}$.*

This is because a lower bound of $V \geq \frac{m}{2b}$ is given by the random Hamiltonian patrol in $K_{b,b}$, which simplifies to $K_{a,b}$.

The star graph, S_n is a graph with a centre and n nodes adjacent to the centre.

Corollary 4.6 (Star)**.** *The value of $S_n \equiv K_{1,n}$ is $V = \frac{m}{2n}$.*

5 Line graph

6 New Bounds

6.1 Problem and Correction of diametric attack

Consider the patroller strategy against a diametric attack that simply oscillates between the two diametric points.

The total number of attacks the attacker is making under this diametric strategy is $2(T - m + 1)$, we will now measure how many the simple strategy for the patroller gets.

We will divide the set of captured attacks, depending on what is happening. This division shall be into start captures, middle captures and end captures.

The start captures are captures catching less than m attacks in the early times, i.e before the middle. The middle captures are captures catching exactly m attacks. The end captures are captures catching less than m attacks in the late times.

Example 6.1 (Problem with diametric attack)**.** Consider the graph L_5 so $\bar{d} = 4$ for $T = 20$,$m = 6$, then under the diametric attack, the patroller oscillating between diametrics points gets.

Start Capture $1 + 5 = 6$ attacks initially.

Middle Capture $6 + 6 = 12$ attacks when arriving at node 5.

End Capture 4 attacks when finishing at node 1.

Giving 22 out of $2(20 - 6 + 1) = 30$ attacks, a better than $\frac{3}{4}$ value.

First off it is worth considering that the number of end attacks that are going to be caught will come from two values (if more than these would be in the middle). We could suggest that waiting at the start is more preferable to "stabilize" into the middle quicker. the cost for doing so is to remove one from each end value, if one of the end values is 1, then the penultimate middle is also reduced by 1 and thus becomes an end value.

While each time we decide to wait gains us 1 for each node in the start values, until it becomes a middle value. As $m \geq \bar{d}$, we are guaranteed that there are at least two start values (as we are looking at times 0 and $\bar{d} - 1 < m$). Therefore it is certain that waiting at the start (at least while there are two start values) is not worse than the just oscillating strategy.

Therefore we can wait until $t = m - (\bar{d} - 1)$ (which as $m > \bar{d} - 1$ means its is always possible), then this is only the start.

Let us count the pattern under this strategy,

Start: Capture $m - \bar{d}$ attacks initially by waiting.

Middle: Capture $m \times (\lfloor \frac{T-2m+1}{\bar{d}} \rfloor + 1)$ attacks in the middle cycles (if any middle times are possible, otherwise zero if negative).

End: Capture $T - 1 - (m - 1 + (\lfloor \frac{T-2m+1}{\bar{d}} \rfloor + 1)\bar{d})$ (at the penultimate node visit (if possible, this really is zero if its negative) and $T - 1 - (m - 1 + (\lfloor \frac{T-2m+1}{\bar{d}} \rfloor + 2)\bar{d})$ at the final node visit (again zero if negative).

This gives

$$m - \bar{d} + \left(m \times (\left\lfloor \frac{T-2m+1}{\bar{d}} \right\rfloor + 1)\right)_+ +$$
$$\left(T - (m - 1 + (\left\lfloor \frac{T-2m+1}{\bar{d}} \right\rfloor + 1)\bar{d})\right)_+ + \left(T - (m - 1 + (\left\lfloor \frac{T-2m+1}{\bar{d}} \right\rfloor + 2)\bar{d})\right)_+$$

We will call $\alpha = \lfloor \frac{T-2m+1}{\bar{d}} \rfloor$

Lemma 6.2 (Condition on T for bound to hold)**.** *When $T = m - 1 + (k+1)\bar{d}$ for some $k \in \mathbb{N}_0$ then the diametric bound holds. Otherwise as $T \to \infty$ then the diametric bound holds.*

Proof. Using $T = m - 1 + (k+1)\bar{d}$ in the formula gives, $\alpha = \left\lfloor \frac{(k+1)\bar{d}-m}{\bar{d}} \right\rfloor = (k+1) + \lfloor \frac{-m}{\bar{d}} \rfloor = (k+1) - 2 = (k-1)$ (the final part is because $2 > \frac{-m}{\bar{d}} \geq -1$ and we will assume that $m > \bar{d}$ here otherwise waiting at one node is just as good as the bound we are trying to achieve)

$$m - \bar{d} + (m + m(k-1))_+ + \big(m - 1 - (m - 1 + (k - 1 + 1)\bar{d})\big)_+ + \big(m - 1 - (m - 1 + (k+1)\bar{d})\big)_+$$

which is $m - \bar{d} + (mk)_+ + \big((k+1)\bar{d} - k\bar{d}\big)_+ + \big((k+1)\bar{d} - (k+1)\bar{d}\big)_+$ giving $m - \bar{d} + mk + \big(\bar{d}\big)_+ + (0)_+ = m(k+1)$. Giving the fraction of $\frac{m(k+1)}{2(k+1)\bar{d}} = \frac{m}{2\bar{d}}$.

For the second part, first we seek to prove that within the choice of T from $m - 1 + (k+1)\bar{d} + r$ where $0 \leq r < \bar{d}$ is the maximum when $r = m$ (i.e $T = 2m - 1 + (k+1)\bar{d}$).

As the choice of r only affects the final 3 parts (middle and ends values), we can just look at considering these values and seeing what the maximal choice is.

Upon substitution we get that: $\alpha = \left\lfloor \frac{(k+1)\bar{d}+r-m}{\bar{d}} \right\rfloor = (k+1) + \lfloor \frac{r-m}{\bar{d}} \rfloor$

so formula becomes $(m(\alpha+1))_+ + \big((k+1)\bar{d} + r - (\alpha+1)\bar{d}\big)_+ + \big((k+1)\bar{d} + r - (\alpha+2)\bar{d}\big)_+$. To decide r we need to know if middle values or end values are non-zero.

Note. The second end value will never be non-zero as $(k+1)\bar{d} + r - (\alpha+2)\bar{d} = (k+1)\bar{d} + r - ((k+1) + \lfloor \frac{r-m}{\bar{d}} \rfloor + 2)\bar{d} = r - (\lfloor \frac{r-m}{\bar{d}} + 2 \rfloor \bar{d} < r - (-1+2)\bar{d} = r - \bar{d} < 0$.

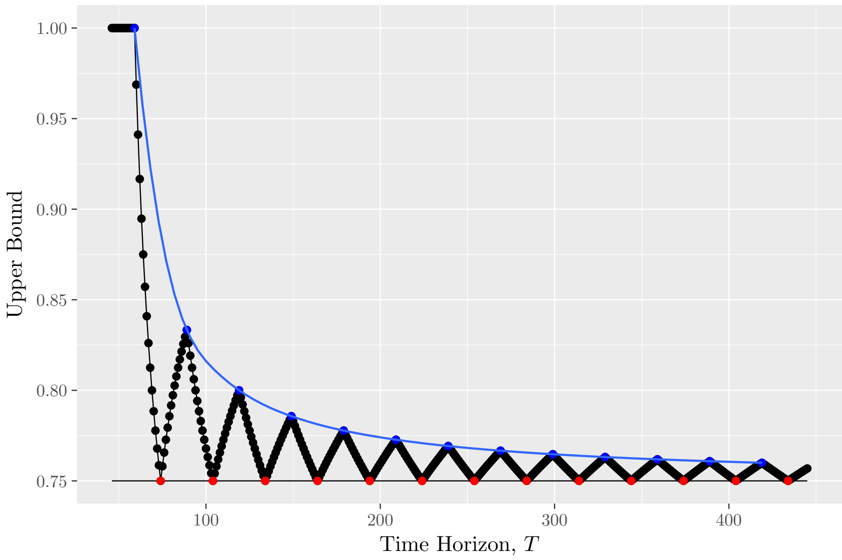


Figure 6.1: Best Upper Bound achievable under the diametric strategy

- No middle values and no end values is impossible assuming $k \in \mathbb{N}_0$.
- Middle values but no end values. As we really want to maximize the end value, increasing r up to the point where α increases (giving a raise of m attacks captured) but increases the number of total attacks by 2 each time it is raised. Hence minimal r is chosen to increase α. This is when $r - m = -\bar{d}$ i.e $r = m - \bar{d}$ changes α by 1 (as $r < m - \bar{d}$ gives an -1 to α, but critical point is when equal to).
- Middle values and end value. As we are looking at $(m - \bar{d})(\alpha + 1) + (k+1)\bar{d} + r$, we still want to increase α without increasing the number of attacks too much, i.e as above.

Then we show that the maximal subsequence tends to the bound as $T \to \infty$, i.e as $k \to \infty$. When substituted, we get that $\alpha = (k+1)$ and so the formula becomes $m - \bar{d} + (m + m(k+2))_+ + \big(2m - 1 - (m - 1 + (k+2)\bar{d})\big)_+ + \big(2m - 1 - (m - 1 + (k+3)\bar{d})\big)_+$ giving $m - \bar{d} + (k+3)m + \big(m - (k+2)\bar{d}\big)_+ + \big(m - (k+3)\bar{d}\big)_+$. As $m < 2\bar{d}$ then we get $m(k+4) - \bar{d}$ caught out of $2(m + k\bar{d})$ giving a fraction of $\frac{m(k+4)-\bar{d}}{2(m+k\bar{d}}$ $\to \frac{m}{2\bar{d}}$.

Hence as the maximal subsequence tends down to the bound, it implies the result. $\square$

Fixing the diametric attack A possible "fix" to the problem of the excess time is to limit the diametric attacks window in which attacks are placed.

Definition 6.3 (Timed diametric attack)**.** Attacking at a pair of diametric nodes equiprobably for the times I,$I+1$...,$I+\bar{d}-1$ (i.e starting attacks at

$\tau, \tau+1, ..., \tau+\bar{d}-1)$ is called the *timed diametric attack*.

Note. The timed diametric attack is only feasible is $T \geq m+\bar{d}-1$.

Lemma 6.4. *When $T \geq m+\bar{d}-1$, the diametric bound $V \leq \max\{\frac{1}{2}, \frac{m}{2\bar{d}}\}$ is valid.*

Proof. First consider all the pure patrolling strategies, $W_i \in \mathcal{W}$, Then as the attacker is only attacking two ends, henceforth called n_1 and $n_{\bar{d}}$, any patrol not starting at n_1 or $n_{\bar{d}}$ is dominated by one that does. This is because the patrol will not capture any attacks until they visit either n_1 or $n_{\bar{d}}$, and then capture a set of attacks that started there previously. The patrol might as well wait there up until this point and do at least as good as arriving there for the first time.

Formally, assume that n_1 is the end node first reached by a patrol, $W(t)$ at time, $t_1 = \min\{t \,|\, W(t) = n_1\}$, then we can form the patrol, $U(t) = \begin{cases} n_1 \text{ for } t \leq t_1, \\ W(t) \text{ for } t > t_1. \end{cases}$ and $P(U, \phi) \geq P(W, \phi)$ where ϕ is the timed diametric attack (or infact the normal diametric attack.

Now we are restricted to patrols starting at end points, it is similar to see when leaving an end point, there is no other decision as you must travel to the other end point, assumed to be $n_{\bar{d}}$. Hence the question becomes when to leave n_1 and travel to $n_{\bar{d}}$. Obviously it should only be undertaken if the journey can be made and more attacks can be caught by doing so.

WLOG assume that $\tau = 0$ (other just wait longer initially, as attacks haven't started), then our choice is what leaving time (last time before moving): $t_l \in \{0, 1, ..., m-2\}$, to pick to maximize the number of attacks caught; or $t_L = \infty$, never leaving to get $\bar{d}$ attacks.

Leaving: Choosing $t_L \in \{0, 1, ..., m-2\}$ gives the patroller $\frac{m}{2\bar{d}}$ as,

Leaving at t_L gives us $\min(t_L+1, \bar{d})$ attacks caught at n_L, and $\min(m+\bar{d}-2-(t_L+\bar{d})+1, \bar{d}) = \min(m-1-t_L, \bar{d})$.

Now choosing $t_L > \bar{d}-1$, doesn't improve the first value and possibly lowers the second value. Hence we restrict ourselves to leave if we catch all attacks, i.e $t_L \leq \bar{d}-1$. Now in this region lowering t_L lowers it by 1 and raises it only raises the second on by 1 if $m-1-t_L \leq \bar{d}$ (i.e $t_L \geq m-1-\bar{d}$ or any t_L if $m-1-\bar{d} \leq 0$). Hence any choice of $\left(m-1-\bar{d}\right)_+ \leq t_L \leq \bar{d}-1$ is equally as good. This gives a number of attacks caught as $t_L+1+m-1-t_L = m$ out of $2\bar{d}$ placed attacks. Hence giving $V \leq \frac{m}{2\bar{d}}$.

Staying: Choosing $t_L = \infty$ gives the patroller $\frac{1}{2}$

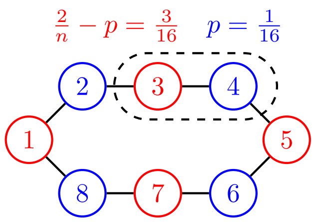
Hence as the patroller can pick from these two options, it gives $V \leq \max\{\frac{1}{2}, \frac{m}{2\bar{d}}\}$. More explicity it gives $V \leq \frac{1}{2}$ if $m < \bar{d}$, and $V \leq \frac{m}{2\bar{d}}$ is $m \geq \bar{d}$. □

6.2 Extension of diametric attack

6.3 Extension of Hamiltonian patrols

We seek to find a larger class of optimal patrols for hamiltonian graphs, by using groups of nodes.

Definition 6.5 (Alternating Random Hamiltonian Patrol(ARHP))**.** An *Alternating Random Hamiltonian Patrol* (*ARHP*) is a mixed strategy following the Hamiltonian cycle but with a probability p of starting at "even" nodes and a probability of $\frac{2}{n} - p$ of starting at "odd" nodes.



Example Figure 6.1: C_8 with the blue nodes being "even" nodes started at with probability $\frac{1}{16}$ and the red nodes being "odd" nodes started at with probability $\frac{3}{16}$.

Lemma 6.6. *When n and m are both even, following the Alternating Random Hamiltonian Patrol, if feasible, gives the same lower bound as the random Hamiltonian patrol, i.e $V \geq \frac{m}{n}$.*

Proof: C.1

If m is odd, say $m = 2m' + 1$ then in the above we get two possibilities for each node depending on the interval choice either $p + \frac{m-1}{n}$ or $\frac{m+1}{n} - p$. So choosing anything other than $p = \frac{1}{n}$ (which is the Random Hamiltonian Patrol strategy) gives a worse result for the patroller.

While not getting a better lower bound, the ARHP does give some control on how to perform optimally in a Hamiltonian graph. The idea of distributing the probability $\frac{2}{n}$ between two types of nodes can be extended to the idea of distributing the probability $\frac{k}{n}$ between k types of nodes (as seen in appendix **??**).

7 Elongated star graph

7.1 Introduction

7.2 Solution for $m \geq 2(k+1)$

8 General star graph

9 Dual star graphs

10 Linked Star graphs

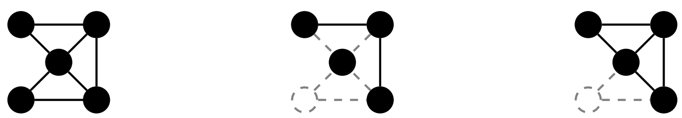
Appendices

A Graph Definitions

Definition A.1 (Graph)**.** A *graph*, $G = G(N, E)$, is made up of: a set of *nodes* (also called *vertices* or *points*), N, which are places ; and a set of *edges* (also called *arcs* or lines), E, which are connections between places, so elements of E must be two-element subsets of N.

Definition A.2 (Subgraph)**.** A graph $Q' = (N', E')$ is said to be a *subgraph* of $Q = (N, E)$ if $N' \subset N$ and $E' \subset E$.

A subgraph is said to be *induced* by N' (or *edge-preserving*) if E' contains all edges (from E) that have both end points in N'.



(a) Q (b) Q_1 (c) Q_1

Figure A.1: Q_1 is a subgraph of Q. However it is not induced as it is missing possible edges connecting nodes that existed in Q. Q_2 shows the induced subgraph on the chosen set of nodes.

Definition A.3 (Walk,Path,Trail,Cycle)**.** A sequence of nodes $(n_0, n_1, ..., n_l)$ is a *walk* of length l if $e_{n_i,n_{i+1}} \in E \ \forall i = 0, ..., l-1$. Corresponding to a walk is the sequnece of l edges $(e_{n_0,n_1}, e_{n_1,n_2}, ..., e_{n_{l-1},n_l})$.

A walk becomes a trail if each edge in the walk is distinct, i.e $e_{n_i,n_{i+1}} \neq e_{n_j,n_{j+1}} \forall i \neq j$. A trail becomes a path if each node in the walk is distinct (except possibly the start and final node), i.e $n_i \neq n_j \forall i \forall i < j \geq l-1$.

A walk,trail or path is said to be *closed* if the start and end nodes are the same. A *cycle* is a closed path with length, $l \geq 3$ (with the special case of $l = 3$ being called a *triangle*).

Definition A.4 (Hamiltonian cycle)**.** A *Hamiltonian cycle* is a cycle which contains every node on the graph, i.e it is a cycle of length $l = |N|$. A graph that exhibits a Hamiltonian cycle is called *Hamiltonian*

Example A.5. For the graph Q as in Figure A.1:

- An example of a walk is $(1, 2, 1, 5, 4, 2)$
- An example of a trail is $(1, 2, 5, 3, 4, 5, 1)$

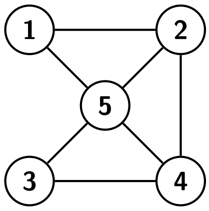


Figure A.1: Graph, Q

- An example of a path is $(1, 2, 4, 3)$
- An example of a Hamiltonian cycle is $(1, 2, 4, 3, 5, 1)$

Hence we would call the graph Q Hamiltonian.

Definition A.6 (Complete graphs)**.** The *complete graph*, K_n, is a graph of n nodes, in which all edges are present, i.e $e_{i,i'} \in E\ \forall i, i' \in N$.

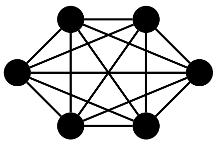
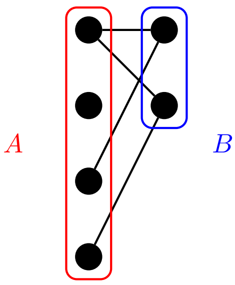


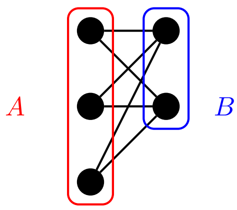
Figure A.2: The complete graph of 6 nodes,K_6.

Definition A.7 (Bipartite)**.** A graph is said to be *bipartite* if $N = A \cup B$, where $A \cap B = \emptyset$,and $e_{i,i'} \notin E\, \forall i, i' \in A$, $e_{i,i'} \notin E\, \forall i, i' \in B$.

Definition A.8 (Complete bipartite)**.** The *complete biparite graph*, $K_{a,b}$, is a bipartite graph of $a + b$ nodes (where $|A| = a$,$|B| = b$), in which all edges are present, i.e $e_{i,i'} \in E\ \forall i \in A\, \forall i' \in B$ and $e_{i,i'} \in E\ \forall i \in B\, \forall i' \in A$.



(a) Q



(b) $K_{3,2}$

Figure A.3: A.3a is an example of a bipartite graph, Q. A.3b is the complete bipartite graph with set sizes of 3 and 2.

Definition A.9 (Subdivision,Smoothing)**.** A *Subdivision* (or *expansion*) of a graph, G, is a new graph G' which is made by subdividing a chosen edge. The subdivision of an edge $\{u, v\}$ yields a graph with a new node w and the splitting of the edge $\{u, v\}$ into $\{u, w\}$ and $\{w, v\}$.

The reverse process is called *Smoothing* of a graph, G, is a new graph G' which is made by smoothing between two nodes. The smoothing out of a node pair (u, v), with $d(u, v) = 2$ and with w between them, then w is removed along with the edges $\{u, w\}$ and $\{v, w\}$, then the edge $\{u, v\}$ is placed to connect u and v.

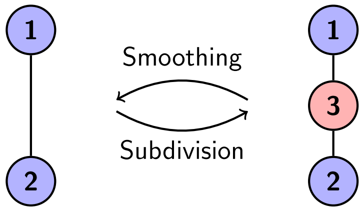


Figure A.4: Subdivision and Smoothing of the edge $\{1, 2\}$

B Old Statements about Old diametric

Ignoring the problem with the old diametric bound, the diametric idea from [**?**] can be extended when their are multiple diametric pairs.

Definition B.1 (Diametrical set)**.** Let $D = \{i \in N \,|\, d(i, i') = \bar{d} \quad \forall i' \in D\}$

Lemma B.2. *By the attacker playing equally likely all nodes in D, at a random time interval, called the extended diametric attack gives* $V \leq \max\left\{\frac{m}{|D|\bar{d}}, \frac{1}{|D|}\right\}$

Proof. The options for the patroller are to wait at a node and have probability $\frac{1}{|D|}$ or to move between a selection of these nodes and then wait(for some time) or move on.

The comparison of m to $\bar{d}$ is important (if $m \leq \bar{d}$ waiting at any given node is good), moving is only beneficial if more attacks can be intercepted by moving rather than waiting that is that $m > \bar{d}$, as this means moving to another node claims m attacks each time compared to waiting which would just give the $\bar{d}$(as this is the time to move), so in this case moving is always better than waiting.

When moving, moving to a the node visited the furtherest in the best will give us the best chance of no-overlap, hence a cycle is formed.

So forming a cycle between the $|D|$ nodes each a distance $\bar{d}$ apart means the probability of intercepting is $\frac{m}{|D|\bar{d}}$.

□

Example B.3. For S as seen in Figure B.1. By the attacker attacking with equal probability on the diametric set of the four 3 nodes, which have $\bar{d} = 4$. This gives that $V \leq \max\{\frac{m}{4\times4}, \frac{1}{4}\} = \max\{\frac{m}{16}, \frac{1}{4}\}$.

By its simplification from C_{16} , we can get that $V \geq \frac{m}{16}$.

Hence in the region of $m \geq 4$ we get that $V = \frac{m}{16}$

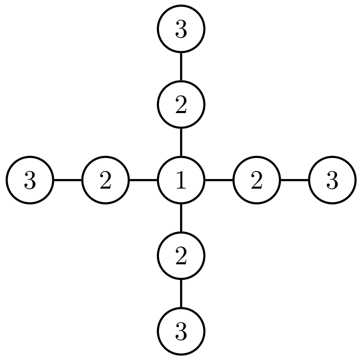


Figure B.1: Graph S used in example

C Proof's

C.1 Alternating Random Hamiltonian proof

Proof. During any attack interval I which is of even length, then $W(I)$ contains m' "even" and m' "odd" nodes for a total of $m = 2m'$ nodes. Therefore by following the Alternating Random Hamiltonian Patrol, $\boldsymbol{\pi}_{ARHP}$, with probability p at "even" nodes and probability $\frac{2}{n} - p$ at "odd" nodes. Then

$$P(\boldsymbol{\pi}_{ARHP}, [i, I]) \geq \underbrace{\overbrace{p}^{\text{even node}} + \overbrace{\frac{2}{n} - p}^{\text{odd node}} + p + \frac{2}{n} - p + ... + p + \frac{2}{n} - p}_{m=2m' \text{ elements}}$$

$$= m'p + m'(\frac{2}{n} - p) = \frac{2m'}{n} = \frac{m}{n} \quad \forall i \in N \quad \forall I \subseteq \mathcal{T}$$

Hence as it holds for all pure attacks

$$P(\boldsymbol{\pi}_{ARHP}, \boldsymbol{\phi}) \geq \frac{m}{n} \quad \forall \boldsymbol{\phi} \in \Phi$$

Hence $V \geq \frac{m}{n}$. $\square$

If m is odd, say $m = 2m' + 1$ then in the above we get two possibilities for each node depending on the interval choice either $p + \frac{m-1}{n}$ or $\frac{m+1}{n} - p$. So choosing anything other than $p = \frac{1}{n}$ (which is the Random Hamiltonian Patrol strategy) gives a worse result for the patroller.