Summary of Star Graph Results

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Contents

1 Summary on S_n^k results

1.1 Full Solution for $m \geq 2(k+1)$

- The patroller follows a hamiltonian path giving the bound $V \geq \frac{m}{2(n+k)}$
- The attacker follows the 'time-delayed' attack (attacking at times $0, ..., 2k+1$ at elgonated node and times $k, k+1$ at normal external nodes) giving the bound $V \leq \frac{m}{2(n+k)}$

Hence we have $V = \frac{m}{2(n+k)}$.

State of proof: Needs reworking to be more accessible

1.2 Full solution for $m = 2k+1, 2k$

We further split this into two regions

- $m = 2k$ or $m = 2k+1$ and $m \geq 2n-1$: First it is worth noting that $R = \emptyset$, $M = \emptyset$.
  - The patroller follows the Combinatorial Improvement, giving the bound $V \leq \frac{\lfloor \frac{m}{2} \rfloor}{\lfloor \frac{m}{2} \rfloor + n - 1} \frac{2k}{2k+2(n-1} = \frac{m}{m+2(n-1)}$
  - The attacker 'augments' their 'time-delayed' attack on $m \geq 2(k+1)$ to place attacks at times $0, ..., 2k-1$ at the elongated node and times $k-1, k$ at normal external nodes, giving the bound $V \geq \frac{2k}{2k+2(n-1)}$

  Hence we have $V = \frac{2k}{2k+2(n-1)}$. Noting: This means that for $m = 2k \implies V = \frac{m}{m+2(n-1)}$.

  State of proof: Needs to be done

- $m = 2k+1$ and $m \leq 2(n-1)$: It is worth noting that from the above, the transition has happened as the attackers strategy is no longer true.
  - The patroller will follow a decomposition into S_{n-1} and L_{k+1} giving $V \geq \frac{1}{\frac{1}{V(S_{n-1})} + \frac{1}{V(L_{k+1})}} = \frac{1}{1 + \frac{2(n-1)}{2k+1}} = \frac{2k+1}{2k+1+2(n-1)} = \frac{m}{m+2(n-1)}$.
  - The attacker will 'augment' their 'time-delayed' attack to placed attacks at times $0, ..., 2k$ at the elongated node and times $k, k+1$ (or $k-1, k$ is proposed to work just as well), giving the bound $V \leq \frac{2k+1}{2k+1+2(n-1)} = \frac{m}{m+2(n-1)}$.

  Hence we have $V = \frac{m}{m+2(n-1)}$. Noting: This type of patroller strategy is just as good for the case of $m = 2k$ when $m \leq 2(n-1)$ and gives the same bound.

  State of proof: Needs to be done

2 Summary on $S_n^{\boldsymbol{k}}$ results

2.1 Full Solution for $m \geq 2(k_{\max} + 1)$

- The patroller follows a hamiltonian path giving, $V \geq \frac{m}{2(n+\sum k_i)}$
- The attacker uses the 'time-delayed' attack, attacking an external node which is i away from the origin at times $k_{\max} - (i-1), ..., k_{\max} + i$ giving, $V \leq \frac{m}{2(n+\sum k_i)}$

Hence we have $V = \frac{m}{2(n+k)}$.

State of proof: Needs reworking to be more accessible (It also covers the S_n^k case in Subsection 1.1)

2.2 Proposition for $m = 2k_{\max} + 1, 2k_{\max}$

A proposed similarity is drew to Subsection 1.2. By augmenting the 'time-delayed' attack and by using a Combinatorial Improvement (which needs to be developed). The similar logic should hold for the 'time-delayed' augmentation, but the Combinatroial Improvement may not yield the same results.

The main issue with getting this is how to change the Combinatorial Improvement to factor in the variety of branches. We may have multiple sets of M type nodes and R type node depending on the length of branches. A set may exist for each branch. Then we need to decide to how to use end-ensuring cycles ?

One suggestion might be to use one on each node where return to the centre is not possible, i.e Has some M or R , then the ones in which we can include in the choosing we 'bundle' together to use the minimum number of end-ensuring cycles.

2.3 $S_n^{\overbrace{(k,...,k)}_{n \textbf{ times}}}$

We will use the fact that the graph is bipartite with some number in the biggest group, depending on if k is odd or even

- If k odd: Then we get $n \times \frac{k+1}{2}$ along the branches and 1 from the centre. Getting $\frac{n(k+1)+2}{2}$ so $V \leq \frac{m}{n(k+1)+2}$.

  Note. This does not align well when $m \geq 2(k+1)$ as we are getting a much weaker upper bound here.

- If even: Then we get $n \times (\frac{k}{2} + 1)$ along the branches (none in centre). Getting $\frac{n(k+2)}{2}$ so $V \leq \frac{m}{n(k+2)}$.

  Note. This aligns perfectly when $m \geq 2(k+1)$ with $\sum k_i = \sum k = nk$, providing the attacker a differnt attack

3 Summary on $S_n \overset{l}{-} S_n$

Appendices

A First appendix section