POLITECNICO
MILANO 1863

Financial Engineering 2024/25

Report Assignment 3 FE

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QUESTIONS:

Introduction

The main goals of this assignment are: computing the Asset Swap Spread over Euribor 3m for a 5-year annual bond, calculating the CDS, and computing the probabilities using both the exact bootstrap method and two approximated approaches (neglecting the accrual term and using the Jarrow-Turnbull method). Furthermore, given the survival probability, the default time has been simulated and the survival probability fitted. Finally, the problem of pricing a Mezzanine Tranche and an Equity Tranche has been addressed.

1) Asset Swap Spread Over Euribor 3m

To compute the Asset Swap Spread over Euribor 3m, starting from the discounting curve on January 31, 2023, at 10:45 CET, and considering a bond with a maturity date of March 31, 2028 (issued on March 31, 2022), a fixed annual rate of 4.8%, and a floating rate linked to Euribor 3m, the following formula was used:

$$S_{\text{asw}} = \frac{C(0) - \bar{C}(0)}{\text{BPV}_f(0)}$$

where $C(0)$ is the price of the coupon bond in the interbank market, $\bar{C}(0)$ is the dirty price (obtained by adding the computed accrual term to the provided clean price), and $\text{BPV}_f(0)$ is the basis point value of the floating leg.

S_{ASW}	87.429 bps

Table 1: Spread in Asset Swap

The obtained asset swap spread indicates an higher perceived risk compared to a risk-free benchmark.

2) CDS Bootstrap

Considering the data obtained via the bootstrap of the previous case study, ENI as obligor, a recovery equal to 40% and the following CDS spreads (annual bond):

Years	1	2	3	4	5	6
CDS Spread (bps)	40	44	47	49	51	52

Table 2: CDS Spreads

we compute the intensities and the survival probabilities using three different approaches:

1. Neglecting the accrual term, with $\lambda(t)$ piecewise constant.
2. Considering the accrual term, with $\lambda(t)$ piecewise constant.
3. Jarrow-Turnbull approximation, considering a constant λ and continuously paid CDS spread.

2.1 Neglecting the accrual

To address Method 1, the survival probabilities and the intensities were computed by applying the following obtained formulas:

$$P(0,T_N) = \frac{(1-\pi)\,E_{\text{cum},N-1}(0) + (1-\pi)\,B(0,T_N)P(0,T_{N-1}) - \bar{s}\,\overline{BPV}_{N-1}(0)}{(1-\pi)B(0,T_N) + \bar{s}\delta_{N-1,N}B(0,T_N)}$$

$$\lambda(T_N) = \frac{1}{\delta(T_N,T_{N-1})}\ln\left(\frac{P(0,T_{N-1})}{P(0,T_N)}\right)$$

where:

- $\overline{BPV}_N = \sum_{i=1}^{N} \delta_{i-1,i} B(0,T_i)P(0,T_i)$
- $E_{\text{cum},N}(0) = \sum_{i=1}^{N} B(0,T_i)[P(0,T_{i-1}) - P(0,T_i)]$

With this Method we attained the following results:

t_i	1y	2y	3y	4y	5y	6y
$P(t_0,t_i)$	0.9868	0.9783	0.9692	0.9600	0.9501	0.9409
λ_i	66 bps	87 bps	94 bps	96 bps	103 bps	97 bps

Table 3: Survival probabilities and intensities neglecting the accrual

These results suggest that, as ENI is a solid company, the survival probabilities remain high throughout the entire time horizon considered.

2.2 Impact of the accrual

In Method 2, the only difference compared to the previously presented approach is the introduction of the accrual term, which modifies the survival probabilities and consequently the intensities as follows:

$$P(0,T_N) = \frac{\left[(1-\pi)E_{\text{cum},N-1}(0) - \bar{s}E^{(\delta)}_{\text{cum},N}(0) + \left(1-\pi-\bar{s}\frac{\delta_{N-1,N}}{2}\right)B(0,T_N)P(0,T_{N-1}) - \bar{s}\overline{BPV}_{N-1}(0)\right]}{(1-\pi)B(0,T_N) + \bar{s}\delta_{N-1,N}B(0,T_N) - \bar{s}\frac{\delta_{N-1,N}}{2}B(0,T_N)}$$

where:

$$E^{(\delta)}_{\text{cum},N}(0) = \sum_{i=1}^{N} \frac{\delta_{i-1,i}}{2} B(0,T_i)[P(0,T_{i-1}) - P(0,T_i)]$$

With this Method we attained the following results:

t_i	1y	2y	3y	4y	5y	6y
$P(t_0,t_i)$	0.9867	0.9782	0.9690	0.9597	0.9499	0.9406
λ_i	67 bps	87 bps	94 bps	96 bps	103 bps	98 bps

Table 4: Survival probabilities and intensities considering the accrual

In order to compare the results obtained using Methods 1 and 2, we compute the absolute differences between the exact and approximated intensities and default probabilities. The errors are as follows:

Error Type	**Max Error**
Default Probability Error (neglecting accrual)	3.07×10^{-4}
Intensities Error (neglecting accrual)	5.40×10^{-5}

Table 5: Maximum Errors Due to Neglecting the Accrual Term

By comparing the tables, we observe that the error between the first method (approximate) and the second one (exact) is on the order of 10^{-4} and 10^{-5}, indicating a very low level of error. This is because the accrual term can be in general neglected, as it is two orders of magnitude smaller than the other term.

2.3 Jarrow-Turnbull approximation

Lastely we consider the Jarrow-Turnbull approximation, according to which the intensities are constant in the time horizon taken under consideration, in particular:

$$\lambda_{JT,i} = \frac{s_i}{1+\pi}$$

and $\lambda_{JT,n} = \frac{1}{n}\sum_{i=1}^{n} \lambda_i$ The results we get are as follow:

t_i	1y	2y	3y	4y	5y	6y
$P(t_0,t_i)$	0.9867	0.9782	0.9691	0.9600	0.9502	0.9410
λ_i	67 bps	87 bps	93 bps	95 bps	101 bps	97 bps

Table 6: Survival probabilities and intensities

In order to compare the results obtained using Methods 3 and 2, we compute the absolute differences between the exact and approximated intensities and default probabilities.

Error Type	**Max Error**
Probability Error (JT approximation)	4.85×10^{-4}
Intensities Error (JT approximation)	1.81×10^{-4}

Table 7: Maximum Errors with JT Approximation

It is evident that the difference between the approximated JT model and the exact one remains minimal. This result aligns with expectations, given that the JT model is traditionally used as a rule of thumb.

Comparisons

The intensities and survival probabilities obtained by applying the 3 different Methods are shown in the following plots:

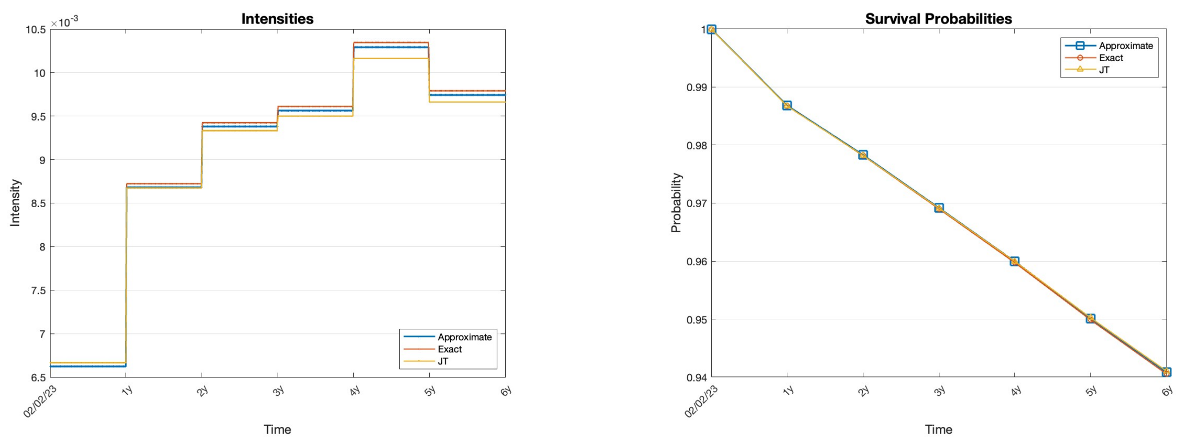

Figure 1: On the left, intensities. On the right, survival probabilities

From figure 3, we can observe that the results are quite similar, indicating that the approximate methods also perform well. Furthermore, as expected from theory, the survival probability decreases over time: uncertainty increases as time horizon increases.

Looking at Tables 5 and 7, we can notice that the approximation of Jarrow & Turnbull tends to be less precise. The errors obtained suggest that this approach is less accurate than the first one, which neglects the accrual term.

3) Credit Simulation

Given the survival probability up to 30 years:

$$P(t,T) = e^{-\int_t^T \lambda_s \, ds}$$

where

$$\lambda_s = \begin{cases} \lambda_1 & \text{if } s \leq \theta, \\ \lambda_2 & \text{if } s > \theta, \end{cases}$$

with $\lambda_1 = 5\,\text{bps},\ \lambda_2 = 9\,\text{bps},\ \theta = 4\,\text{years}$ we perform the credit simulation.

Simulation of the default time τ

We simulate the default time τ, considering a two-regime model factor for default risk intensity, acording to which the intensity of default changes at a specific threshold time θ, resulting in two distinct hazard rates. A random seed is set to ensure reproducibility. A vector u of M = 10^5 simulation from a uniform distribution in [0,1] is generated. In particular,

$$u = e^{-\int_{t_0}^{\tau} \lambda(s)\, ds}$$

with $t_0 = 0$ that represents the current time. Given the provided shape of the intensity function, that is piecewise constant, we get the corresponding τ as follows:

$$\tau = \begin{cases} -\frac{\ln(u)}{\lambda_1} & \text{if } u \in \left(e^{-\lambda_1 \theta}, 1\right), \\ \frac{-\ln(u) + \theta(\lambda_2 - \lambda_1)}{\lambda_2} & \text{otherwise} \end{cases}$$

We plot the simulated survival probabilities on a log-log scale.

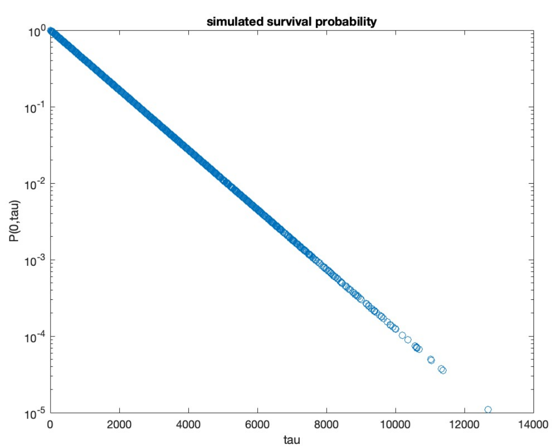

Figure 2: simulated survival probability

Fit the survival probability

In order to fit the survival probabilities, we compute the Maximum Likelihood Estimators using:

$$\hat{\lambda}_1 = \frac{n_1}{\sum_{i:\tau_i \leq \theta} \tau_i + n_2\theta} \tag{1}$$

$$\hat{\lambda}_2 = \frac{n_2}{\sum_{i:\tau_i > \theta} (\tau_i - \theta)} \tag{2}$$

We then compute their 95% confidence intervals by constructing the Fisher Information Matrix. The lengths of the intervals are calculated to assess precision. The results obtained are shown in the following table:

λ_1^{MLE}	5.3305×10^{-4}
λ_2^{MLE}	9.0002×10^{-4}
CI λ_1	$10^{-4} \times [4.615, 6.046]$
CI λ_2	$10^{-4} \times [8.944, 9.056]$

Table 8: MLE Estimates and Confidence Intervals for λ_1 and λ_2

Looking at the values in Table 8, we can observe that the actual values of $\lambda_1 = 5\,\text{bps}$ and $\lambda_2 = 9\,\text{bps}$ lie within their respective confidence intervals, as do their MLE estimators. Furthermore, we apply an empirical

method in which the experimental survival probabilities are computed using a cumulative sum approach, normalized by the total number of simulations M.

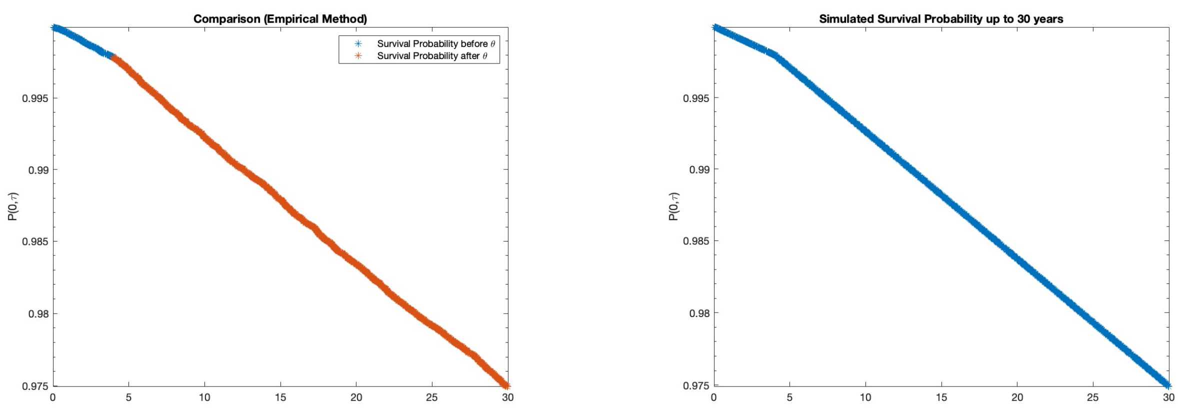

Figure 3: On the left, comparison of survival probabilities before and after θ. On the right, simulated survival probability

Figure 3 suggests how survival probabilities evolve over time. It is evident that there is an exponential relationship between the time to default and the survival probability.

4) MBS Pricing

The goal is price a Mezzanine tranche and an equity tranche using three different techniques:

- the Exact solution
- the approximated solution (Kullback Leibler approximation)
- the Large Homogeneus Portfolio (LHP) solution.

The mezzanine tranche and the equity tranche are notes issued by an SPV to finance a reference portfolio. The equity tranche has a higher risk and return than the Mezzanine tranches. In fact, payments are received only after the Senior Tranche and all the Mezzanine Tranches have been paid.

Mezzanine tranche:

For the the Exact solution and the Kullback Leibler approximation techniques, the number of obligors I changes between 10 and $2 * 10^4$. In Figure 4 we can see the plot of the results on a log-lin scale. In particular we observe that, in agreement of the Central Limit Theorem, with the increase of the obligors I the prices of the two first techniques converge towards the price computed with the LHP methods, which is **0.7527**.

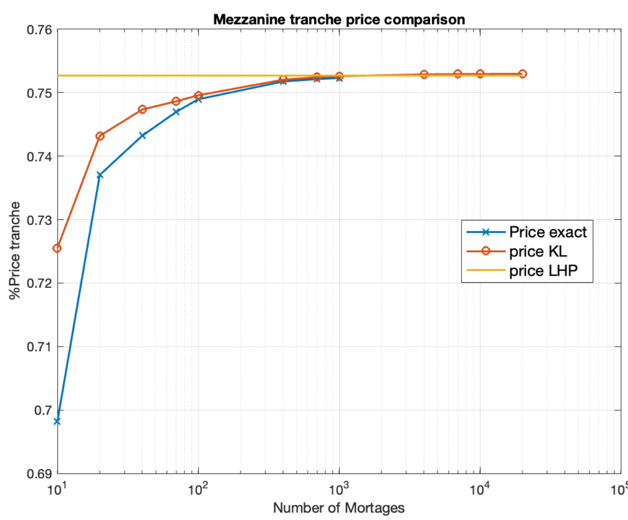

Figure 4: Mezzanine Tranche price Comparison

In the exact solution there is a factorial, so we can compute the price of the mezzanine tranche up to I = 10^3.

Equity tranche:

Now we consider an equity tranche and, computing its value as we did with the mezzanine tranche, we obtain the plot in Figure 5.

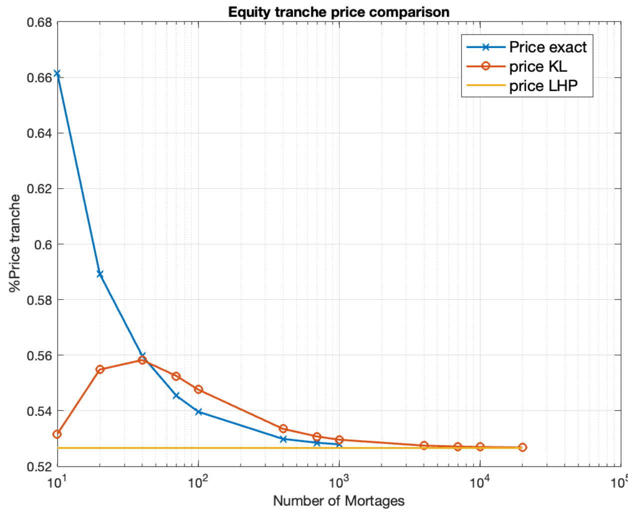

Figure 5: Equity Tranche Price Comparison - KL approximation is inadeguate

We can see that the KL approximation is not adequate. In fact, the price of the stock erodes starting with the first defaulted obligor, but this causes problems because the Stirling formula used in calculating the approximate solution does not work near 0. The discrepancy between the exact solution and the KL approximation is particularly evident for small I.

A possible way to solve this issue is to calculate the expected loss of the equity as the difference between the expected loss of the reference portfolio and the loss of the Mezzanine tranche opposite to the equity. In particular, we consider the tranche with detachment points $K_d = 0.05$ and $K_u = 1$. It is important to pay attention to the notional that are different between the reference portfolio, the equity and the mezzanine.

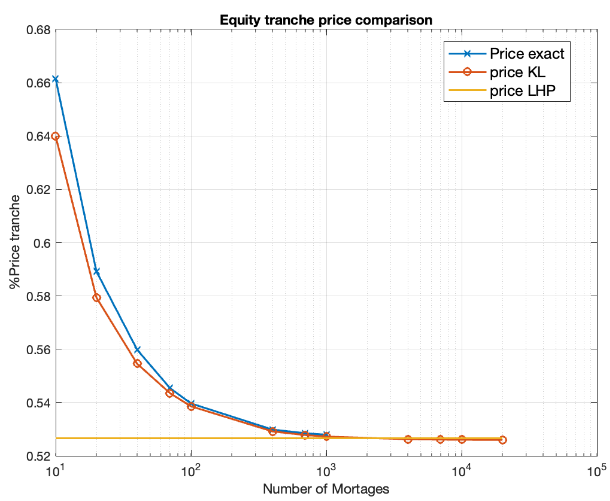

Figure 6: Equity Tranche Price Comparison

We can see in Figure 6 how with this trick the KL approximation becomes correct and near the exact solution. The price obtained with this two techniques with the increase of the obligors I converge towards the LHP solution that has price **0.5266**.