POLITECNICO
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Report Assignment 1

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Contents

QUESTIONS:

Introduction

This report provides an overview of the methodologies used to price a European call option. It covers analytical methods such as the Black formula, the Cox-Ross-Rubinstein (CRR) binomial model, and the Monte Carlo approach. A comparative analysis is conducted to assess convergence speed and approximation error. Before proceeding with the numerical analysis, we outline the different pricing methods employed. We have 1 million contracts; however, in the following discussion, we will present the results for just one of them.

Parameters

The parameters used for pricing a European call option are as follows:

- $S_0 = 1$ (Initial underlying price, 1 Euro)
- $K = 1.05$ (Strike price)
- $r = 0.025$ (Annualized risk-free interest rate)
- $TTM = \frac{1}{3}$ (Time to maturity in years, i.e., 4 months)
- $\sigma = 0.21$ (Annualized volatility)
- $flag = 1$ (Option type: 1 for Call, -1 for Put)
- $d = 0.02$ (Dividend yield)

The discount factor and the forward price are given by:

$$B = e^{-r\cdot \mathrm{TTM}} \quad \text{and} \quad F_0 = S_0 \cdot e^{(r-d)\cdot \mathrm{TTM}}$$

a. Pricing an European Option

• Closed Black Formula

The Black-Scholes pricing model is implemented using the function `EuropeanOptionClosed`:

$$\text{optionPrice} = \texttt{EuropeanOptionClosed}(F_0, K, B, T, \sigma, flag)$$

- **Cox-Ross-Rubinstein Binomial Tree**

The CRR method divides the time to expiration into N steps. At each step, the underlying asset follows an up/down factor, and the option price is obtained by backtracking through the binomial tree until the root. The model's error decreases approximately as $O(1/N)$.

$$\text{optionPrice} = \texttt{EuropeanOptionCRR}(F_0, K, B, T, \sigma, N, flag)$$

- **Monte Carlo**

In the Monte Carlo approach, M simulated paths of the underlying asset price are generated using standard normal random variables. The option price is determined by averaging the discounted payoffs. The error follows an order of $O(1/\sqrt{M})$. Furthermore, a fixed seed is used to ensure reproducibility.

$$\text{optionPrice} = \texttt{EuropeanOptionMC}(F_0, K, B, T, \sigma, M, flag)$$

In Table 1, we present the results obtained by applying the three different approaches. As can be seen, M=100 is sufficient for the CRR method, while for the Monte Carlo method, more simulations are needed to obtain a more precise result.

Method	**Price**
Closed-Form	0.028874
Cox-Ross-Rubinstein	0.028805
Monte Carlo	0.025368

Table 1: Comparison of option prices using different methods

b. Selection of M

Using the CRR model, increasing N rapidly reduces the error. The first value of N, considering it as a vector of powers of 2, that results in an error below 1bp is $N = 128$. However, further analysis suggests that the optimal number of steps is $N = 80$ as it can be deduced also looking at figure 1.

For the Monte Carlo method, increasing M reduces the standard error following an inverse square root law. The desired accuracy is reached for $M = 409600$ simulations.

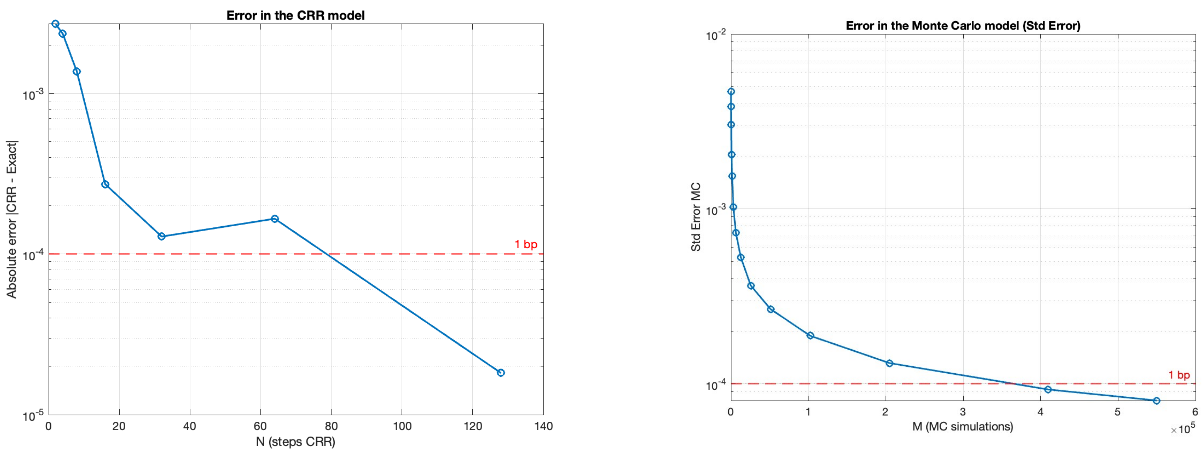


Figure 1: On the left, error CRR. On the right, error MC

The choice between methods depends on the trade-off between computational speed and precision.

To sum up, this implementation compares different European option pricing techniques, evaluating their accuracy and convergence speed against a reference exact price. The accuracy is controlled by adjusting the number of steps N in the CRR model or the number of simulations M in the Monte Carlo approach.

c. Errors

We want to show that the numerical errors for a Call Option scale approximately as $1/M$ for the CRR method and as $1/\sqrt{M}$ for the Monte Carlo method.

Through the function:

$$\text{optionPrice} = \texttt{PlotErrorCRR}(F_0, K, B, T, \sigma)$$

we compute the error of the CRR method. Specifically, for each element of the vector M, which consists of powers of 2, we calculate the error as the absolute difference between the price obtained using the CRR method and the exact price, $\text{Error}_{CRR} = |C_{CRR}(0,T) - C(0,T)|$.

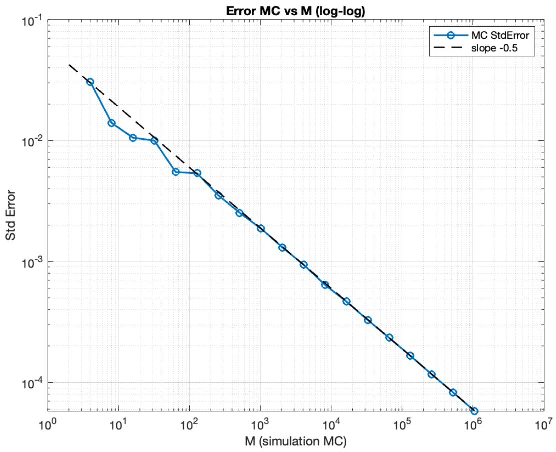
To calculate the error of the Monte Carlo method, we use the function:

$$[\text{M}, \text{stdEstim}] = \texttt{PlotErrorMC}(F_0, K, B, T, \sigma)$$

This function calculates the unbiased standard error of the Monte Carlo method $SE = \frac{\text{std}(\text{Payoff}_{\text{discounted}})}{\sqrt{M}}$ estimated for the vector M composed of powers of 2.

In Figure 2 we see the MC and CRR error graph on a log-log scale, confirming the expected results.

The oscillatory behaviour showed in the plot related to the CRR error is due to numerical approximation. However, by examinating the overall trend, we can appreciate that it behaves like $1/M$ as expected.



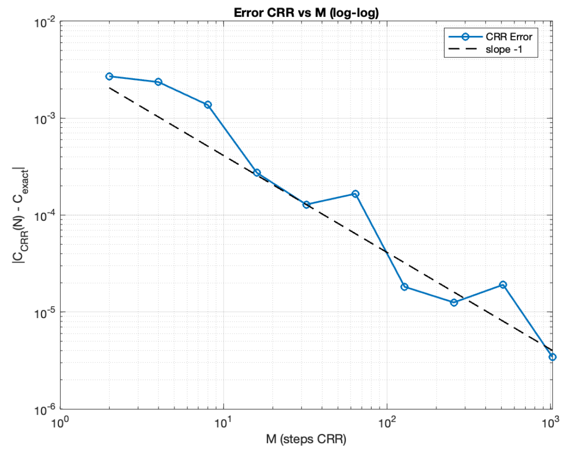


Figure 2: Comparison of numerical errors for MC (on the left) and CRR (on the right) methods

d. Pricing an European Call Option with European Barrier

To price the European Call Option with a European barrier KO, we defined three ad hoc functions: two for numerical methods, namely Monte Carlo and the CRR (Cox-Ross-Rubinstein) method, and one for the computation through the closed formula.

For the Monte Carlo and CRR methods, it is sufficient to proceed as for the standard Call Option, while changing the payoffs to match those of the Barrier Option.

We compute the closed formula by replicating the payoff with a long position on a Call with strike K, a short position on a Call with strike KO (Bull Call Spread) and a Cash or Nothing option with strike KO and princial KO-K and assuming the Absence of Arbitrage. The resulting formula is:

$$C_K - C_{KO} - (KO - K) \cdot B(T) \cdot N(d_{2,KO})$$

We have obtained the following results:

Method	Price
Barrier option Closed Formula	0.027923
Barrier option CRR	0.028133
Barrier option MC	0.027906

Table 2: Prices of the Barrier Option using different methods

For the numerical methods, we used the optimal values of M and N found above. We observe that the price obtained using the CRR approach is slightly inaccurate. This may be due to the fact that we are pricing an exotic option. However, as the number of time steps increases, the value converges to the one obtained via the closed formula. Indeed for N = 130 we get the exact price.

e. Vega of an European Call Option with European Barrier

To compute the vega of the Barrier Option, we have designed one function that can handle the three different computation methods: CRR, Monte Carlo (MC) and the closed formula.

For the two numerical methods, we have used the standard centered numerical approximation after computing the prices with the respective methods. The closed formula is as follows:

$$\nu = \frac{\partial \text{Barrier}}{\partial \sigma} = \nu_K - \nu_{KO} - B \cdot (KO - K) \cdot \frac{e^{-\frac{d_{2,KO}^2}{2}}}{\sqrt{2\pi}} \cdot \left(\frac{\log(KO/F_0)}{\sigma^2 \sqrt{T}} - \frac{1}{2}\sqrt{T} \right) \tag{1}$$

where

$$\nu_K = B \cdot F_0 N'(d_1)\sqrt{T} \quad \text{and} \quad \nu_{KO} = B \cdot F_0 N'(d_{1,KO})\sqrt{T}$$

where the last term in 1 is the derivative with respect to σ of the Cash-or-Nothing option discussed above.

Below, we present the plots of the vega obtained using the three different approaches, assuming the underlying price ranges from 0.65 to 1.45. We observe that the CRR method does much worse in approximating the true value compared to Monte Carlo.

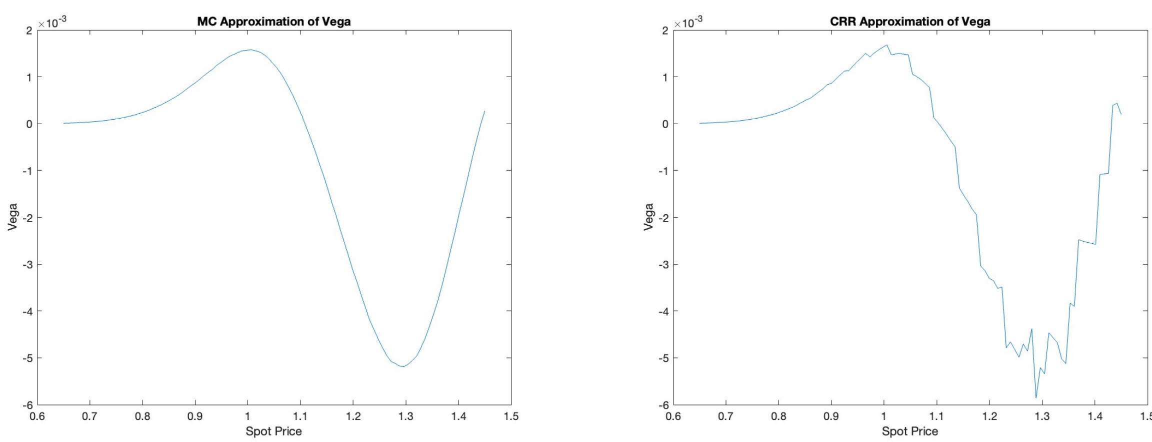


Figure 3: Comparison of Vega estimations with Monte Carlo (on the right) and CRR (on the left)

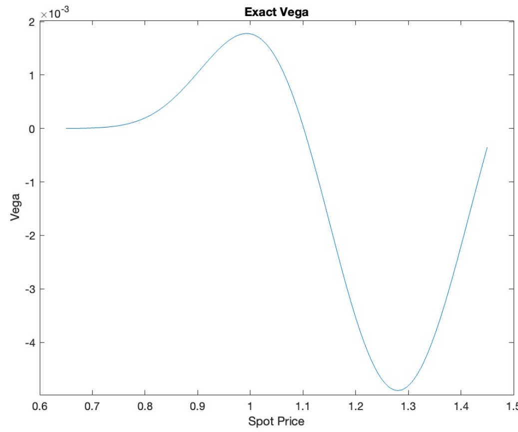


Figure 4: Exact Vega (Closed Formula)

From the plot, we can see that the vega of an up-and-out Call option is positive when the underlying asset price is close to K and negative when it is near to KO. This can be explained by the fact that when the asset approaches K, the option has a higher probability of being in the money at expiry, making the vega positive. Conversely, when the underlying price is close to the barrier KO, the risk of the option being knocked out increases. In this case, higher volatility decreases the option's value because it raises the chance that the asset price will hit the barrier, causing the option to expire worthless.

f. Antithetic Variables MC

The main idea behind the antithetic variables technique is to reduce variance by exploiting a specific relationship between two correlated random variables. In our case, the first variable, g_1, is generated from a standard random sample, while the second variable, g_2, is obtained by inverting the sign of all the random samples of g_1. The final payoff is then computed as the average of the payoffs generated by g_1 and g_2, and this is used to estimate the option price and its standard error. As expected, the figure 5 shows that the convergence rate remains the same, but the use of antithetic variates leads to significantly greater accuracy.

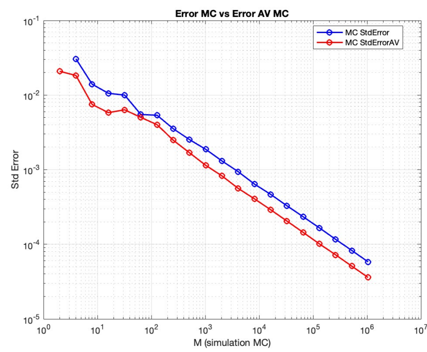


Figure 5: Comparison of MC errors (blue) with MC errors using AV (red)

g. Pricing a Bermudan Option with CRR

To price a Bermudan option, where the holder has the right to exercise the option at the end of each month, we utilized the Cox-Ross-Rubinstein (CRR) approach. Working backward in time, we computed the continuation value at the nodes where early exercise is feasible and compared it with the intrinsic value. Specifically, at each node in the tree where early exercise is possible, we imposed the following condition:

$$CB(t_1, t_2) = \max(S(t_1) - K, C(t_1, t_2))$$

Using a dividend yield of $d = 2\%$, we obtained the following prices: po

$CB(t_0, T)$	$C(t_0, T)$
0.028963	0.028805

As shown in the table above, the price obtained for the Bermudan option is slightly higher than the price of the European option, suggesting that there exists at least one date where early exercise is optimal.

h. Pricing a Bermudan Option and an European Option with different dividend yields

To price the Bermudan option while considering dividend yields ranging from 0% to 5%, we applied the same methodology used in Section *g*. To compare how prices fluctuate with changes in the dividend yields, we plotted both trends and their difference as shown in 6

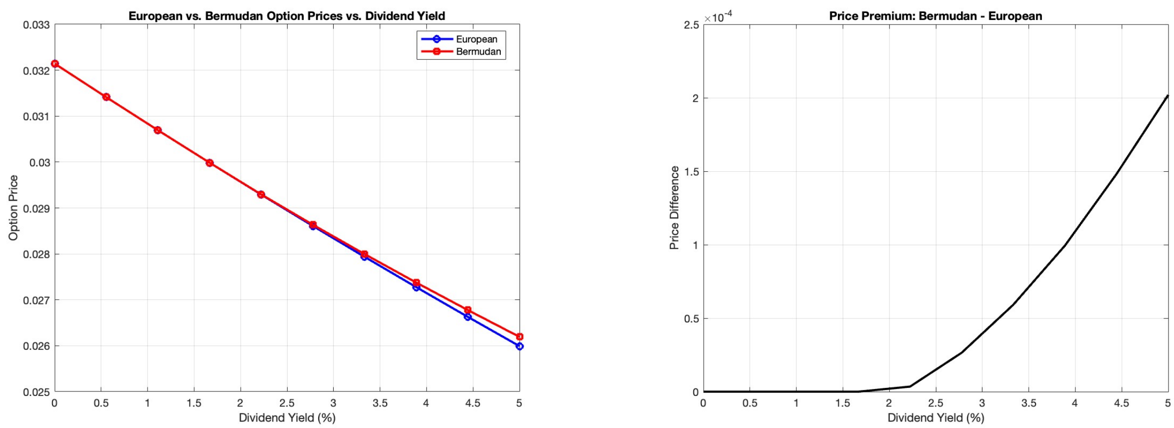


Figure 6: On the left, European Option price and Bermudan Option price wrt dividends. On the right, difference between Bermudan and European price

We observe that when the dividend yield is $d = 0\%$, the two prices coincide. This result is expected, indeed as theoretical models suggest in absence of dividends, early exercise of a Bermudan option is never optimal.

Furthermore, we notice that the price difference increases as the dividend yield d increases, which is consistent with our expectations. This behavior can be attributed to the fact that higher dividends reduce the value of holding the option, making early exercise more attractive.