Financial Engineering 2024/25

Report Final Project 9 B
Pricing of Multi-Name Credit Product

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Abstract

This report presents a Financial Engineering project focused on the valuation and pricing of multi-name credit products, with particular emphasis on the structuring of tranches over a homogeneous mortgage portfolio. The reference portfolio consists of 500 exposures with uniform characteristics (average notional of €2 million, recovery rate of 40%, and a 4-year default probability of 6%). The core of the project involves calibrating a double t-Student copula model under the Large Homogeneous Portfolio (LHP) assumption, with the objective of replicating the market-implied correlations observed across different tranches. Additionally, the impact of the LHP approximation on pricing is investigated, alongside the use of the Kullback-Leibler (KL) expansion to enhance computational efficiency. The analysis also includes a comparison with alternative models such as the Vasicek model and the Gaussian copula model proposed by Li. The entire workflow has been implemented in both MATLAB and Python, ensuring numerical robustness and reproducibility. The methodology closely follows and extends the framework proposed in the reference paper by Hull and White (2004)

Keywords: t-Student copula, LHP assumption, tranche calibration, KL approximation, Gaussian copula, Vasicek model

Introduction

In recent years, the financial industry has increasingly focused on the modelling and management of credit risk in multi-name portfolios. Among the most studied instruments are *Collateralized Debt Obligations (CDOs)*, which allow for the repackaging of credit risk into tranches with different risk-return profiles. The valuation of such instruments critically depends on the *dependence structure among obligors*, typically modelled through copula functions.

This report addresses the pricing of tranches from a homogeneous portfolio of 500 mortgages by employing a *latent factor copula framework*. The project is inspired by the seminal work of **Hull and White (2004)**, which proposes efficient methods to value CDO tranches without relying on Monte Carlo simulation. In particular, we implement and calibrate the *double t-Student copula model*, a generalization of the Gaussian copula that allows for heavier tails and better captures default clustering.

The calibration is performed under the *Large Homogeneous Portfolio (LHP)* assumption, which enables analytical simplifications. Subsequently, the impact of this assumption is assessed by computing exact prices for finite portfolio sizes and comparing them to those obtained via the *Kullback-Leibler (KL)* expansion and the LHP analytical formula. Moreover, *alternative models* such as the *Vasicek one-factor model* and the *Gaussian copula model proposed by Li* are also explored to evaluate their pricing accuracy and consistency.

The computational analysis has been conducted using both **MATLAB** and **Python**, allowing for flexible prototyping and reliable numerical implementation. Each step of the methodology—from calibration to pricing—has been designed with a focus on interpretability, robustness, and consistency with market practices.

The report is structured as follows:

- **Section 1** introduces the reference portfolio and presents the calibration of the double t-Student copula model under the Large Homogeneous Portfolio (LHP) assumption.
- **Section 2** investigates the convergence of tranche prices to the LHP limit as the number of exposures increases, using exact and approximate pricing methods.
- **Section 3** applies the Kullback-Leibler (KL) expansion to perform a more efficient calibration of the copula model and compares the results with those obtained via full numerical integration.
- **Section 4** evaluates the pricing of tranches under the Vasicek Gaussian copula model and contrasts the results with those obtained from the t-Student framework.
- **Section 5** explores the Gaussian copula model proposed by Li, using Monte Carlo simulations to compare its behavior with that of the t-Student model and assess convergence to the LHP benchmark.

Each section builds upon the previous one, gradually increasing the complexity and realism of the pricing framework while ensuring consistency with market practices. The final analysis provides insights into the trade-offs between model accuracy, computational efficiency, and sensitivity to portfolio granularity.

Point A – Calibration of the Double t-Student Model under LHP Assumption

This section presents the calibration of the double t-Student copula model under the *Large Homogeneous Portfolio (LHP)* assumption, which allows for analytical simplifications in tranche pricing for a homogeneous credit portfolio. The objective is to identify the optimal value of the degrees of freedom ν and correlation ρ that best reproduce the market-implied correlations.

Portfolio Setup

The portfolio consists of $I = 500$ homogeneous exposures, each characterized by:

- a notional amount of 2 million€,
- a recovery rate of 40%,
- a default probability over 4 years equal to 6%.

The market-implied correlations for the tranches are:

Ku (%)	Market Correlation
3.00	0.230
6.00	0.260
9.00	0.291
12.00	0.323
22.00	0.355

Method 1 – Calibration from Equity Tranche

In the first approach, the model is calibrated to exactly reproduce the price of the *equity tranche* (0–3%). For each candidate value of ν, a single correlation ρ is estimated such that the price of the equity tranche under the double t-Student model matches that of the Vasicek model with the market-implied correlation.

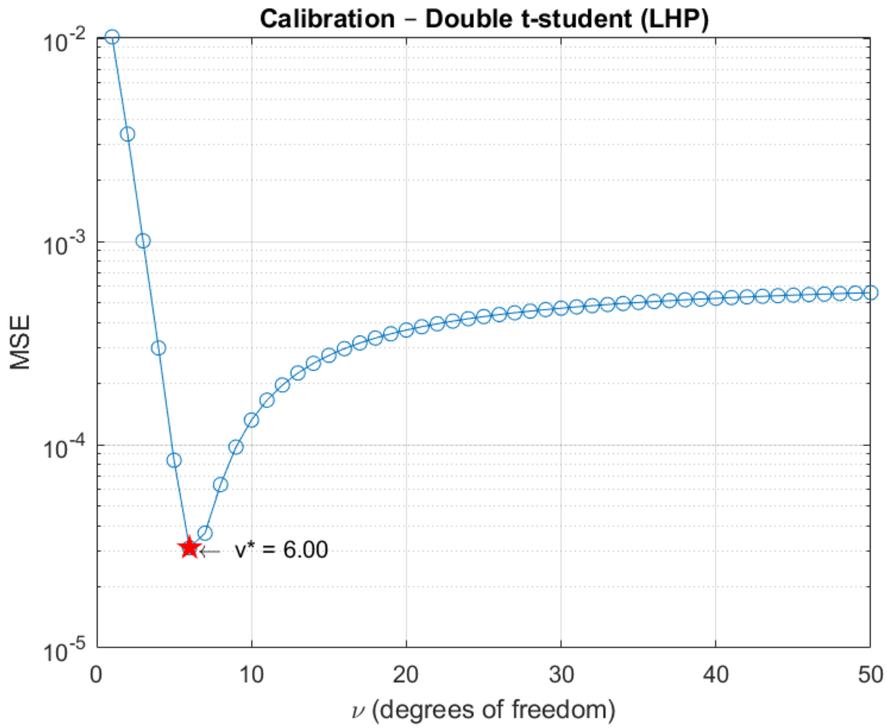

Figure 1: MSE vs ν – Calibration based on equity tranche (Method 1)

Once calibrated, the model-implied correlations for the remaining tranches are computed and compared to the market-implied ones. A curve of MSE(ν) is then constructed, and the optimal ν^* is identified as the minimizer.

The result of this calibration is :

- $\nu^* = 6$
- $\rho^* = 0.32$
- $\text{MSE} = 3.108 \times 10^{-5}$

This method ensures a perfect fit to the equity tranche, but shows limitations in matching the senior tranches.

Method 2 – Tranche-by-Tranche Calibration

In the second approach, we would like to give a correlation for every cumulative tranche with the double-t-student model as it is done in the market. Then, different correlation ρ_j^* is calibrated for each tranche and for each value of ν. For every cumulative tranche $(0, K_u)$, the correlation is estimated such that the price produced by the double t-Student model matches the price obtained with the Vasicek model under the corresponding market-implied correlation.

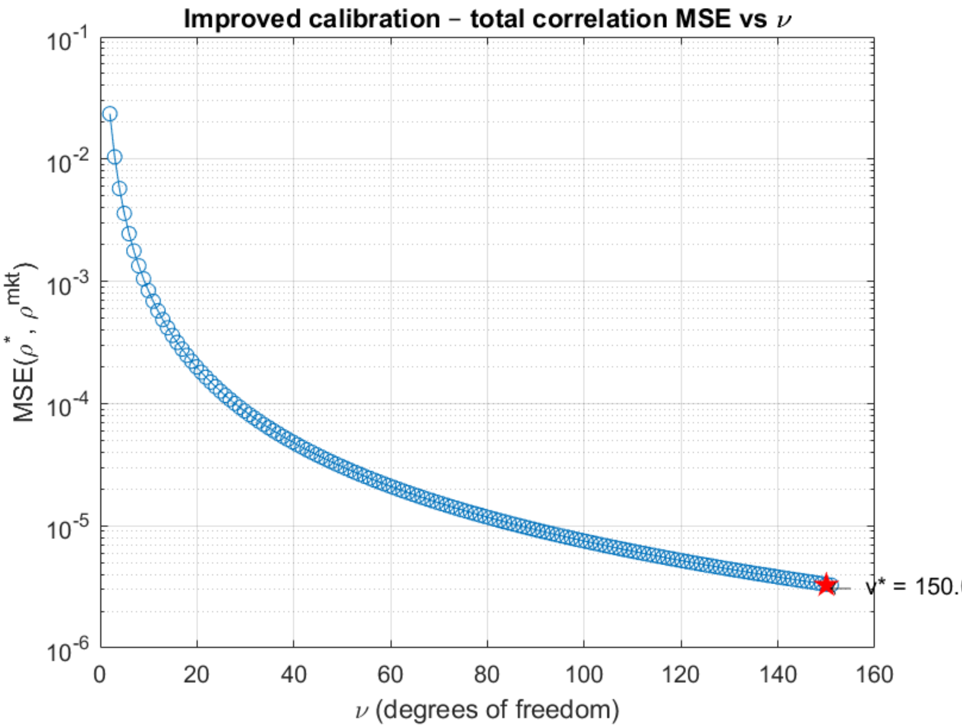

Figure 2: Improved calibration – MSE of total correlation vs ν (Method 2)

The optimal ν is then the one that minimizes the mean squared error across all tranches. The result of this calibration is:

- $\nu^* = 150$
- $\text{MSE} = 3.27 \times 10^{-6}$

The high value of ν^* indicates a behavior that closely resembles a Gaussian distribution, but the fit is significantly improved:

Ku (%)	**Market ρ**	**Calibrated ρ^***
3.00	0.230	0.233
6.00	0.260	0.262
9.00	0.291	0.292
12.00	0.323	0.323
22.00	0.355	0.355

Method 3 – Global Joint Calibration (ν, ρ)

This time, instead of calibrating the correlation to only match the market equity tranche price, we would like to minimize the entire MSE of every cumulative tranche by finding the couple (ν, ρ) which minimizes the global MSE. Then, third method performs a simultaneous optimization of both the degrees of freedom ν

and the copula correlation ρ, minimizing the global Mean Squared Error across all tranches. This is a truly joint calibration that avoids giving priority to any single tranche. The optimization is conducted using the `fminsearch` function in MATLAB, starting from an initial guess $(\nu_0, \rho_0) = (10, 0.30)$. This method performs an unconstrained, derivative-free minimization of the global error function.

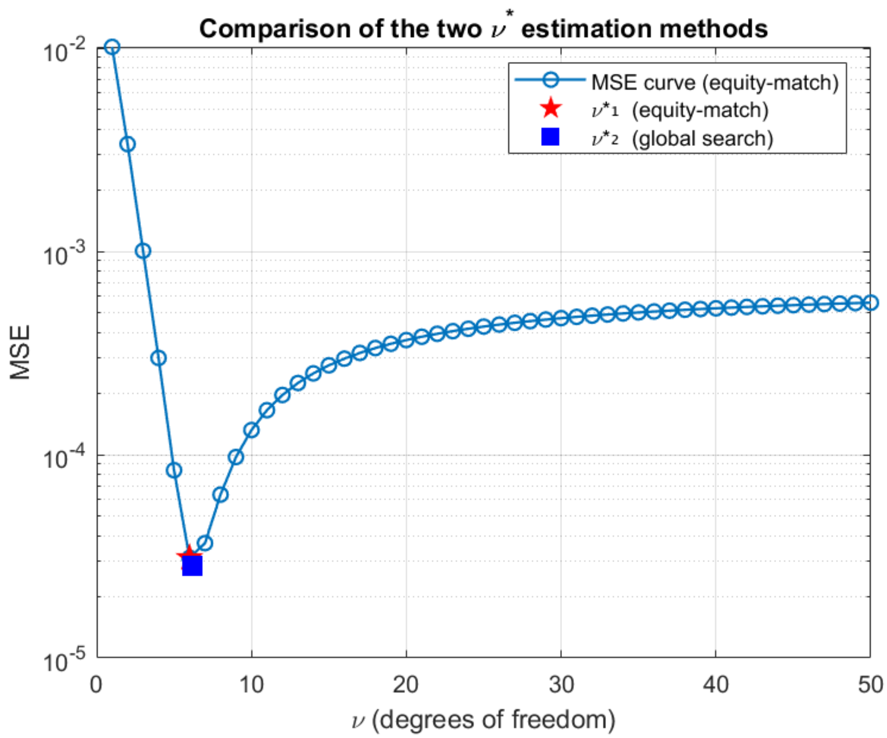

Figure 3: Comparison of ν^* estimates – Method 1 vs Global (Method 3)

The loss function is the average squared deviation between market prices and those implied by the double t-Student model evaluated over all tranches:

$$\text{MSE}(\nu, \rho) = \frac{1}{n} \sum_{j=1}^{n} \left(p_j^{\text{model}}(\nu, \rho) - p_j^{\text{market}} \right)^2$$

Compared to Method 1, which focuses only on equity, this method attempts to find a pair (ν^*, ρ^*) that provides the best overall fit.

The optimal solution obtained is:

- $\nu^* = 6.19$
- $\rho^* = 0.316$
- $\text{MSE} = 3.58 \times 10^{-5}$

Interestingly, the optimal values coincide with those found using Method 1.

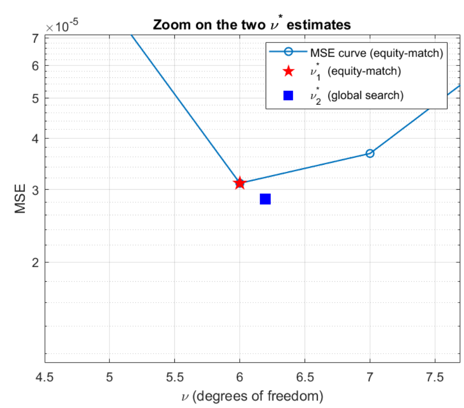

Figure 4: Zoom around optimal values of ν^*
Method 1 vs Method 3

This confirms that the equity tranche is the most influential in the global loss function due to its higher sensitivity to the shape of the copula. However, Method 3 validates the result from a different optimization standpoint, providing a consistency check. Moreover, a closer inspection of the error function (Figure 4) shows that the global optimum lies very close to the equity-match solution, with marginal gain in MSE.

Conclusion of Point A

Calibrating the model based solely on the equity tranche yields very good results, as reflected by the low MSE obtained. However, in point 3, we explored an alternative approach by minimizing the pair (ν^*, ρ^*), which produced results very close to those obtained through equity tranche calibration. This confirms that calibrating on the equity tranche remains a simple yet highly effective method for aligning the model with market prices of the product.

Point B – Convergence of Tranche Prices to the LHP Limit

This section evaluates the impact of the Large Homogeneous Portfolio (LHP) assumption on the pricing of synthetic CDO tranches under the double t-Student copula model.

The reference portfolio remains the same as in Point A, with homogeneous exposures characterized by a notional amount of €2 million, a recovery rate of 40%, and a 4-year default probability of 6%. However, the number of exposures I, previously fixed to 500, is now allowed to vary across a wide range, from 10 to 10,000, in order to assess convergence to the LHP limit.

The objective is to quantify how tranche prices evolve as I increases and to compare three different pricing approaches:

- the exact numerical solution (for values of I where computation is feasible),
- the Kullback–Leibler (KL) approximation,
- the analytical LHP formula (valid in the limit as $I \to \infty$).

Methodology

We use the optimal degrees of freedom $\nu^* = 6$ from Point A, along with the calibrated correlation $\rho = 0.32$, and compute tranche prices for a range of portfolio sizes $I \in [1, 1000]$ using three distinct approaches:

- **Exact (HP)** pricing, based on full numerical integration;
- **Kullback-Leibler (KL)** approximation, faster but less accurate for small I;
- **LHP analytical formula**, representing the theoretical asymptotic limit.

We consider the first three cumulative tranches of the CDO structure:

- Equity: 0–3%
- Mezzanine 1: 0–6%
- Mezzanine 2: 0–9%

Results and Interpretation

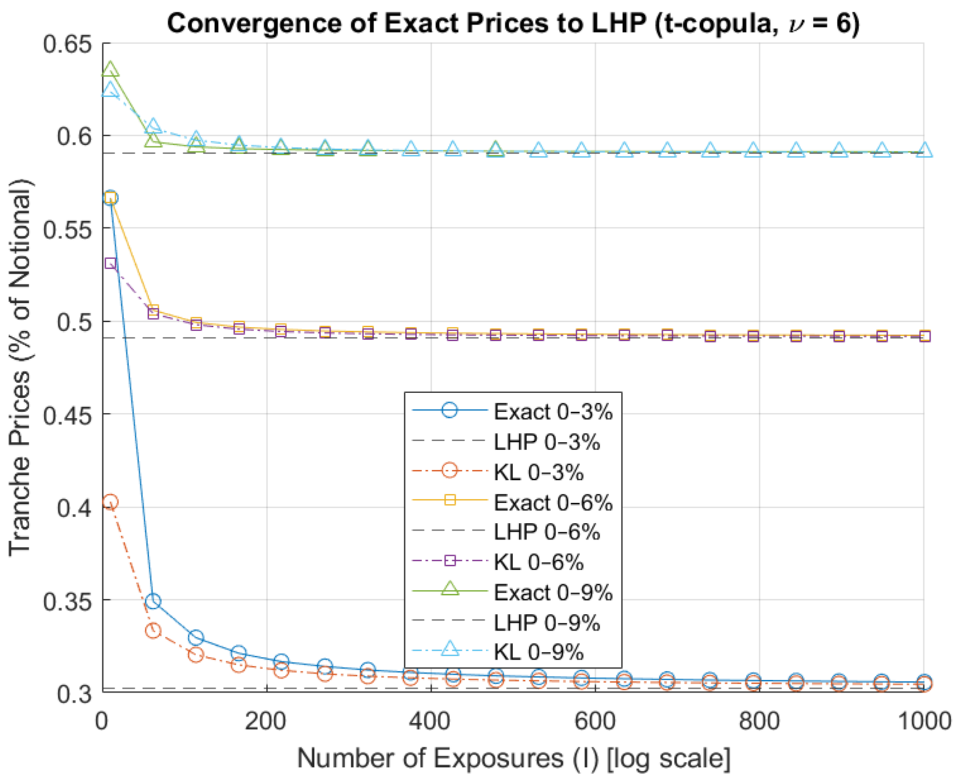

Figure 5: Convergence of exact and KL prices to LHP values (double t-Student, $\nu = 6.19$)

Figure 5 shows the prices of the three tranches as the number of exposures I increases (x-axis in logarithmic scale). Each method is represented with different line styles:

- Solid lines with markers: exact prices;
- Dashed lines: LHP prices;
- Dash-dot lines: KL approximation.

The convergence is clearly visible: both the exact and KL prices approach the LHP benchmark as I increases. For $I > 200$, the difference between the methods becomes negligible. The KL approximation, in particular, closely follows the exact result and proves to be a computationally efficient and reasonably accurate alternative, especially for mid-to-large portfolios. Deviations are more visible for small I, especially in the equity tranche, due to its higher sensitivity to granularity effects.

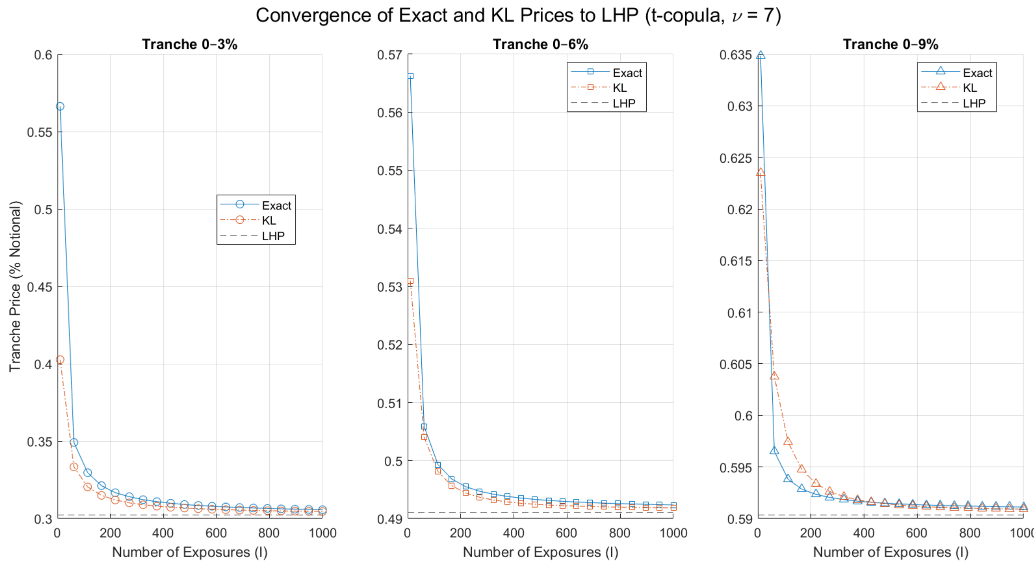

Figure 6: Convergence of tranche prices to the LHP limit under the double t-Student copula model with $\nu = 6.19$ and $\rho = 0.316$

Figure 6 illustrates the price evolution of the three main cumulative tranches (Equity 0–3%, Mezzanine 1 0–6%, Mezzanine 2 0–9%) as a function of the number of exposures I, shown on a logarithmic scale. Solid lines with markers represent exact prices obtained via numerical integration; dashed horizontal lines indicate the theoretical LHP benchmark; and dash-dotted lines correspond to the Kullback–Leibler (KL) approximation.

Several key insights emerge:

- As I increases, both the exact and KL prices converge rapidly to the LHP limit.
- The KL approximation becomes remarkably accurate for $I > 200$, with negligible deviation from the exact computation.
- The equity tranche is the most sensitive to granularity effects, showing larger discrepancies at low I values.

This convergence analysis supports the theoretical soundness of the LHP framework for large portfolios and highlights the KL expansion as a valuable trade-off between accuracy and computational efficiency, especially for medium-to-large exposure sizes.

Conclusion of Point B

This convergence analysis confirms the theoretical validity of the LHP assumption and the KL expansion for large portfolios. While exact pricing remains the most accurate, the KL approximation offers a strong balance between precision and computational efficiency. The results support the use of LHP-based pricing methods in the next stages of the project.

Point C – Calibration Using KL Approximation

This section describes the calibration of the double t-Student copula model using the *Kullback-Leibler (KL)* approximation. The goal is to reproduce the results of Point A, but through a more computationally efficient method, leveraging analytical approximations to speed up the calibration process.

Objective and Context

In Point A, the model was calibrated by pricing the equity tranche exactly using full numerical methods. While accurate, such methods are computationally demanding, especially when scanning across many candidate values of ν. Here, we instead apply the KL expansion to estimate tranche prices analytically and perform calibration accordingly.

Methodology

The calibration follows a grid search over different values of ν. For each candidate value:

- A correlation ρ^* is determined such that the price of the equity tranche under the KL approximation matches the market price.
- Using this (ν, ρ^*) pair, model-implied prices for all tranches are computed.
- Prices are then calculated with this correlation and compared to market values.
- The Mean Squared Error (MSE) between model-implied and market-implied correlations is calculated.

The optimal degrees of freedom ν^* is selected as the one minimizing this MSE.

Results

The calibration process yields the following results:

- Optimal $\nu^* = 4.92$
- Optimal $\rho^* = 0.3396$
- Minimum MSE $= 4.76 \times 10^{-4}$

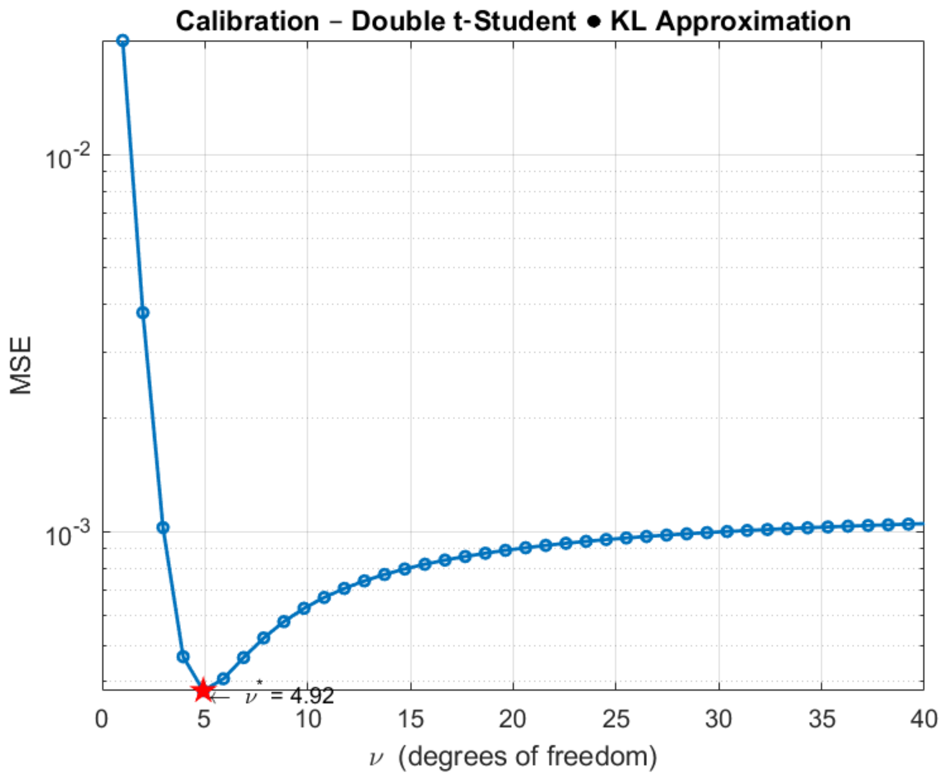

Figure 7: KL-based calibration of the double t-Student model. The red marker denotes the optimal value ν^*.

Figure 7 shows the curve of the MSE as a function of the degrees of freedom ν. The y-axis is in logarithmic scale to better visualize the variation of the error. The red star highlights the point where the minimum MSE occurs, indicating the best compromise between model complexity and calibration accuracy.

The plot clearly shows that the error drops significantly in the range $3 < \nu < 6$, and increases steadily afterwards. This supports the suitability of the KL approximation in producing reliable results for small to moderate values of ν, where the tail behavior of the t-distribution has significant impact on tranche pricing.

Conclusion of Point C

The KL-based calibration method is able to efficiently recover parameters that yield a good fit to market data. Although slightly less accurate than full numerical calibration, the KL approach offers a remarkable trade-off between precision and computational speed. The calibrated values of ν^* and ρ^* will be used in the subsequent model comparisons.

Point D – Pricing with the Vasicek Model

This section explores the pricing of CDO tranches under the **Vasicek one-factor Gaussian copula model**, using three different methods: exact integration, KL approximation, and the analytical LHP limit. The aim is to compare the consistency and accuracy of these techniques in a simplified but widely-used framework.

Model Setup

The Vasicek model assumes a Gaussian dependence structure driven by a single systemic risk factor, with a fixed pairwise correlation. For consistency, we use the market-implied correlation for the equity tranche:

- $\rho_{\text{equity}} = 0.230$
- Number of exposures: $I = 500$

Pricing Methods Compared

We compute the tranche prices using the following methods:

1. **Exact pricing** (mode 0): full numerical integration for finite I.
2. **KL approximation** (mode 1): fast, analytical method based on Kullback-Leibler expansion.
3. **LHP limit** (mode 2): closed-form pricing under the Large Homogeneous Portfolio assumption ($I \to \infty$).

Results and Interpretation

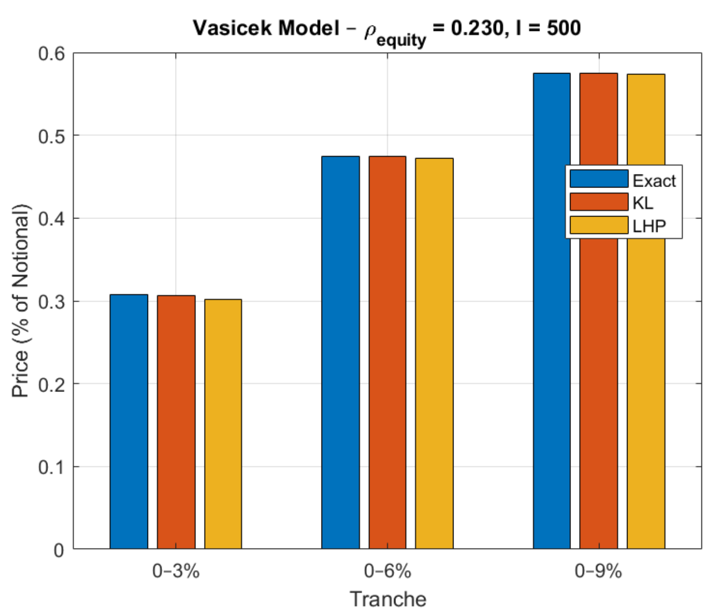

Figure 8: Tranche pricing under the Vasicek model for $I = 500$ and $\rho = 0.230$.

Figure 8 presents a side-by-side comparison of tranche prices obtained using the three methods for the first three tranches: 0–3%, 3–6%, and 6–9%.

- All methods yield nearly identical results for each tranche, confirming the robustness of the KL approximation and the LHP formula under Gaussian assumptions.
- Differences are minimal, even for $I = 500$, suggesting that the LHP and KL approximations are sufficiently accurate for practical use.
- This contrasts with the heavier-tailed t-Student model, where deviations between exact and approximate pricing were more noticeable.

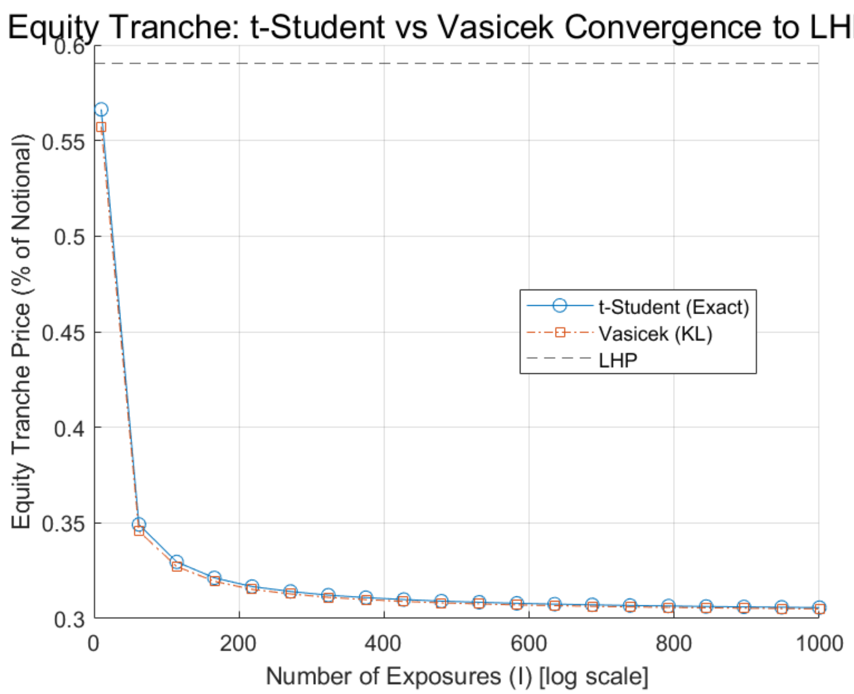

Figure 9: Equity tranche pricing: comparison between t-Student (Exact) and Vasicek (KL) models as I increases. The dashed line represents the LHP limit.

Figure 9 illustrates the convergence of the equity tranche price (0–3%) under two different copula models: the exact pricing from the double t-Student model and the KL approximation under the Vasicek (Gaussian) model. The horizontal dashed line marks the Large Homogeneous Portfolio (LHP) limit.

The results confirm that both models converge to the same asymptotic price as I increases. However, we observe that:

- The t-Student model, due to its heavier tails, starts at a higher price for small I, capturing more extreme joint defaults.
- The Vasicek model exhibits slightly lower tranche prices in the small-I regime, consistent with its lighter tail structure.

- As $I \to 1000$, both models become indistinguishable, validating the use of LHP approximations even across different copula frameworks.

This graph serves as a bridge between the t-Student and Vasicek approaches, showing that despite structural differences, their outputs align in large portfolios. It further strengthens the rationale for using simpler models like Vasicek in practical applications when computational resources are limited.

Conclusion of Point D

The Vasicek model demonstrates high numerical stability and fast convergence to the LHP limit. The KL approximation is almost indistinguishable from the exact method in this setting, supporting its use in Gaussian-based pricing frameworks when computational efficiency is desired. These results also offer a benchmark to compare against the more complex t-Student copula-based approaches used earlier in the report.

Point E – Comparative Analysis of Copula–Based Tranche Pricing Models

Introduction

The purpose of this section is to scrutinise how three alternative dependence structures — two Gaussian implementations of Li's one–factor model and a double−t copula — behave when the size of the underlying credit portfolio varies between $I = 10$ and $I = 1\,000$. All engines price the first three cumulative tranches (Equity 0–3%, Mezz–1 0–6%, Mezz–2 0–9%) with identical marginal inputs (default probability $p = 6\%$, recovery $R = 40\%$, maturity $T = 4$ y, discount curve as calibrated previously). Monte–Carlo (MC) simulations use 10^5 paths per grid point so that sampling noise is negligible relative to model risk. Long–horizon (LHP) analytical prices serve as the asymptotic benchmark whenever available.

Gaussian Copula with ρ from the Double t-Student Calibration

Figure 10 juxtaposes MC tranche prices (solid markers) to the analytical LHP limit. The correlation fed into the Gaussian engine originates from the heavy–tailed calibration performed under the double−t copula.

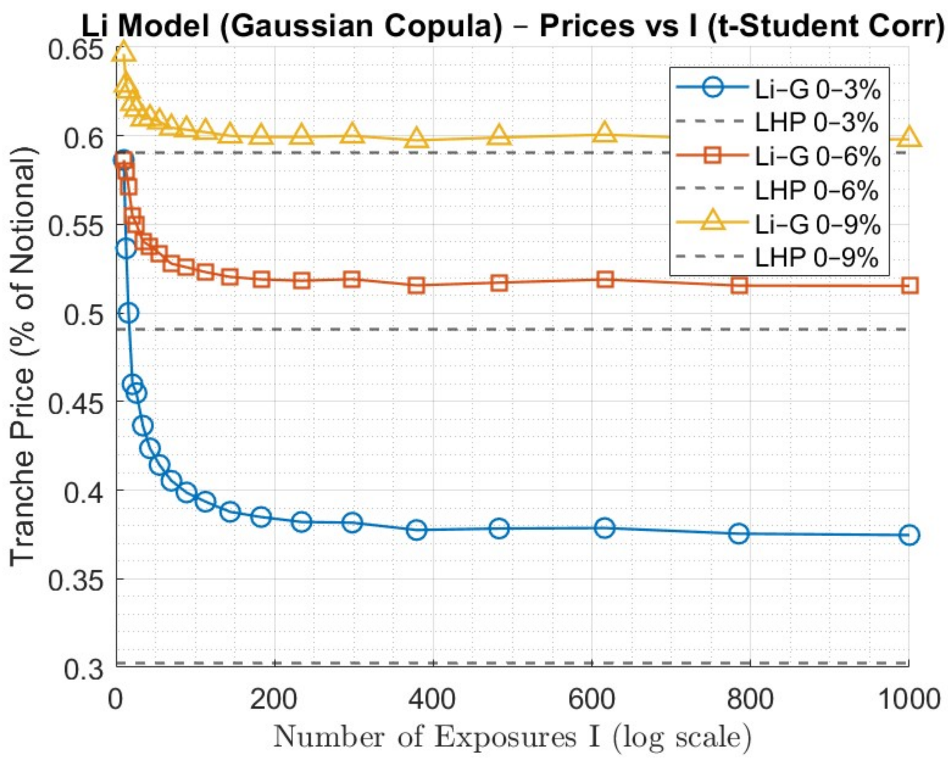

Figure 10: Li model (Gaussian copula) priced with the correlation obtained from a double−t calibration. Prices are expressed as a percentage of tranche notional; the abscissa is logarithmic in I.

In this plot, we observe that the Gaussian Copula price does not converge to the LHP Vasicek price. This discrepancy arises because, in this case, we used the correlation parameter calibrated from the double t-Student copula model to price with a Gaussian copula. For consistency and convergence toward the LHP

Vasicek equity tranche price as $I \to \infty$, we should have used the correlation implied by the Vasicek model.

Later in the analysis, we will ensure consistency in the choice of parameters, allowing us to observe convergence between the different pricing methods.

Double t-Student Copula with $\nu = 6$ and t-student correlation

Figure 11 reports MC prices, the finite–I Hermite–Polynomial (HP) approximation and the LHP formula. The HP curve hugs the MC mean within the 95 % confidence band across the entire grid, validating the semi–closed solution. Granularity matters: the equity price falls from roughly 52 bp at $I = 10$ to 31 bp at $I = 1\,000$ — a 40 % contraction against just 8 % for Mezz–2.

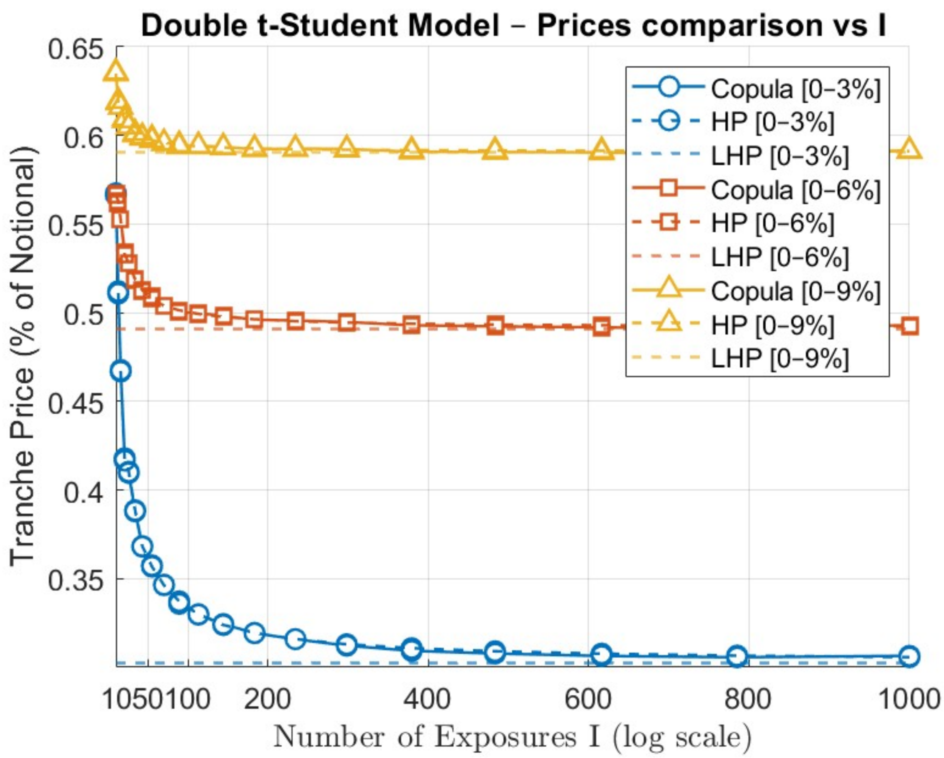

Figure 11: Double t-Student copula (degrees of freedom $\nu = 6$). Comparison of MC, HP and LHP valuations.

The heavier tails increase the probability of clustered defaults at small I, pushing tranche prices above the Gaussian counterparts. Convergence to LHP is therefore slower, especially for equity.

Gaussian Copula with Vasicek Equity–Implied ρ

When the equity–implied correlation from the Vasicek calibration ($\rho = 0.23$) feeds the Gaussian engine (Figure 12), convergence accelerates markedly: the equity tranche deviates by less than 2 bp from its LHP limit already for $I \gtrsim 200$. Again, HP and MC estimates are statistically indistinguishable.

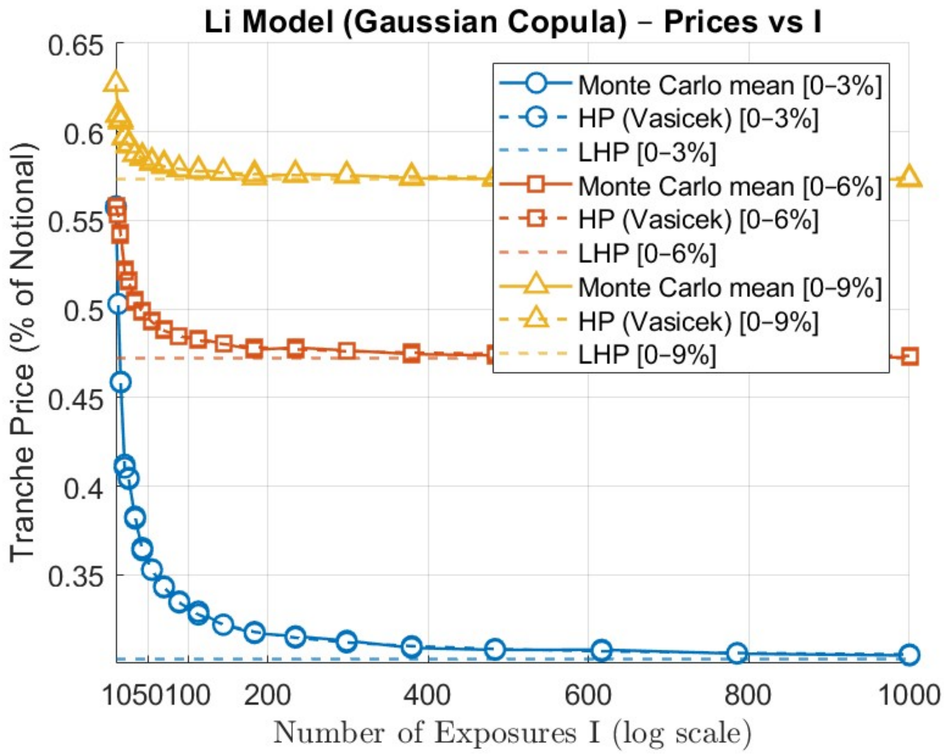

Figure 12: Li model (Gaussian copula) priced with the Vasicek equity–implied correlation.

Monte Carlo and Incertitude Interval

Figure 13 gathers the equity, Mezz–1 and Mezz–2 trajectories produced by the double $-t$ copula together with their HP approximation and the LHP benchmark. The shaded bands represent 95 % MC confidence intervals.

We compare here the t-Student Monte Carlo prices with the HP double t-Student prices, taking into account the Monte Carlo confidence intervals. We observe that the HP price is almost always contained within these intervals, which indicates that the t-Student Monte Carlo method provides a very good fit to the market prices.

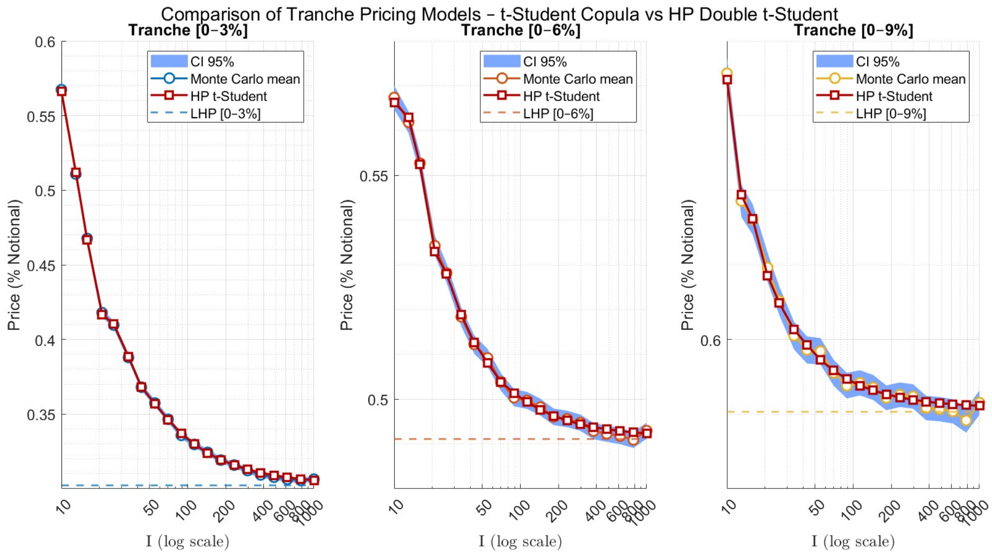

Figure 13: Side–by–side comparison of Monte–Carlo means (blue), HP double−t prices (red squares) and the analytical LHP limit (gold dashed) across the three tranches.

Conclusion

In this project, we have implemented a pricing framework for credit tranches using a double t-Student copula model, as an alternative to the classical Gaussian copula in the Vasicek framework. This approach allowed us to better capture tail dependencies and the joint default behavior observed in real credit portfolios. We highlighted the impact of the copula choice on the pricing of equity and mezzanine tranches, showing that the double t-Student copula produces heavier tails and therefore higher risk premiums. These findings underscore the importance of model selection in structured credit pricing and provide a solid foundation for future research into more realistic and robust dependence structures.

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