27/3/25

Funzioni Olomorfe e eq. Cauchy-Riemann

Sia $f: G \to \mathbb{C}$ con $G \subseteq \mathbb{C}$ aperto. Pongo $f(z) = u(x,y) + i\,v(x,y)$ con $z = x+iy$ e $u,v: \mathbb{R}^2 \to \mathbb{R}$. Se f é derivabile in $z_0 \in G \Rightarrow u$ e v ammettono derivate parziali in (x_0, y_0) e soddisfano

$$\begin{cases} \left.\dfrac{\partial u}{\partial x}\right|_{z_0} = \left.\dfrac{\partial v}{\partial y}\right|_{z_0} \\[2ex] \left.\dfrac{\partial u}{\partial y}\right|_{z_0} = -\left.\dfrac{\partial v}{\partial x}\right|_{z_0} \end{cases}$$

Eq. di Cauchy-Riemann

e $f'(z_0) = \left.\dfrac{\partial u}{\partial x}\right|_{z_0} + i \left.\dfrac{\partial v}{\partial x}\right|_{z_0}$. Il teorema vale anche al contrario.

Inoltre u e v sono funzioni armoniche, ovvero $\nabla^2 u = \nabla^2 v = 0$

Dim

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = 0$$

#

Se $f: G \subseteq \mathbb{C} \to \mathbb{C}$, é differenziabile $\forall z \in G$, f si dice olomorfa o analitica in G

Se $f: \mathbb{C} \to \mathbb{C}$ é olomorfa in tutto $\mathbb{C}$, f si dice intera

Una proprietá interessante é che se f é olomorfa $\Rightarrow \dfrac{\partial f}{\partial \bar z} = 0$

Dim considero gli operatori differenziali:

$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \bar z} = \frac{\partial x}{\partial \bar z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \bar z}\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

$$z = x + iy, \quad \bar z = x - iy \longrightarrow x = \frac{z + \bar z}{2}, \quad y = \frac{z - \bar z}{2i}$$

Se f é analitica $\to$ soddisfa CR.

$$\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(u(x,y) + i\,v(x,y)\right) = \frac{1}{2}\underbrace{\cancel{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}}_{=0} + \frac{i}{2}\underbrace{\cancel{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)}}_{=0} = 0$$

#

Trovare la funzione analitica $f(z) = u(x,y) + i\, v(x,y)$ per

i) $u(x,y) = x^2 - y^2 - x$

ii) $v(x,y) = e^{-2xy} \sin(x^2-y^2)$

iii) $u(x,y) = (e^{2x} + e^{-2x}) \cos(2y) + x^2 - y^2$

A lezione avete visto $v(x,y) = \int_{(x_0,y_0)}^{(x,y_0)} dx \left(-\frac{\partial u}{\partial y}\right) + \int_{(x,y_0)}^{(x,y)} dy \left(\frac{\partial u}{\partial x}\right) + v(x_0,y_0)$

i) $-\frac{\partial u}{\partial y} = 2y \quad , \quad \frac{\partial u}{\partial x} = 2x - 1$

$$\Rightarrow \quad v(x,y) = \int_{(x_0,y_0)}^{(x,y_0)} dx\; 2y + \int_{(x,y_0)}^{(x,y)} dy\; (2x-1) + v(x_0,y_0)$$

$$= 2y_0(x - x_0) + (2x-1)(y-y_0) + v(x_0,y_0)$$

Posso scegliere $(x_0,y_0) = (0,0)$ e la costante $v(x_0,y_0)$ è arbitraria $\rightarrow v(x_0,y_0) = 0$

$$v(x,y) = 2xy - y$$

<u>check</u>:

$$\frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y} \qquad \frac{\partial v}{\partial y} = 2x - 1 = \frac{\partial u}{\partial x}$$

$$\rightarrow \quad f(z) = x^2 - y^2 - x + i(2xy - y) = z^2 - z$$

ii) $v(x,y) = e^{-2xy} \sin(x^2-y^2)$

- $\frac{\partial v}{\partial x} = -2y\, e^{-2xy} \sin(x^2-y^2) + e^{-2xy}\, 2x \cos(x^2-y^2)$

- $\frac{\partial v}{\partial y} = -2x\, e^{-2xy} \sin(x^2-y^2) - 2y\, e^{-2xy} \cos(x^2-y^2)$

e usiamo

$$u(x,y) = \int_{(x_0,y_0)}^{(x,y_0)} dx \, \frac{\partial v}{\partial y} + \int_{(x,y_0)}^{(x,y)} dy \left(-\frac{\partial v}{\partial x}\right) + \overset{=0}{\cancel{u(x_0,y_0)}}$$

osserviamo che

$$\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}\left(e^{-2xy}\cos(x^2-y^2)\right)$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(e^{-2xy}\cos(x^2-y^2)\right)$$

$$\Longrightarrow u(x,y) = e^{-2xy}\cos(x^2-y^2)\Big|_{(x_0,y_0)}^{\cancel{(x,y_0)}} + e^{-2xy}\cos(x^2-y^2)\Big|_{\cancel{(x,y_0)}}^{(x,y)} =$$

$$= e^{-2xy}\cos(x^2-y^2) - \cancel{1}$$

u è definita a meno di una costante

$$f(z) = e^{-2xy}\cos(x^2-y^2) + i\,e^{-2xy}\operatorname{sen}(x^2-y^2) = e^{-2xy}\left(\cos(x^2-y^2) + i\operatorname{sen}(x^2-y^2)\right) =$$

$$= e^{-2xy}\,e^{i(x^2-y^2)} = e^{i(x^2-y^2+2ixy)} = e^{iz^2}$$

iii) $u(x,y) = (e^{2x}+e^{-2x})\cos 2y + x^2 - y^2$

$$\frac{\partial u}{\partial x} = 2(e^{2x}-e^{-2x})\cos 2y + 2x = \frac{\partial}{\partial y}\left[(e^{2x}-e^{-2x})\operatorname{sen} 2y + 2xy\right]$$

$$\frac{\partial u}{\partial y} = -2(e^{2x}+e^{-2x})\operatorname{sen} 2y - 2y = -\frac{\partial}{\partial x}\left[(e^{2x}-e^{-2x})\operatorname{sen} 2y + 2xy\right]$$

$$\Longrightarrow v(x,y) = \int_{(x_0,y_0)}^{(x,y_0)} dx\left(-\frac{\partial u}{\partial y}\right) + \int_{(x,y_0)}^{(x,y)} dy\left(\frac{\partial u}{\partial x}\right) + v(x_0,y_0) =$$

$$= \dots = (e^{2x}-e^{-2x})\operatorname{sen} 2y + 2xy$$

$$f(z) = (e^{2x}+e^{-2x})\cos 2y + x^2 - y^2 + i(e^{2x}-e^{-2x})\operatorname{sen} 2y + 2ixy$$

$$= e^{2x}\left(\overset{e^{i2y}}{\cos 2y + i \operatorname{sen} 2y}\right) + e^{-2x}\left(\overset{e^{-i2y}}{\cos 2y - i \operatorname{sen} 2y}\right) + x^2 - y^2 + 2ixy =$$

$$= e^{2z} + e^{-2z} + z^2 = 2\cosh 2z + z^2$$

Esercizio funzione olomorfa

i) Determinare $a \in \mathbb{R}$ t.c. $u(x,y) = \cos x\,(e^{ay} + e^{-y})$ sia la parte reale di una funzione olomorfa $f(z)$ su $\mathbb{C}$.

ii) Determinare $f(z)$ per i valori $a > 0$ trovati.

i) Perché u sia la parte reale di una funzione olomorfa devono valere CR. Tuttavia sappiamo che se valgono CR, allora u e v sono armoniche.

$\nabla^2 u = 0$

$$\frac{\partial u}{\partial x} = -\sin x\,(e^{ay} + e^{-y}) \qquad \frac{\partial u}{\partial y} = \cos x\,(a e^{ay} - e^{-y})$$

$$\frac{\partial^2 u}{\partial x^2} = -\cos x\,(e^{ay} + e^{-y}) \qquad \frac{\partial^2 u}{\partial y^2} = \cos x\,(a^2 e^{ay} - e^{-y})$$

$$0 = \cos x\left[a^2 e^{ay} \cancel{+e^{-y}} - e^{ay} \cancel{-e^{-y}}\right] = \cos x\, e^{ay}(a^2 - 1) \longrightarrow a = \pm 1$$

ii) Se $a = 1 \longrightarrow u(x,y) = 2\cos x \cosh y$

$$\longrightarrow v(x,y) = \int_{(0,0)}^{(x,0)} dx\left(-\frac{\partial u}{\partial y}\right) + \int_{(x,0)}^{(x,y)} dy\left(\frac{\partial u}{\partial x}\right) + v(0,0)$$

$$-\frac{\partial u}{\partial y} = -2\cos x \sinh y$$

$$\frac{\partial u}{\partial x} = -2\operatorname{sen} x \cosh y$$

$$v(x,y) = \int_{(0,0)}^{(x,0)} dx\left(-2\cos x \sinh y\right) + \int_{(x,0)}^{(x,y)} dy\left(-2\operatorname{sen} x \cosh y\right) =$$

$$= -2\,\mathrm{sen}\,x \,\sinh y \,\Big|_{\cancel{(0,0)}}^{\cancel{(x,0)}} - 2\,\mathrm{sen}\,x\,\sinh y\,\Big|_{\cancel{(x,0)}}^{(x,y)} = -2\,\mathrm{sen}\,x\,\sinh y$$

$$f(z) = 2\cos x \cosh y - 2i\,\mathrm{sen}\,x\,\mathrm{senh}\,y =$$

$$= \cos x\left(e^{y}+e^{-y}\right) - i\,\mathrm{sen}\,x\left(e^{y}-e^{-y}\right) =$$

$$= e^{y}\left(\cos x - i\,\mathrm{sen}\,x\right) + e^{-y}\left(\cos x + i\,\mathrm{sen}\,x\right) =$$

$$= e^{y-ix} + e^{-y+ix} = e^{-iz} + e^{iz} = 2\cos z$$

PROVA D'ESAME DEL 19/7/2019

Dato $f(z) = \frac{1}{z}$

i) Dimostrare che valgono CR in $z \neq 0$ e mostrare esplicitamente che la derivata è $-\frac{1}{z^2}$

$$f(z) = \frac{1}{x+iy} = \frac{1}{(x+iy)}\frac{x-iy}{x-iy} = \frac{x-iy}{x^2+y^2} = \underbrace{\frac{x}{x^2+y^2}}_{u(x,y)} - i\underbrace{\frac{y}{x^2+y^2}}_{v(x,y)}$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\ \frac{\partial v}{\partial y} &= \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \end{aligned}\right\} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= -\frac{2xy}{(x^2+y^2)} \\ \frac{\partial v}{\partial x} &= \frac{2xy}{(x^2+y^2)} \end{aligned}\right\} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Per calcolare $f'(z)$ possiamo o usare la definizione

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{z+h}-\frac{1}{z}\right) = \lim_{h\to 0}\frac{\cancel{z}-\cancel{z}-h}{(z+h)z\,h} = -\frac{1}{z^2}$$

oppure

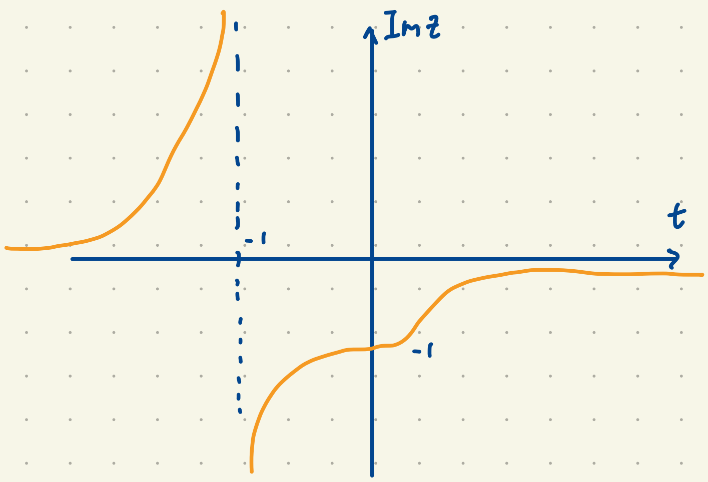
$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2} = \frac{y^2-x^2+2ixy}{|z|^4} = -\frac{(x-iy)^2}{|z|^4} =$$

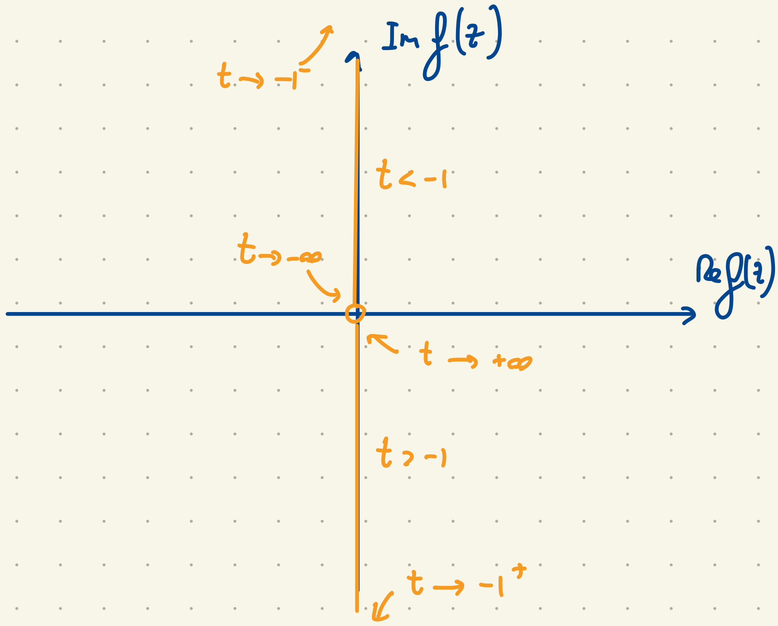
$$= -\frac{\bar{z}^2}{|z|^2} = -\frac{1}{z^2}$$

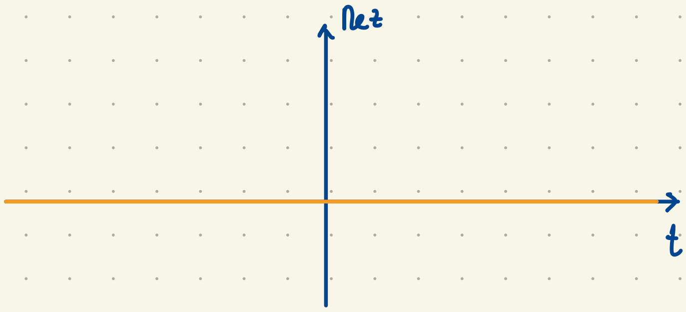
$z\bar{z} = |z|^2$

ii) Data la mappa $w = f(z)$ trovare l'immagine della retta $z(t) = i + it^3$, $t \in \mathbb{R}$. Cosa si può dire di modulo e direzione dei vettori tangenti a $f(z(t))$?

$$f(z(t)) = \frac{1}{i(1+t^3)} = -\frac{i}{1+t^3}$$





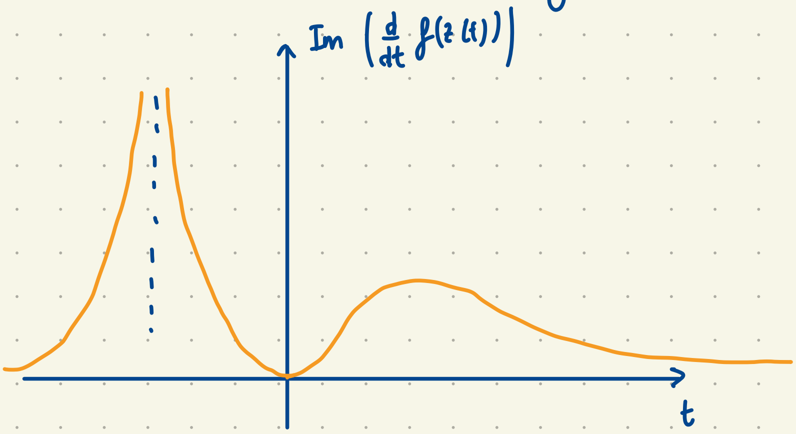


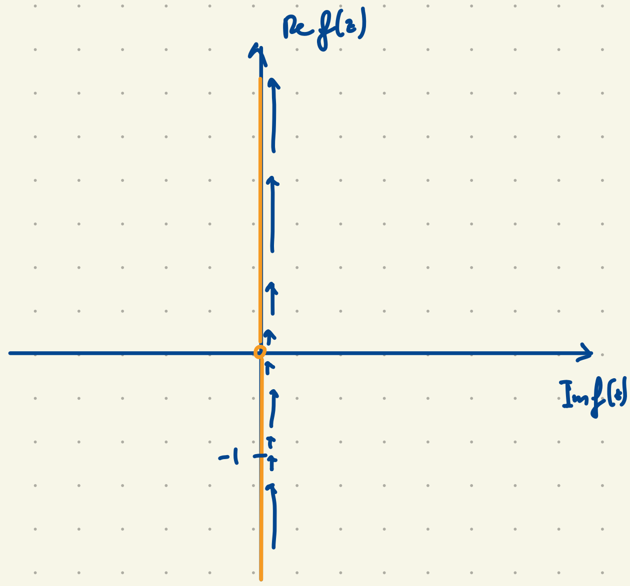
I vettori tangenti alla curva $f(z(t))$ sono dati da

$$\frac{d}{dt} f(z(t)) = \frac{d}{dz} f(z(t)) \cdot \frac{dz(t)}{dt} = -\frac{1}{i^2(1+t^3)^2} \cdot i3t^2 = i\,\frac{3t^2}{(1+t^3)^2}$$

chain-rule

La direzione è puramente immaginaria





Prova d'esame 11/9/17

Determinare l'immagine della circonferenza unitaria e degli assi $\mathrm{Re}\, z = 0$, $\mathrm{Im}\, z = 0$ per la mappa $w = \frac{z}{1+z}$

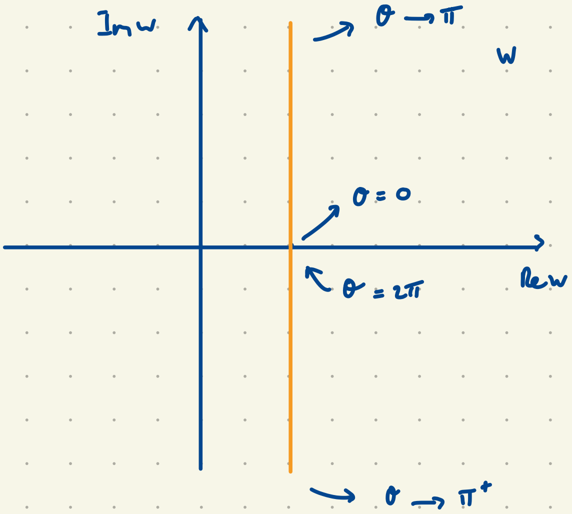
circonferenza

$$z = e^{i\theta} \qquad \theta \in [0, 2\pi)$$

$$w = \frac{e^{i\theta}}{1+e^{i\theta}} = \frac{e^{i\theta}}{1+e^{i\theta}} \, \frac{1+e^{-i\theta}}{1+e^{-i\theta}} =$$

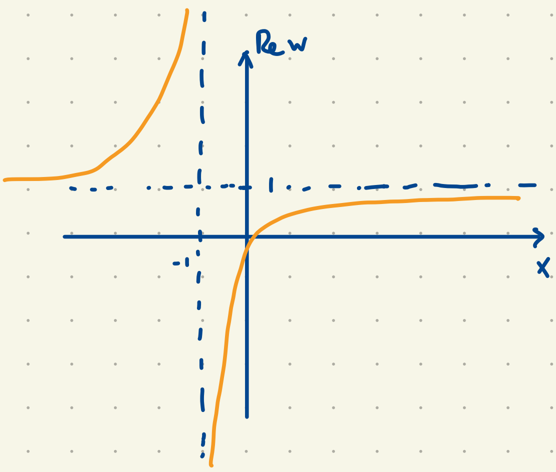
$$= \frac{e^{i\theta}+1}{1+e^{i\theta}+e^{-i\theta}+1} = \frac{1+e^{i\theta}}{2+2\cos\theta} = \frac{1}{2} + i \frac{\operatorname{sen}\theta}{2(1+\cos\theta)}$$

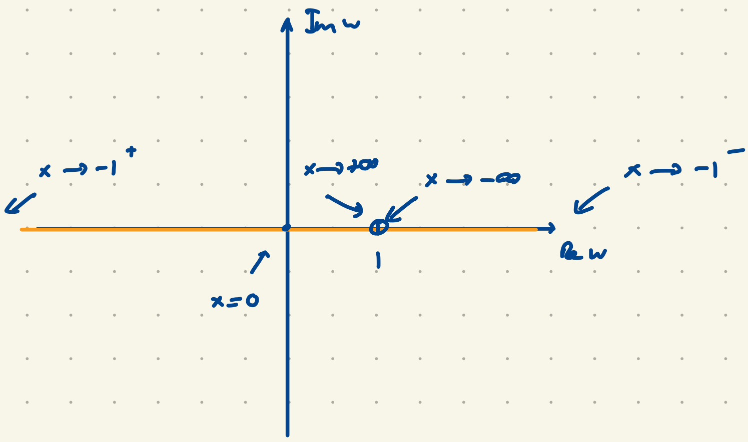
$\tan\frac{\theta}{2}$



Retta $\mathrm{Im}\, z = 0 \longrightarrow z = x$

$$w = \frac{x}{1+x}$$ $\longrightarrow$ viene mappato nell'asse reale





Asse immaginario $\mathrm{Re}\, z = 0 \longrightarrow z = iy$

$$w = \frac{iy}{1+iy} = \frac{iy}{1+iy} \, \frac{1-iy}{1-iy} = \frac{y^2}{1+y^2} + i \frac{y}{1+y^2}$$

- se $y \to -\infty$ $\quad w = 1$
- se $y = -1$ $\quad w = \frac{1}{2} - \frac{i}{2}$
- se $y = 0$ $\quad w = 0$
- se $y = 1$ $\quad w = \frac{1}{2} + \frac{i}{2}$
- se $y \to +\infty$ $\quad w = 1$

Sembra una circonferenza centrata in $\frac{1}{2}$ e di raggio $\frac{1}{2}$.

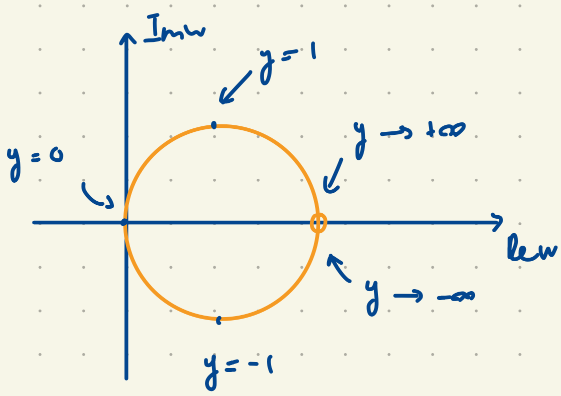
$$w = \frac{1}{2} - \frac{1}{2} + \frac{y^2}{1+y^2} + \frac{i}{2}\,\frac{2y}{1+y^2} = \frac{1}{2} + \frac{1}{2}\left[\frac{y^2-1}{y^2+1} + i\,\frac{2y}{1+y^2}\right]$$

Se questa é una circonferenza devo poterla scrivere come $\frac{1}{2} + \frac{1}{2}e^{i\theta}$

Osservo che

$$\left(\frac{y^2-1}{y^2+1}\right)^2 + \left(\frac{2y}{1+y^2}\right)^2 = \frac{y^4 - 2y^2 + 1 + 4y^2}{(1+y^2)^2} = \frac{(1+y^2)^2}{(1+y^2)^2} = 1$$

$\Rightarrow$ posso identificare $\frac{y^2-1}{y^2+1} = \cos\theta$ e $\frac{2y}{1+y^2} = \operatorname{sen}\theta$



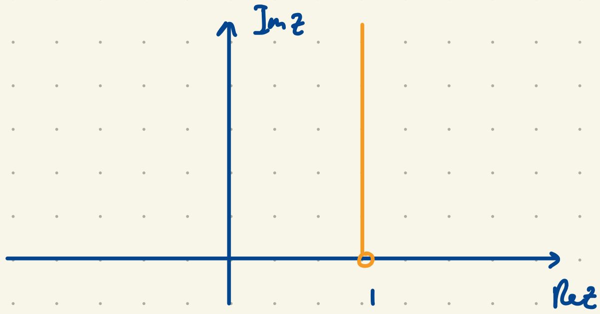
PROVA D'ESAME 29/1/19

i) Determinare l'immagine della semiretta $\mathrm{Re}\, z = 1$, $\mathrm{Im}\, z \geq 0$ attraverso la mappa $u + iv = \mathrm{Log}\, z$

ii) Determinare vettore tangente in ogni punto e tracciare qualitativamente il grafico della curva nel piano (u, v)

$$w = \mathrm{Log}\, z = \log|z| + i \,\mathrm{Arg}\, z \qquad \text{e} \qquad -\pi < \mathrm{Arg}\, z \leq \pi$$

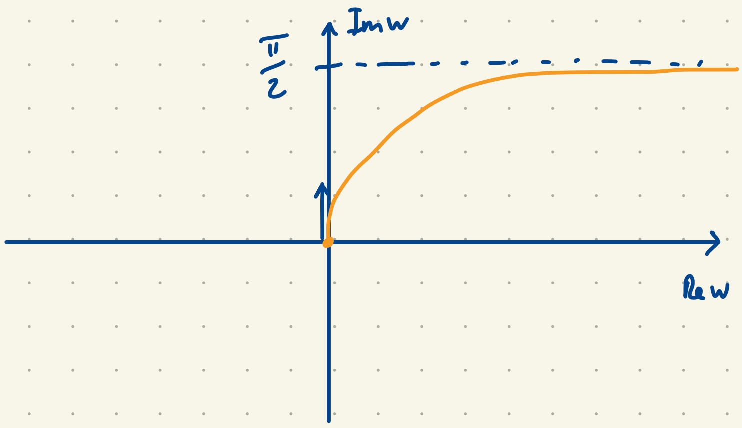
i) sia $z = 1 + it \qquad t \in [0, +\infty)$



$$\mathrm{Log}(z(t)) = \log|1+it| + i\,\mathrm{Arg}(1+it)$$

se $t > 0 \implies \mathrm{Arg}(1+it) = \mathrm{arctg}(t)$

$$\implies \mathrm{Log}(z(t)) = \log(1+t^2) + i\,\mathrm{arctg}(t)$$



ii) $\frac{d}{dt} \log z(t) = \frac{2t}{1+t^2} + \frac{i}{1+t^2}$

$t = 0 \longrightarrow \frac{d}{dt} \mathrm{Log}(z(t)) = i$

$t \to \infty \longrightarrow \frac{d}{dt} \mathrm{Log}(z(t)) \sim \frac{2}{t} + \frac{i}{t^2}$

La componente immaginaria va a zero più velocemente di quella reale $\implies$ é un vettore orizzontale.