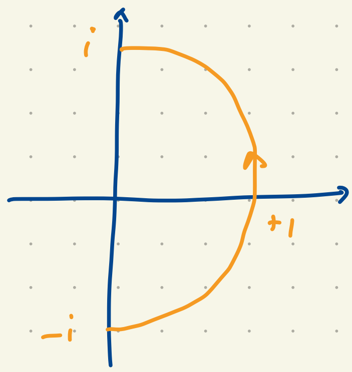
Tema d'esame 20/6/18

10/6/25

Sia γ la semicirconf. orientata da $-i$ a i e passante per $+1$. Calcolare

$$\int_\gamma f(z)\,dz$$



i) $f(z) = z + |z|$

ii) $f(z) = \mathrm{Log}(z)$

iii) $f(z) = \sin z$

parametrizzo $\gamma(t) = e^{it}$

$$= \cos t + i \sin t \,, \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\gamma'(t) = i e^{it} = -\sin t + i \cos t$$

i) $$\int_\gamma z + |z|\, dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(e^{it} + |e^{it}|\right) i e^{it}\, dt =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(e^{it} + 1\right) i e^{it}\, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} i e^{2it} + i e^{it}\, dt =$$

$$= \frac{i}{2i} e^{2it} + \frac{i}{i} e^{it} \Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -\frac{1}{2} + i - \left(-\frac{1}{2} - i\right) = 2i$$

ii) $$\int_\gamma \mathrm{Log}\, z\, dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{Log}\, e^{it}\, i e^{it}\, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} it\, i e^{it}\, dt =$$

$$= -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t e^{it}\, dt = *$$

$$\int t e^{it} dt = t\frac{e^{it}}{i} - \int \frac{e^{it}}{i} dt = \frac{te^{it}}{i} - \frac{1}{i}\frac{e^{it}}{i} = -i te^{it} + e^{it} =$$

$$f \; g'$$

$$= e^{it}(1-it)$$

$f' = 1$

$g = \frac{e^{it}}{i}$

$$* = (-1+it)e^{it}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \left(-1+i\frac{\pi}{2}\right)i - \left(-1-i\frac{\pi}{2}\right)(-i) = -i-\frac{\pi}{2} - i + \frac{\pi}{2} = -2i$$

Metodo 2

Osservo che $\mathrm{Log}\, z$ é olomorfa e $\quad \mathrm{Log}\, z = \frac{d}{dz}\left(z\,\mathrm{Log} z - z\right)$

$$\Rightarrow \int_\gamma \mathrm{Log}\, z \, dz = z\,\mathrm{Log}\, z - z\Big|_{-i}^{i} = i\frac{\pi}{2}i - i - \left(-i\left(-i\frac{\pi}{2}\right) + i\right) = -2i$$

iii) $\sin z$ é olomorfa sulla curva

$$\Rightarrow \int_\gamma \sin z \, dz = -\cos z\Big|_{-i}^{+i} = -\cos i + \cos(-i) = 0$$

Integrale di Fresnel

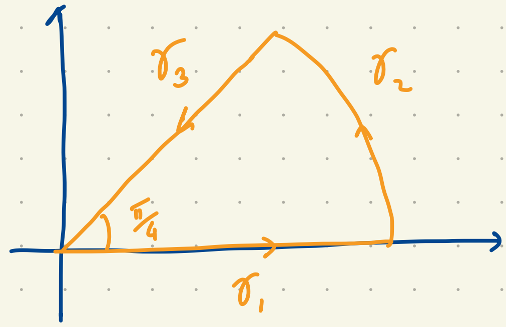
$$\int_0^\infty \cos x^2 \, dx = \int_0^\infty \sin x^2 \, dx = \frac{\sqrt{2\pi}}{4}$$

Considero $f(z) = e^{iz^2}$ e il cammino chiuso $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$

$\gamma_1(t) = t \qquad 0 \le t \le R$

$\gamma_2(t) = Re^{it} \qquad 0 \le t \le \frac{\pi}{4}$

$\gamma_3(t) = te^{i\frac{\pi}{4}} \qquad R \le t \le 0$



Ho scelto $f(z)$ perché

$$\int_\gamma f(z)\,dz = \int_\gamma e^{iz^2}\,dz = \int_\gamma \cos z^2\,dz + i\int_\gamma \sin z^2\,dz$$

Osservo che $f(z)$ é intera su $\gamma(t)$ e al suo interno

$$\Rightarrow \int_\gamma f(z) = 0$$

Calcoliamo i vari contributi

γ_1 :
$$\int_{\gamma_1} f(z)\,dz = \int_0^R dt\, e^{it^2} = \int_0^R \cos x^2\,dx + i\int_0^R \sin x^2\,dx$$

γ_3 :
$$\int_{\gamma_3} f(z)\,dz = \int_R^0 dt\, e^{it^2 e^{2i\frac{\pi}{4}}} \cdot \underbrace{e^{i\frac{\pi}{4}}}_{\gamma_3'(t)} = -e^{i\frac{\pi}{4}}\int_0^R e^{-t^2}\,dt$$

$\Rightarrow$ é una metá gaussiana nel limite $R \to \infty$

$\left(\int_{-\infty}^{+\infty} e^{-a(x+b)^2} = \sqrt{\frac{\pi}{a}}\right) \Rightarrow$ il nostro integrale per $R \to \infty$ fa $\frac{\sqrt{\pi}}{2}$

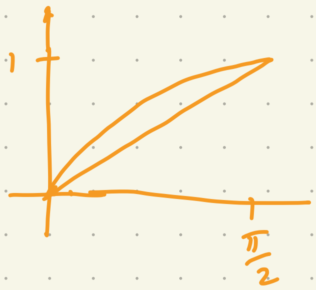
γ_2:
$$\int_{\gamma_2} f(z)\,dz = \int_0^{\frac{\pi}{4}} dt\, iRe^{it}\, e^{iR^2 e^{2it}}$$

osservo

$$\left|\int_{\gamma_2} f(z)\,dz\right| \le \int_{\gamma_2} |f(z)|\,dz = \int_0^{\frac{\pi}{4}} dt\; R \left| e^{iR^2(\cos 2t + i \operatorname{sen} 2t)}\right| =$$

$$= \int_0^{\frac{\pi}{4}} R\, e^{-R^2 \operatorname{sen} 2t}$$

ma per $0 < t < \frac{\pi}{2}$ $\quad \operatorname{sen} t \ge \frac{2t}{\pi}$



$$\le \int_0^{\frac{\pi}{4}} dt\; R\, e^{-R^2 \frac{4t}{\pi}} = -\frac{\pi}{4R^2} R\, e^{-R^2\frac{4t}{\pi}} \Big|_0^{\frac{\pi}{4}} = -\frac{\pi}{4R}\left(e^{-R^2} - 1\right) \xrightarrow[R\to\infty]{} 0$$

Per $R \to \infty$

$$\int_{\gamma_1} \longrightarrow \int_0^{\infty} \cos x^2\, dx + i \int_0^{\infty} \operatorname{sen} x^2\, dx$$

$$\int_{\gamma_2} \longrightarrow 0$$

$$\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}$$

$$\int_{\gamma_3} \longrightarrow -e^{i\frac{\pi}{4}}\,\frac{\sqrt{\pi}}{2}$$

$$\int_{\gamma} = 0$$

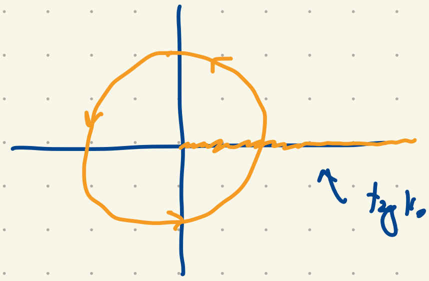
$$\Longrightarrow \int_0^{\infty} \cos x^2\, dx + i \int_0^{\infty} \sin x^2\, dx = \frac{\sqrt{2\pi}}{4} + i\,\frac{\sqrt{2\pi}}{4}$$

Calcolare l'integrale

$$\int_\gamma z^{-1} \log z \, dz \qquad \text{con} \qquad \gamma(\theta) = e^{i\theta} \qquad 0 < \theta < 2\pi$$

Per il log scegliamo il ramo $\log z = \log r + i\theta$ con $\theta \in (0, 2\pi)$

$z^{-1} \log z$ é continua su γ



Allora

$$\int_\gamma z^{-1} \log z \, dz = \int_0^{2\pi} e^{-i\theta} (\cancel{\log 1} + i\theta) i e^{i\theta} \, d\theta =$$

$$= \int_0^{2\pi} i^2 \theta \, d\theta = -2\pi^2$$

Alternativamente

$$z^{-1} \log z = \frac{d}{dz}\left(\frac{1}{2} \log^2 z\right) \quad \text{ad eccezione del taglio}$$

considero $\gamma_\epsilon(\theta) = e^{i\theta} \qquad \epsilon \leq \theta \leq 2\pi - \epsilon$

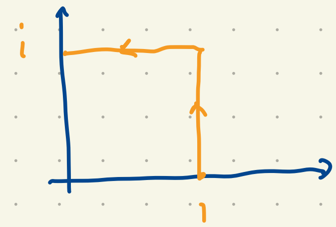
$$\int_\gamma z^{-1} \log z \, dz = \lim_{\epsilon \to 0} \int_{\gamma_\epsilon} z^{-1} \log z \, dz = \lim_{\epsilon \to 0} \frac{1}{2} \log^2 z \Big|_{e^{i\epsilon}}^{e^{i(2\pi - \epsilon)}} =$$

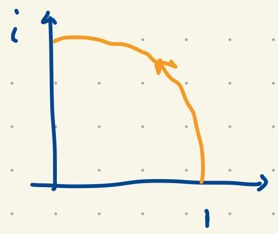
$$= \lim_{\epsilon \to 0} \frac{1}{2} \left(i^2 (2\pi - \epsilon)^2 - i^2 \epsilon^2 \right) = -2\pi^2$$

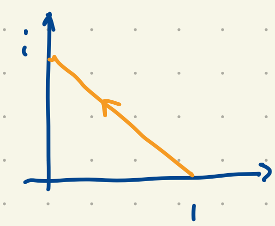
Calcolare i seguenti integrali

i) $\int_\gamma \mathrm{Re}\, z \, dz$ $\qquad$ ii) $\int_\gamma z^2 dz$

sulle 3 curve

a) 

b) 

c) 

ia) $$\int_0^1 1 \cdot i \, dt - \int_0^1 \mathrm{Re}\,(i+t)\, dt = i - \int_0^1 t \, dt = -\frac{1}{2} + i$$

ib) $$i \int_0^{\frac{\pi}{2}} e^{it} \, \mathrm{Re}\,(e^{it})\, dt = i \int_0^{\frac{\pi}{2}} \cos^2 t \, dt - \int_0^{\frac{\pi}{2}} \cos t \sin t \, dt = i\frac{\pi}{4} - \frac{1}{2}$$

ic) $\gamma(t) = (1-t) + it$

$$\int_\gamma \mathrm{Re}\, z = \int_0^1 \mathrm{Re}\, ((1-t)+it)(i-1)\, dt$$

$$= (i-1)\int_0^1 1-t \, dt = \frac{i-1}{2}$$

ii a) $$\int_\gamma z^2 dz = -i \int_0^1 t^2 dt + \int_1^0 t^2 dt = -\frac{1+i}{3}$$

b) $$\int_\gamma z^2 dz = i \int_0^{\frac{\pi}{2}} e^{2it} e^{it} = i \int_0^{\frac{\pi}{2}} e^{3it} dt = -\frac{1}{3}(1+i)$$

e) $\int_\gamma z^2 dz = \int_0^1 \left((1-t)+it\right)^2 (i-1)\, dt =$

$= (i-1)\left(\int_0^1 dt - 2\int_0^1 t\, dt + 2i \int_0^1 t\, dt - 2i \int_0^1 t^2 dt \right) =$

$= -\frac{1+i}{3}$