Tutorato, Lezione 2, 03/04/2025

- C.R. in coordinate polari
- funzioni multivalued, costruzione fogli di Rienman
- integrazione in campo complesso

# FUNZIONE OLOMORFA IN COORDINATE POLARI

Sia $u(r,\theta) = r\cos\theta$, dimostrare che è la parte reale di una funzione analitica e trovare la sua parte immaginaria tale che $v(r,0)=0$.

i) devo far vedere che $u(r,\theta)$ è armonica
$\Rightarrow$ parte reale di una funzione analitica

- in coordinate cartesiane

$$u(r,\theta) = r\cos\theta = x \Rightarrow \nabla^2 u = 0 \quad \checkmark$$

- in coordinate polari

$$\nabla^2_{pol}\, u(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(r\cos\theta) + \frac{-1}{r^2}\frac{\partial}{\partial \theta}(r\sin\theta)$$

$$= \frac{1}{r}\cos\theta - \frac{1}{r}\cos\theta = 0 \quad \checkmark$$

ii) Trovare la parte immaginaria in coordinate polari

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\Rightarrow \frac{\partial f}{\partial r} = \cos\theta \frac{\partial f}{\partial x} + \sin\theta \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial \theta} = -r\sin\theta \frac{\partial f}{\partial x} + r\cos\theta \frac{\partial f}{\partial y}$$

$$u_r + i v_r = \cos\theta(u_x + i v_x) + \sin\theta(u_y + i v_y)$$

$$(C.R) = \cos\theta(u_x - i u_y) + \sin\theta(u_y + i u_x) \quad \text{I}$$

$$u_\theta + i v_\theta = -r\sin\theta(u_x - i u_y) + r\cos\theta(u_y + i u_x) \quad \text{II}$$

$+ri$ I $=$ II ← Osservo che

$$\Rightarrow (u_r + i v_r)\, i r = u_\theta + i v_\theta$$

$$-v_r r + i r u_r = u_\theta + i v_\theta$$

$$\Rightarrow \boxed{\begin{cases} u_\theta = -v_r r \\ v_\theta = r u_r \end{cases}}$$

CR in coordinate polari

Quindi $u(r,\theta) = r\cos\theta$

$v_r = -u_\theta / r = +\frac{r\sin\theta}{r} = \sin\theta$

$v_\theta = r \cdot u_r = r\cos\theta$

$v_\theta = r\cos\theta \rightsquigarrow v(r,\theta) = r\sin\theta + A(r)$

ma $v(r,0) = A(r) = 0$ dalle condizioni al contorno.

$\Rightarrow v(r,\theta) = r\sin\theta$

$\Rightarrow f(z) = u(r,\theta) + i v(r,\theta) = r\cos\theta + i r\sin\theta = z$

# ESERCIZI SU BRANCH POINTS / CUTS

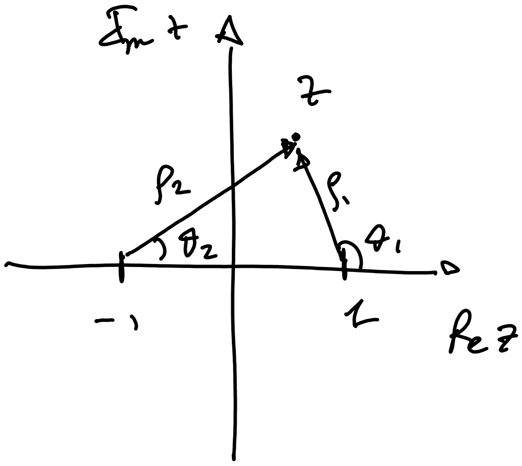
## Prova d'esame 20/06/2024, Esercizio 2B

Disegnare i fogli di Riemann della seguente funzione polidroma e scriverne la forma polare:

$$f(z) = \left(\frac{z+1}{z-1}\right)^{1/3}$$

**Sol:**

Branch points $z = \pm 1$ $\Rightarrow$ scriviamo in forma polare intorno a $z = \pm 1$




$$f(z) = \left(\frac{\rho_2\, e^{i\theta_2 + 2n\pi i}}{\rho_1\, e^{i\theta_1 + 2m\pi i}}\right)^{1/3}$$

$$= \left(\frac{\rho_2}{\rho_1}\right)^{1/3} e^{i\left[\frac{\theta_2 - \theta_1}{3} + \frac{2\pi}{3} i (n-m)\right]}$$

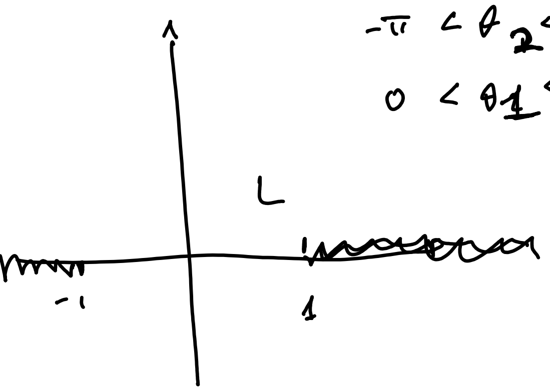
Osservo che:

- se $n \to n+1$ e $m \to m+1$ la funzione torna allo stesso valore

$\Rightarrow$ $\infty$ non è un branch point

$-\pi < \theta_2 < \pi$

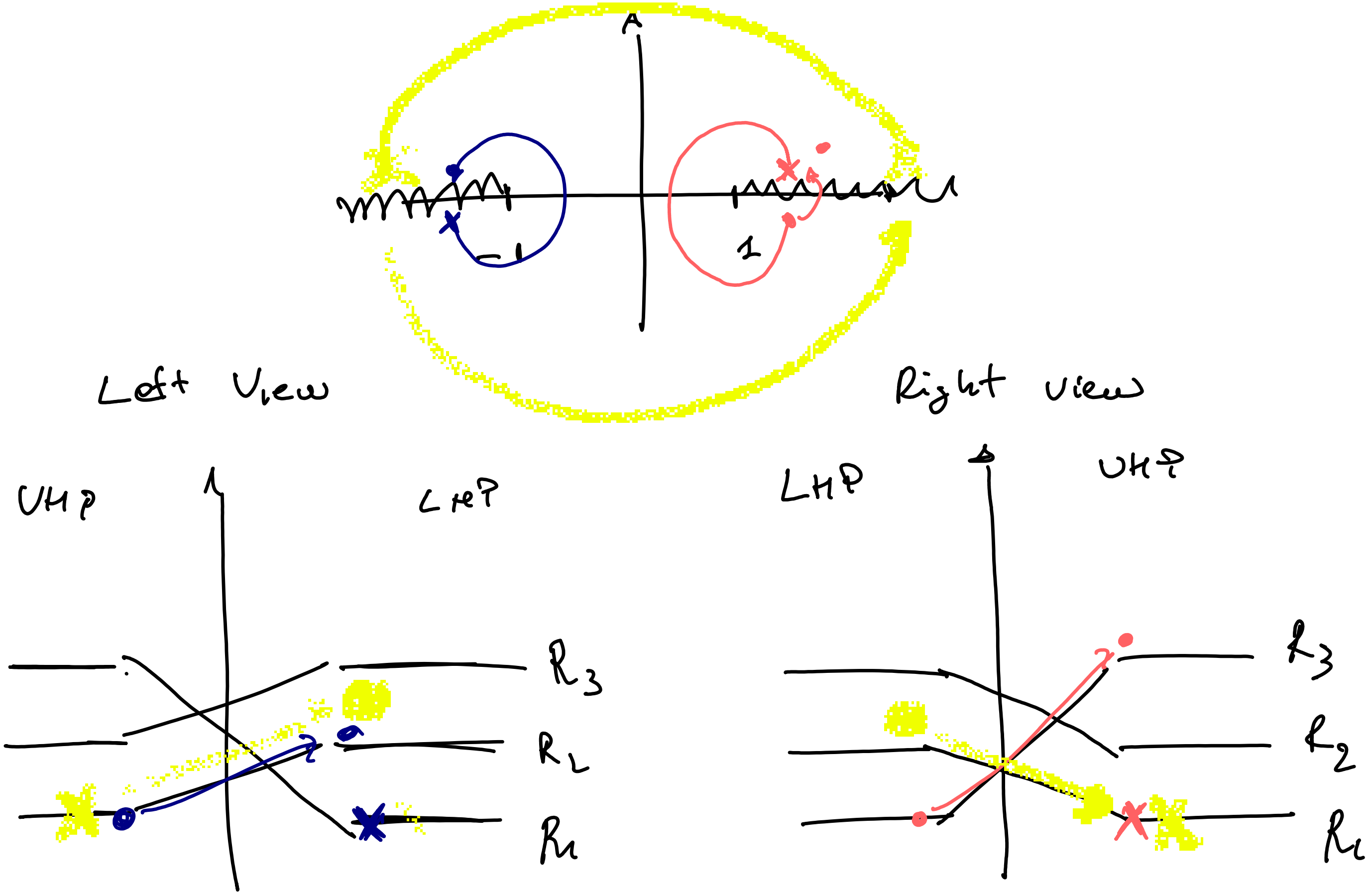
$0 < \theta_1 < 2\pi$




$$R_1 : \left(\frac{\rho_2}{\rho_1}\right)^{1/3} e^{i\frac{(\theta_2 - \theta_1)}{3}}$$

$$R_2 : \left(\frac{\rho_2}{\rho_1}\right)^{1/3} e^{i\frac{(\theta_2 - \theta_1)}{3} + \frac{2}{3}\pi i}$$

$$R_3 : \left(\frac{\rho_2}{\rho_1}\right)^{1/3} e^{i\frac{\theta_2 - \theta_1}{3} + \frac{4\pi}{3} i}$$



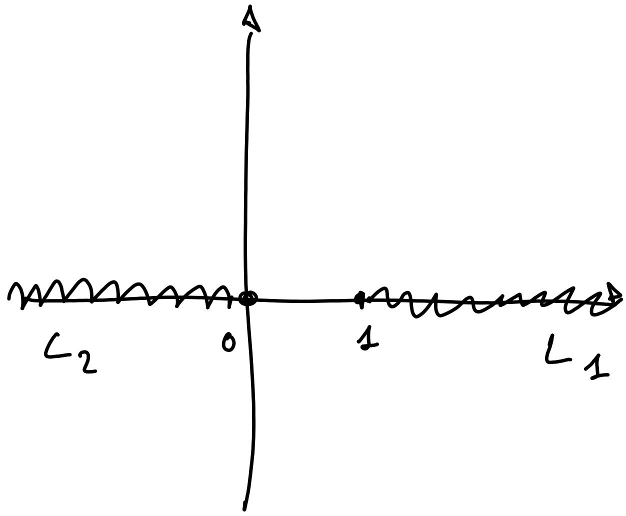
-1
1
Left View
UHP
LHP
R3
R2
R1
Right view
LHP
UHP
R3
R2
R1

Disegnare i fogli di Riemann di

Es. 8.35 Wong

$f(z) = \sqrt{z+1}\,\ln z$

Branch points in $z=0$, $z=1$ e $z=\infty$




$\ln z \longrightarrow$ infiniti Branch

$\sqrt{z+1} \longrightarrow$ 2 branch

$\Rightarrow$ infiniti Branches ma ognuno è raddoppiato dalla radice

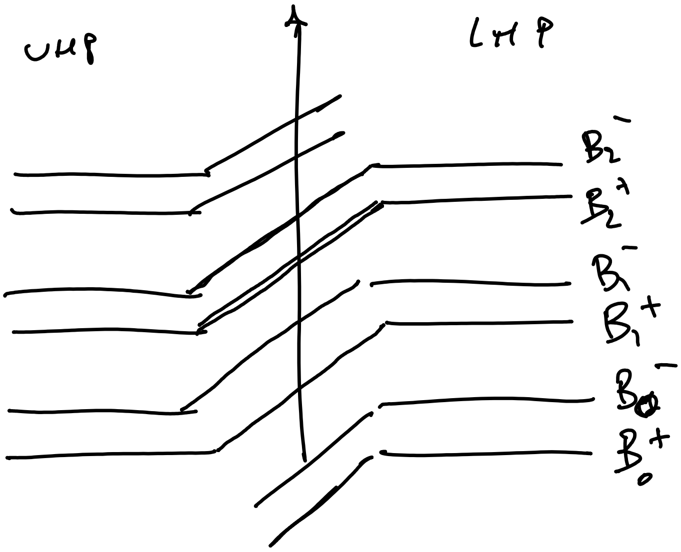
Chiamiamo i Branches $B_n^{\pm}$

$z = \rho e^{i\vartheta}$

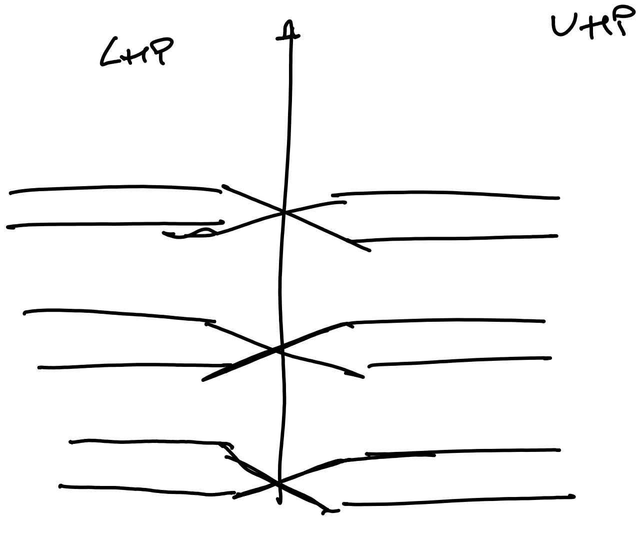
$z+1 = \rho_1 e^{i\vartheta_1}$

$$B_n^{\pm} : f(z) = \pm\sqrt{\rho_1}\, e^{i\frac{\vartheta_1}{2}}\left(\ln\rho + i\vartheta + 2n\pi\right)$$

Left View




Right View



# INTEGRALI COMPLESSI

$$\int_\gamma f(z)\,dz$$

**Def:** sia $\gamma(t) : [a,b] \to \mathbb{C}$ un cammino regolare a tratti e $f : D \to \mathbb{C}$ funzione continua con $\mathbb{C} \supset D \supset \{\gamma\}$

$$\Rightarrow \int_\gamma f(z)\,dz \equiv \int_a^b f(\gamma(t))\,\gamma'(t)\,dt \,.$$

$\gamma(t) = x(t) + iy(t)$

$\gamma'(t) = \frac{dx}{dt}(t) + i\frac{dy}{dt}(t)$

utile anche scriverlo come

$$\int_\gamma f(z)\,dz = \int_\gamma \left[u(x,y) + iv(x,y)\right](dx + i\,dy)$$

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**Oss 1:** l'integrale non dipende dalla parametrizzazione della curva

$\gamma(t) \longrightarrow \gamma(s) \qquad s = g(t)$

$$\int_{g(a)}^{g(b)} f(\gamma(s))\,\gamma'(s)\,ds = \int_{g(a)}^{g(b)} f(\gamma(s(t)))\,\frac{d}{ds}\gamma(s(t))\,ds$$

$\gamma(t(s)) \longrightarrow \begin{cases} \frac{d}{ds}\gamma = \frac{d\gamma}{dt}\cdot\frac{dt}{ds} \\ \int_a^b f(\gamma(t))\,\dot{\gamma}\,dt \end{cases}$

$\boxed{\frac{dt}{ds}\cdot ds}$

formula cambiamento variabile.

Oss2

Se $f(z)$ ammette primitiva, ovvero $F(z)$ olomorfa t.c.

$F'(z) = f(z)$

$$\Rightarrow \int_\gamma dz\, f(z) = \int_a^b \dot\gamma(t)\, f(\gamma(t))$$

$$= \int_a^b \dot\gamma(t) \quad F'(\gamma(t))\, dt$$

$$= \int_a^b \frac{d}{dt} F(\gamma(t)) \quad dt = F(z = \gamma(b)) - F(z = \gamma(a))$$

$\Rightarrow$ Se $\exists F(z)$ $\quad \oint_\gamma dz\, f(z) = 0$

$$\boxed{\begin{array}{l}\text{Thm (Cauchy): } f \text{ analitica su e all'interno di } \gamma \\ \text{curva chiusa e regolare a tratti} \\ \Rightarrow \oint_\gamma f(z)\, dz = 0\end{array}}$$

$\rightsquigarrow \displaystyle\int_{\gamma_1} dz\, f(z) = \int_{\gamma_2} dz\, f(z)$ se $\begin{array}{l}\gamma_1(a) = \gamma_2(a) \\ \gamma_1(b) = \gamma_2(b)\end{array}$

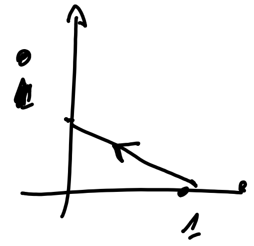
$\forall \gamma_1, \gamma_2$

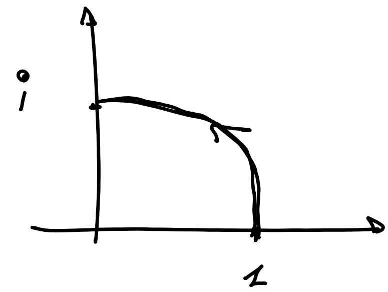
Esercizio: calcolare

i) $\int_\gamma dz \, \mathrm{Re}\, z$

sulle curve

ii) $\int_\gamma dz \, z^2$

a) 

b) 

a) $\gamma(t) = (1-t) + it$ $\qquad t \in [0;1]$

$\gamma'(t) = -1 + i$

b) $\gamma(\theta) = e^{i\theta}$ $\qquad \theta \in [0;\pi/2]$

$\gamma'(t) = i e^{i\theta}$

ia) $$\int_\gamma dz \, \mathrm{Re}(z) = \int_0^1 \mathrm{Re}(\gamma(t)) \, \gamma' \, dt$$

$$= \int_0^1 (1-t)(-1+i) \, dt = \int_0^1 (-1 + i + t - it) \, dt$$

$$= \left( -t + it + \frac{t^2}{2} - \frac{it^2}{2} \right) \Big|_0^1$$

$$= -1 + i + \frac{1}{2} - \frac{i}{2} = -\frac{1}{2} + \frac{i}{2}$$

ib) $\int dz\, Re(z) = \int_0^{\pi/2} Re(e^{i\theta})\, i e^{i\theta} d\theta$

$= \int_0^{\pi/2} i \cos\theta\, e^{i\theta} d\theta = i \int_0^{\pi/2} \frac{e^{i\theta} + e^{-i\theta}}{2} e^{i\theta} d\theta$

$= \frac{i}{2} \int_0^{\pi/2} (e^{2i\theta} + 1) d\theta$

$= \frac{i}{2} \left( \frac{e^{2i\theta}}{2i} + \theta \right) \Big|_0^{\pi/2} = \frac{e^{i\pi}}{4} + \frac{i}{2}\frac{\pi}{2} - \frac{1}{4}$

$= -\frac{1}{4} + \frac{i\pi}{4} - \frac{1}{4} = -\frac{1}{2} + \frac{i\pi}{4}$

Notare che ia $\neq$ ib perché Re(z) non è olomorfa.

iia) $\int dz\, z^2 = \int_0^1 dt\, (1-t+it)^2 (-1+i)$

$= \int_0^1 dt\, (1 + \cancel{t^2} - \cancel{t^2} + 2it - 2t - 2it^2)(-1+i)$

$= \int_0^1 dt\, (-1 \underbrace{-2it + \cancel{2t} + 2it^2 + i - \cancel{2t} - 2i\cancel{t}}_{} + 2t^2$

$= \int_0^1 dt\, (-1 - 4it + i + 2it^2 + 2t^2)$

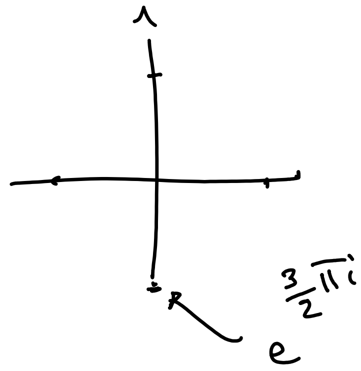
$= \left( -t - \frac{4it^2}{2} + it + \frac{2it^3}{3} + \frac{2}{3}t^3 \right) \Big|_0^1$

$= \left( -1 - 2i + i + \frac{2i}{3} + \frac{2}{3} \right) = -\frac{1}{3} - \frac{i}{3}$ ✓

iib) $\int dz\, z^2 = \int_0^{\pi/2} d\theta\, i e^{i\theta} e^{2i\theta} = i\int_0^{\pi/2} d\theta\, e^{3i\theta}.$

$$i \left.\frac{e^{3i\theta}}{3i}\right|_0^{\pi/2} = \cancel{i}\,\frac{e^{\frac{3\pi}{2}i} - 1}{3\cancel{i}}$$

$$e^{\frac{3}{2}\pi i} = -i$$



$$= \frac{-i-1}{3}$$

Con primitiva

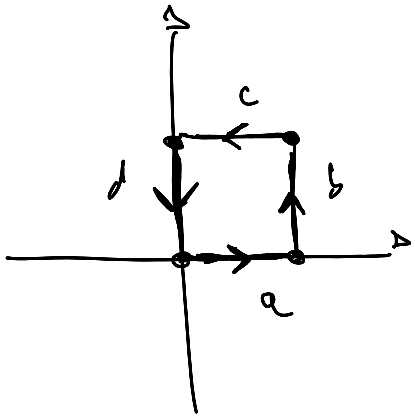
$$\int_\gamma dz\, z^2 = \left.\frac{z^3}{3}\right|_{\gamma(a)}^{\gamma(b)} = \left.\frac{z^3}{3}\right|_1^i = -\frac{i}{3} - \frac{1}{3} \quad \checkmark$$

Notare come in questo caso invece iia = iib conseguenza dell'analiticità di $z^2$.

# Tema d'esame 26/09/2018 Es. 1

Calcolare $\int_{\square} dz\,|z|$ dove $\square$ è il contorno antiorario del quadrato di vertici $0, 1, 1+i, i$

Nota $\rightarrow$ $|z|$ non è olomorfa $\Rightarrow$ l'integrale non è nullo!




a) $z = t \qquad dz = dt \qquad |z| = t$

b) $z = 1+it \qquad dz = i\,dt \qquad |z| = \sqrt{1+t^2}$

c) $z = t+i \qquad dz = dt \qquad |z| = \sqrt{1+t^2}$

$t$ da $1$ a $0$

d) $z = it \qquad dz = i\,dt \qquad |z| = t$

$t$ da $1$ a $0$

$$\int_{\square} dz\,|z| = \int_a |z|\,dz + \int_b |z|\,dz + \int_c |z|\,dz + \int_d |z|\,dz$$

$$= \int_0^1 t\,dt + \int_0^1 i\,dt\,\boxed{\sqrt{1+t^2}} + \int_1^0 dt\,\boxed{\sqrt{1+t^2}} + \int_1^0 t\,i\,dt$$

Radice positiva perché dal calcolo del modulo di z

Calcoliamo

$\int \sqrt{1+t^2}\,dt$ sostituzione $t = \sinh v \qquad \mathrm{Ch}\,t = \sqrt{1+t^2}$

$dt = \cosh v\,dv$

$\rightarrow \int \sqrt{1+\sinh^2 v}\,\cosh v\,dv$

$\cosh^2 v - \sinh^2 v = 1$

$\int \cosh^2 v\,dv = \sinh v \cosh v - \int \sinh^2 v\,dv = \mathrm{Sh}\,v\,\mathrm{Ch}\,v - \int (\mathrm{Ch}^2 v - 1)$

$\int \mathrm{Ch}^2 v\,dv = \int \sqrt{1+t^2}\,dt = \dfrac{\mathrm{Sh}\,v\,\mathrm{Ch}\,v - v}{2} = \dfrac{t\sqrt{1+t^2}}{2} - \dfrac{\mathrm{Sh}^{-1}t}{2}$

$$= \int_0^1 t\,dt + \int_0^1 i\,dt\sqrt{1+t^2} + \int_1^0 dt\sqrt{1+t^2} + \int_1^0 t\,i\,dt$$

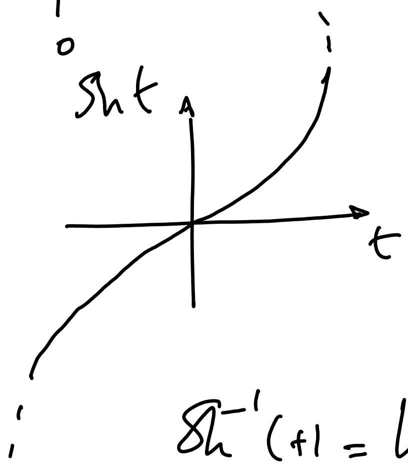
quindi

$$= \frac{1}{2} + \frac{1}{2}i\left(t\sqrt{1+t^2} - Sh^{-1}t\right)\Big|_0^1 + \frac{1}{2}\left(t\sqrt{1+t^2} - Sh^{-1}t\right)\Big|_1^0 - \frac{i}{2}$$

$$Sh\,t = \frac{e^t - e^{-t}}{2}$$

$$Sh^{-1}(0) = 0$$

$$Sh^{-1}(1) = \ln(1+\sqrt{2})$$




$$Sh^{-1}(t) = \ln\left(t + \sqrt{1+t^2}\right)$$

Quindi

$$= \frac{1}{2} + \frac{i}{2}\left(\sqrt{2} - \ln(1+\sqrt{2})\right) - \frac{1}{2}\left(\sqrt{2} - \ln(1+\sqrt{2})\right) - \frac{i}{2}$$

$$= \frac{1}{2}\left[1 - \sqrt{2} + \ln(1+\sqrt{2})\right] + \frac{i}{2}\left[-1 + \sqrt{2} - \ln(1+\sqrt{2})\right].$$