CS-E5740 Complex Networks,
Answers to exercise set 3

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Problem 1

The python implementation of the generative function for Barabási-Albert networks is provided in the attached Jupyter notebook.

a) When generating a BA network with $N = 250$ and $m = 2$ starting from a 4-clique network, the following results were obtained:

$$Maximum\ degree\ of\ a\ node = 35$$

$$Total\ number\ of\ edges = 506$$

The visualized network is presented in Figure 1.

b) When generating a BA network using parameters $N = 10^4$ and $m = 4$, and comparing its logarithmically binned probability density function to the theoretical prediction of $P(k)$, the graph reported in Figure 2 was created.

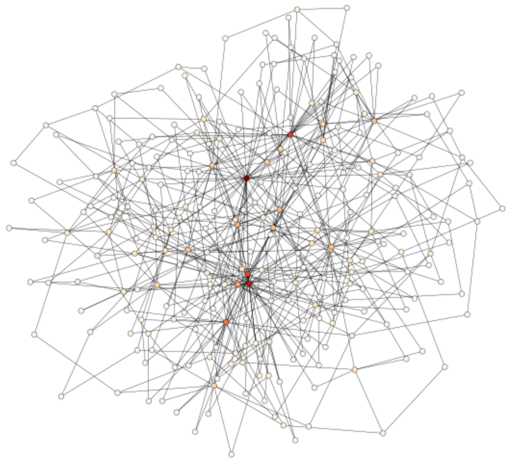

Figure 1: visualized BA network.

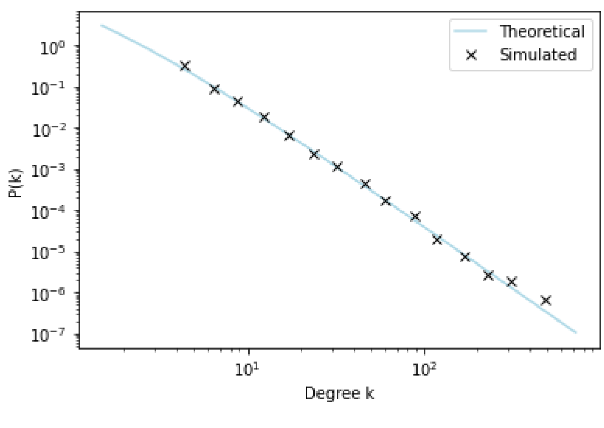

Figure 2: comparison between experimental and theoretical $P(k)$.

Problem 2

a) The sum of node degrees for all nodes in the network is equal to the number of nodes in the network multiplied by its average degree.
Thus we can write

$$\sum_{j=1}^{N} k_j = N \cdot < k >= N \cdot 2m$$

Then, since the probability that a new edge attaches to *any* vertex of degree k is the sum of the probabilities for it to attach to each of the vertices of degree k, and since

$$\pi_i(k) = \frac{k_i}{\sum k_j}$$

we can write

$$\Pi(k) = \sum_{i=1}^{\mu_{k,N}} \pi_i(k) = \pi(k) \cdot \mu_{k,N} = \pi(k) \cdot N \cdot f_{k,N} = \frac{kNf_{k,N}}{2mN} = \frac{kf_{k,N}}{2m}$$

b) For $k = m$, $\mu_k^+ = 1$ because each node that is added has, by definition, m edges, and since the starting network must be a fully-connected network of at least m nodes, this also means that there can be no nodes with $m-1$ edges which, if chosen for the edges when adding a node, would be "promoted" to degree m.
Since in general

$$\mu_k^- = \frac{1}{2} k f_{k,N}$$

we can write:

- For $k = m$:

$$(N+1)f_{m,N+1} - Nf_{m,N} = \mu_m^+ - \mu_m^- = 1 - \frac{1}{2} m f_{m,N}$$

- For $k > m$, knowing that the number of new nodes with degree k is equal to the number of nodes of degree $k-1$ that were chosen for the edges of the newly added node:

$$(N+1)f_{k,N+1} - Nf_{k,N} = \mu_k^+ - \mu_k^- = \frac{1}{2}(K-1)f_{k-1,N} - \frac{1}{2} k f_{k,N}$$

c) For $N \to \infty$ we have that $f_{k,N+1} = f_{k,N} = f_k$, $f_{k-1,N} = f_{k-1}$, and $f_{m,N+1} = f_m$ because a variation of 1 node between one iteration and the next becomes negligible, as there exist ∞ nodes of every degree. Then, for $N \to \infty$ we can write:

– For $k = m$:

$$(N+1)f_{m,N+1} - Nf_{m,N} \xrightarrow{N\to\infty} (N+1)f_m - Nf_m = f_m$$

and

$$1 - \frac{1}{2}mf_{m,N} \xrightarrow{N\to\infty} 1 - \frac{1}{2}mf_m$$

and thus

$$f_m = 1 - \frac{1}{2}mf_m = \frac{2}{m+2}$$

– For $k > m$:

$$(N+1)f_{k,N+1} - Nf_{k,N} \xrightarrow{N\to\infty} (N+1)f_k - Nf_k = f_k$$

and

$$\frac{1}{2}(K-1)f_{k-1,N} - \frac{1}{2}kf_{k,N} \xrightarrow{N\to\infty} \frac{1}{2}(K-1)f_{k-1} - \frac{1}{2}kf_k$$

and thus

$$f_k = \frac{1}{2}(K-1)f_{k-1} - \frac{1}{2}kf_k = \frac{k-1}{k+2}f_{k-1}$$

d) To derive a formula for f_k as function of only k and m, we start by listing the formulas for f_{m+1}, f_{m+2}, and f_{m+3}:

– For $k = m+1$:

$$f_{m+1} = \frac{m+1-1}{m+1+2} \cdot f_{m+1-1} = \frac{m}{m+3} \cdot f_m = \frac{m}{m+3} \cdot \frac{2}{m+2}$$

which is

$$f_k = \frac{2m}{(k+1)(k+2)}$$

– For $k = m+2$:

$$f_{m+2} = \frac{m+2-1}{m+2+2} \cdot f_{m+2-1} = \frac{m+1}{m+4} \cdot \frac{m}{m+3} \cdot \frac{2}{m+2}$$

which is

$$f_k = \frac{2m(m+1)}{k(k+1)(k+2)}$$

– For $k = m+3$:

$$f_{m+3} = \frac{m+3-1}{m+3+2} \cdot f_{m+3-1} = \frac{m+2}{m+5} \cdot \frac{m+1}{m+4} \cdot \frac{m}{m+3} \cdot \frac{2}{m+2}$$

which, by cancelling out the $m+2$ terms, is again

$$f_k = \frac{2m(m+1)}{k(k+1)(k+2)}$$

From this we can see that going forward the formula will remain unchanged, and thus the general formula for f_k is:

$$f_k = \frac{2m(m+1)}{k(k+1)(k+2)}$$