Computer Vision Ex 1

① a) $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ ⟶ in homogeneous coordinates

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

so:

$$ax + by + c = 0 \rightsquigarrow \underbrace{\begin{bmatrix} a & b & c \end{bmatrix}}_{\mathbf{l}^T} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{\mathbf{x}} = 0$$

$$\rightsquigarrow \boxed{\mathbf{l}^T \mathbf{x} = 0}$$

---

b) since $(\mathbf{a} \times \mathbf{b})^T \mathbf{c} = 0$ if $\mathbf{a} // \mathbf{b}$, $\mathbf{b} // \mathbf{c}$, or $\mathbf{a} // \mathbf{c}$

given $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$

we have:

$$\mathbf{x}^T \mathbf{l} = (\mathbf{l} \times \mathbf{l}')^T \mathbf{l} = 0 \rightsquigarrow \boxed{\mathbf{x} \in \mathbf{l}}$$

$$\mathbf{x}^T \mathbf{l}' = (\mathbf{l} \times \mathbf{l}')^T \mathbf{l}' = 0 \rightsquigarrow \boxed{\mathbf{x} \in \mathbf{l}'}$$

so assuming $\mathbf{l}$ and $\mathbf{l}'$ not parallel, $\mathbf{x}$ belongs to both lines (is intersection)

c) given $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$

we have:

$$\mathbf{l}^T \mathbf{x} = (\mathbf{x} \times \mathbf{x}')^T \mathbf{x} = 0 \rightsquigarrow \boxed{\mathbf{x} \in \mathbf{l}}$$

$$\mathbf{l}^T \mathbf{x}' = (\mathbf{x} \times \mathbf{x}')^T \mathbf{x}' = 0 \rightsquigarrow \boxed{\mathbf{x}' \in \mathbf{l}}$$

so both $\mathbf{x}$ and $\mathbf{x}'$ belong to $\mathbf{l}$

---

d) given $\mathbf{y} = \alpha \mathbf{x} + (1-\alpha)\mathbf{x}'$

and $\mathbf{x}^T \mathbf{l} = 0$, $\mathbf{x}'^T \mathbf{l} = 0$

we have:

$$\mathbf{y}^T \mathbf{l} = [\alpha \mathbf{x} + (1-\alpha)\mathbf{x}']^T \mathbf{l}$$

$$= \alpha \underbrace{\mathbf{x}^T \mathbf{l}}_{0} + (1-\alpha) \underbrace{\mathbf{x}'^T \mathbf{l}}_{0} = 0$$

$$\boxed{\mathbf{y} \in \mathbf{l}}$$

② a) • Translations:

$$x' = H_T x = \begin{bmatrix} I & t \\ \mathbf{0}^T & 1 \end{bmatrix} x \quad \text{with } t = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

• Euclidean transformations:

$$x' = H_E x = \begin{bmatrix} R & t \\ \mathbf{0}^T & 1 \end{bmatrix} x$$

$$\text{with } R = \begin{bmatrix} \varepsilon \cos\theta & -\sin\theta \\ \varepsilon \sin\theta & \cos\theta \end{bmatrix}$$

• Similarity tranformations:

$$x' = H_S x = \begin{bmatrix} sR & t \\ \mathbf{0}^T & 1 \end{bmatrix} x$$

• Affine transformations:

$$x' = H_A x = \begin{bmatrix} A & t \\ \mathbf{0}^T & 1 \end{bmatrix} x$$

$$\text{with } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$$

- Projective transformations:

$$x' = H_p x = \begin{bmatrix} A & t \\ \mathbf{v} & v \end{bmatrix} \quad \text{with } \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

---

b) Translation: 2 dof

Euclidean: 3 dof

Similarity: 4 dof

Affine: 6 dof

Projective 8 dof

---

c) Because homogeneous vectors are invariant to scaling by constant factors

$$x' = H_p x \longrightarrow k' \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = H_p k \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \frac{k}{k'} H_p \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

From which we can deduce that H_p is also invariant to scaling, as k/k' can assume any real value then if $H_p = \kappa \begin{bmatrix} A & t \\ v & v \end{bmatrix}$ it can also be

written as $H_p = \kappa \begin{bmatrix} A & t \\ v & 1 \end{bmatrix}$ if we divide by v

$\rightarrow$ this means that H_p actually contains only $\boxed{8 \text{ dof}}$, as κ can assume any value

---

③ a) since $l^T x = 0$ and $x' = Hx$

we have:

$$l^T x = l^T I x = l^T H^{-1} \underbrace{H x}_{x'}$$

$$= \underbrace{l^T H^{-1}}\, x'$$

if we call this l'^T, we have that $x' \in l'$ (where l' is a line)

then:

$$l'^T = l^T H^{-1} \rightsquigarrow \boxed{l' = H^{-T} l}$$

b) If we transform x_1, x_2, l_1, l_2:

$$\frac{(l_1'^T x_1')(l_2'^T x_2')}{(l_1'^T x_2')(l_2'^T x_1')} = \frac{(l_1^T \cancel{H^{-1}} \cancel{H} x_1)(l_2^T \cancel{H^{-1}} \cancel{H} x_2)}{(l_1^T H^{-1} H x_2)(l_2^T H^{-1} H x_1)}$$

$$= \frac{(l_1^T x_1)(l_2^T x_2)}{(l_1^T x_2)(l_2^T x_1)}$$

If we removed any of the factors, invariance to scaling would not hold anymore

For example, by removing $(l_2^T x_1)$ and scaling $x_{1,2}$ by a factor of $k_{1,2}$ we have

$$I = \frac{(l_1^T k_1 x_1)(l_2^T \cancel{k_2} x_2)}{(l_1^T \cancel{k_2} x_2)}$$

as we can see, the scaling factor k_1 cannot be eliminated.