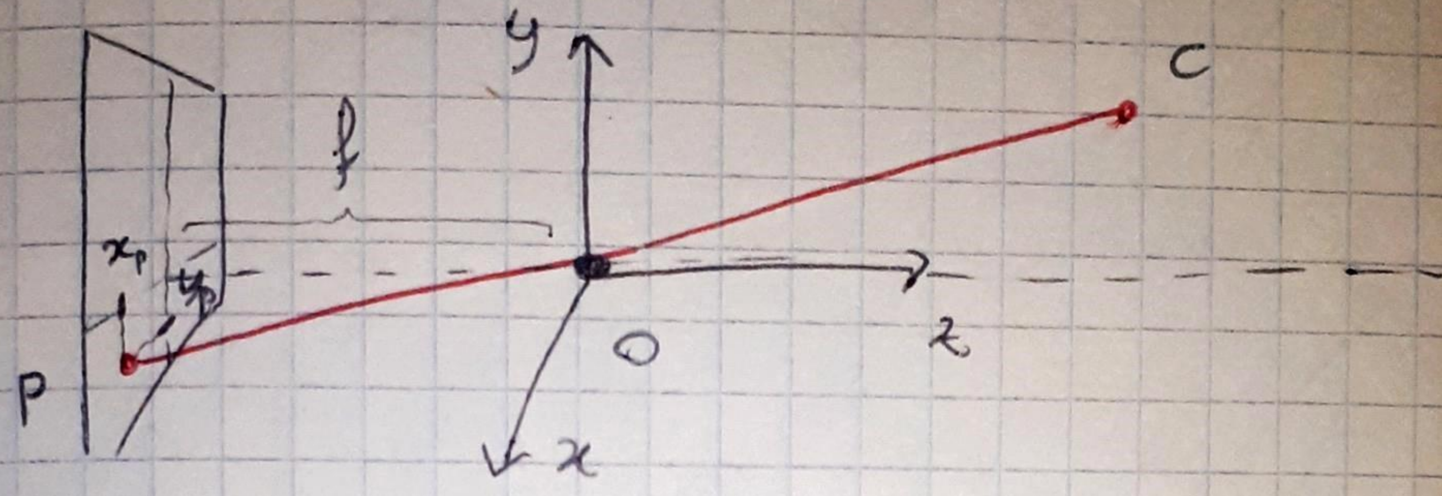
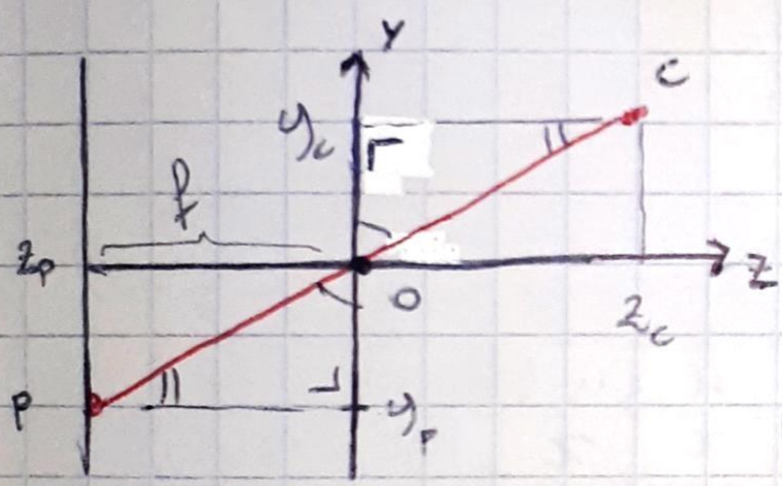
Computer Vision Assignment 2

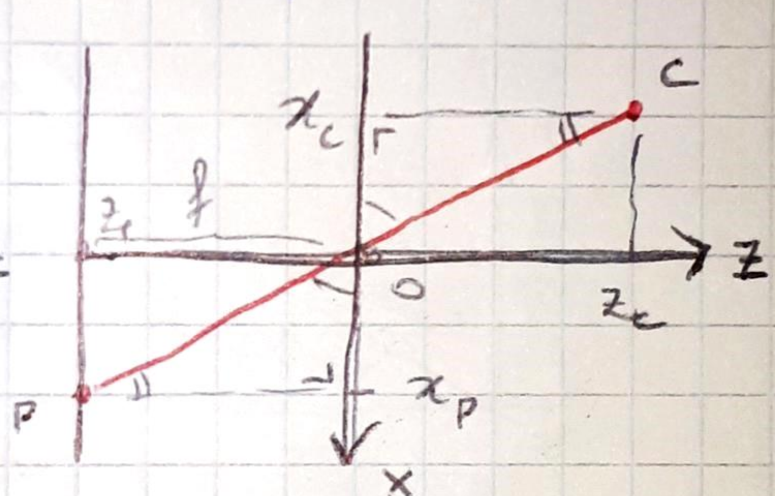
Ex 1) Given a pinhole camera:



We can split the view into 2 perspectives:

- side view
- top view





In each case, the triangles are similar ($P y_p O$ with $C y_c O$, and $P x_p O$ with $C x_c O$, respectively) because their angles are congruent. Then:

$$\frac{y_p}{f} = \frac{y_c}{z_c} \qquad \text{because of proportions between their sides} \qquad \frac{x_p}{f} = \frac{x_c}{z_c}$$

$$\boxed{y_p = f\,\frac{y_c}{z_c}} \qquad \boxed{x_p = f\,\frac{x_c}{z_c}}$$

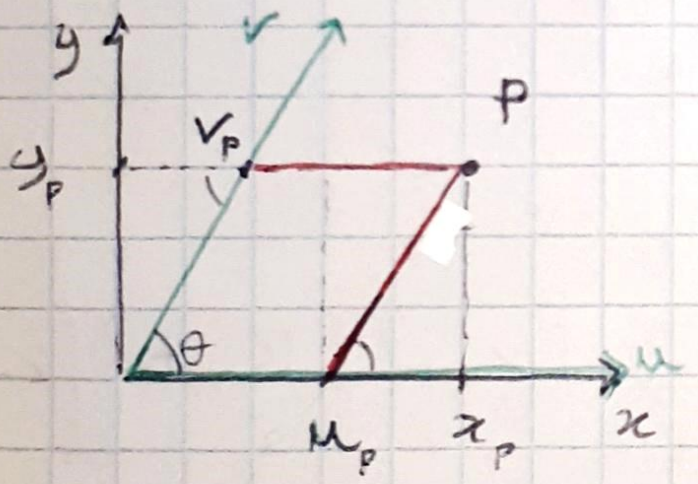
Ex 2)

a) When $x \parallel u$ and $v \parallel y$:

$$p = (u, v) = (u_0 + x_p \cdot m_u \,,\, v_0 + y_p \cdot m_v)$$

assuming that the principal point is offset from the image center.

b) When $x \parallel u$ and $\angle(u, v) = \theta$



we can calculate u_p and v_p using trigonometry

$$\Rightarrow \boxed{v_p} = \boxed{\frac{y_p}{\sin\theta}} \qquad \boxed{u_p} = x_p - \frac{y_p}{\sin\theta} \cdot \cos\theta$$

$$= \boxed{x_p - y_p \cot g\, \theta}$$

So we have:

$$P = (u, v) = \left(u_0 + \left(x_p - y_p \cot g\, \theta \right) m_u \,,\, v_0 + \frac{y_p}{\sin\theta} m_v \right)$$

Ex 3) By using homogeneous coordinates, case 2.a) can be represented as:

$$\tilde{p} = K \boldsymbol{x}_c$$ where $\tilde{p}$ is p in homogeneous coordinates, $\boldsymbol{x}_c = (x_c, y_c, z_c)$, and

$$\boxed{K} = \boxed{\begin{bmatrix} f \cdot m_u & 0 & u_0 \\ 0 & f \cdot m_v & v_0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \text{so} \quad \tilde{p} = \begin{bmatrix} f m_u x_c + u_0 z_c \\ f m_v y_c + v_0 z_c \\ z_c \end{bmatrix}$$

Ex 4) We can write:

$$\boldsymbol{x}_c = R \boldsymbol{x}_w + t \longrightarrow \begin{bmatrix} \boldsymbol{x}_c \\ 1 \end{bmatrix} = \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_w \\ 1 \end{bmatrix}$$

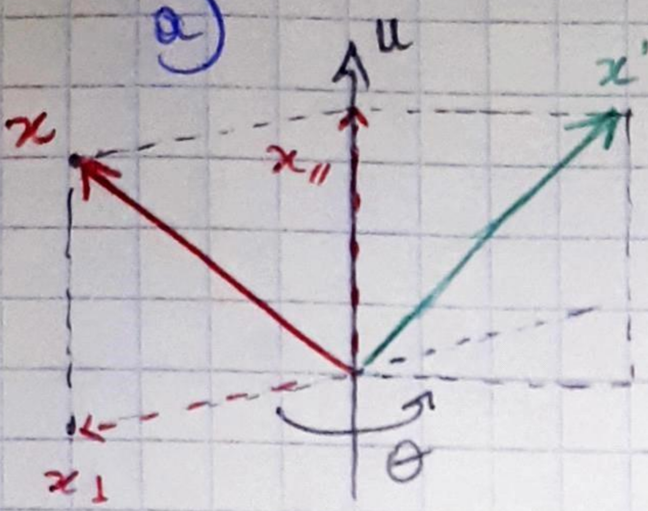
then:

$$\tilde{p} = [K \mid 0] \begin{bmatrix} \boldsymbol{x}_c \\ 1 \end{bmatrix} = \underbrace{[K \mid 0] \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}}_{P} \begin{bmatrix} \boldsymbol{x}_w \\ 1 \end{bmatrix}$$

So $$\boxed{P = K [R \mid t]}$$

Ex 5) If the rotation is defined by axis u:

a)



we decompose x in a perpendicular and parallel component wrt u:

$$x = x_{//} + x_\perp$$

where:

- $x_{//} = (x \cdot u)u$ (vector projection)
- $x_\perp = x - x_{//} = -u \times (u \times x)$ (vector rejection)

Through the rotation, $x_{//}$ will remain unchanged:

$$x'_{//} = x_{//}$$

Meanwhile, $x_\perp$ will maintain its magnitude but be rotated according to θ:

$$|x'_\perp| = |x_\perp| \quad \text{and} \quad \underbrace{x'_\perp = \cos\theta \cdot x_\perp + \sin\theta \cdot u \times x_\perp}_{\text{this can be rewritten as}}$$

$$x'_\perp = \cos\theta \cdot x_\perp + \sin\theta \cdot u \times x$$

because $u \,//\, x_{//}$, meaning

$$u \times V_{//} = 0 \implies u \times x_\perp = u \times x$$

by substituting the definitions for $x_\perp$ and $x_{\parallel}$ in the initial equation we get:

$$\boxed{x'} = x'_{\parallel} + x'_{\perp} = x_{\parallel} + \cos\theta \cdot x_{\perp} + \sin\theta \cdot u \times x$$

$$= x_{\parallel} + \cos\theta \, (x - x_{\parallel}) + \sin\theta \cdot u \times x$$

$$= \cos\theta \cdot x + (1 - \cos\theta) x_{\parallel} + \sin\theta \cdot u \times x$$

$$= \boxed{\cos\theta \cdot x + (1 - \cos\theta)(u \cdot x) u + \sin\theta \cdot u \times x}$$

which is the common form of Rodrigues' formula

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b) $$u \cdot x = [u_1 \; u_2 \; u_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = u_1 x_1 + u_2 x_2 + u_3 x_3$$

$$u \times x = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = (u_2 x_3 - u_3 x_2)\hat{i} + (u_3 x_1 - u_1 x_3)\hat{j} + (u_1 x_2 - u_2 x_1)\hat{k}$$

· Starting from Rodrigues' formula:

$$x' = Rx = \cos\theta \, x + (1 - \cos\theta)(u \cdot x) u + \sin\theta \, u \times x$$

$$= \cos\theta \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + (1 - \cos\theta)(u_1 x_1 + u_2 x_2 + u_3 x_3) \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} +$$

$$+ \sin\theta \begin{bmatrix} u_2 x_3 - u_3 x_2 \\ u_3 x_1 - u_1 x_3 \\ u_1 x_2 - u_2 x_1 \end{bmatrix}$$

$$Rx = \begin{bmatrix} x_1\cos\theta \\ x_2\cos\theta \\ x_3\cos\theta \end{bmatrix} + \begin{bmatrix} (1-\cos\theta)(u_1x_1+u_2x_2+u_3x_3)\,u_1 \\ (1-\cos\theta)(u_1x_1+u_2x_2+u_3x_3)\,u_2 \\ (1-\cos\theta)(u_1x_1+u_2x_2+u_3x_3)\,u_3 \end{bmatrix} +$$

$$+ \begin{bmatrix} \sin\theta\,(u_2x_3-u_3x_2) \\ \sin\theta\,(u_3x_1-u_1x_3) \\ \sin\theta\,(u_1x_2-u_2x_1) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \cos\theta+(1-\cos\theta)u_1^2 & (1-\cos\theta)u_1u_2-\sin\theta\,u_3 & (1-\cos\theta)u_1u_3+\sin\theta\,u_2 \\ (1-\cos\theta)u_2u_1+\sin\theta\,u_3 & \cos\theta+(1-\cos\theta)u_2^2 & (1-\cos\theta)u_2u_3-\sin\theta\,u_1 \\ (1-\cos\theta)u_3u_1-\sin\theta\,u_2 & (1-\cos\theta)u_2u_3+\sin\theta\,u_1 & \cos\theta+(1-\cos\theta)u_3^2 \end{bmatrix}}_{\boxed{R(\theta,u_1,u_2,u_3)}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$