Politecnico di Torino

Master of Science in Energy and Nuclear Engineering
01FJIXY - Computational Thermomechanics
A.Y. 2025/2026

Homework 03 - Cylinder with Internal Pressure

Tommaso COGOZZO

1 Problem statement and given data

1.1 Thick cylinder

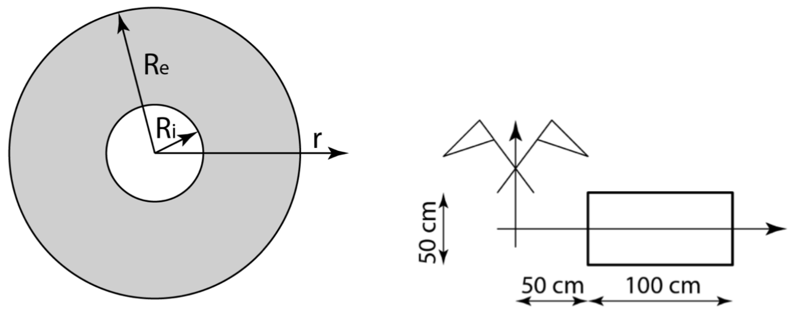


Figure 1: Thick cylinder with internal pressure

Data:

- Internal radius: $Ri = 0.5\,m$
- External radius: $Re = 1.5\,m$
- Thickness: $t = 1\,m$
- Mesh height considered: $h = 0.5\,m$
- Elasticity modulus: $E = 1.0e10\,N/m^2$
- Poisson's ratio: $\nu = 0.2$
- Internal pressure: $P = 1.0e6\,N/m^2$

1.2 Thin cylinder

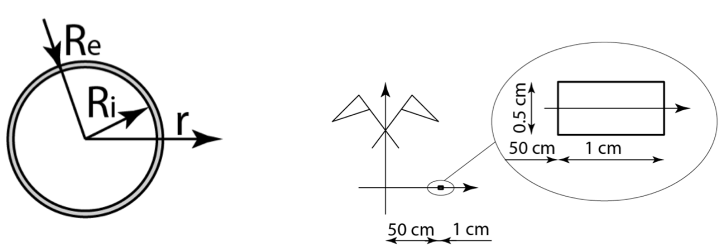


Figure 2: Thin cylinder with internal pressure

Data:

- Internal radius: $Ri = 0.50\,m$
- External radius: $Ri = 0.51\,m$
- Thickness: $t = 0.01\,m$
- Mesh height considered: $h = 0.005\,m$
- Elasticity modulus: $E = 1.0e10\,N/m^2$
- Poisson's ratio: $\nu = 0.2$
- Internal pressure: $p_i = 1.0e6\,N/m^2$

2 Objective

The problem of a cylinder subjected to an internal p_i pressure and an external pressure p_o (equal to atmospheric pressure), commonly referred to as Lamé's problem, can be solved analytically. The radial and circumferential stress components are obtained from the following equations:

$$\sigma_r = -\frac{(p_i - p_o) R_i^2 R_e^2}{(R_e^2 - R_i^2)\, r^2} + \frac{p_i R_i^2 - p_o R_e^2}{R_e^2 - R_i^2}$$

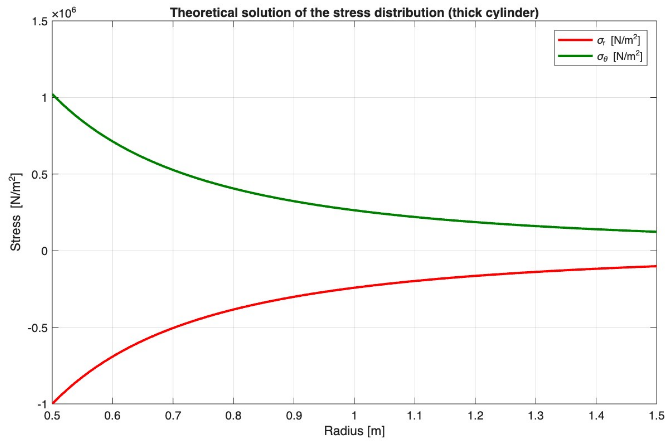
$$\sigma_\theta = +\frac{(p_i - p_o) R_i^2 R_e^2}{(R_e^2 - R_i^2)\, r^2} + \frac{p_i R_i^2 - p_o R_e^2}{R_e^2 - R_i^2}$$

By implementing the two equations in MATLAB and evaluating them for both cylinder configurations, the analytical solution for the stress distribution through the cylinder wall thickness is obtained as reported in Fig.3.

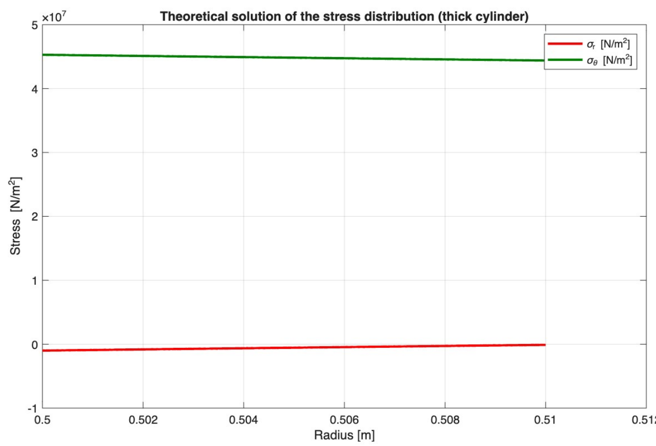
The objective of this report is therefore to analyze the problem in both thickness configurations of the cylinder using the Finite Element Analysis Program (FEAP). The study will be conducted using different discretization strategies:

- Triangular plane-stress elements;
- Rectangular plane-stress elements.

For each element type, the analyses will begin with a coarse mesh and will subsequently be refined in both the vertical and horizontal directions. Finally, the convergence of the computed stress values at the tip toward the analytical solution will be evaluated.



(a) Analytical stress distribution through thickness (thick cylinder)



(b) Analytical stress distribution through thickness (thin cylinder)

Figure 3: Stress analytical distribution through thickness

3 Finite Element Analysis with Triangular Plane Stress Elements

The thick cylinder was initially analyzed using linear triangular elements. With a relatively coarse mesh (4×2 nodes), the resulting solution is presented in the Fig.04.

It is immediately apparent that a mesh refinement of 4×2 leads to a result that is insufficiently accurate when compared to the analytically computed solution. The values of the functions σ_r and σ_θ at the domain boundaries differ

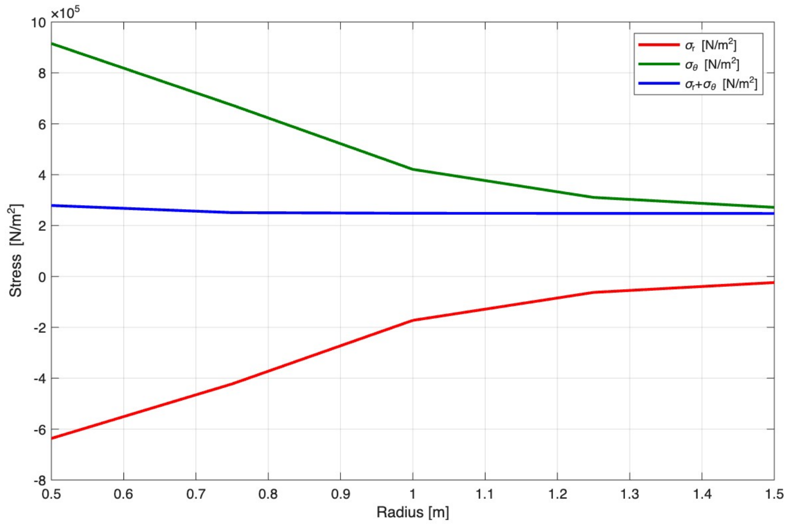


Figure 4: Numerical (4x2 linear triangular elements) radial and circumferential stresses distribution.

significantly from those shown in Figure 3(a). To obtain a numerical solution that better converges toward the analytical values, it is necessary to refine the mesh by increasing the number of finite elements.

Figure 5 shows the numerical distributions of σ_r, σ_θ, and $\sigma_r + \sigma_\theta$, obtained using a 10×2 mesh of linear triangular elements. Although the numerical solution does not yet exactly match the analytical solution, it is evident that the results are significantly more accurate compared to those obtained with the 4×2 mesh. This second analysis, in fact, yields values of radial and circumferential stress near the domain boundaries that exhibit a much lower relative error with respect to the theoretical solution.

By further refining the mesh to improve the convergence of the numerical solution toward the theoretical one, a 400×2 mesh of linear triangular elements was adopted. The resulting distributions of σ_r and σ_θ across the thickness of the thick cylinder are presented in Figure 6, clearly demonstrating an excellent qualitative agreement with the analytical solution.

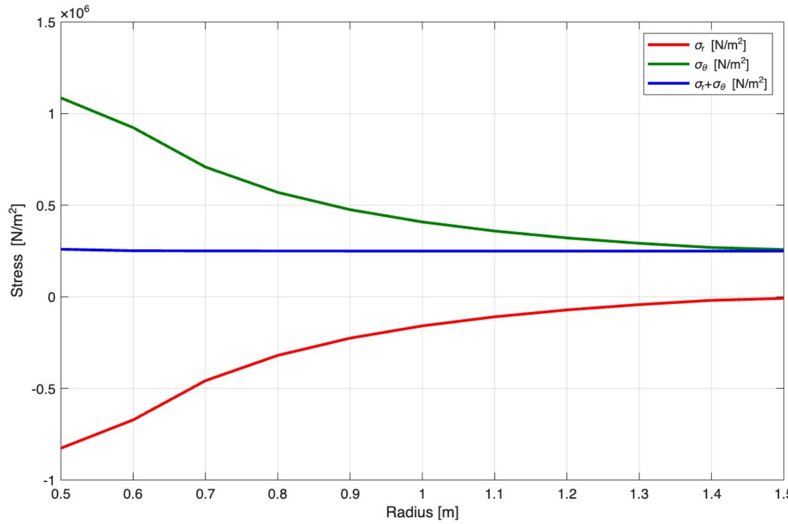


Figure 5: Numerical (10x2 linear triangular elements) radial and circumferential stresses distribution.

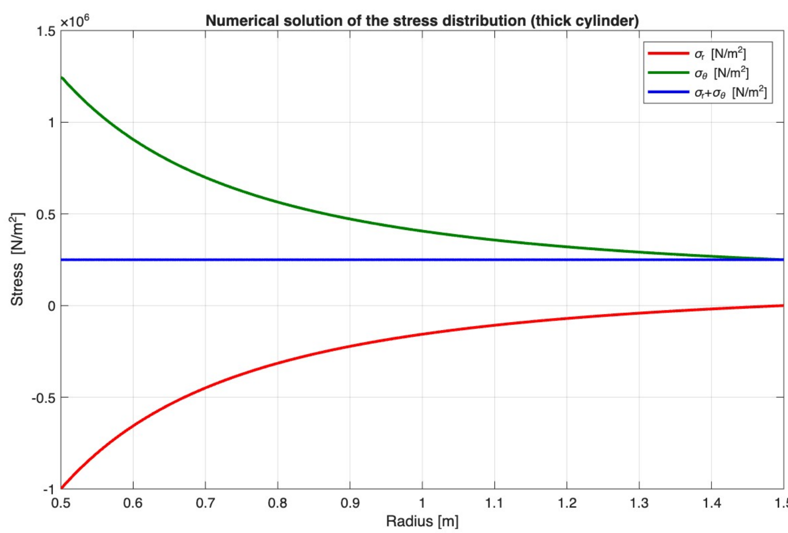


Figure 6: Numerical (400x2 linear triangular elements) radial and circumferential stresses distribution.

The same procedure was also applied to the thin cylinder problem. Unlike the thick cylinder case, a mesh discretization of only 4×2 linear triangular elements (Figure 7) appears to yield results that are both qualitatively and quantitatively satisfactory. This difference can be attributed to the reduced thickness of the cylinder, which decreases the gradient of stress and displacement across the radial direction, making the solution less sensitive to mesh refinement.

Consequently, the analysis demonstrates that for thinner geometries, it is possible to achieve accurate numerical results even with relatively coarse meshes. This has important practical implications, as it allows for a significant reduction in computational cost while maintaining sufficient accuracy. Nevertheless, it should be noted that while coarse meshes may be acceptable for thin cylinders, a careful evaluation of convergence and error is still recommended, especially near boundaries or regions where stress concentrations may occur.

Overall, these findings highlight the interplay between geometry, mesh refinement, and numerical accuracy in finite element analyses. They suggest that an appropriate choice of mesh density should consider both the geometric characteristics of the structure and the expected variation of the field variables, thereby optimizing computational efficiency without compromising reliability.

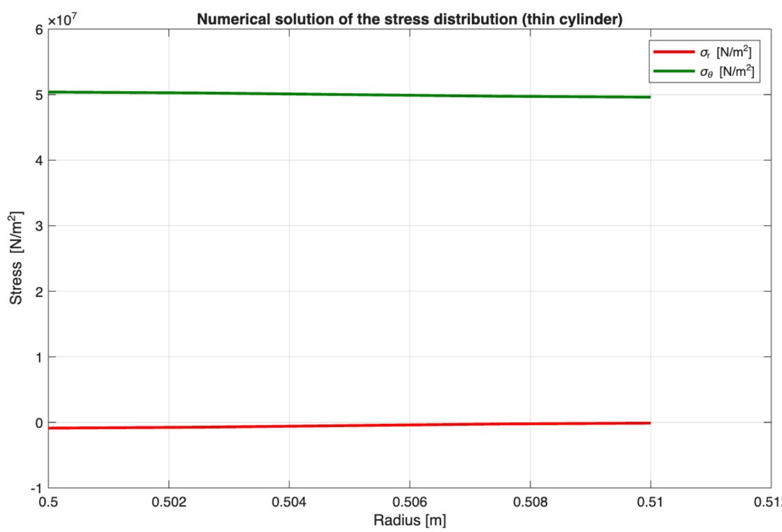


Figure 7: Numerical (4x2 linear triangular elements) radial and circumferential stresses distribution in thin cylinder.

4 Finite Element Analysis with Rectangular Plane Stress Elements

The second phase of this finite element analysis, performed on cylinders with varying thicknesses subjected to internal pressure, involves a discretization of the domain using linear rectangular elements (4 nodes). The computational solution obtained with a 4×2 mesh is shown in Figure 8. This result deviates significantly from the analytically computed values of σ_r and σ_θ, both quantitatively and qualitatively.

The observed discrepancy can be primarily attributed to the coarse mesh, which is insufficient to capture the stress gradients accurately, particularly near the boundaries and in regions where stress variation is more pronounced. Linear rectangular elements, while computationally efficient, may also introduce additional approximation errors when used in very coarse discretizations.

These findings highlight the critical importance of mesh refinement in finite element analyses. To achieve results that converge toward the analytical solution, a denser mesh with a higher number of elements is required. This allows for a more accurate representation of the stress distribution across the cylinder, reducing both numerical error and qualitative deviation.

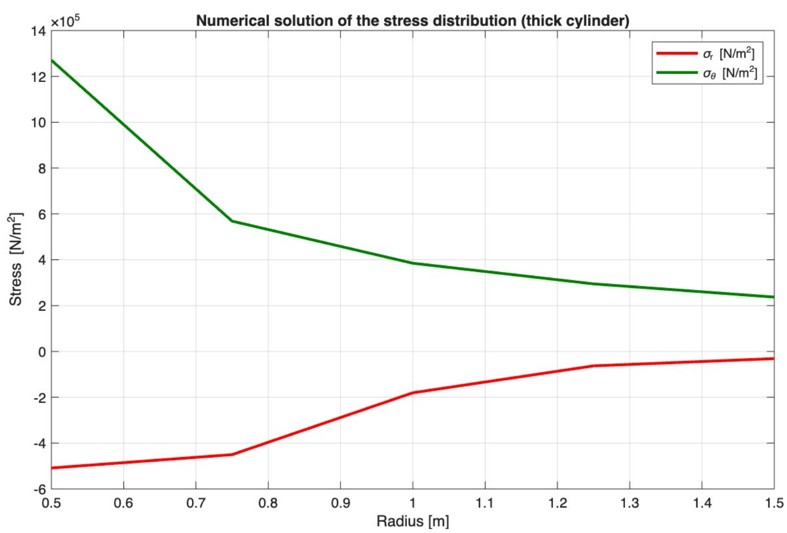


Figure 8: Numerical (4x2 linear rectangular elements) radial and circumferential stresses distribution in thick cylinder.

The result shown in Figure 9 was obtained using a discretization with linear rectangular elements and a refined mesh of 400×2 elements. The increased mesh density allows for a more accurate representation of the stress fields across the cylinder, significantly improving both the qualitative and quantitative agreement with the analytical solution. This demonstrates the importance of mesh refinement in capturing the correct stress distribution, particularly in regions near boundaries.

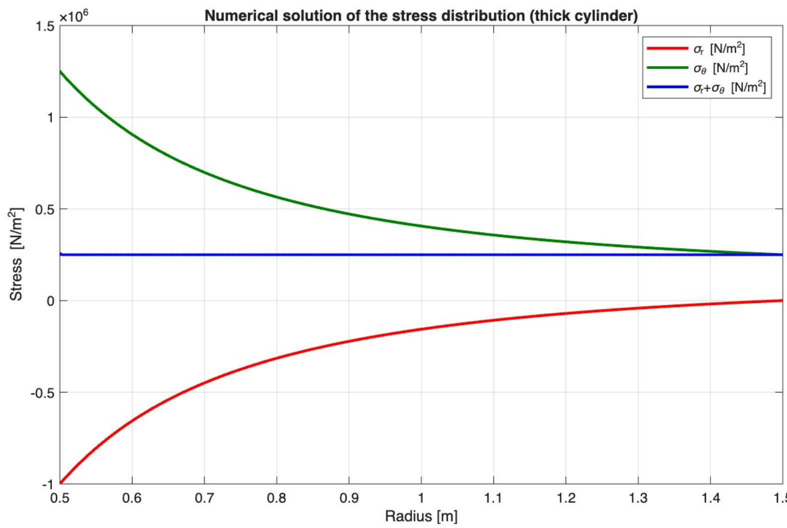
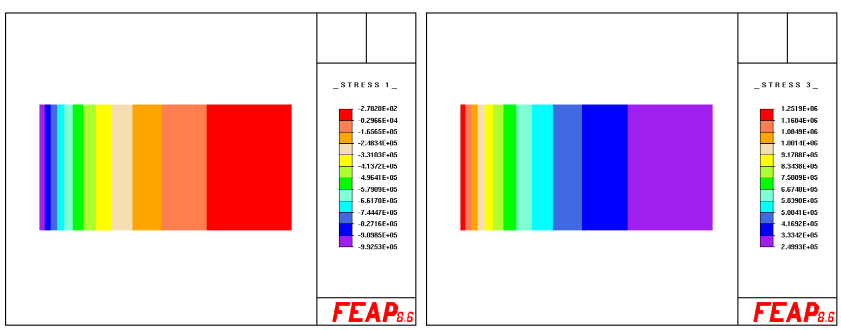


Figure 9: Numerical (400x2 linear rectangular elements) radial and circumferential stresses distribution in thick cylinder.



(a) Radial stress (b) Circumferential stress

Figure 10: Stress numerical color map with linear rectangular elements, 400x2

The figure 10 shows the color maps for the radial stress σ_r and the circumferential stress σ_θ resulting from the discretization just implemented. It is evident that the values near the boundaries are consistent with those reported in the plot shown in Figure 9. Furthermore, the distribution across the cylinder thickness appears smooth and continuous, indicating that the mesh is sufficiently refined to capture the main features of the stress fields. This provides additional confidence in the accuracy and reliability of the numerical solution.

Finally, the same analysis was performed for the thin cylinder. Even using linear rectangular elements, the solution obtained with a 4×2 mesh (Figure 11) already demonstrates remarkable agreement with the theoretical solution. This high level of accuracy, achieved despite the relatively coarse mesh, can be attributed to the reduced thickness of the cylinder, which leads to smaller stress gradients across the radial direction. As a result, further mesh refinement is not necessary, making the computational effort for this case considerably lower while still providing reliable results.

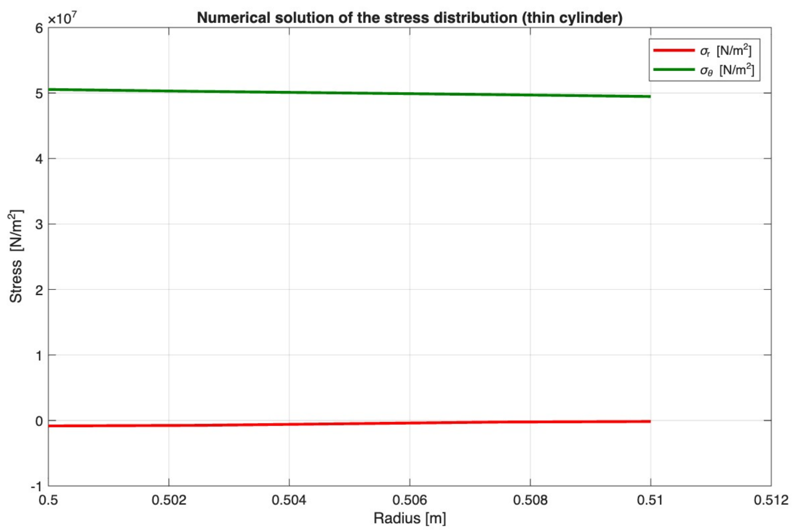


Figure 11: Numerical (4x2 linear rectangular elements) radial and circumferential stresses distribution in thin cylinder.

5 Conclusion

The finite element analyses conducted on cylinders of varying thickness under internal pressure highlight several important observations. First, finer mesh discretizations consistently produce results that are closer to the analytical solution, both quantitatively and qualitatively. This demonstrates the critical importance of mesh refinement in capturing stress gradients accurately, particularly in regions near boundaries or where variations are steep.

Second, a comparison between linear triangular and linear rectangular elements indicates that, for the same mesh density, triangular elements provide slightly better accuracy and closer agreement with the analytical solution. This is likely due to their greater flexibility in representing the curved geometry of the cylinder and capturing stress variations more effectively, whereas rectangular elements, although efficient, may be less precise in regions with pronounced gradients.

Third, for cylinders with reduced thickness, accurate and reliable numerical solutions can be obtained even with relatively coarse meshes. In such cases, the smaller domain leads to less pronounced stress gradients, allowing minimal discretizations to produce excellent agreement with theoretical values. This suggests that for thin-walled structures, computational efficiency can be maximized without compromising accuracy.