Sentiment-Driven Speculation in Financial Markets with Heterogeneous Beliefs: a Machine Learning approach *

Tommaso Di Francesco
t.difrancesco@uva.nl
University of Amsterdam
Ca'Foscari University of Venice

Cars Hommes
University of Amsterdam
Bank of Canada
Tinbergen Institute

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Abstract

We study an heterogenous asset pricing model in which different classes of investors coexist and evolve, switching among strategies over time according to a fitness measure. In the presence of boundedly rational agents, with biased forecasts and trend following rules, we study the effect of two types of speculation: one based on fundamentalist and one on rational expectations. While the first is only based on knowledge of the asset, the second takes also into account the behavior of other investors. We bring the model to data by estimating it on the Bitcoin Market with two contributions. First, we construct the Bitcoin Twitter Sentiment Index (BiTSI) to proxy a time varying bias. Second, we propose a new method, based on a Recurrent Neural Network, for the estimation of nonlinear rational expectations models. We show that the switching finds support in the data and that while fundamentalist speculation amplifies volatility, rational speculation has a stabilizing effect on the market.
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1 Introduction

Agents expectations play a crucial role in economics. Aggregate variables depend on the interaction and decisions of multiple individuals. And individuals take into consideration a future which is uncertain when making decisions. The way in which expectations about the future are formed, bears therefore an effect on the aggregate variables themselves. Financial markets are a prime example of this *expectation feedback* since the price of an asset includes expectations regarding its performance, or its discounted cash flow, in the future. The most successful approach to model expectations in economics is that introduced by Muth (1961) of rational expectations (RE). The rational expectations hypothesis posits that agents form expectations which are model consistent and based on all available information. Moreover classical models assumes a representative agent. While idiosyncratic errors across agents are permitted, on average the economy acts as if populated by a representative rational agent. This assumption was challenged theoretically at least in two directions. First Sargent (1993) remarks that it seems implausible for agents to have full knowledge of the model when in reality economists themselves have to form hypothesis regarding the model structure and test them with econometric tools. Second it seems unlikely that no systematic heterogeneity is present individual forecasts. Finance has been ahead of economics in this regard, since, as Shalen (1993) *"it is well understood that speculative trade in financial markets depends on divergent beliefs"*. On this note, papers like De Long et al. (1990) introduce the concept of *noise traders*, a group on investors systematically acting on some signal uncorrelated with an asset fundamental. In a similar spirit Harris and Raviv (1993) and Hong and Stein (2003) study the effect on the market of differences of opinion among investors. A common theme of all these paper is that the different beliefs are exogenous determined. Brock and Hommes (1998) is a noticeable attempt to endogenize divergence of beliefs by proposing an Heuristic Switching Model (HSM). Investors can form their beliefs by choosing among different predictors according to a fitness measure.

For an agent to be perfectly rational in such an environment, they need to consider the impact of these behavioral biases on market participants' decisions and factor this into their own predictions. However in the literature, when modelling the interaction of heterogeneous agents, it is often assumed that the competitors of irrational investors are so called fundamentalist traders, that act as in a way which would be rational only in an homogeneous rational world. The reason for this assumption is that measuring the irrationality of others can be challenging as it involves understanding and predicting the impact of behavioral biases on market behavior. In this paper we propose to measure this irrationality by sentiment analysis of textual data, gathered from the popular social network Twitter. The idea behind using sentiment analysis is that tweets can provide insight into the emotions and opinions of market participants, which can be indicative of irrational behavior. By analyzing the sentiment of tweets that mention a specific asset, an investor can get an idea of how market participants feel about the asset. For example, a high volume of negative sentiment tweets may suggest that market participants are becoming fearful or bearish about the asset, which could be a sign of irrational selling pressure. On the other hand, a high volume of positive sentiment tweets may suggest that market participants are becoming overly optimistic or bullish, which could be a sign of irrational buying pressure. In the next section we introduce an asset pricing model with heterogeneous and boundedly rational agents. Section (3) introduces two types of speculators into the baseline model. The first type is a fundamentalist trader. The second type uses machine learning in its forecast to approximate the unknown functional form of the model. Section (4) presents the index we construct to measure market sentiment, focusing on the Bitcoin Market. In section (5) we estimate the theoretical models presented on empirical data. Section (6) concludes and discusses possible future extensions.

2 The model

The economy is populated by I investors with heterogeneous beliefs to be specified below. Time is discrete. We assume that the fundamental value of the asset is equal to 0. This implies that every positive price can be seen as price deviation from the fundamental value. The equilibrium price of Bitcoin is

$$Rp_t = \sum_{j=1}^{J} n_{j,t}\mathbb{F}_{j,t}(p_{t+1}) + \varepsilon_t, \tag{1}$$

where $\mathbb{F}_{j,t}(p_{t+1})$ is the subjective forecast of future price in period t. Timing is important and we assume that the current equilibrium price p_t is not realized and therefore not available to the agents when forming their beliefs. In other words they are making a two period ahead forecast. Equation (1) simply states that the current price is a weighted average of the different J beliefs, weighted by the fraction of the population $n_{j,t}$ that embraces the belief at time t. ε_t is normal, identical and independently distributed noise. Fractions are updated every period according to a fitness measure that is public knowledge and is given by past returns in excess of the risk-free rate

$$\pi_{j,t} = (p_t - Rp_{t-1})\left(\mathbb{F}_{j,t-1}(p_t) - Rp_{t-1}\right). \tag{2}$$

The fraction of agents choosing strategy j is given by the multinomial logit model Manski and McFadden (1981)

$$n_{j,t} = \frac{e^{\beta\pi_{j,t-1}}}{\sum_{j=1}^{J} e^{\beta\pi_{j,t-1}}}. \tag{3}$$

The parameter β represents the intensity of choice. When $\beta = 0$, agents simply randomize in their predictor's choice, and fractions are constant at $1/J$. When $\beta \to \infty$ agents immediately switch to the most profitable strategy, and all but the fraction associated with the best strategy are zero. Equations (1) and (3) jointly determine the full price and fractions evolution. A derivation based on micro-foundations is given in Appendix (A). It is important to notice that realized profit and forecast accuracy are not perfectly proportional. When an individual has a perfect forecast, their profits are guaranteed to be positive and given by $(p_t - Rp_{t-1})^2$. However for an individual with incorrect forecast to earn more than this quantity it is sufficient to be inaccurate in the "right direction". Consider the quantity $(p_t - Rp_{t-1})(\mathbb{F}_{j,t-1}(p_t) - p_t)$ which represents the numerator of the difference in realized profits obtained by a perfectly accurate individual, and a generic individual employing strategy j. For this quantity to be positive, thus consisting of more profits for the agent using the "incorrect forecast", it is sufficient that

$$\text{sgn}\,(p_t - Rp_{t-1}) \cdot \text{sgn}\,(\mathbb{F}_{j,t-1}(p_t) - p_t) = 1,$$

with $\text{sgn}(x) = x/|x|$ is the sign function. The intuition is that the forecasting error has the same direction of the price change. The reason why even with a perfect forecast an individual does not purchase or sell unlimited quantities of the risky asset is the bound imposed by their risk aversion and the variance of the risky asset. Incorrect investors are overly optimistic or pessimistic, therefore purchasing or selling more then they should. Sometimes, by chance, their overconfidence pays off.

2.1 Expectations

We assume that all forecasting strategies are given by a linear function of past and future prices and of a bias process.

$$\mathbb{F}_{j,t}(p_{t+1}) = h_{j,t}(\{p_{t-l}\}_{l=1}^{L}, \{\mathbb{E}_t(p_{t+j})\}_{j=0}^{K}, \{w_{t-l}\}_{l=1}^{L}).$$

with w_t being an observable i.i.d stochastic process independent of prices. The functional form allows for simple but typical typical beliefs supported by multiple experimental evidence as in Hommes et al. (2005) or for more sophisticated ones. The formulation so far is general enough to accommodate for a multiplicity of strategies as in Brock, Hommes and Wagener (2005), but we will focus on 4 that are representative of behaviors we expect to see in the market and are summarized below.

Trend chasers. Sometimes also referred to as chartists. They form their forecast by an analysis of past prices. We consider the simplest form given by

$$h_{j,t}(\{p_{t-l}\}_{l=1}^{L}, \{\mathbb{E}_t(p_{t+j})\}_{j=0}^{K}, \{w_{t-l}\}_{l=1}^{L}) = gp_{t-1}, \quad g > 0. \tag{4}$$

In forming their expectations they extrapolate from last period price deviation. The parameter g being greater then 0 implies that they expect positive price deviations to continue. The magnitude of g determines the degree of trend following with the case $g > R$ usually referred to as *strong trend chasing*. The presence of this category of investors in the system is consistent with evidence provided by the literature about the existence of a momentum factor in stock markets, and originated by the seminal paper of Jegadeesh and Titman (1993).

Pure bias. This class of investors base their beliefs on some process which in the empirical part we will think of as sentiment, unrelated to the asset fundamentals or prices. Their forecast is

$$h_{j,t}(\{p_{t-l}\}_{l=1}^{L}, \{\mathbb{E}_t(p_{t+j})\}_{j=0}^{K}, \{w_{t-l}\}_{l=1}^{L}) = bw_{t-1}, \quad b > 0. \tag{5}$$

Fundamentalists. Investors of this type base their expectations on the fundamental value of the asset. They would be rational in a rational world in the sense that if the market would be populated only by fundamentalists, then their forecast will be the correct one. They do however fail to take into consideration the presence of agents with different beliefs in the market. Recalling that the fundamental value in this setting is equal to 0 we have

$$h_{j,t}(\{p_{t-l}\}_{l=1}^{L}, \{\mathbb{E}_t(p_{t+j})\}_{j=0}^{K}, \{w_{t-l}\}_{l=1}^{L}) = 0. \tag{6}$$

Rational expectations. They are the more sophisticated investors having knowledge of the underlying model regulating the asset price evolution and the composition of the market. Their forecast is the correct one and coincides with the one the modeler would obtain.

$$h_{j,t}(\{p_{t-l}\}_{l=1}^{L}, \{\mathbb{E}_t(p_{t+j})\}_{j=0}^{K}, \{w_{t-l}\}_{l=1}^{L}) = \mathbb{E}_t(p_{t+1}. \tag{7}$$

We now investigate some markets to investigate the dynamics that can emerge from the interactions of these simple strategies. The first case we consider is a two type market.

2.2 Trend chasers vs pure bias

We take the first strategy to be trend chasing and the second one to be pure bias. We start by considering deterministic simulations and assume that w_{t-1} is constantly equal to its expected value which we fix to 1 for convenience. We also keep the noise component ε fixed at its expected value of 0.

The full model is therefore given by the following two equations:

$$Rx_t = n_{1,t}gx_{t-1} + (1 - n_{1,t})b, \tag{8}$$

$$n_{1,t} = \{1 + exp(\beta(x_{t-1} - Rx_{t-2})(b - gx_{t-3}))\}^{-1}. \tag{9}$$

We provide analytical solutions for the steady states of the system, for extreme values of the intensity of choice parameter β.

Lemma 1 (Existence of Steady States for the two type model) *For $\beta = 0$ the model has a unique positive steady state $x = \frac{b}{2R-g}$. For $\beta \to +\infty$ there are the following possibilities. If $g = R$ then any $0 < x \leq \frac{b}{R}$ is a positive steady state. If $g > R$ there exists a unique and positive steady state $p = \frac{b}{R}$. If $g < R$ there are no steady states.*

The lemma shows that the even in the neoclassical limit in which agents immediately switch to the best performing strategy, pure bias agents are not pushed out of the market. Indeed the most interesting case is the one in which $g > R$ ans trend chasing is *strong*.

This behavior of trend followers is labeled as *strong*. In this situation the steady state is a positive deviation from the fundamental value, the magnitude of which depends on b. When β is strictly positive, the steady state equation can only be derived implicitly. We do this in appendix (C) while below we use numerical simulations to highlight the global dynamics that this system may generate.

2.3 Numerical Simulations: b = 1.0, g = 1.3, R = 1.01

We fix all parameters but the intensity of choice β. We then study the global dynamics of the model. First, we report a Bifurcation diagram for increasing value of the intensity of choice parameter. For each value of β we simulate 10000 iterations of the system, and then visualize the last 2000 price realisations, in order to eliminate the effect of initial conditions. In panel (a) of figure (1) we can observe that the system has a stable steady state for values of the intensity of choice lower than approximately 6.725. After the parameter crosses this value, a bifurcation occurs. In order to characterize this bifurcation and the system dynamics after bifurcation, we use maximum lyapunov exponents[1] and a plot of the modulus of the system eigenvalues. The maximum lyapunov exponent is a convenient tool to detect chaos in dynamical system, which is associated with positive values of the exponent. As we can see in panel (b) of figure (1) the exponent is negative for low values of β and becomes equal to 0 after the bifurcation. Therefore the system exhibits periodic and quasi periodic orbits, never producing chaos. In panel (c) we plot the modulus of the eigenvalue of the system, that we compute in section (B.5) of the appendix. We can see that for values of $\beta \approx 6.725$ the two complex eigenvalues cross the unit circle. We conclude therefore that an Hopf bifurcation occurs. Finally in panel (d) we show the creation of stable invariant circles after the bifurcation, allowing us to classify the bifurcation as super-critical.

[1]The computation is an in-built function of the package Julia package DynamicalSystems.jl, that relies on the method by Benettin et al. (1980).

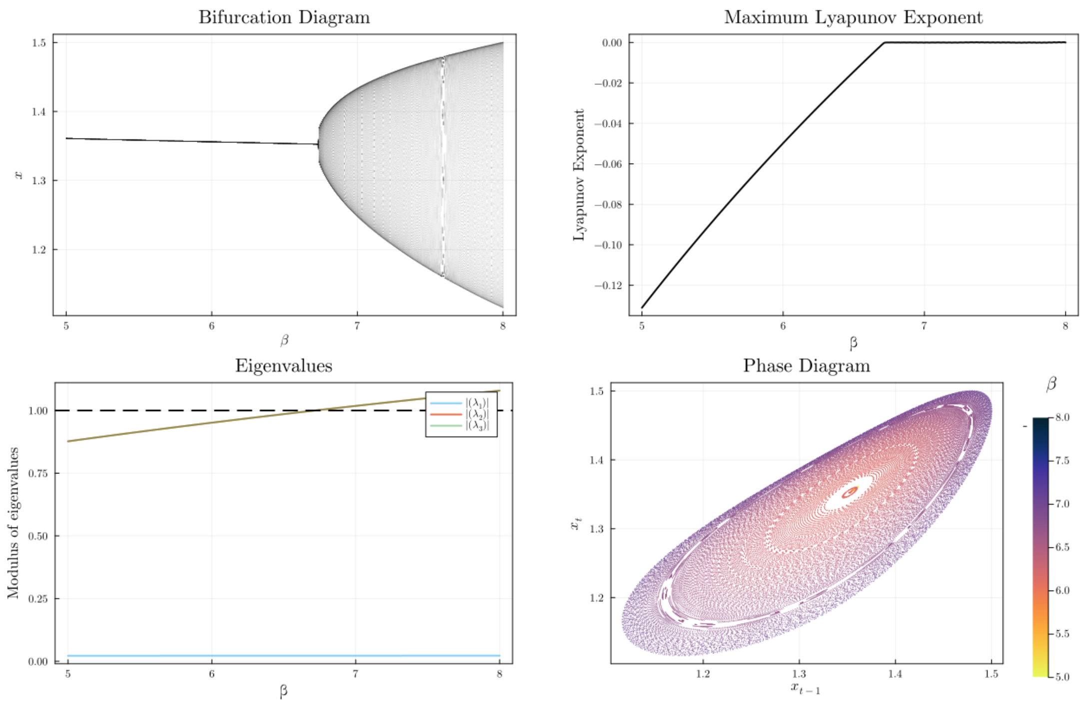


Figure 1: Numerical simulations for the two type system, other parameters are $g = 1.3$, $b = 1$, $R = 1.01$. An Hopf bifurcation occurs for $\beta \approx 6.725$, as the two complex eigenvalues have modulus $|\lambda_2| = |\lambda_3| = 1$. After the bifurcation periodic and quasi periodic orbits are created.

We then investigate the mechanism responsible for the orbit observed, by plotting the trajectory of the system and the fractions evolution for a value of the intensity of choice $\beta = 7.0$, in figure (2). The bubble and burst behavior is caused by oscillating periods of optimism and pessimism. Biased agents act in a way which is qualitatively similar to fundamentalists in the original Brock and Hommes (1998) paper, but their bias implies that the fluctuations are translated upward.

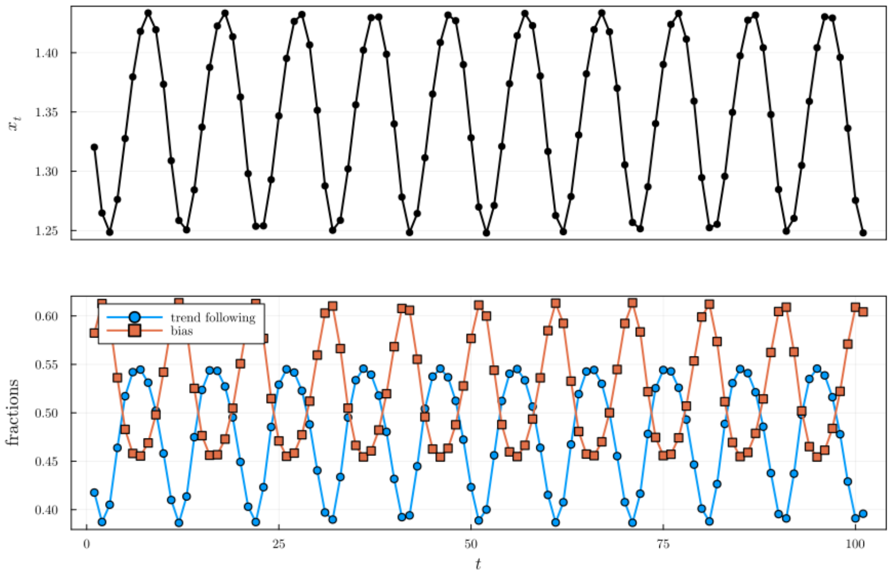


Figure 2: Time series of price deviation (top) and fractions (bottom) for $\beta = 7.0$

3 Introducing Speculation

This section analyzes the effect of speculation in the heterogeneous model. Given the presence of non rational behaviors among investors, classical economic theory would predict that a rational speculator may earn positive returns with its activity, until irrational investors are pushed out of the market, and prices return to rational levels. This is in essence the Friedman hypothesis Friedman (1953). In order to test whether rational investors do possess this capability in the model, one must first define a forecasting strategy for the rational investor. Generally, rationality in asset pricing has been associated with so called fundamentalist traders. Fundamentalist traders are fully aware of the dynamics characterizing an asset fundamentals, and therefore able to compute its fundamental price. They use this information to make investment decisions based on the belief that the market will eventually reflect the fundamental value of an asset. However, to put it in the words of Keynes: *"Markets can stay irrational longer than you can stay solvent"*. This is especially true in our model in which the fundamental value of the asset is zero which implies that every strictly positive price can be seen as "irrational". It must be remarked that being rational in an irrational world however, is crucially different than being fundamentalist. Moving forward therefore we will compare the two cases: when speculators are fundamentalists and when they take into account the "irrationality" of others.

3.1 Fundamentalist Speculators

Fundamentalist agents have knowledge of the underlying model determining the asset price, but fail to take into consideration the existence of agents with different forecasting strategies in the market. Behaving as if they were the representative agent, their forecast is given by

$$f_{3,t}(\{x_{t-l}\}_{l=1}^{L}, \{x_{t+j}\}_{j=0}^{K}, \{w_{t-l}\}_{l=1}^{L}) = 0, \tag{10}$$

meaning that they always predict the asset to be at the fundamental value. The resulting three type model is then given by

$$Rx_t = n_{1,t} g p_{t-1} + n_{2,t} b, \tag{11}$$

with

$$n_{1,t} = \frac{exp\{\beta \pi_{1,t-1}\}}{\sum_{j=1}^{3} exp\{\beta \pi_{j,t-1}\}}, \quad n_{2,t} = \frac{exp\{\beta \pi_{2,t-1}\}}{\sum_{j=1}^{3} exp\{\beta \pi_{j,t-1}\}}. \tag{12}$$

As before, we offer a lemma about the existence of steady states of the model.

Lemma 2 (Existence of Steady States of the model with fundamental speculators) *For $\beta = 0$, the model with the fundamentalist speculator has a positive and unique steady state $x = \frac{b}{3R-g}$. For $\beta \to +\infty$ there are no positive steady states.*

We observe that when no switching is present in the model, the resulting steady state is lower then the one in the benchmark two types model. This is consistent with the price reflecting the fundamentalist expectations of the third category of investors. The impact of these agents on global dynamics however is not so straightforward, and we turn again to numerical simulations to analyze it.

3.2 Numerical Simulations: b = 1.0, g = 1.3, R = 1.01

Just as before we fix all parameters but the intensity of choice β. In panel (a) of figure (3) we can observe that the system has a stable steady state for values of the intensity of choice lower than approximately 5.6. After the parameter crosses this value, a bifurcation occurs. Again, in order to characterize this bifurcation and the system dynamics after bifurcation, we use maximum lyapunov exponents and a plot of the modulus of the system eigenvalues. As we can see in panel (b) of figure (3) the exponent is negative for low values of β and becomes equal to 0 after the bifurcation. Therefore the system exhibits periodic and quasi periodic orbits, never producing chaos. In panel (c) we plot the modulus of the eigenvalues of the system, that we compute in section (B.5) of the appendix. We can see that just as before for values of $\beta \approx 5.6$ the two complex eigenvalues cross the unit circle and a Hopf bifurcation occurs. Finally in panel (d) we show the creation of stable orbits after the bifurcation, again allowing us the bifurcation as super-critical. Two consideration must be made at this point: first, the presence of the fundamentalist type, pushes the overall price closer to the fundamental one. Second, disagreement is added in the system. This causes periodic and quasi periodic orbits to occur for lower values of the intensity of choice, and orbits to have higher amplitude then with respect to the two types model.

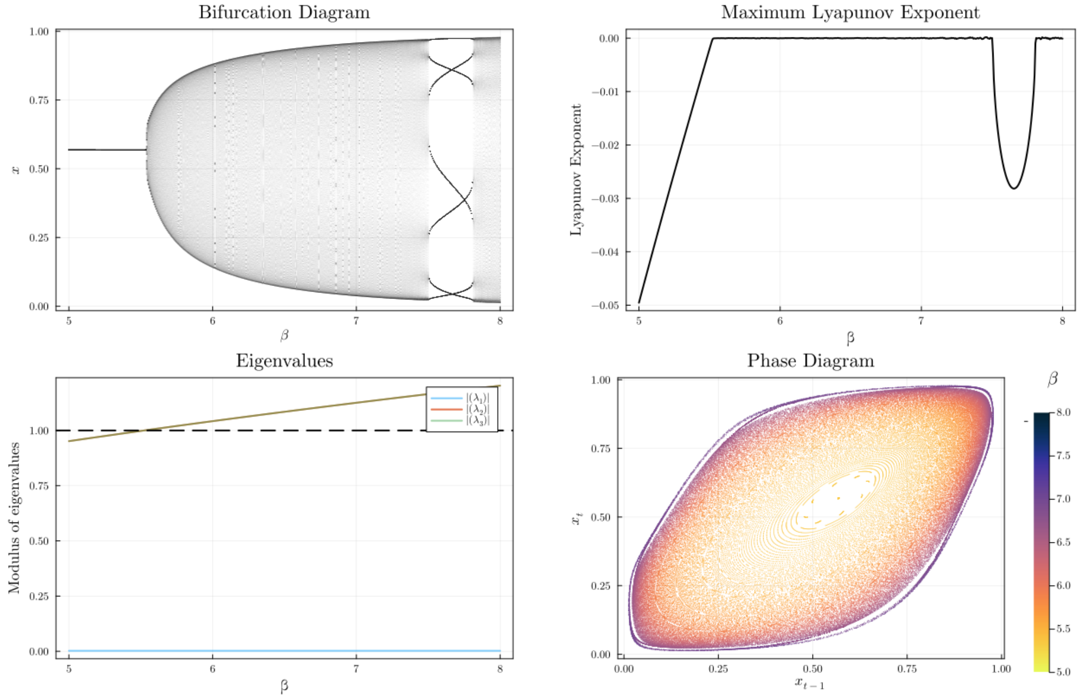


Figure 3: Numerical simulations for the fundamentalist speculation system, other parameters are $g = 1.3$, $b = 1$, $R = 1.01$. An Hopf bifurcation occurs for $\beta \approx 5.6$, as the two complex eigenvalues have modulus $|\lambda_2| = |\lambda_3| = 1$. After the bifurcation periodic and quasi periodic orbits are created.

Figure (4) shows that the dynamic is mainly led by the fundamentalist and the bias categories of investors, whit the trend following strategy never being adopted by more then 40 % of the model's population.

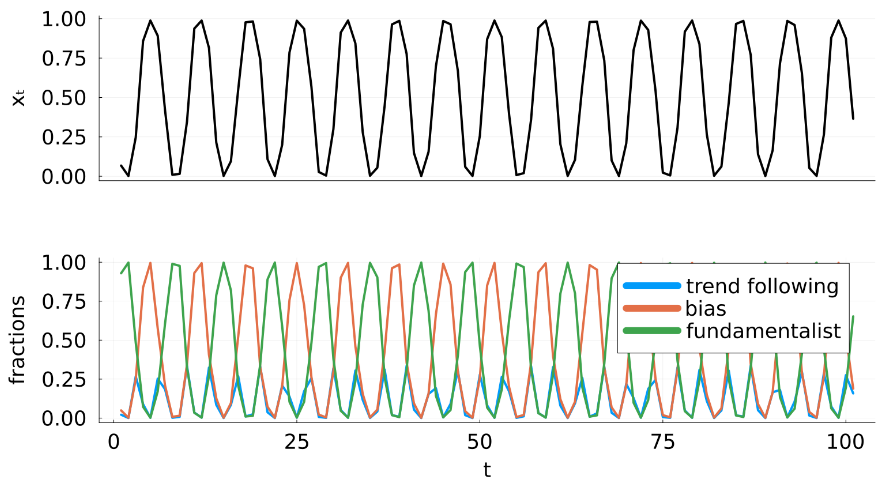


Figure 4: Time series of price deviation (top) and fractions (bottom) for $\beta = 7.0$

3.3 Rational Speculators

Rational speculators have not only knowledge of the underlying asset pricing model but also of the forecasting strategies adopted by other investors. In a setting in which there is no unobservable noise, they have perfect foresight. Their forecast is given by

$$f_{3,t}(\{x_{t-l}\}_{l=1}^{L}, \{x_{t+j}\}_{j=0}^{K}, \{w_{t-l}\}_{l=1}^{L}) = x_{t+1}, \tag{13}$$

which gives the following full model

$$Rx_t = n_{1,t}gx_{t-1} + n_{2,t}b + n_{3,t}x_{t+1}, \tag{14}$$

with

$$n_{1,t} = \frac{exp\{\beta\pi_{1,t-1}\}}{\sum_{j=1}^{3} exp\{\beta\pi_{j,t-1}\}}, \quad n_{2,t} = \frac{exp\{\beta\pi_{2,t-1}\}}{\sum_{j=1}^{3} exp\{\beta\pi_{j,t-1}\}}, \quad n_{3,t} = \frac{exp\{\beta\pi_{3,t-1}\}}{\sum_{j=1}^{3} exp\{\beta\pi_{j,t-1}\}}. \tag{15}$$

We offer the usual lemma for the existence of steady states for extreme values of the intensity of choice.

Lemma 3 (Existence of Steady States of the model with rational speculators) *Assume $\beta = 0$, then the model with the rational speculator has a positive and unique steady state $x = \frac{b}{3R-g-1}$. When $\beta \to \infty$, then the model has no positive steady state.*

As before the first observation comes from the case of no switching among strategies. The presence of rational speculators, implies a steady state value slightly lower than in the baseline model. However compered with the three type model with fundamentalist speculator, we observe that the steady state value is much larger. The need for the rational speculator of considering the bias in its forecast, makes so that the model steady state is further from the fundamental. While in the previous models we were able to provide numerical simulations effortlessly, this is not possible in this case. The presence of the x_{t+1} term on the right hand side of equation (14) makes it unfeasible to simulate the model by leveraging established methods from difference equations. To derive rational expectations we then rely on the Extended Path (EP) method proposed by Fair and Taylor (1983)[2] While the EP algorithm is quite famous, we report it in appendix (D) for convenience and because we slightly adapted to deal with the specific problem at hand.

3.4 Numerical Simulations:b = 1.0, g = 1.3, R = 1.01

As with the previous two models we study global dynamics of the system for fixed values of the parameters b, g and R, and varying the intensity of choice β. In panel (a) of figure (5) we plot the bifurcation diagram for increasing values of the intensity of choice β. We

[2] Boehl and Hommes (2021) recently proposed a new algorithm that achieves higher accuracy compared to the EP in a similar setting. In our specific case we find that the EP is sufficient and as such using more sophisticated method is not needed.

can observe that the system exhibits a stable steady state for values of the intensity of choice approximately close to 6.7. After this value a bifurcation occurs, and the system enters an unstable region. Unfortunately the presence of the rational agents prevents us to analytically compute the eigenvalues of the system, and we can not fully characterize the bifurcation. We can observe however by means of a phase plot in panel (b), that the system exhibits quasi periodic orbits with longer periodicity than with respect to the system with the fundamentalist speculator. In fact the dynamics are extremely similar to those of the two type model.

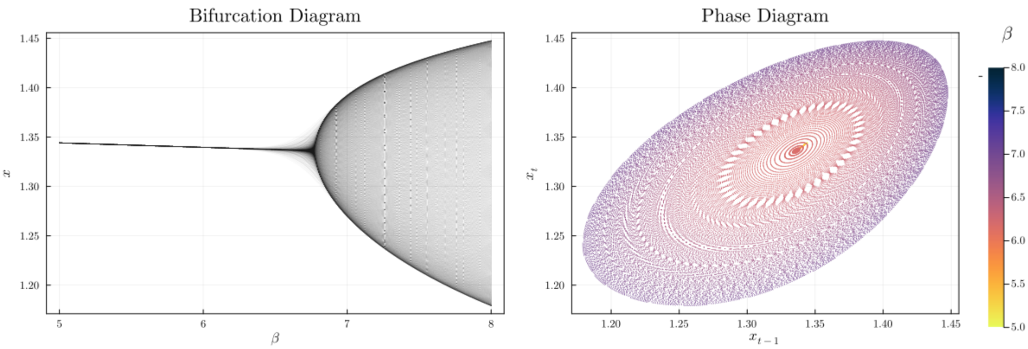


Figure 5: Numerical simulations for the model with LSTM speculator, other parameters are g = 1.3, b = 1, R = 1.01.

The main consideration to make is that unlike the case with the fundamentalist speculators, the system does not approach the fundamental value, even after the bifurcation occurs. To get a better sense of this phenomenon, in figure (6) we plot the fractions evolution for a value of the intensity of choice $\beta = 8.0$.

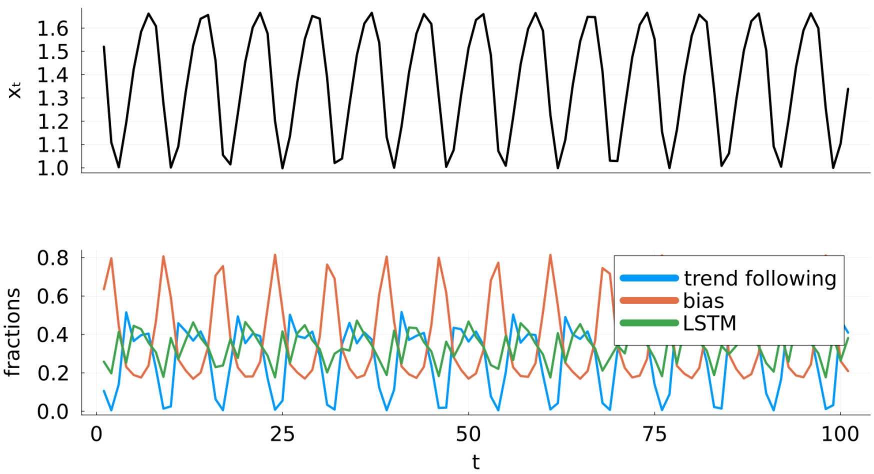
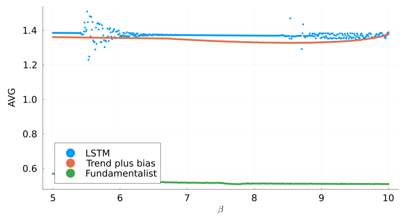


Figure 6: Time series of price deviation (top) and fractions (bottom) for $\beta = 7.0$

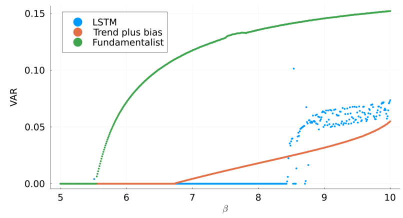
We can see that the fraction of agents using the rational forecast fluctuates significantly less then the other two. Therefore there are two reasons why the resulting price is further away from the fundamental value. The first one is that in order to produce an accurate forecast, rational speculators must take into account the other two agents behavior. This results in a forecast significantly higher then the one a fundamentalist agent would make. The second one is that, as we already remarked, realized profits and forecast accuracy are not perfectly correlated. Speculators using the rational forecast are always present in the system, but they are not able to drive other strategies out of the market.

3.5 Speculation Effects

As we can see in figure (7) the two types of speculation have different effects on the market.



(a) Mean price deviation for increasing values of the intensity of choice



(b) Variance of price deviations for increasing values of the intensity of choice

Figure 7: Comparison of the mean and variance for the three types model with fundamentalist and LSTM speculator for different values of β. Other parameter values are g = 1.3, b = 1, R = 1.01.

When speculators are fundamentalist, they drive prices towards the fundamental benchmark. Their presence though results in increased disagreement, which in turn causes the model to enter the unstable region for lower values of the intensity of choice. Moreover in the unstable region, the resulting volatility is substantially increased. When speculators are rational, we can observe that prices are on average slightly lower than compered with the no speculation case, but still far from the fundamentalist benchmark. An interesting result is that speculation decreases volatility in the unstable region with respect to the two type model. Rational speculation can have a positive effect on reducing volatility, at the cost of keeping prices further away from the fundamental price that would be achieved in an homogeneous rational world.

4 Taking the model to the data

In this section, we focus on empirically validating the models previously discussed, utilizing real data from the Bitcoin market—the pioneering and most renowned cryptocurrency. Our selection of Bitcoin is driven by two principal factors. Firstly, the vast availability of data: Bitcoin is among the most extensively reported financial markets in both the media and online forums, offering a rich repository of textual data, including social media contributions and online dialogues. Secondly, the Bitcoin market exhibits a lesser degree of efficiency compared to other financial markets, a phenomenon attributable to various aspects such as regulatory absence, pronounced volatility, and its nascent developmental phase. These attributes render the market more prone to speculative bubbles and other market inefficiencies, potentially leading to substantial investor losses and presenting numerous arbitrage opportunities, as evidenced in studies like Urquhart (2016) and Makarov and Schoar (2020).

Our objective is to transition from a static bias to a stochastic measure of bias or sentiment w_t. To construct a proxy for market sentiment towards Bitcoin, we utilize data from Twitter (currently known as X), where tweets serve as a barometer for the financial mood, revealing public sentiment towards specific stocks, markets, or the economy at large. Given their often subjective, informal nature, and potential to be influenced by emotions, tweets present a particularly relevant source of data for our analysis.

Data collection involved scraping the site for all posts containing the term "Bitcoin" from November 1, 2019, to December 30, 2022. To derive a meaningful sentiment index from this textual data, standard preprocessing steps were applied to eliminate stopwords and non-alphabetic characters. We then employed the widely-used VADER sentiment analysis library, as introduced by Hutto and Gilbert (2014), in which the methodology is evaluated in its performance on a dataset of social media postings. VADER determines the sentiment of text using a mix of lexical heuristics and a sentiment lexicon, and it can handle idiomatic language and sarcasm. Sentiment scores, ranging from -1 (highly negative) to +1 (highly positive), with 0 indicating neutrality, were computed for each tweet based on the presence of positive, negative, and neutral words. These scores were aggregated daily to form an index labeled Bitcoin Twitter Sentiment Index (BiTSI). Figure (8) shows the evolution of the index and the daily closing price of Bitcoin in US Dollars. To contextualize the index's peaks and troughs, we highlight the headlines of articles associated with the three highest and lowest BiTSI values, offering insights into the news events likely driving these sentiment shifts.

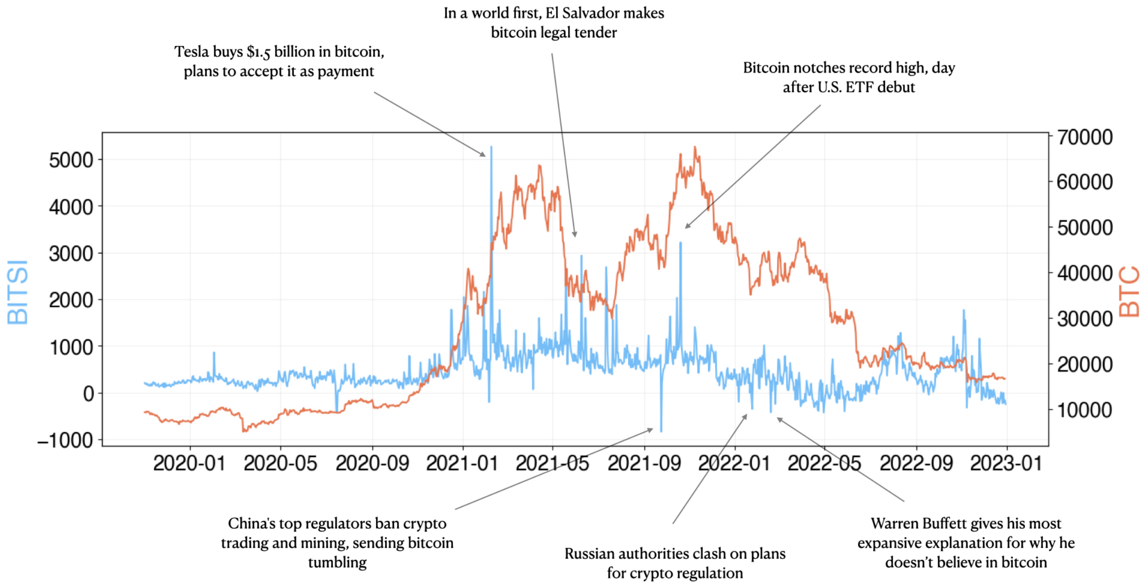


Figure 8: Bitcoin and BITSI time series, with relevant events

This is not the first work to propose to capture investor attention through textual data, it differs however from prominent examples in the literature, for the use of Natural Language Processing to measure the "direction" and not only the magnitude of attention. Urquhart (2018) and Liu and Tsyvinski (2020) proxy investors attention with google trends, computing an index based on queries counting the word Bitcoin. On a similar note Shen, Urquhart and Wang (2019*a*) use the sheer number of occurrences on Twitter. These methodologies have demonstrated a predictive capability regarding Bitcoin's trading volume and volatility, indicating a significant investor response to their respective proxies. Our approach enhances this by introducing a qualitative dimension, enabling the classification of sentiment as either positive or negative, rather than merely quantifying attention levels. Furthermore, this sentiment data informs a structural model designed to predict the price change direction, in addition to its magnitude. The index, and the sheer number of tweets per day statistics, are reported in table (1).

Table 1: Descriptive statistics for the BiTSI and number of Tweets per day

	BiTSI	**Volume**
mean	455.85	8626.90
std	426.51	5160.04
min	-841.15	650
max	5264.47	56179
skew	2.58	1.66
kurtosis	18.58	8.06
obs	1117	1117

As a first analysis we show that our index is not capturing information from other sources highlighted by the literature as having explanatory power on the cryptocurrency return. In a similar fashion to the traditional factor models, Shen, Urquhart and Wang (2019*b*) and Liu, Tsyvinski and Wu (2022) both show that the cross section of cryptocurrencies' excess returns can be substantially explained by three factors: market, size and momentum. In figure (9) we show correlation among the index and other variables that should proxy at daily level the role of the three factors mentioned above. Excess return is measured as the difference between daily Bitcoin returns and daily risk free rate, represented by the Market Yield on U.S. Treasury Securities at 1-Year Constant Maturity and available on FRED.[3] We then have two proxies for momentum, given by the first and second difference of daily closing price. Finally 'Volume' represents the daily volume of Bitcoin. All Bitcoin data are gathered from Yahoo Finance. In order to check for the possibility of delayed impact of these factors on the index, we also reported lagged values of all the variables, represented by the '(t-1)' label at the end of the symbol.

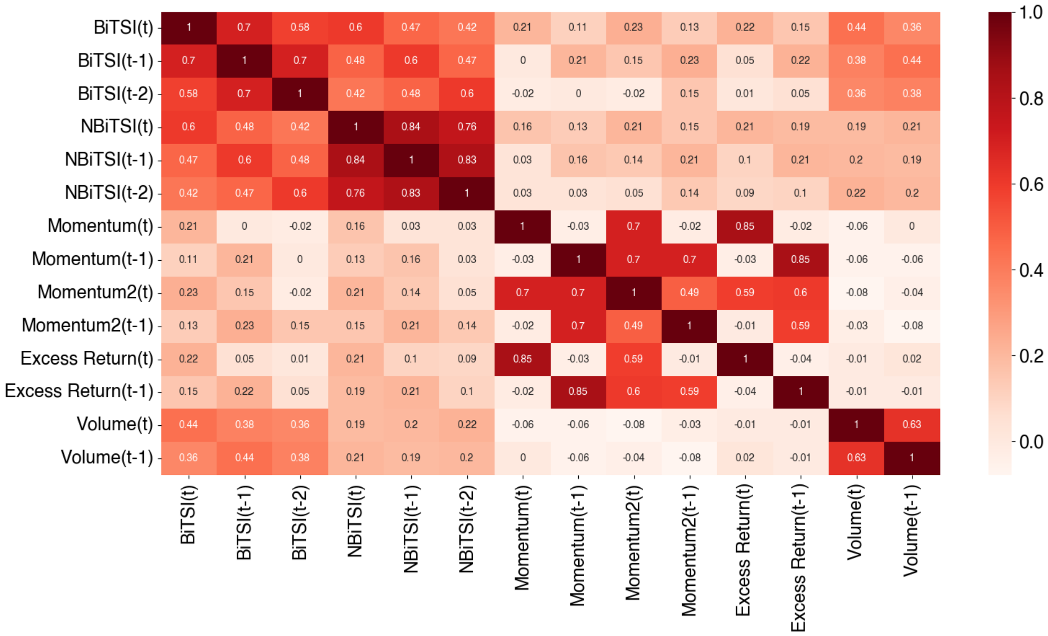


Figure 9: Correlation of the BiTSI and other variables

The correlation between the index and all the other variables is mostly positive and relatively small, except the one with Volume. The result is however explained by the correlation, already documented in the literature, between Bitcoin volume and Tweet volume. This is why we also include the Normalised BiTSI, where the normalization factor is the volume of Tweets per day. After removing the magnitude effect, the index appears to be only mildly correlated to the variables.

[3]The yearly rate is then transformed in the corresponding daily one by $r_{daily} = r_{yearly}/365$. The choice is motivated by the yearly capitalization of the instrument.

5 Estimation

We turn now to the estimation of the three models introduced above. While the first two are straightforward to estimate via Non Linear Least Squares (NLS), the model with rational speculators require to obtain rational expectations first. In their paper Fair and Taylor (1983) propose also a maximum likelihood estimation which is build on the EP algorithm. The idea is to first solve the model for rational expectations given a choice of the parameters. After obtaining the rational expectations vector one can compute the log-likelihood associated with the parameters. In order to obtain the optimal estimates one has therefore to apply the EP algorithm for every possible combinations of parameters in a specific grid. This can be computationally expensive as in our case we are interested in three parameters, g, b and β and would like to require a high grid density given the highly non-linear structure of the problem. To summarize their procedure suffers from the *cures of dimensionality* as every extra parameters to be estimated exponentially increases the computational time. To deal with this we propose a procedure which consists of two ingredients. The first is to ensure that we can express the state variable as a function of past states only. The objective function we are interested in is of the form

$$x_t = \mathcal{F}(\{x_{t-l}\}_{l=0}^{L}, \{\mathbb{E}_{t+k-1}(x_{t+k})\}_{k=-L}^{K}, \{w_{t-l}\}_{l=0}^{L}; \alpha) + \eta_t \tag{16}$$

and we want to estimate the parameter vector α. Defining by $\mathcal{F}^{(n)}$ the function obtained from $\mathcal{F}$ by iterating all future expectations of the state(s) variable n-1 times. We can easily see that if

$$\lim_{n\to+\infty} \frac{\partial \mathcal{F}^{(n)}}{\partial \mathbb{E}_{t+k-1}(x_{t+k})} \to 0, \forall k > 0,$$

and there exist a finite M such that

$$x_{t+j}^{e} \in [-M, M], \forall j > 0,$$

then there exist a N and an $\epsilon > 0$ such that for all $n > N$

$$\left| \mathcal{F}^{(n)}(\{x_{t-l}\}_{l=0}^{L}, \{\mathbb{E}_{t+k-1}(x_{t+k})\}_{k=-L}^{K}, \{w_{t-l}\}_{l=0}^{L}; \alpha) - \mathcal{H}(\{x_{t-l}\}_{l=0}^{L}, \{\mathbb{E}_{t+k-1}(x_{t+k})\}_{k=-L}^{0}, \{w_{t-l}\}_{l=0}^{L}; \alpha) \right| < \epsilon$$

This ensures that iterating the function with respect to future variables, we eventually reach a point after which the impact of the future on today is negligible. Unfortunately showing that a general function $\mathcal{F}$ satisfies the assumption is not trivial in non linear cases.[4] We can only assess that in practice this condition is likely to be met for realistic values of the parameters.

The second ingredient is that of obtaining an approximation to the unknown function $\mathcal{H}$ via a Deep Neural Network. In our specific case we have

$$x_t = \mathcal{H}(\{x_{t-l}\}_{l=0}^{L}, \{\mathbb{E}_{t+k-1}(x_{t+k})\}_{k=-2}^{0}, \{w_{t-l}\}_{l=0}^{L}; \alpha) + \eta_t. \tag{17}$$

[4] For a detailed discussion we can refer to Boehl and Hommes (2021)

This implies that

$$\mathbb{E}_t\left(x_{t+1}\right) = \mathcal{G}(\{x_{t-l}\}_{l=0}^{L}, \{\mathbb{E}_{t+k-1}(x_{t+k})\}_{k=-2}^{0}, \{w_{t-l}\}_{l=0}^{L}; \alpha), \tag{18}$$

and substituting backwards to get rid of the expectations operator

$$\mathbb{E}_t\left(x_{t+1}\right) = \mathcal{J}(\{x_{t-l}\}_{l=0}^{L}, \{w_{t-l}\}_{l=0}^{L}; \alpha), \tag{19}$$

The problem is that the functional form $\mathcal{J}$ is unknown and that the lags L are time varying. To solve the first challenge we propose to use a Deep Neural Network. We deem his appropriate as it was shown in Hornik, Stinchcombe and White (1989) that multilayer feedforward networks are universal approximators. So we reparamterize the function to have

$$\mathbb{E}_{3,t}\left(x_{t+1}\right) = \widehat{\mathcal{J}}(\{x_{t-l}\}_{l=0}^{L}, \{w_{t-l}\}_{l=0}^{L}; \theta), \tag{20}$$

where θ are the parameters of the Neural Network. To deal with the time varying lags to be included as input variables, we resort to a Long-Short Term Memory Model (LSTM) to deal sequentially with the dependence on lags past the horizon $L = 3$. In essence we have

$$\mathbb{E}_{3,t}(x_{t+1}) = \widehat{\mathcal{J}}\left(\{x_{t-l}\}_{l=1}^{3}, \{w_{t-l}\}_{l=1}^{3}; \theta\right), \tag{21}$$

We provide an overview of the model in appendix (E). After obtaining a vector of rational expectations we can proceed with estimation. We must then accommodate some changes to allow the incorporation of financial data into the theoretical model. First, we incorporate a time varying risk-free rate, as proxied by the Market Yield on U.S. Treasury Securities at 1-Year Constant Maturity discussed above so that instead of having a fiexed R we have a time varying R_t. Second we deal with the non-stationarity of the data. It was already shown that the fundamental price in this setting is constant and equal to zero. This implies that price deviations from the fundamental will also be non stationary. We therefore follow ter Ellen, Hommes and Zwinkels (2021) in rewriting the model in deviation from a moving average

$$\text{MA}_t = \frac{\sum_{i=W}^{1} p_{t-i}}{W}$$

where W is the window size, so that our state variable is $x_t = p_t - \text{MA}_t$. In the baseline we fix it to 40 days. We discuss the sensitivity of the estimation to this choice in appendix (H). Third, we need to deal with numerical errors in the estimation caused by the having to take exponents of large profits when computing the fractions. We do this by assuming that the subjective variance of the variable our agents needs to forecast is homogeneously equal to the square of the last available moving average $(\text{MA}_{t-3})^2$. Lastly we deal with heteroskedasticity of residuals by rewriting the model in percentage deviation from the moving average fundamental, by simply dividing both sides of the pricing equation by the fundamental value itself.

The model we take to the data is then the following

$$R_t \frac{x_t}{\mathrm{MA}_t} = n_{1,t} g \frac{x_{t-1}}{\mathrm{MA}_t} + n_{2,t} b \frac{w_{t-1}}{\mathrm{MA}_t} + n_{3,t} \frac{\mathbb{E}_{3,t}(x_{t+1})}{t}, \tag{22}$$

where $n_{3,t}$ will be 0 in the two types model, and $\mathbb{E}_{3,t}(x_{t+1})$ will be 0 in the fundamental case and the output of the LSTM model in the rational one.

Fractions are given as the usual multinomial logit, where now the fitness measure is given by risk adjusted profits of the form

$$\pi_{1,t} = \frac{1}{\mathrm{MA}^2_{t-3}} \left(x_t - R_{t-2} x_{t-1}\right) \left(g x_{t-3} - R_{t-2} x_{t-1}\right), \tag{23}$$

$$\pi_{2,t} = \frac{1}{\mathrm{MA}^2_{t-3}} \left(x_t - R_{t-2} x_{t-1}\right) \left(b w_{t-3} - R_{t-2} x_{t-1}\right), \tag{24}$$

$$\pi_{3,t} = \frac{1}{\mathrm{MA}^2_{t-3}} \left(x_t - R_{t-2} x_{t-1}\right) \left(\mathbb{E}_{3,t-2}(x_{t-1}) - R_{t-2} x_{t-1}\right), \tag{25}$$

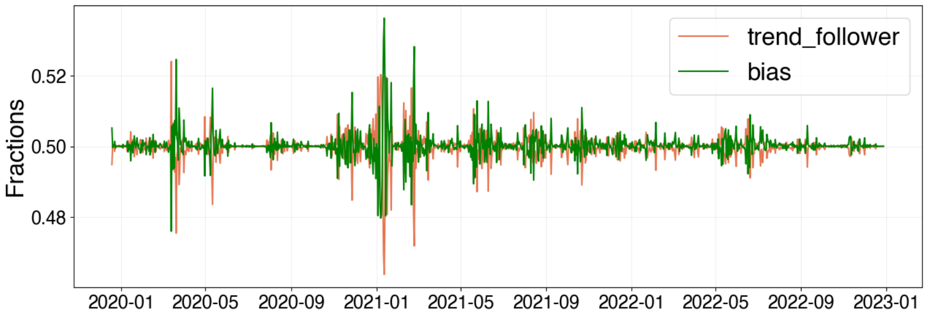
We plot and test the variables for stationarity in section (G) of the appendix.

We finally estimate the parameters g, b and β for the three model presented by NLS. Results are reported in table (2).

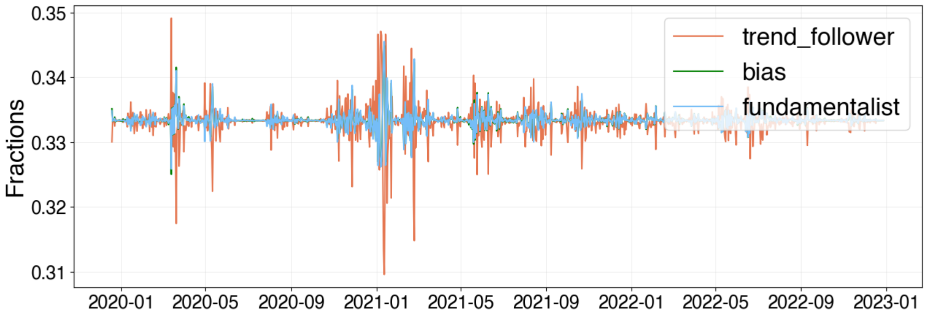
Table 2: Estimation results

	Trend Follower + Bias	**TF + B + Fundamentalist**	**TF + B + LSTM**
g	1.91	2.86	1.92
	(121.69)	(121.53)	(86.59)
b	0.58	0.87	0.75
	(6.5)	(6.51)	(5.94)
β	1.05	0.52	1.41
	(2.23)	(2.23)	(2.12)
$r2_{adj}$	0.94	0.94	0.95
F	5.05	5.17	4.64
p	(0.02)	(0.02)	(0.03)
het	0.17	0.17	0.21

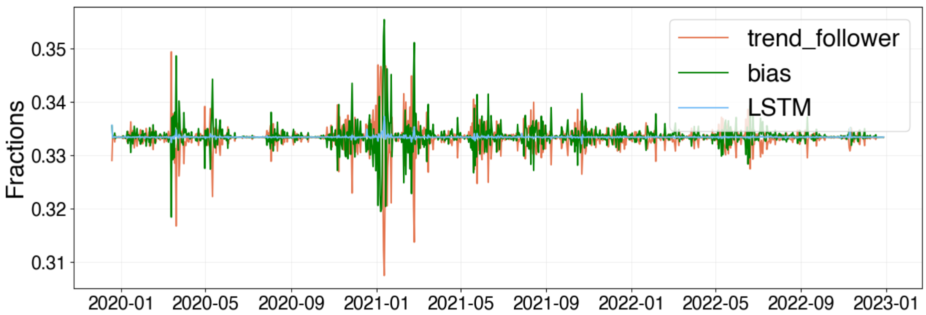
We report the point estimate of the parameters and the associated t-statistic in parentheses. We also report the adjusted R-squared value, and the value and associated p-value of the F-test for the significance of the non-linear model with respect to a linear one. This can be seen as nested in the non-linear model and corresponding to a value of the intensity of choice β equal to 0. Lastly we report the p-value associated with the Engle's Test for Autoregressive Conditional Heteroscedasticity on the estimation residuals. To corroborate the significance of the parameters we obtain confidence interval by bootstrap in appendix (I.) Following the terminology of Brock and Hommes (1998) the first type of investors are strong trend chasers, expecting deviations from fundamentals to continue with increased magnitude in the subsequent periods. This is in line with psychological aspects, like "fear of missing out" (FOMO) which is extremely relevant in the cryptocurrency market and documented for example by Baur and Dimpfl (2018). Their extrapolation becomes highest in the model with fundamentalist agent. This can be explained by the presence of a type of investor in the model that always forecasts the fundamental price, therefore absorbing part of what before we might have classified as mild trend chasers. The second category of investors can be classified as weakly sentiment follower, as they constantly expect a small deviation from the fundamental moving average which is positively proportional to the sentiment index. The F-test confirms in every case evidence of non-linearities, at least at the 5% level. The small but significance estimate for β implies long periods of coexistence of different strategies in the market. However as we can see from the figure () there are periods of substantial switching. We conjecture that the result is due to the proximity of the intensity of choice to the bifurcation. Large enough shocks can temporarily affect the stability of the steady state and drive the model in the unstable region, with a resulting volatility amplification that can last for several days.



(a) Trend Followers + Bias, (iii)



(b) Trend Followers + Bias + Fundamentalist, (i)



(c) Trend Followers + Bias + LSTM, (i)

Figure 10: Fractions evolution for estimated parameters

6 Conclusion and Discussion

In this paper we have studied an heterogeneous asset pricing model, with different categories of investors endogenously evolving over time. The presence of boundedly rational investors in the market, with trend following and bias forecasting rules, raises the question about speculation opportunities. We have introduced two different types of speculation. The first one is associated with fundamentalist traders, that know the true underlying process of the asset and behave accordingly. We remarked however that to have perfect foresight in our model one has to take into account the possibly incorrect strategies of other investors. Once one accepts the inclusion of multiple behaviors into the model, rational expectations are not analytically available nor it seems plausible to accommodate investors with perfect knowledge of other agents investment decisions. We therefore proposed an approximation of perfect foresight by the use of a machine learning algorithm in the form of a LSTM model. This approach has two main advantages. By making use of the relevant variables in the determination of the next state, and by being able to approximate any functional form, the model is correctly specified. Moreover the model does not require perfect knowledge of the system, but only of the relevant variables concurring to the price realization. This assumptions seems more realistic to make and in line with the adaptive learning literature, by which agents could use econometric methods to test about the relevance of certain variables. We show that when speculators are of the second type, the resulting price is further away from the fundamental price but volatility is reduced compared to the fundamentalist speculation benchmark. The second part of the paper is devoted to provide empirical support to our model. We opted to use data from the Bitcoin market, and provided a way of capturing the bias of a category of investors by using NLP on Twitter data. This allowed us to create the BiTSI index, that we used as proxy of a time evolving bias. The BiTSI index is shown to capture an aspect of the market which is uncorrelated with the main factors highlighted by the literature in explaining Bitcoin returns. After estimating the models by Non Linear Least Squares we found evidence of the coexistence of trend following and biased category of investors in the market. Strong evidence supporting the evolutionary switching mechanism was however provided only for the two types model. For future directions, regarding the theoretical part, one might focus on mechanisms to select the relevant variables. In this paper we assumed that speculators know the relevant variables to include in their forecast. This assumption could be released to involve, for example, the use of regularization techniques, dimensionality reduction methods, or ensemble learning approaches. On the empirical part we believe that our BiTSI index could be useful in a variety of applications. This is confirmed by our finding that a simple LSTM model using past prices and our index, achieves an extremely good performance in explaining daily excess returns. An obvious extension could be that of constructing a sentiment index for the whole cryptocurrency market and not limited to the only Bitcoin, and evaluate its performance as a possible factor in the cross section of the sector returns.

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A Micro-foundation of the Brock and Hommes (1998) Model

Consider an overlapping generation model. The economy is populated by N agents living two periods. When young agents receive w_0 units of consumption good. When old they consume all of their wealth, with a utility function given by

$$U(c_{t+1}) = -e^{-ac_{t+1}},$$

where $a > 0$ is the coefficient of absolute risk aversion. Agents can chose between two types of securities to transfer wealth from the first period to the second. They can use a riskless asset, which pays fixed interest $R > 1$ for each unit of saved good or alternatively they can use a risky asset. Agents pay price p_t to purchase the asset at time t, and when old they obtain the payoff $y_{t+1} = p_{t+1} + d_{t+1}$. This is given by the price at which they can sell the asset p_{t+1} plus a dividend claim d_{t+1} which is assumed to be constant plus normal white noise. Agents need to chose their demand of the risky asset, defined by z_t^d, in order to maximize their utility, subject to the following budget constraint

$$c_{t+1} = R(w_0 - p_t z_t^d) + z_t^d y_{t+1}.$$

All agents assume that c_{t+1} is normally distributed, then their utility maximization problem, is equivalent to

$$\max_{\{z_t^d\}} \left(-exp \left\{ -a\mathbb{E}_t(c_{t+1}) + \frac{a^2}{2}\mathbb{V}_t(c_{t+1}) \right\} \right). \tag{26}$$

Exploiting the budget constraint we have that

$$\mathbb{E}_t(c_{t+1}) = R(w_0 - p_t z_t^d) + z_t^d \mathbb{E}_t(y_{t+1}),$$

and

$$\mathbb{V}_t(c_{t+1}) = (z_t^d)^2 \mathbb{V}_t(y_{t+1}).$$

Then the optimal choice of the risky asset must satisfy the first order condition of equation (26).

$$z_t^d = \frac{\mathbb{E}_t(y_{t+1}) - Rp_t}{a\mathbb{V}_t(y_{t+1})}.$$

To get a solution for the price p_t we impose the equilibrium condition that demand must equal supply z_t^s. Finally we assume net supply of the risky asset is 0 to get

$$\frac{\mathbb{E}_t(y_{t+1}) - Rp_t}{a\mathbb{V}_t(y_{t+1})} = 0. \tag{27}$$

From equation (27) we can notice that if agents had homogenous beliefs, therefore sharing the same expected value and variance of the asset, the pricing equation would be

$$Rp_t = \mathbb{E}_t(y_{t+1}). \tag{28}$$

Equation (28) has two class of solutions, usually referred to as fundamentalist and rational bubble solution. Under the assumption $d_t \sim \mathcal{N}(d, \sigma_d^2) \forall t$, the fundamental price is $p_t \equiv \bar{p} = d/r$ in each time step, where $r = R - 1$ is the net free-rate. Whereas the rational bubble price is $\tilde{p}_t = \zeta_t R^{-t}$ where ζ_t is any martingale process. Now consider J different types of agents, with different expectation formation processes regarding the future price of the risky asset. The equilibrium pricing equation (27) becomes

$$\sum_{j=1}^{J} \left(n_{j,t} \cdot \frac{\mathbb{E}_{j,t}(y_{t+1}) - Rp_t}{a\mathbb{V}_{j,t}(y_{t+1})} \right) = 0,$$

where $\mathbb{E}_{j,t}(y_{t+1})$ and $\mathbb{V}_{j,t}(y_{t+1})$ indicate the subjective expectation and variance of agent j and $n_{j,t}$ represents the fraction of agents using strategy j. Finally we make the assumption of constant beliefs on variance: $\mathbb{V}_{j,t}(y_{t+1}) = \sigma_y^2, \forall j$. [5] With this assumption the equilibrium price is given by

$$Rp_t = \sum_{j=1}^{J} n_{j,t} \cdot \mathbb{E}_{j,t}(y_{t+1}). \tag{29}$$

Fractions are updated every period according to a fitness measure that is public knowledge and is given by an exponential moving average of past profits

$$U_{j,t} = \eta \pi_{j,t} + (1 - \eta) U_{j,t-1}, \tag{30}$$

with $\eta \in (0, 1]$ being the memory parameter.

Profits or returns in excess of the risk-free rate are given by

$$\pi_{j,t} = (y_t - Rp_{t-1})\, z_{t-1}^d. \tag{31}$$

It is important to notice that realized profit and forecast accuracy are not perfectly proportional. When an individual has a perfect forecast, their profits are guaranteed to be positive and given by $(y_t - Rp_{t-1})^2 / a\sigma_y^2$. However for an individual with incorrect forecast to earn more than this quantity it is sufficient to be inaccurate in the "right direction". Consider the quantity $(y_t - Rp_{t-1})(\mathbb{E}_{j,t-1}(y_t) - y_t)$ which represents the numerator of the difference in realized profits obtained by a perfectly accurate individual, and a generic individual employing strategy j. For this quantity to be positive, thus consisting of more profits for the agent using the "incorrect forecast", it is sufficient that

$$\text{sgn}\,(y_t - Rp_{t-1}) \cdot \text{sgn}\,(\mathbb{E}_{j,t-1}(y_t) - y_t) = 1,$$

[5] Although we shall overlook this second-order effect, it should be noted that heterogeneity in conditional expectations does, in fact, cause heterogeneity in conditional variance. Again we refer to Brock and Hommes (1998) for a more thorough discussion.

with $\text{sgn}(x) = x/|x|$ is the sign function. The intuition is that the forecasting error has the same direction of the price change. The reason why even with a perfect forecast an individual does not purchase or sell unlimited quantities of the risky asset is the bound imposed by their risk aversion and the variance of the risky asset. Incorrect investors are overly optimistic or pessimistic, therefore purchasing or selling more then they should. Sometimes, by chance, their overconfidence pays off.

The fraction of agents choosing strategy j is given by the multinomial logit model Manski and McFadden (1981)

$$n_{j,t} = \frac{e^{\beta U_{j,t-1}}}{\sum_{j=1}^{J} e^{\beta U_{j,t-1}}}. \tag{32}$$

The parameter β represents intensity of choice. When $\beta = 0$, agents simply randomize in their predictor's choice, and fractions are constant at $1/J$. When $\beta \to \infty$ agents immediately switch to the most profitable strategy, and all but the fraction associated with the best strategy are zero.

It is convenient at this point to introduce an additional assumption on the subjective expectations. Specifically we assume that all agents have

$$\mathbb{E}_{j,t}(y_{t+1}) = \bar{y} + f_j(\{x^e_{t-l}\}_{l=1}^{L}, \{x_{t+j}\}_{j=0}^{K}, \{w_{t-l}\}_{l=1}^{L}).$$

with $\bar{y} = R\bar{p}$, $x_t = p_t - \bar{p}$ and w_t being an observable i.i.d stochastic process independent of prices and dividends, which represents bias. This implies that it is possible to express all forecasting strategies as the sum of two components. The first one is the payoff prediction implied by the fundamental price and the second is a subjective and linear function of past and future price deviations from the fundamental and of a bias process. Whit this assumption we can rewrite the pricing equation (29) as

$$Rx_t = \sum_{j=1}^{J} n_{j,t} \cdot f_j(\{x_{t-l}\}_{l=1}^{L}, \{x_{t+j}\}_{j=0}^{K}, \{w_{t-l}\}_{l=1}^{L}), \tag{33}$$

where now profits in equation (31) are given by

$$\pi_{j,t} = \frac{1}{a\sigma_y^2}\left(x_t - Rx_{t-1} + \delta_t\right)\left(f_j(\{x_{t-l-1}\}_{l=1}^{L}, \{x_{t+j-1}\}_{j=0}^{K}, \{w_{t-l-1}\}_{l=1}^{L}) - Rx_{t-1}\right), \tag{34}$$

with $\delta_t \sim \mathcal{N}(0, \sigma_d^2)$ as shown in Appendix (B.1).

B Proofs

B.1 Profits for the model in deviation form

Start from equation (31).

$$\pi_{j,t} = (y_t - Rp_{t-1})\, z^d_{t-1} = \frac{(y_t - Rp_{t-1})\,(\mathbb{E}_{j,t-1}(y_t) - Rp_{t-1})}{a\sigma_y^2}.$$

Using also the assumption on forecast form we have

$$\pi_{j,t} = \frac{1}{a\sigma_y^2}(y_t - Rp_{t-1})\,(R\bar{p} + f_j(x_t, x_t - 2, b_t - 2) - Rp_{t-1}) = (y_t - Rp_{t-1})\,(f_j(x_t, x_{t-2}, b_{t-2}) - Rx_{t-1}).$$

The first term in parenthesis can be expanded as

$$(y_t - Rp_{t-1}) = (p_t + d_t - Rp_{t-1}) = (x_t + d_t - Rx_{t-1} - \bar{p} + R\bar{p})\,.$$

Notice then that $R\bar{p} - \bar{p} = Rd/r - d/r = d$ and define $\delta_t \equiv d_t - d$ to have the final result.

B.2 Proof of Lemma 1 (See page 5)

Steady states of the system must solve

$$Rx = n_1 gx + (1 - n_1)b,$$

where

$$n_1 = \frac{1}{1 + exp\{\beta(x - Rx)(b - gx)\}}.$$

Now when $\beta = 0$, $n_1 = \frac{1}{2}\forall x \implies x = \frac{b}{2R-g}$. When $\beta \to \infty$ we must make the following considerations. Define $\Delta\pi = (x - Rx)(b - gx)$, as the difference in realized profits between the bias traders and the trend following traders. We have the following

$$\begin{cases} \lim_{\beta\to\infty} \bar{n}_1 = \frac{1}{2} & \text{for} \quad \Delta\pi = 0 \\ \lim_{\beta\to\infty} \bar{n}_1 = 1 & \text{for} \quad \Delta\pi < 0 \\ \lim_{\beta\to\infty} \bar{n}_1 = 0 & \text{for} \quad \Delta\pi > 0 \end{cases}$$

Now in order to prove the statement we proceed by cases. First consider $g = R$, then if $\Delta\pi = 0$ we have $x = \frac{b}{2R-g} = \frac{b}{R}$. This in turn implies $\Delta\pi = (\frac{b}{R} - b)(b - b) = 0$. If $\Delta\pi > 0$ we have $n_1 = 0$, which implies again $p = \frac{b}{R}$. However now this leads to a contradiction, since as in the previous case it would imply $\Delta\pi = 0$. If $\Delta\pi < 0$, then $n_1 = 1$, any $x \geq 0$ would be a solution. However if $x = 0$, then $\Delta\pi = 0$, leading to a contradiction. If $x > 0$ then x must be such that $\Delta\pi = (x - Rx)(b - gx) < 0$. Since $(x - Rx) < 0$, then it must be $(b - gx) > 0 \implies x < \frac{b}{g} = \frac{b}{R}$. When $g > R$ with the same arguments as before we see that the only steady state is $x = \frac{b}{R}$ and when $g < R$ no steady states exist.

B.3 Proof of Lemma 2 (See page 9)

Steady states of the system must solve

$$R\bar{p} = \bar{n}_1 g\bar{p} + \bar{n}_2 b,$$

where

$$\bar{n}_1 = \frac{exp\{\beta\pi_1\}}{exp\{\beta\pi_1\} + exp\{\beta\pi_2\} + exp\{\beta\pi_3\}}, \quad \bar{n}_2 = \frac{exp\{\beta\pi_2\}}{exp\{\beta\pi_1\} + exp\{\beta\pi_2\} + exp\{\beta\pi_3\}},$$

and

$$\pi_1 = (\bar{p} - R\bar{p})(g\bar{p} - R\bar{p}), \quad \pi_2 = (\bar{p} - R\bar{p})(b - R\bar{p}), \quad \pi_3 = (\bar{p} - R\bar{p})(-R\bar{p}).$$

For $\beta = 0$ then $\bar{n}_1 = \bar{n}_2 = \frac{1}{3}$, which implies $\bar{p} = \frac{b}{3R-g}$.

For $\beta \to +\infty$ we must reason by cases following the same procedure as in the previous proof. It is still the case that if one of the three realized profit is strictly greater then all others, then all agents immediately switch to that strategy. We then make use of the fact that

$$\max\{a, b, c\} = \min\{a/d, b/d, c/d\},$$

for any strictly negative d. Consider the quantity $(\bar{p} - R\bar{p})$. Given a greater than unity risk free interest rate R, the quantity is strictly negative for positive values of the steady state, and 0 for $\bar{p} = 0$. We can therefore analyze the two cases. $\bar{p} = 0$ would imply the profits of all three strategies to be equal to 0, causing fractions to be equal to $1/3$. This would in turn imply a steady state value of $b = \frac{b}{3R-g}$, which is different than 0 for $b > 0$ therefore leading to a contradiction. When $\bar{p} > 0$ then we have that

$$argmax\{\pi_1, \pi_2, \pi_3\} = argmin\{(g\bar{p} - R\bar{p}), (b - R\bar{p}), (-R\bar{p})\} = argmin\{g\bar{p}, b, 0\} = 3,$$

where the first equality holds since we have divided the arguments of the argmax by the negative quantity $d = (\bar{p} - R\bar{p})$ and the last since g and b are both greater than 0. This implies $\bar{n}_3 = 1$, causing the steady states value to be $\bar{p} = 0$, again leading to a contradiction.

B.4 Proof of Lemma 3 (See page 11)

Steady states of the system must solve

$$R\bar{p} = \bar{n}_1 g\bar{p} + \bar{n}_2 b + \bar{n}_3 \bar{p},$$

where

$$\bar{n}_1 = \frac{exp\{\beta\pi_1\}}{exp\{\beta\pi_1\} + exp\{\beta\pi_2\} + exp\{\beta\pi_3\}}, \quad \bar{n}_2 = \frac{exp\{\beta\pi_2\}}{exp\{\beta\pi_1\} + exp\{\beta\pi_2\} + exp\{\beta\pi_3\}},$$

and

$$\pi_1 = (\bar{p} - R\bar{p})(g\bar{p} - R\bar{p}), \quad \pi_2 = (\bar{p} - R\bar{p})(b - R\bar{p}), \quad \pi_3 = (\bar{p} - R\bar{p})(\bar{p} - R\bar{p}).$$

For $\beta = 0$ then $\bar{n}_1 = \bar{n}_2 = \bar{n}_3 = \frac{1}{3}$, which implies $\bar{p} = \frac{b}{3R-g-1}$.

For $\beta \to +\infty$ we must reason by cases following the same procedure as before. It is still the case that if one of the three realized profit is strictly greater then all others, then all agents immediately switch to that strategy. We make use again of the fact that

$$\max\{a, b, c\} = \min\{a/d, b/d, c/d\},$$

for any strictly negative d. Consider the quantity $(\bar{p} - R\bar{p})$, given a greater than unity risk free interest rate R, the quantity is strictly negative for positive values of the steady state, and 0 for $\bar{p} = 0$. We can therefore analyze the two cases. $\bar{p} = 0$ would imply the profits of all three strategies to be equal to 0, causing fractions to be equal to $1/3$. This would in turn imply a steady state value of $b = \frac{b}{3R-g-1}$, which is different than 0 for $b > 0$ therefore leading to a contradiction. When $\bar{p} > 0$ then we have that

$$argmax\{\pi_1, \pi_2, \pi_3\} = argmin\{(g\bar{p} - R\bar{p}), (b - R\bar{p}), (\bar{p} - R\bar{p})\} = argmin\{g\bar{p}, b, \bar{p}\} = argmin\{b, \bar{p}\},$$

since $g > 1$. This implies that for positive values of the steady state the first strategy never yields the greater profit, limiting the analysis to the consideration of the following three cases. $b > \bar{p} \implies \bar{n}_3 = 1$, causing the steady states value to be $\bar{p} = 0$, leading to a contradiction. $b < \bar{p} \implies \bar{n}_2 = 1 \implies \bar{p} = \frac{b}{R} < b$, leading to a contradiction.
$b = \bar{p} \implies \bar{n}_2 = \bar{n}_3 = \frac{1}{2} \implies \bar{p} = \frac{b}{2R-1}$, leading to a contradiction. Therefore no positive steady states exist.

B.5 Eigenvalues for the two type model

It is convenient to rewrite the model in (8) and (9) transforming it from a univariate third order difference equation to a first order difference equation with three states. We use the following change of variables $(p_t, p_{t-1}, p_{t-2}) = (x_{t+1}, w_{t+1}, z_{t+1})$ to rewrite the model as the following system:

$$\begin{cases} x_{t+1} = \frac{g}{R} x_t \left\{1 + exp[\beta(x_t - Rw_t)(b - gz_t)]\right\}^{-1} + \frac{b}{R}\left(1 - \left\{1 + exp[\beta(x_t - Rw_t)(b - gz_t)]\right\}^{-1}\right) \\ w_{t+1} = x_t \\ z_{t+1} = w_t \end{cases} \tag{35}$$

Taking first order derivatives we get the following Jacobian matrix

$$\begin{bmatrix} A & B & C \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

with

$$A = \frac{\left(\left(\beta g^2 z_t - b\beta g\right) x_t - b\beta g z_t + g + b^2\beta\right) \mathrm{e}^{\beta\cdot(b - gz_t)(x_t - Rw_t)} + g}{R \cdot \left(\mathrm{e}^{\beta\cdot(b - gz_t)(x_t - Rw_t)} + 1\right)^2},$$

$$B = -\frac{\beta \cdot (gx_t - b)\,(gz_t - b)\,\mathrm{e}^{\beta\cdot(b-gz_t)(x_t-Rw_t)}}{\left(\mathrm{e}^{\beta\cdot(b-gz_t)(x_t-Rw_t)} + 1\right)^2},$$

$$C = \frac{\beta g \cdot (x_t - Rw_t)\,(gx_t - b)\,\mathrm{e}^{\beta\cdot(x_t-Rw_t)(b-gz_t)}}{R \cdot \left(\mathrm{e}^{\beta\cdot(x_t-Rw_t)(b-gz_t)} + 1\right)^2}.$$

To evaluate the stability of the steady state, one can observe the eigenvalues of the Jacobian matrix, which in this case are given by:

$$\lambda_1 = \frac{\sqrt[3]{2A^3 + 3\sqrt{3}\sqrt{4A^3C - A^2B^2 + 18ABC - 4B^3 + 27C^2} + 9AB + 27C}}{3\sqrt[3]{2}} -$$

$$-\frac{\sqrt[3]{2}\left(-A^2 - 3B\right)}{3\sqrt[3]{2A^3 + 3\sqrt{3}\sqrt{4A^3C - A^2B^2 + 18ABC - 4B^3 + 27C^2} + 9AB + 27C}} + \frac{A}{3}$$

$$\lambda_2 = -\frac{1}{6\sqrt[3]{2}}(1 - i\sqrt{3})\sqrt[3]{2A^3 + 3\sqrt{3}\sqrt{4A^3C - A^2B^2 + 18ABC - 4B^3 + 27C^2} + 9AB + 27C} +$$

$$+\frac{(1 + i\sqrt{3})\left(-A^2 - 3B\right)}{3 \cdot 2^{2/3}\sqrt[3]{2A^3 + 3\sqrt{3}\sqrt{4A^3C - A^2B^2 + 18ABC - 4B^3 + 27C^2} + 9AB + 27C}} + \frac{A}{3}$$

$$\lambda_3 = -\frac{1}{6\sqrt[3]{2}}(1 + i\sqrt{3})\sqrt[3]{2A^3 + 3\sqrt{3}\sqrt{4A^3C - A^2B^2 + 18ABC - 4B^3 + 27C^2} + 9AB + 27C} +$$

$$+\frac{(1 - i\sqrt{3})\left(-A^2 - 3B\right)}{3 \cdot 2^{2/3}\sqrt[3]{2A^3 + 3\sqrt{3}\sqrt{4A^3C - A^2B^2 + 18ABC - 4B^3 + 27C^2} + 9AB + 27C}} + \frac{A}{3}$$

B.6 Eigenvalues for the model with fundamentalist

As before we rewrite the model as the "following" system:

$$\begin{cases} x_{t+1} = \frac{g}{R}x_t n_{1,t} + \frac{b}{R}n_{2,t} \\ w_{t+1} = x_t \\ z_{t+1} = w_t \end{cases} \tag{36}$$

with

$$n_{1,t} = \frac{exp[\beta(x_t - Rw_t)(gz_t - Rw_t)]}{Z_t},$$

$$n_{2,t} = \frac{exp[\beta(x_t - Rw_t)(by_{t-3} - Rw_t)]}{Z_t},$$

$$n_{3,t} = \frac{exp[\beta(x_t - Rw_t)(-Rw_t)]}{Z_t},$$

$$Z_t = exp[\beta(x_t - Rw_t)(gz_t - Rw_t)] + exp[\beta(x_t - Rw_t)(by_{t-3} - Rw_t)] + exp[\beta(x_t - Rw_t)(-Rw_t)].$$

Taking first order derivatives we get the following Jacobian matrix:

$$\begin{bmatrix} A & B & C \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A = \frac{\mathrm{e}^{R\beta w\cdot(x-Rw)} \cdot \left(g\mathrm{e}^{2\beta\cdot(gz-Rw)(x-Rw)+R\beta w\cdot(x-Rw)} + \mathrm{e}^{\beta\cdot(gz-Rw)(x-Rw)} \cdot \left(\left(\left(\beta g^2 z - b\beta g\right) x - b\beta gz + g + b^2\beta\right) \mathrm{e}^{\beta\cdot(b-Rw)(x-Rw)+R\beta w\cdot(x-Rw)} + \beta g^2 zx + g\right) + b^2\beta \mathrm{e}^{\beta\cdot(b-Rw)(x-Rw)}\right)}{R\cdot\left(\mathrm{e}^{\beta\cdot(gz-Rw)(x-Rw)+R\beta w\cdot(x-Rw)} + \mathrm{e}^{\beta\cdot(b-Rw)(x-Rw)+R\beta w\cdot(x-Rw)} + 1\right)^2},$$

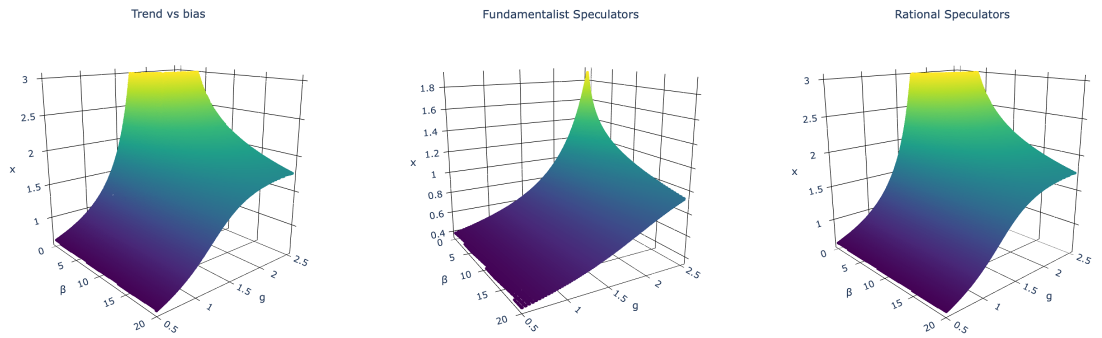
$$B = -\frac{\beta \mathrm{e}^{R\beta w\cdot(x-Rw)} \cdot \left(\mathrm{e}^{\beta\cdot(x-Rw)(gz-Rw)} \cdot \left(\left(\left(g^2 x - bg\right) z - bgx + b^2\right) \mathrm{e}^{\beta\cdot(b-Rw)(x-Rw)+R\beta w\cdot(x-Rw)} + g^2 xz\right) + b^2 \mathrm{e}^{\beta\cdot(b-Rw)(x-Rw)}\right)}{\left(\mathrm{e}^{\beta\cdot(x-Rw)(gz-Rw)+R\beta w\cdot(x-Rw)} + \mathrm{e}^{\beta\cdot(b-Rw)(x-Rw)+R\beta w\cdot(x-Rw)} + 1\right)^2},$$

$$C = \frac{\beta g\cdot(x-Rw)\left(\left(gx-b\right)\mathrm{e}^{\beta\cdot(b-Rw)(x-Rw)+R\beta w\cdot(x-Rw)} + gx\right)\mathrm{e}^{\beta\cdot(x-Rw)(gz-Rw)+R\beta w\cdot(x-Rw)}}{R\cdot\left(\mathrm{e}^{\beta\cdot(x-Rw)(gz-Rw)+R\beta w\cdot(x-Rw)} + \mathrm{e}^{\beta\cdot(b-Rw)(x-Rw)+R\beta w\cdot(x-Rw)} + 1\right)^2}.$$

The general form of the eigenvalues computed in section (B.5) is still valid, so that we can obtain them by simply replacing the formulae for A, B and C.

C Steady States

In the figure below we numerically solve for the implicit function defining the steady state of the systems. To obtain a plot in three dimension we fix the value of the parameters $R = 1.01$ and $b = 1$ and vary the parameters g and β.



D The Extended Path (EP) algorithm

Before describing the algorithm we find it useful to highlight what we are after and why we need the algorithm in the first place. Start by assuming that expectations formed by the rational agent in the past are indeed rational, that is $\mathbb{E}_{t-3}(x_t - 1) = x_{t-1}$. Then the rational agents needs to choose $\mathbb{E}_{t-1}(x_t + 1)$ considering the following. After the choice is made, a realization of x_t is going to realize by equation (14). Given this and $\mathbb{E}_t(x_t + 2)$ one can shift 14) forward one period to obtain the realization x_{t+1}. In order for an agent to have rational expectations we then require $\mathbb{E}_{t-1}(x_t + 1) = x_{t+1}$. We can see then that when forming expectations at time $t - 1$ the agents must consider how expectations are going to be formed at time t. At time t this is still true, and the agent will have to consider what their expectations are going to be at time $t+1$ and so on. Essentially the whole future *path* of expectations is relevant for the choice of expectations today and hence for actual realization of the state variable. We are now ready to describe the algorithm. Assume we want to simulate the model for T periods, then

(i) Choose an integer k, an initial guess at the number of periods beyond the horizon T, in order to obtain a solution which differs from rational expectations below a tolerance level ϵ. Generate an initial expectations vector $\{\tilde{x}_{t+j}\}_{j=1}^{T+k}$;

(ii) Use (14) to obtain a vector $\{x_{t+j}\}_{j=0}^{T+k-1}$ and this in turn to obtain the implied $\{x_{t+j}\}_{j=1}^{T+k-1}$;

(iii) Compute the sum of the absolute deviation between $\{x_{t+j}\}_{j=1}^{T+k-1}$ and $\{\tilde{x}_{t+j}\}_{j=1}^{T+k-1}$. If this is less then ϵ set $\{\tilde{x}_{t+j}\}_{j=1}^{T+k-1}$ = $\{x_{t+j}\}_{j=1}^{T+k-1}$ and return to step (i) Theses iterations are called *Type 1*. Call e(k) the vector obtained after convergence;

(iv) Repeat steps (i) to (iii) by replacing k with $k+1$. Call e(k+1) the vector obtained after convergence. Compute the sum of the absolute deviation between the first e(k) and the first k elements of e(k+1). Iterate until it holds that $\sum |e(k+i) - e(k+i+1)| < \epsilon$ for some i. These iterations are called *type 2*;

(v) The rational expectations vector is given by the first T elements of the vector $e(k+i)$.

Now as Fair and Taylor (1983) notice in the original paper "for a general nonlinear model there is no guarantee that any of the iterations will converge. If convergence is a problem, it is sometimes helpful to "damp" the successive solution values." In practice this will be true in our case when the system enters the unstable region. To deal with this we change the updating mechanism such that the new vector is equal to 0.5 times the original vector times 0.5 times the new one. Finally in order to avoid the algorithm to run forever, we set a maximum of 1000 *type 1* iterations and 100 *type 2* iterations. The parameters of the algorithm that we set are then $T = 2000$, $\epsilon = e^{-14}$ and $k = 100$. In figure () we report the sum of absolute deviations between the rational expectation vector and the realizations of state variables. We can see that although convergence is not reached, the the error is almost always lower than 8×10^{-15} implying an average error lower than 4×10^{-18}.

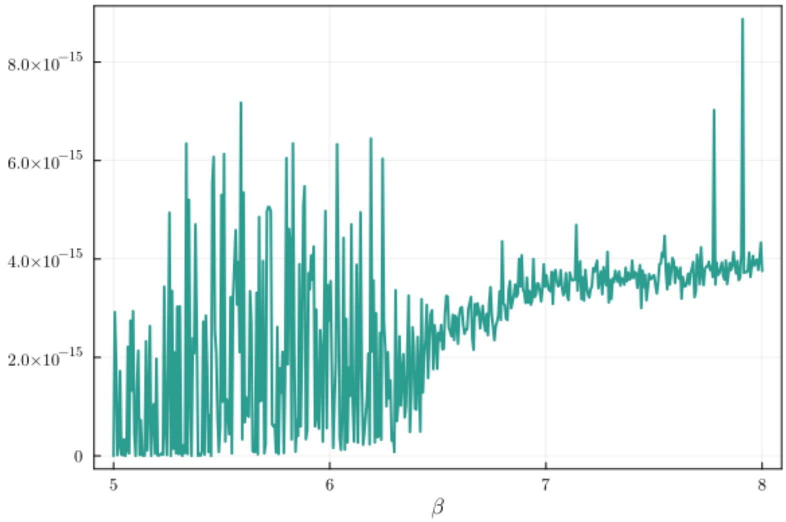


Figure 12: Sum of absolute errors

E LSTM Model

The dataset on which we train the LSTM model is then composed of the following input vector

$$[x_{t-1}/\mathrm{MA}_t, x_{t-2}/\mathrm{MA}^2_{t-3}, x_{t-3}/\mathrm{MA}^2_{t-3}, w_{t-1}/\mathrm{MA}_t, w_{t-2}/\mathrm{MA}^2_{t-3}, w_{t-3}/\mathrm{MA}^2_{t-3}, R_t, R_{t-1}, R_{t-2}],$$

to predict x_{t+1}/MA with

$$x_t = \left(p_t \Big/ \frac{\sum_{i=W}^{1} p_{t-i}}{W} - 1 \right).$$

The LSTM layer takes an input vector of size 9 and processes it through a recurrent LSTM cell with 96 hidden units. The LSTM cell has the following equations[6]:

Input gate

$$i_t = \sigma(W_i x_t + U_i h_{t-1} + b_i), \tag{37}$$

forget gate

$$f_t = \sigma(W_f x_t + U_f h_{t-1} + b_f), \tag{38}$$

output gate

$$o_t = \sigma(W_o x_t + U_o h_{t-1} + b_o), \tag{39}$$

cell state update

$$c'_t = tanh(W_c x_t + U_c h_{t-1} + b_c), c_t = f_t \odot c_{t-1} + i_t \odot c'_t, \tag{40}$$

$$h_t = o_t \odot tanh(c_t). \tag{41}$$

[6]Note that the notational conventions employed in equations from (37) to (45) will be confined exclusively to this particular segment of the paper.

Where x_t is the input at time t, h_t is the hidden state at time t, c_t is the cell state at time t, and σ is the sigmoid function, $\odot$ represents the Hadamard product. The weight matrices W and U, and the bias vectors b are the learnable parameters of the LSTM layer.

We then have three dense layers, with input and output size equal to 96, with tanh activation functions. Dense Layer 1,

$$y_1 = tanh(W_1 h_t + b_1). \tag{42}$$

Where W_1 is the weight matrix of shape (96, 96), b_1 is the bias vector of shape (96,1), and h_t is the output from the LSTM layer.

Dense Layer 2,

$$y_2 = tanh(W_2 y_1 + b_2). \tag{43}$$

Where W_2 is the weight matrix of shape (96, 96), b_2 is the bias vector of shape (96,1), and y_1 is the output from Dense Layer 1.

Dense Layer 3,

$$y_3 = tanh(W_3 y_2 + b_3). \tag{44}$$

Where W_3 is the weight matrix of shape (96, 96), b_3 is the bias vector of shape (96,1), and y_2 is the output from Dense Layer 2.

Finally we have the output layer, represented by a Dense layer with input size 96 and output size 1,

$$y_4 = W_4 y_3 + b_4. \tag{45}$$

Where W_4 is the weight matrix of shape (1, 96), b_4 is the bias vector or in this case a scalar, and y_3 is the output from Dense Layer 3.

The output y_4 is the final output of the model. It is now clear that $y_4 = \widehat{\mathcal{J}}\left(\{x_{t-l}\}_{l=1}^{L}, \{w_{t-l}\}_{l=1}^{L}; \theta\right)$

F LSTM on simulated data

We use this section to show the estimation based on the LSTM model is able to the recover the parameters of the true data generating process in which rational expectations are obtained by EP. To do this we first generate a deterministic time series and then add some idiosyncratic noise to generate a noise version that we use for estimation. We do this for fixed values of $b = 1.0$, $g = 1.3$ $R = 1.01$ and $\beta = 7.2$, corresponding to the stable and unstable region respectively. We generate a series of length 2000 and add to normal noise with mean 0 and standard deviation 0.1. Then we train the LSTM model to produce the rational forecast. The r-squared obtained by the model is 0.25. Despite the low noise environment the estimates of the NLS reported in table (3) are relatively close to the true values, hence confirming the validity of our approach.

Table 3: Estimation on simulated data

	Point estimate	Standard Error
g	1.25	0.02
b	1.04	0.02
β	6.75	1.14

G Stationarity tests

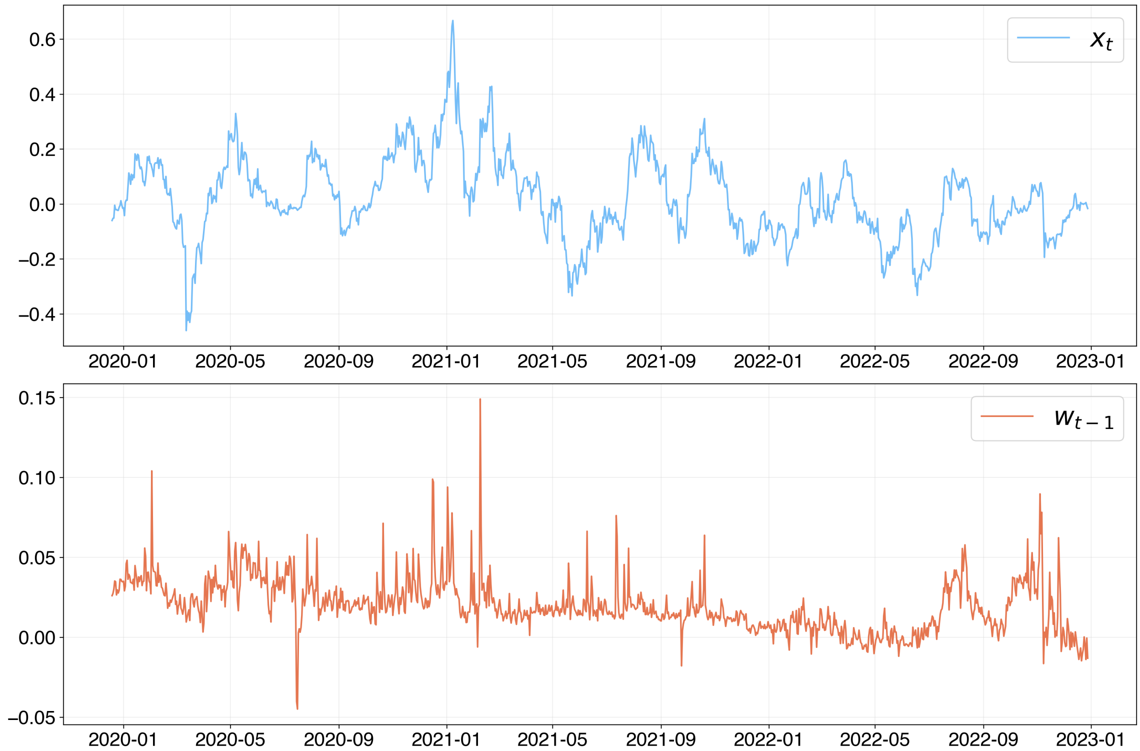


Figure 13: Variable for the estimation

We test the presence of a unit root for the two time series related to the percentage deviation form the moving average fundamental by means of the augmented Dickey–Fuller test (ADF). The associated p-values are: 1.27e-06 for x_t and 0.037 for w_{t-1}. We can therefore reject the null hypothesis of unit root of the ADF.

H Robustness to different windows

In this section we check the robustness of the estimation to a different choice of the window length in the moving average fundamental value. We repeat the analysis for each value between 1 and 99. A window choice of 1 implies that we are actually estimating the model on daily returns. For each choice we re estimate the two types model with trend followers and bias and plot the point estimate of the parameters with associated standard errors in the top panel of figure (14), the dashed line is at 0. The bottom panel shows the p-value for the F-statistic of significance of the non-linear model with respect to the linear one and the adjusted r-squared. The dashed line marks the 5% significance level.

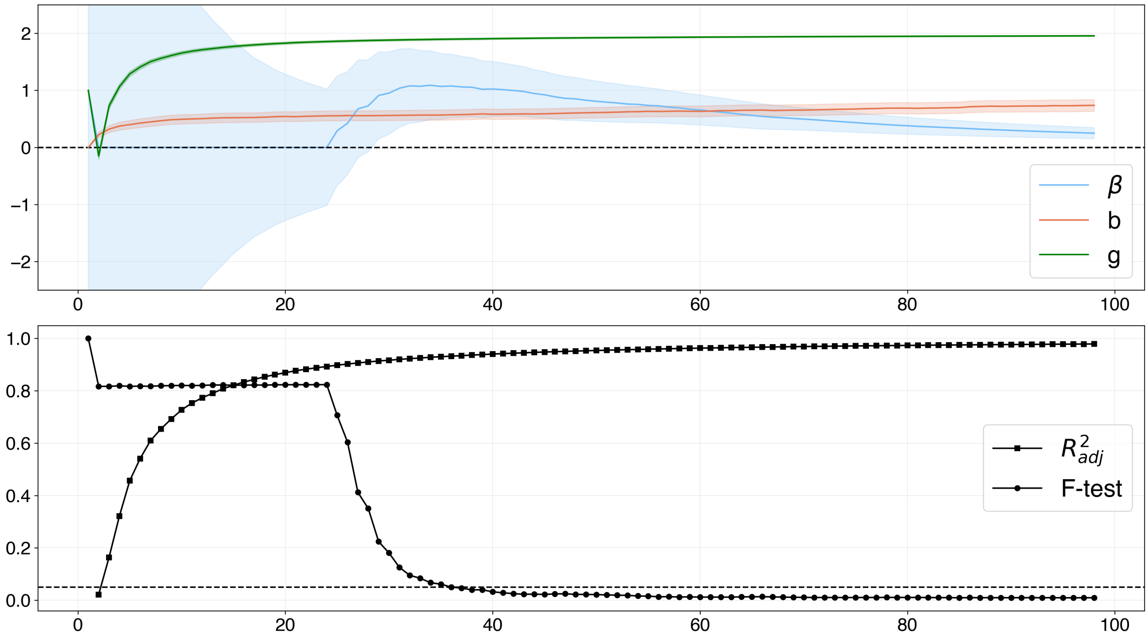


Figure 14: Effect of different windows

Daily returns (window length = 1) are as one would expect unpredictable, with an adjusted r-squared close to 0. For a window length smaller then 25 the intensity of choice β is not significantly different then 0, For values lower then 7 we get estimates ranging so much that we had to impose limits on the axis in order to obtain a meaningful visualization. For values higher then 30 however we start to observe results that are quantitatively similar with high r-square, positive and significant βs and preference for the non-linear model as conveyed by the p-value of the F-test.

I Bootstrapped standard errors

In this section we show the robustness of the standard errors by computing them via bootstrap. The p-value of the test for Conditional Heteroscedasticity does not reject homoskedasticity in the residuals. Nonetheless we follow the approach of Gonçalves and Kilian (2004). That is for 2000 bootstrap replications we estimate generate a series of pseudo residuals by multiplying an original residual by a random number drawn from a Stdandard Normal Distribution We then create a pseudo time series using the associated nonlinear model by replacing the actual residuals with the pseudo residuals. Finally we re-estimate the model using the pseudo time series to obtain a new set of parameter estimates and report the values of the associated confidence interval at the .99 percent level in table (4).

Table 4: 99% bootstrapped confidence intervals

	Lower Bounds	**Upper Bounds**
Trend followers + Bias:	[1.88 0.4 0.08]	[1.94 0.75 1.99]
Trend followers + Bias + Fundamentalist:	[2.81 0.62 0.06]	[2.9 1.13 0.98]
Trend followers + Bias + LSTM:	[1.88 0.49 0.12]	[1.97 0.98 2.66]