Complex Economic Dynamics

Lecture 2 – 03 November 2021

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using PlutoUI    , Plots    , LaTeXStrings
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using DynamicalSystems
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1-Dimensional dynamical systems

1-D systems dynamics are fairly simple. For monotonic functions in particular we have the following results:

> **Theorem**
>
> if $f:\mathbb{R}\to\mathbb{R}$ is a monotonic and continous function, then an orbit $\{x_t\}$ either:
>
> Diverges
>
> Convergese to a Steady State
>
> Convergese to a 2-cycle (only if decreasing)

Bifurcations

Bifurcations are qualitative changes in dynamics as a prameter varies. For the 1-state, 1-parameter family $f_\lambda \mathbb{R}\to\mathbb{R} \quad \lambda\in\mathbb{R}$ we will analyze 4 types of bifurcations

Period-doubling bifurcations:

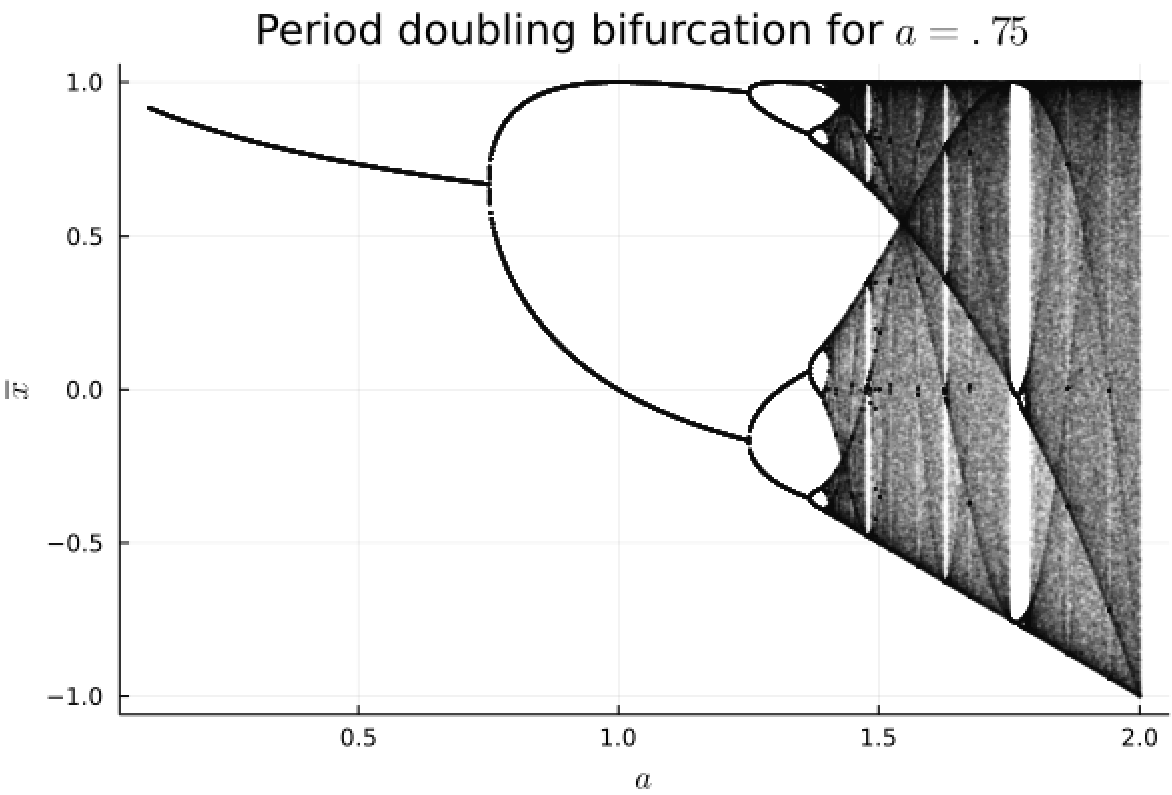
- Before: Steady-state (stable / unstable);
- At: Stability change and creation of a 2-cycle;
- After: unstable / stable Steady-state and stable / unstable 2-cycle (n.b. order matters);

Note, everywhere for "ss" one can read k-cycle and for 2-cycle one reads 2k-cycle

> Necessary condition: at $\lambda=\lambda^*: f'_\lambda(x^*)=-1$

Example: $f_a(x_t)$, where $f_a(x) = 1 - ax^2, \quad a > 0, \quad \text{and with} \quad x_0 \in [-1, +1]$.

We have a period doubling bifurcation for $a = 3/4$

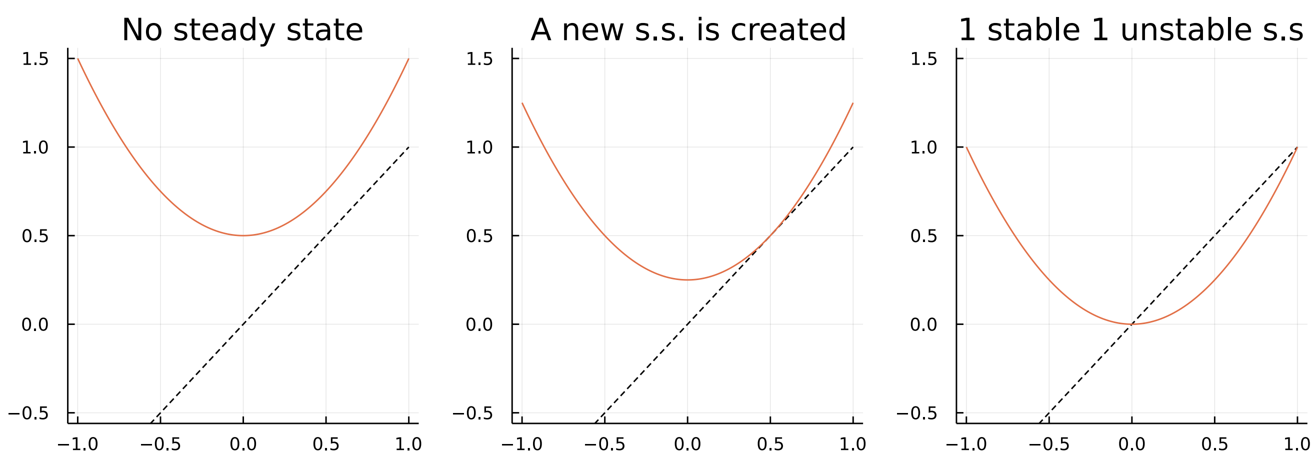


Tangent (saddle-node)

- Before: No Steady-state;
- At: New Staesy-state is created;
- After: There are 2 Steady-state (one stable and one unstable);

Necessary condition: at $\lambda = \lambda^* : f'_\lambda(x^*) = 1$

Example: $f_a(x) = x^2 - a \quad \text{for} \quad a = -.25$

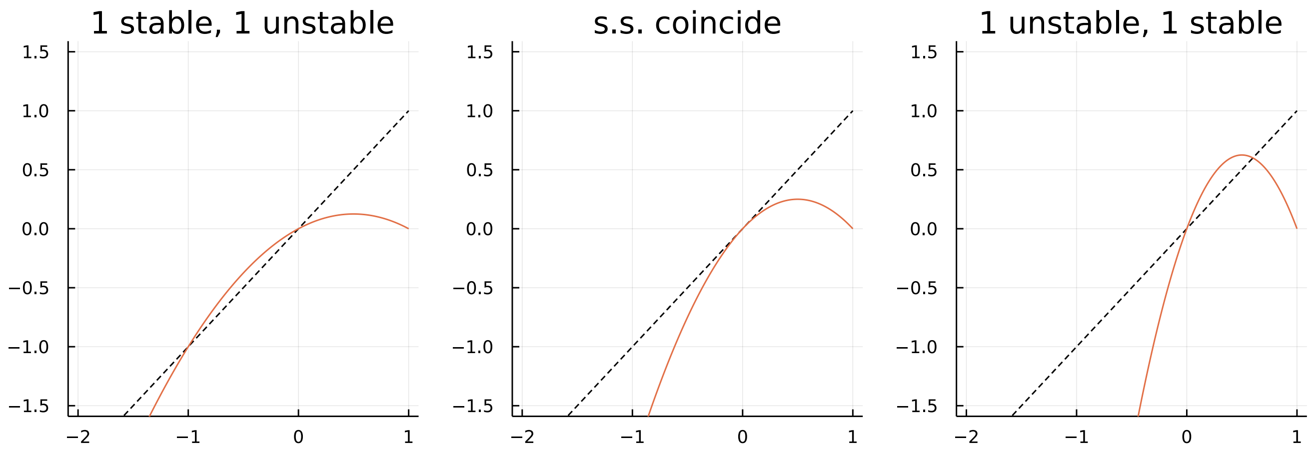


Transcritical bifurcation

- Before: 1 stable and 1 unstable steady state;
- At: Steady states coincide and they exchange stability;
- After: 1 unstable and 1 stable steady state;

Only occurs in system with special structure

Example: $f_a(x) = ax(1-x) \quad \text{for} \quad a = 1$

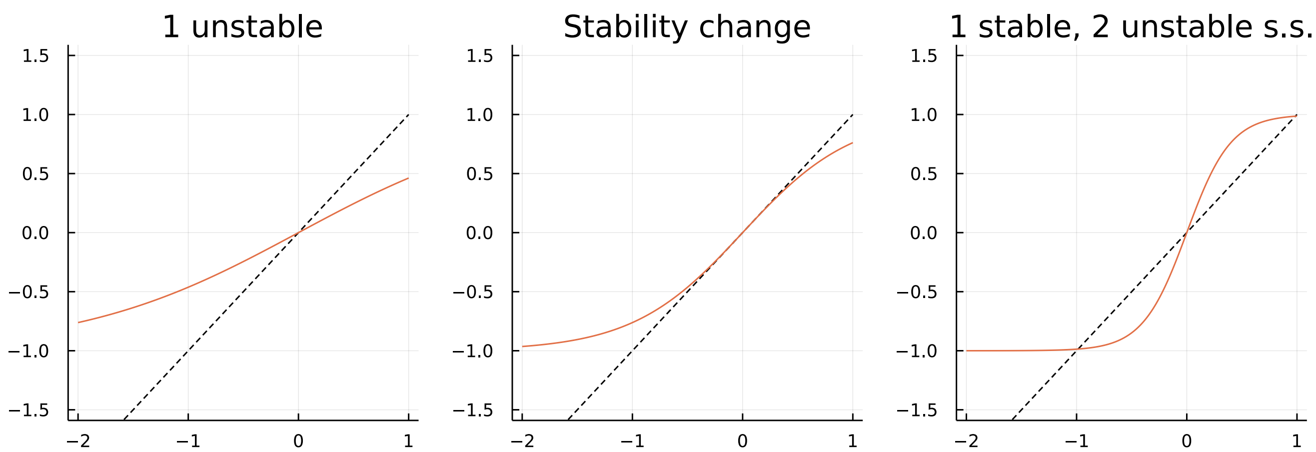


Pitchfork bifurcation

- Before: 1 stable/unstable steady state;
- At: Stability change and two added s.s;
- After: 1 unstable/stable steady state and two added stable/unstable s.s.;

Typically occurs in systems with a reflection symmetry: $f_\lambda(-x) = -f_\lambda(x)$

Example: $f_a(x) = tanh(ax) \quad \text{for} \quad a = 1$



Topological chaos

The dynamics $x_{t+1} = f(x_t)$ is called (topologically) chaotic if the following three properties are satisfied:

- There is an infinite set P of (unstable) periodic orbits with different periods
- There is an uncountable set S of aperiodic points
- f has dependence on inital conditions wrt $\Lambda = P \cup S$

Definition:

A map: $f : \Lambda \rightarrow \Lambda \quad \Lambda \in \mathbb{R}$ has sensitive dependence on inital conditions if there is $c > 0$ s.t. $\forall x_0 \in \Lambda$, and all $\mathscr{U} = (x_0 - \epsilon, x_0 + \epsilon), \quad \epsilon > 0$, there is $y_0 \in \Lambda \cap \mathscr{U}$ and $T > 0$, such that the distance $(x_T, y_T) > c$

Li & York Theorem (period 3 implies chaos)

Given a continous map $f : \mathbb{R} \rightarrow \mathbb{R}$, if there is a point x_0 s.t.

$$f^3(x_0) \leq x_0 < f^(x_0) < f^2(x_0)$$

then, the dynamics are topologically chaotic.

Appendix

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generateinstance (generic function with 1 method)
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computeorbit (generic function with 1 method)
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