Complex Economic Dynamics

Lecture 21 01 November 2021

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Intializing packages

```
begin
    using PlutoUI , Images , Plots
    using HypertextLiteral , LaTeXStrings , Symbolics
end
```

Contents

During the course we will analyze:

- Theory of dynamical systems:
  - Steady states
  - Stability
  - Bifurcations, local and global
- Applications to economic and financial models:
  - Cobweb Market Models
  - Asset Pricing Dynamics

Introduction

Let us pretende there exists a "Standard Economic Theory" with many interconnected markets, we usually look for General Equlibria, that is an equilibrium that can be disturbed by small stochastic shocks (e.g. News), and still return to the equilibrium.

These models are based on symplifing assumptions about the agents, and in particular on **Rationality** and **Homogeneity**.

This course discusses tolls to build and analyze models with **Heterogeneous** and **Boundedly Rational** agents, that can help to explain **Complex Dynamic Phenomena**.

In particular fluctations in Economc systems may arise from **exogenous factors** (e.g. Climate change, innovations, wars, revolutions...) or by **endogenous factors** that can arise from interactions among agents (e.g. Hog cycles, bubbles and crashes in financial systems)

Our main driver is the degree of rationality of the agents and we will focus on Bounded Rationality where agents have:

- Limited information sets
- Simple forecasting rules
- May have an incorrect model of the system

Dynamical Systems

Let

$$x_t \in \mathbb{R}^n$$

be a vector describing the system or the **State Variable**.

A **Dynamical System** is a map from a states to new states:

$$x_{t+1} = f_\lambda(x_t)$$

where:

$$f : \mathbb{R}^n \to \mathbb{R}^n$$

is the evolution law or **law of motion** and

$$\lambda \in \mathbb{R}^m$$

is a set of parameters influencing the system.

Our aim is to understand the long run behavior of the states and its dependence on parameters and the initial state x_0.

Some examples:

Quadratic map:

$$x_{t+1} = \lambda x_t (1 - x_t)$$

Linear Dynamical System:

$$x_{t+1} = \lambda x_t$$

which is a naive case as we can actually derive an explicit solution in:

$$x_t = \lambda^t x_0$$

Hénon Map:

$$x_{t+1} = 1 - a x_t + y_t$$

$$y_{t+1} = b x_t$$

Remarks on notation

An orbit is a sequence of states: $\{x_0, x_1, x_2, \ldots\}$

where:

$$x_1 = f(x_0)$$

$$x_2 = f(x_1) = f(f(x_0)) = (f \circ f) x_0$$

$$x_3 = f(x_2) = f(f(f(x_0))) = (f \circ f \circ f) x_0$$

In general we have:

$$x_n = f(x_{n-1}) = (f \circ f \circ \cdots \circ f) x_0 = f^n(x_0)$$

Note: "∘" reads "follow".

Steady States

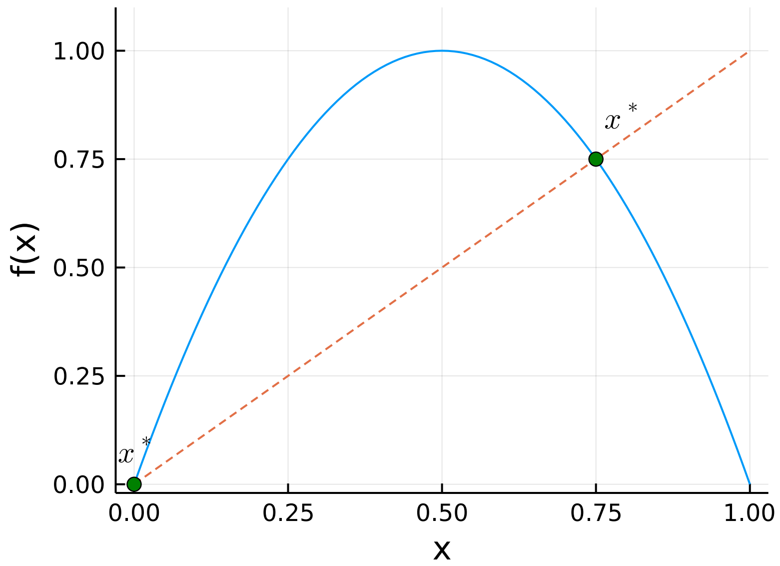
A point x^* is a **Steady State** or a **Fixed point** of the evolution f if:

$$x^* = f(x^*)$$

In a one dimensional setting, the fixed points are graphically represented by the intersection of the map with the line $y = x$. For example if we consider the quadratic map:

$$f_4(x) = 4x(1 - x)$$

we have:



Local Stability

Error in the book

A fixed point x^* is locally stable if there is an $\epsilon > 0$ s.t. $\forall x_0 \in (x^* - \epsilon, x^* + \epsilon)$ we have:

$$\lim_{t\to\infty} x_t = \lim_{t\to\infty} f^t(x_0) = x^*$$

But this is wrong!

Local Stability (actual definition)

A Steady State x^* of a dynamical system is locally (asymptotically) stable if $\forall\delta > 0 \exists\epsilon > 0$ s.t. $\forall x_0 \in (x^* - \epsilon, x^* + \epsilon)$ we have that:

$x_t = f^t(x_0) \in (x^* - \delta, x^* + \delta)$ and $x_t = f^t(x_0) \to x^*$ as $x \to \infty$

Unstable Steady State

Conversely we can state that a Steady State x^* of a dynamical system is **unstable** if $\exists\delta > 0$ s.t. $\forall\epsilon > 0 \quad \exists x_0 \in (x^* - \epsilon, x^* + \epsilon)$ such that we have:

$x_t = f^t(x_0) \notin (x^* - \delta, x^* + \delta)$

> **Local Stability Theorem**
>
> if $f : \mathbb{R} \to \mathbb{R}$ is continously differentiable and x^* is a fixed point of f, then if:
>
> $$|f'(x^*)| < 1 \implies x^* \quad \text{is locally stable}$$
>
> $$|f'(x^*)| > 1 \implies x^* \quad \text{is unstable}$$

```
md"""
>**Local Stability Theorem**
>
> if $f:\mathbb{R} → \mathbb{R}$ is continously differentiable and $x^*$ is a fixed point of $f$, then if:
>
>$|f'(x^*)| < 1 ⇒ x^* \quad \text{is locally stable}$
>
>$|f'(x^*)| > 1 ⇒ x^*\quad \text{is unstable}$
"""
```

Below we can find examples of:

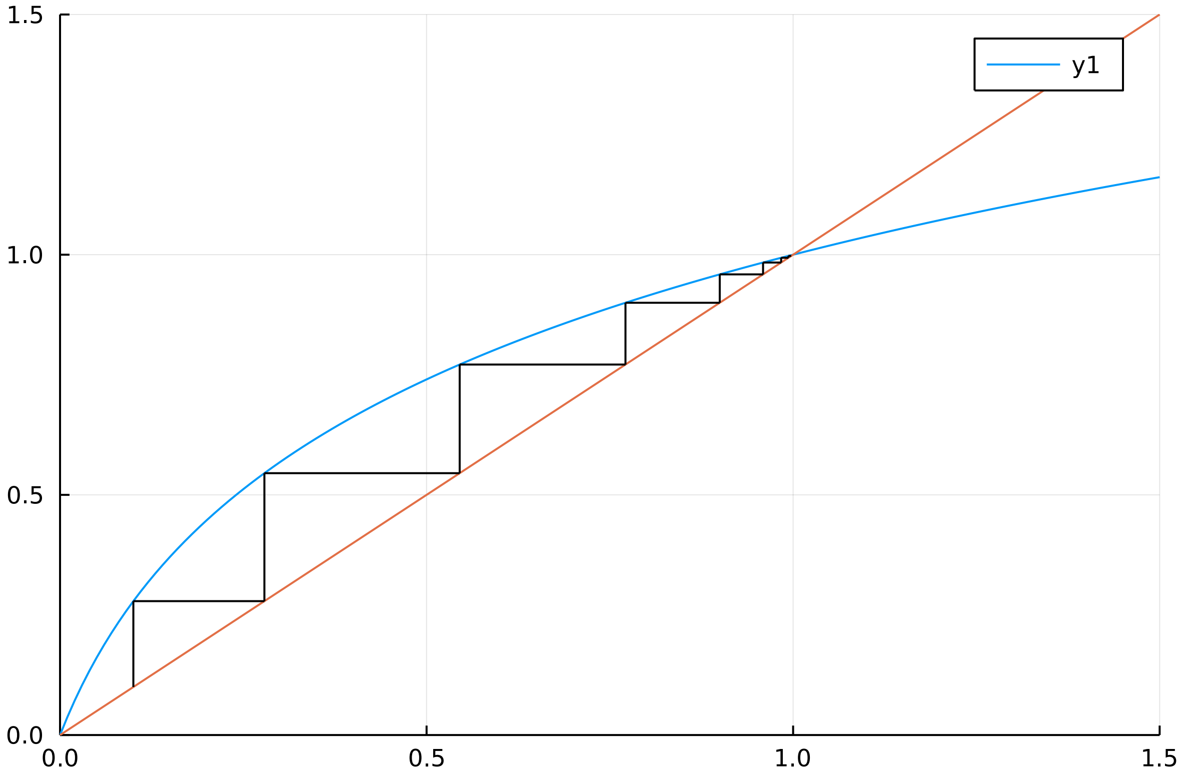
- (a) Monotonic convergence, when $\quad f'(x^*) < 1$
- (b) Monotonic divergence, when $\quad f'(x^*) > 1$
- (c) Oscillatory convergence, when $\quad -1 < f'(x^*) < 0$
- (d) Oscillatory divergence, when $\quad f'(x^*) < -1$

To see some plots, select a type of convergence from the menu:

Monotonic convergence

and the number of iterations from this slider:

8



Periodic Points

A point x is a **periodic point** of period k if:

$$f^k(x) = x \quad \text{and} \quad f^t(x) \neq x \quad 0 < t < k$$

A **periodic orbit** or (**k-cycle**) can be defined as $\{x, f(x), \ldots, f^{k-1}(x)\}$

Note: `a periodic $k$ point of $f$ is a fixed point of $f^k$`

Local Stability for period k points

$$|f^k(x)| < 1 \implies \quad \text{period k point x is locally stable}$$

$$|f^k(x)| > 1 \implies \quad \text{period k point x is unstable}$$

Aperiodic Points

A point is **aperiodic** if:

- Its orbit is bounded;
- It is not a periodic point
- Its orbit does not converge to a periodic orbit

```
cobweb (generic function with 1 method)
```

```
od (generic function with 1 method)
```