École polytechnique fédérale de Lausanne
Applied Mathematics Master

Mean-field disordered systems via Hamilton-Jacobi-Bellman equation

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A.Y. 2024/2025

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1 Introduction

This semester project focuses on analyzing and reviewing a recent approach proposed by Jean-Christophe Mourrat for computing the limiting free energy of mean-field systems [2]. The analysis is centered on two representative models, each exemplifying a broader class of systems: the Curie–Weiss model, a prototypical example of a non-disordered mean-field system, and the Sherrington–Kirkpatrick (SK) model [8], one of the first disordered physical models introduced to describe the behavior of spin glasses.
Mourrat's technique relies on building a more refined version of the free energy of the systems in a suitable way so that it satisfies a partial differential equation (PDE) of Hamilton–Jacobi–Bellman (HJB) type:

$$\partial_t f - H(\nabla f) = 0. \tag{1}$$

Although this principle can be applied for a generic non-linearity $H(x)$, for the two models we focus on $H(x) = x^2$, which is a convex function. This convexity is what allows the HJB equation to admit a variational formulation, making it possible to compute the solution using standard tools from convex analysis.
We note that applying the same approach to models with non-convex interactions is considerably more challenging and remains an active area of research [3].
The project is structured as follows. We begin with an analysis of the Curie–Weiss model, which serves as a toy model relative to the Sherrington–Kirkpatrick one. Its simplicity allows for an explicit computation of the free energy using large deviation arguments. At the same time, it is rich enough to highlight the key structural elements we must identify and preserve in order to extend the HJB approach to more complex systems. In parallel, we aim to understand why this model behaves so well and what essential features are absent in the SK model.
After this, we turn our focus to the SK model istelf, exploring a naïve extension of the Curie–Weiss approach. This attempt reveals the limitations of the method and clarifies what additional structure or tools are needed to make progress.
We proceed by examining the current state of art of the HJB approach for SK and outline how the proper enrichment of the free energy leads to consistency with the Parisi formula in the general temperature regime. We then take a detour to the Random Energy Model (REM), which we consider as another simplification of SK: being the $p \to \infty$ limit of a pure p-spin model with no interaction between energy levels. This suggested that the REM could offer a clean and tractable setting to test whether certain aspects of the HJB approach could carry over to a disordered, yet simplified, context. However, this turned out to be not so fruitful, as the lack of structure in the REM's Hamiltonian yields an ill-posed PDE.
Finally, we highlight the main conceptual and technical obstacles of the HJB framework, focusing in particular on the complications introduced by the infinite dimension of the HJB-PDE in the case of SK and what is still missing to characterize the limiting free energy in this model.

2 Curie–Weiss Model

The Curie–Weiss model is a paradigmatic example of a mean-field spin system consisting of N binary spins $\sigma_i \in \{-1, +1\}$. In this model *mean-field* means that there is invariance under any permutation of the index set, thus the lattice model has the structure of a complete graph and no notion of neighbours.
The energy of a configuration $\sigma \in \{-1, +1\}^N$ at inverse temperature $t \geq 0$ and external field $h \in \mathbb{R}$ is given by the Hamiltonian:

$$H_N(\sigma; t, h) := \frac{t}{N} \sum_{i,j=1}^{N} \sigma_i \sigma_j + h \sum_{i=1}^{N} \sigma_i.$$

From the Hamiltonian, we define the probability measure through which configurations are sampled:

Definition 1 (Gibbs measure)**.** *The Gibbs measure associated with the Hamiltonian $H_N(t, h, \sigma)$ is defined as the probability distribution:*

$$\mathbb{P}(\sigma; t, h) := \frac{\exp\left(H_N(\sigma; t, h)\right)}{Z}, \tag{2}$$

where the partition function Z_N is given by

$$Z_N = \sum_{\sigma \in \{-1,+1\}^N} \exp\left(H_N(\sigma; t, h)\right).$$

The partition function encodes many relevant physical quantities of the model and allows for an understanding of phase transitions, if any. Now we can define the free energy, which is the quantity in which we are interested in computing the limit for $N \to \infty$.

Definition 2 (Free energy for Curie–Weiss)**.** *The finite-volume free energy for the Curie–Weiss model at inverse temperature t and external field h is defined as*

$$F_N(t, h) := \frac{1}{N} \log\left(2^{-N} Z_N\right).$$

The factor 2^{-N}, which corresponds to subtracting by a $\log 2$ factor, has been introduced so that we can interpret $2^{-N} \sum_{\sigma \in \{\pm 1\}^N}$ as integration against the uniform probability measure on $\{\pm 1\}^N$.

2.1 Large deviation principles

As mentioned earlier, the Curie–Weiss model is sufficiently simple for its limiting free energy to be computed by a large deviation argument. This is made clear by noting that, the free energy is just a function of the magnetization $S_N(\sigma) = \sum_{i=1}^N \frac{\sigma_i}{N}$, which is in turn a sum of independent random variables under the uniform measure:

$$F_N(t,h) := \frac{1}{N} \log \left(2^{-N} \sum_{\sigma \in \{-1,+1\}^N} \exp\left(N(t(S_N(\sigma))^2 + hS_N(\sigma)) \right) \right). \tag{3}$$

As a result, it is not difficult to believe that a concentration phenomenon arise in the large N limit.
We do not report the proof here, which can be found in Corollary 2.20 of [2], but we report the result, as the explicit form of the limiting free energy will be essential in what follows.

Theorem 1 (Limiting free energy of the Curie–Weiss model)**.** *The limiting free energy for the Curie–Weiss model is given by*

$$f(t,h) := \lim_{N\to\infty} F_N(t,h) = \sup_{m\in[-1,1]} \left(tm^2 + hm - \psi^*(m) \right), \tag{4}$$

where the function $\psi^* : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *is defined by*

$$\psi^*(m) := \frac{1+m}{2} \log(1+m) + \frac{1-m}{2} \log(1-m), \quad \text{for } m \in [-1,1],$$

with the convention that $0 \log 0 := 0$*, and* $\psi^*(m) = +\infty$ *for* $m \notin [-1,1]$*.*

Provided that differentiation under the supremum in (4) is justified, the derivatives of f can be expressed in terms of any maximizer $m^*(t,h)$, leading to the following result:

Theorem 2. *At every point of differentiability* (t,h) *and for any maximizer* $m^*(t,h)$ *of* (4) *the following identities hold:*

$$\partial_t f(t,h) = m_0^2(t,h), \qquad \partial_h f(t,h) = m_0(t,h),$$

$$\begin{cases} \partial_t f(t,h) - (\partial_h f(t,h))^2 = 0, \\ f(0,h) = \psi(h). \end{cases} \tag{5}$$

This theorem naturally raises the question of whether the limiting free energy $f(t,h)$ can be characterized as the unique solution to equation (5). The answer, however, is not as straightforward as it might appear. Although $f(t,h)$ satisfies the equation in a formal sense, it is not differentiable for all values of t, and therefore cannot be a classical solution. Furthermore, even imposing Lipschitz continuity is not sufficient to guarantee uniqueness, as the equation may admit infinitely many Lipschitz solutions.
With this problem in mind, we now turn our attention back to the finite-volume free energy F_N, with the aim of understanding in what sense the limiting free energy should satisfy the problem.

2.2 Approximate HJB equation

Let us recall the definition of the Gibbs expectation:

$$\langle f(\sigma)\rangle := \frac{\sum_{\sigma\in\{\pm1\}^N} f(\sigma)\exp\left(H_N(t,h,\sigma)\right)}{Z_N},$$

and observe the following identities for the derivatives of the finite-volume free energy:

$$\partial_t F_N(t,h) = \left\langle \left(\frac{1}{N}\sum_{i=1}^N \sigma_i\right)^2 \right\rangle, \qquad \partial_h F_N(t,h) = \left\langle \frac{1}{N}\sum_{i=1}^N \sigma_i \right\rangle.$$

It follows that

$$\partial_t F_N(t,h) - \left(\partial_h F_N(t,h)\right)^2 = \mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^N \sigma_i\right),$$

which highlights that the variance of the magnetization controls the deviation of the finite-N free energy from satisfying the HJB equation exactly.
Moreover, a direct computation yields:

$$\partial_h^2 F_N(t,h) = \left\langle \left(\frac{1}{N}\sum_{i=1}^N \sigma_i\right)^2 \right\rangle - \frac{1}{N}\sum_{i=1}^N \langle\sigma_i\rangle^2 = N\,\mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^N \sigma_i\right),$$

so that we obtain the identity

$$\partial_t F_N(t,h) - \left(\partial_h F_N(t,h)\right)^2 = \frac{1}{N}\partial_h^2 F_N(t,h).$$

At this point we have shown that the limiting free energy satisfies the HJB equation at every point of differentiability and that the finite-volume free energy satisfies an approximate HJB equation with an error term that we expect to vanish in the $N\to\infty$ limit. These two facts lead us to conjecture that we could define an appropriate concept of weak solution such that the equation is well-posed, i.e. the limiting free energy could be defined as the unique solution of the HJB equation in a weak sense.

2.3 Viscosity solutions

In this paragraph we introduce viscosity solutions as a notion of well-suited solution for our problem, based on what was said at the end of the last paragraph.
To achieve this, we define what is a viscosity solution and show that any subsequential limit of the sequence $(F_N)_N$ is a viscosity solution of the Hamilton–Jacobi–Bellman equation. Then, we appeal to a comparison principle to conclude that the viscosity solution is unique, thereby identifying the limiting free energy as the unique solution of the HJB PDE.
We start by giving the definition of viscosity solutions as we use them in this work and then explain briefly the intuition behind them and why they are called in this way:

Definition 3 (Viscosity solution)**.** *Let $u:\mathbb{R}_{\geq 0}\times\mathbb{R}^d\to\mathbb{R}$ be a continuous function. We say that u is a* viscosity solution *of the Hamilton–Jacobi–Bellman equation* (1) *if, for every $(t^*,x^*)\in\mathbb{R}_{>0}\times\mathbb{R}^d$ and every test function $\phi\in C^\infty(\mathbb{R}_{>0}\times\mathbb{R}^d;\mathbb{R})$, the following two conditions hold:*

1. *(Subsolution) If $u - \phi$ attains a local maximum at (t^*, x^*), then*

$$\big(\partial_t \phi - H(\nabla \phi)\big)(t^*, x^*) \leq 0.$$

2. *(Supersolution) If $u - \phi$ attains a local minimum at (t^*, x^*), then*

$$\big(\partial_t \phi - H(\nabla \phi)\big)(t^*, x^*) \geq 0.$$

The observation that Hamilton–Jacobi–Bellman equations, such as (1), may not admit classical solutions even when the initial data are smooth, served as a foundation for defining this type of weak solutions [1]. Since solutions may develop singularities in finite time, a natural approach to overcome this issue is to add a vanishing viscosity term and consider instead the regularized equation

$$\partial_t f_\varepsilon - H(\nabla f_\varepsilon) = \varepsilon \Delta f_\varepsilon,$$

which is uniformly parabolic and admits smooth solutions for all $\varepsilon > 0$.
From a physical point of view, this is equivalent to adding a forcing term that smoothes out irregularities in the profile of f. The term $\varepsilon \Delta f$ acts as a viscous force that pushes the value of f towards local uniformity, as the Laplacian Δf measures how much f deviates from its local average. In this way, it keeps the solution from developing singularities.
Under suitable assumptions, the family $(f_\varepsilon)_{\varepsilon > 0}$ converges locally uniformly to a continuous function f as $\varepsilon \to 0$, which can be shown to be a viscosity solution in the sense of the definition given above. Thus "viscosity solution" are called like this from the limit behavior of smooth approximations governed by an artificial viscosity term.

Theorem 3 (every subsequential limit is a viscosity solution [2])**.** *Let f be a subsequential limit of the free energy for the Curie–Weiss model $(F_N)_N$ in the topology of local uniform convergence. Then the function f is a viscosity solution to* (5)*.*

Proof. As shown earlier by the identities involving the derivatives of F_N, the sequence $(F_N)_N$ is equicontinuous and uniformly bounded on every compact subset of $\mathbb{R}_{\geq 0} \times \mathbb{R}$. Thus, by the Arzelà–Ascoli theorem, we may extract a subsequence (which we continue to denote by $(F_N)_N$ for simplicity) that converges locally uniformly to a continuous function f.
Now fix a smooth function $\phi \in C^\infty(\mathbb{R}_{>0} \times \mathbb{R}; \mathbb{R})$ such that $f - \phi$ has a strict local maximum at some point $(t^*, h^*) \in \mathbb{R}_{>0} \times \mathbb{R}$. Since $F_N \to f$ locally uniformly, there exists a sequence $(t_N, h_N)_{N \geq 1} \subset \mathbb{R}_{>0} \times \mathbb{R}$ converging to (t^*, h^*), such that for all N large enough, $F_N - \phi$ attains a local maximum at (t_N, h_N). In particular, since $t^* > 0$, we have $t_N > 0$ for all sufficiently large N, so the first-order derivatives of $F_N - \phi$ vanish at (t_N, h_N), and we also have

$$\partial_h^2 (F_N - \phi)(t_N, h_N) \leq 0.$$

Using the identity

$$\partial_t F_N - (\partial_h F_N)^2 = \frac{1}{N} \partial_h^2 F_N,$$

we deduce that

$$\big(\partial_t \phi - (\partial_h \phi)^2\big)(t_N, h_N) = \frac{1}{N} \partial_h^2 F_N(t_N, h_N) \leq \frac{1}{N} \partial_h^2 \phi(t_N, h_N).$$

Since ϕ is smooth its second derivative is bounded and we may let $N \to \infty$ to conclude

$$\left(\partial_t \phi - (\partial_h \phi)^2\right)(t^*, h^*) \leq 0$$

so that f is a viscosity subsolution of the HJB equation. The argument for the supersolution property is analogous and yields that f is indeed a viscosity solution.
Moreover, the initial condition is inherited from the fact that

$$F_N(0, h) = F_1(0, h) = \log\cosh(h) \quad \forall N \in \mathbb{N}$$

thus $f(0, h) = \log\cosh(h)$ as well. □

At this stage, it could still happen that different subsequences of $(F_N)_N$ converge to different limits f, g, both of which are viscosity solutions of the same equation.
Therefore, to identify the limit uniquely, we need a result ensuring that two viscosity solutions with the same initial condition must coincide. This is called a comparison principle and since the proof is very technical we just refer to [2] for full details and just report the statement.

Theorem 4. *Let $\mathcal{L}$ be the space of uniformly Lipschitz function in space*

$$\mathcal{L} = \left\{ u : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}, u \text{ is continuous with } \sup_{t \geq 0} \|u(t, \cdot)\|_{\text{Lip}} < +\infty \right\}.$$

If $u, v \in \mathcal{L}$ are two viscosity solution to (5)*, then $u = v$*

Intuitively, the idea is that the HJB equation "preserves the ordering" of initial data: if two solutions start in the same position and one is never allowed to overtake the other, then they must coincide.
This is made more clear noting that the behaviour of these solutions is controlled through test functions that touch them from above or below and that a viscosity subsolution cannot rise above a supersolution without violating the defining inequalities at a point of contact. As a result, once an ordering is established at the initial time, it is preserved throughout the evolution, ensuring that two solutions with the same initial data must coincide.
Thus, using this comparison principle, we can establish that the limiting free energy of the Curie–Weiss model can be fully characterized as the unique viscosity solution to the corresponding HJB equation. At this point we recap what we did: from theory of large deviations we were able to compute the limiting free energy of Curie–Weiss model and this hinted that heuristically it should satisfy an HJB equation. Then by leveraging the theory of viscosity solution we were able to identify this limiting free energy as the unique viscosity solution to the HJB equation, thus obtaining the result in a new independent way. What is left to show is that the solutions produced by these two different approaches yields the same result, i.e. we want to prove the consistency of the solutions.
This is achieved by exploiting the convexity of the Hamiltonian:

Theorem 5. *(Hopf formula [2]) The unique viscosity solution of* (5) *is given by the following variational representation:*

$$f(t, h) = \sup_{m \in \mathbb{R}} \inf_{y \in \mathbb{R}} \left\{ \psi(y) + m(h - y) + t m^2 \right\}, \tag{6}$$

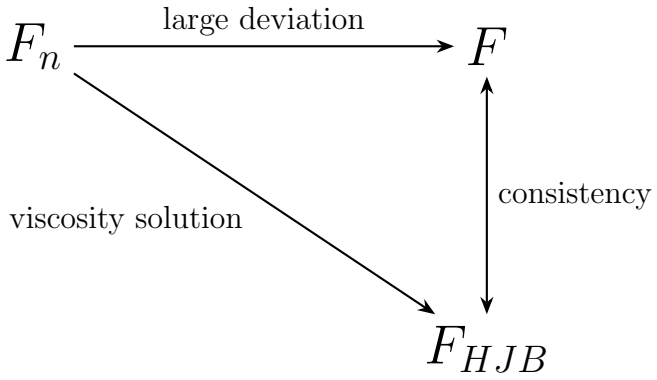
Theorem 6. *The formulations given by* (6) *and* (4) *are equivalent*

Proof. Recognizing that $\psi^*(m) = \sup_{y\in\mathbb{R}}\{\psi(y) - my\}$ we get:

$$f(t,h) = \sup_{m\in\mathbb{R}} \inf_{y\in\mathbb{R}} \left\{\psi(y) + m(h-y) + tm^2\right\} = \sup_{m\in\mathbb{R}}(mh + tm^2 - \psi^*(m))$$

and then we restrict ourselves to the set $[-1,1]$ since outside $\psi^*(m) = +\infty$ □

To recap, everything worked smoothly in the Curie–Weiss case due to the fact that the model naturally satisfies an approximate finite-dimensional Hamilton–Jacobi–Bellman equation.
For this reason, we could exploit the theory of viscosity solutions to characterize the limiting free energy without relying on any other known result.
The following diagram summarize the logical passages we pursued:



In contrast, the situation is significantly more delicate in the Sherrington–Kirkpatrick model: the appropriate enrichment that preserves enough structure to derive an analogous HJB equation, is intrinsically infinite-dimensional and this fact leads to an infinite-dimensional PDE, whose analysis is substantially more intricate.

3 Sherrington–Kirkpatrick model

The Sherrington–Kirkpatrick (SK) model can be seen as a natural generalization of the Curie–Weiss model. In the SK model, the interaction between each pair of spins (σ_i, σ_j) is modulated by a random coupling parameter $g_{ij} \sim \mathcal{N}(0,1)$, independently sampled for each pair.

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{i,j} g_{ij}\sigma_i\sigma_j. \tag{7}$$

These gaussian couplings encode the strength and sign of the interaction between individual spins, introducing disorder into the system.
While this modification may seem minor, it fundamentally changes the nature of the model: the randomness in the couplings breaks the full permutation symmetry (but it is still mean-field meaning that the law of H_N is invariant under permutation) and now the problem of finding extrema of the energy is a NP-hard problem [4]. This means that the problem of analyzing or maximizing the Hamiltonian is substantially more difficult than in the Curie–Weiss case.
At this point in analogy with (3) our quantity of interest is

$$\overline{F}_N(t) := -\frac{1}{N}\mathbb{E}\log\left(2^{-N}\sum_{\sigma\in\{\pm1\}^N}\exp\left(\sqrt{2t}H_N(\sigma) - Nt\right)\right), \tag{8}$$

where now the expected value outside the logarithm is taken over the randomness of the parameters g_{ij} and the additional $-Nt$ acts as a normalizing factor such that

$$\mathbb{E}\exp\left(\sqrt{2t}H_N(\sigma) - Nt\right) = 1,$$

which aims to simplify the further analysis. We also recall a fundamental quantity that will appear repeatedly in the analysis of the Sherrington–Kirkpatrick (SK) model.

Definition 4 (Overlap)**.** *Given two independent samples drawn from the Gibbs measure with fixed coupling parameters, i.e. samples from* $\mathbb{P}(\sigma) = \frac{\exp(\beta H_N(\sigma))}{Z}$ *(also called replicas)* $\sigma^1, \sigma^2 \in \{-1,+1\}^N$*, the* overlap *is defined as*

$$R_{1,2} \;=\; \frac{1}{N}\,\sigma^1\cdot\sigma^2 \;=\; \frac{1}{N}\sum_{i=1}^{N}\sigma_i^1\sigma_i^2.$$

The overlap is a normalized measure of the correlation of two configurations and it characterizes the covariance of the gaussian family $\big(H(\sigma)\big)_\sigma$:

$$\mathbb{E}H_N(\sigma^1)H_N(\sigma^2) = \frac{1}{N}\sum_{i,j=1}^{N}\sigma_i^1\sigma_j^1\sigma_i^2\sigma_j^2 = N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^1\sigma_i^2\right)^2 = NR_{1,2}^2. \tag{9}$$

3.1 Cavity method

Contrary to the Curie–Weiss case, and as suggested by the previous discussion, the task of computing the limiting free energy in the Sherrington–Kirkpatrick model is significantly more challenging even without invoking the HJB framework. A rigorous derivation has been obtained only recently by Talagrand in [9] through the use of a cavity argument. We recall the result, which is expressed by the celebrated Parisi formula.

Theorem 7 (limiting free energy for SK)**.** *Let $\overline{F}_N(\beta)$ be defined as:*

$$\overline{F}_N(\beta) := \frac{1}{N}\mathbb{E}\log\sum_{\sigma\in\{\pm1\}^N}\exp\left(\beta H_N(\sigma)\right) \tag{10}$$

where $H_N(\sigma)$ is the one of (7)*, then it holds the following:*

$$\lim_{N\to+\infty}\overline{F}_N(\beta) = \inf_{\zeta\in\mathcal{D}[0,1]}\left(\Phi_\zeta(0,0) - \beta^2\int_0^1 t\zeta(t)\,\mathrm{d}t + \log 2\right),$$

where $\Phi_\zeta : [0,1]\times\mathbb{R}\to\mathbb{R}$ is the solution of the parabolic equation

$$\begin{cases} -\partial_t\Phi_\zeta(t,x) = \beta^2\left(\partial_x^2\Phi_\zeta(t,x) + \zeta(t)\left(\partial_x\Phi_\zeta(t,x)\right)^2\right), & \text{on } [0,1]\times\mathbb{R}, \\ \Phi_\zeta(1,x) = \log\cosh(x), & \text{for } x\in\mathbb{R}, \end{cases}$$

solved backward in time from $t=1$ to $t=0$, and the infimum is taken over the space

$$\mathcal{D}[0,1] := \{\zeta : [0,1]\to[0,1] \mid \zeta \text{ is right-continuous, non-decreasing, and } \zeta(1)=1\}.$$

We will show later how this expression is consistent with the solution of the HJB equation in the high-temperature regime, where the functional order parameter ζ is a step function. This corresponds to a setting in which the variance of the overlap vanishes in the $N\to\infty$ as shown in [11], recovering what is known as the original Sherrington–Kirkpatrick ansatz [8]

$$\lim_{N\to\infty}\overline{F_N}(\beta) = \left(\log(2) + +\mathbb{E}_z\left[\log\cosh\left(\beta\sqrt{2q}\,z\right)\right] + \frac{\beta^2}{2}(1-q)^2\right),$$

where q is a parameter such that it solves the following autoconsistent equation:

$$q = \mathbb{E}_z\left[\tanh^2\left(\beta\sqrt{2q}\,z\right)\right]. \tag{11}$$

We remark the fact that this is not true in general but only up to some $q = q_{critical}$.
We will not prove how Mourrat, building a much more complex refinement of the free energy, proves this consistency results for all temperatures, but the aim of our discussion will be to understand what are the results missing in this framework that we had instead for the Curie–Weiss model.

3.2 Naïf enriching

The idea is to build a refined version of (8) with some additional terms depending on a new parameter in such a way that could let us write an approximate HJB equation for this new free energy, i.e. we aim to find some terms that are similar to $H_N(\sigma)$ such that we can compensate small variations of t with small variations of h. We may try adding a term analogous to the one added in Curie-Weiss with the exception that now we have to account for the random couplings. The most natural way to introduce an external magnetic field in the SK model is to add an extra spin and connect it to all existing spins in the same way as the original couplings, i.e. with independent standard gaussian random variables.
So that given $z = (z_1,\dots,z_N)\in\mathbb{R}^N$ standard gaussian vector, we have the following analogy:

SK model	**Curie–Weiss model**
$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(\sum_{j=1}^{N} J_{ij}\sigma_j \right) \sigma_i$	$H_N(\sigma) = \frac{1}{N} \sum_{i=1}^{N} \left(\sum_{j=1}^{N} \sigma_j \right) \sigma_i$
$\sum_{i=1}^{N} z_i \sigma_i$	$\sum_{i=1}^{N} \sigma_i$

and the corresponding enriched free energy is:

$$\overline{F}_N(t,h) := -\frac{1}{N}\mathbb{E}\log\left[2^{-N}\sum_{\sigma\in\{\pm1\}^N}\exp\left(\sqrt{2t}H_N(\sigma) - Nt + \sqrt{2h}z\cdot\sigma - Nh\right)\right].$$

By gaussian integration by parts we get:

$$\partial_h \overline{F}_N(t,h) = -\frac{1}{N}\mathbb{E}\left\langle \frac{1}{\sqrt{2h}} z\cdot\sigma - N \right\rangle = \mathbb{E}\langle R_{1,2}\rangle, \tag{12}$$

$$\partial_t \overline{F}_N(t,h) = -\frac{1}{N}\mathbb{E}\left\langle \frac{1}{\sqrt{2t}} H_N(\sigma) - N \right\rangle = \mathbb{E}\langle R_{1,2}^2\rangle, \tag{13}$$

$$\partial_t \overline{F}_N(t,h) - \left(\partial_h \overline{F}_N(t,h)\right)^2 = \mathrm{Var}(R_{1,2}).$$

It can be shown that for a low-temperature regime the overlap concentrates around the solution of the self-consistent equation (11):

Theorem 8. *Let $R_{1,2}$ denote the overlap in the SK model. Then for all $t < \frac{1}{4}$, we have*

$$\lim_{N\to\infty} \mathbb{E}[(R_{1,2} - q)^2] = 0.$$

Sketch of Latala argument [10]. For a fixed t, introduce an interpolated Hamiltonian:

$$\mathcal{H}_s(\sigma) = \sqrt{2st}\, H_N(\sigma) + \sqrt{2h}\, z\cdot\sigma - Nst - Nh, \qquad s\in[0,1],$$

such that at $s = 0$ the SK disorder is off, while at $s = 1$ we recover the original Hamiltonian. This trick is known as Guerra interpolation and exploits the fact that when the disorder is off, the measure factorizes over the spins. Indeed, at $s = 0$, spins are independent (conditionally on z), and the free energy becomes the log–Laplace transform of a sum of independent gaussian variables. Thus, the overlap concentration admits a uniform sub-gaussian bound when the disorder is off.
Next, compute ∂_s of the interpolated free energy via gaussian integration by parts. Replica symmetry and Hölder's inequality show that

$$\partial_s \mathbb{E}\left\langle e^{\lambda N (R_{1,2}-q)^2} \right\rangle_{s,t,h} \le 0 \quad \text{if} \quad 0 \le s < \frac{\lambda}{4t}.$$

Where $\langle\cdot\rangle_{s,t,h}$ is the expectation w.r.t. to the gibbs measure associated with the interpolated hamiltonian. This inequality implies that the exponential moment of the centered overlap is non-increasing in s. Hence, the sub-gaussian estimate at $s = 0$ transfers to $s = 1$.
Finally, the uniform exponential control implies that for $t < \frac{1}{4}$, we have $\mathrm{Var}(R_{1,2}) = O(N^{-1})$, so $R_{1,2}$ concentrates in L^2 around its mean in the thermodynamic limit, concluding the proof. □

3.3 Consistency of HJB and Parisi for small t

Now we show the consistency of the solution of (14) with the Parisi formula in the low-t regime, i.e. when the infinite-volume limit yields exactly the HJB equation:

$$\partial_t \overline{f}(t,h) - \left(\partial_h \overline{f}(t,h)\right)^2 = 0 \tag{14}$$

We want exploit the convexity of the non-linearity and use an Hopf-Lax formula, thus we start computing the initial condition:

$$\overline{F_N}(0,h) = -\frac{1}{N}\mathbb{E}\left[\log \sum_{\sigma\in\{\pm1\}^N} 2^{-N}\exp(\sqrt{2h}z\cdot\sigma - Nh)\right],$$

now exploiting the fact that at $t=0$ it factorizes over the spins we get

$$\sum_{\sigma\in\{\pm1\}^N}\exp(\sqrt{2h}z\cdot\sigma) = \sum_{\sigma\in\{\pm1\}^N}\prod_{i=1}^{N}\exp(\sqrt{2h}z_i\cdot\sigma_i) = \prod_{i=1}^{N}2\cosh(\sqrt{2h}z_i\cdot\sigma_i),$$

and by log properties and linearity of expectation

$$\psi(h) = f(0,h) = \lim_{N\to\infty}\overline{F_N}(0,h) = h - \mathbb{E}\left[\log\cosh(\sqrt{2h}z)\right].$$

Now by an hopf-lax formula we can write:

$$f(t,0) = \sup_h\left\{\psi(h) - \frac{h^2}{4t}\right\} = \sup_h\left\{h - \mathbb{E}\left[\log\cosh(\sqrt{2h}z)\right] - \frac{h^2}{4t}\right\},$$

making the change of variable $h \to 2qt$

$$f(t,0) = \sup_q\left\{2qt - \mathbb{E}\left[\log\cosh(\sqrt{4qt}z)\right] - q^2t\right\},$$

and reminding the link between (8) and (10):

$$\lim_{N\to\infty}\overline{F_N}(\beta) = -f(\frac{\beta^2}{2},0) + \log(2) + \frac{\beta^2}{2},$$

we get:

$$\lim_{N\to\infty}\overline{F_N}(\beta) = \inf_q\left(\log(2) + \mathbb{E}_z\left[\log\cosh\left(\beta\sqrt{2q}\,z\right)\right] + \frac{\beta^2}{2}(1-q)^2\right).$$

Now taking the derivative with respect to q and setting it to zero yields:

$$\frac{\beta}{\sqrt{2q}}\mathbb{E}_z\left[z\cdot\tanh\left(\beta\sqrt{2q}\,z\right)\right] - \beta^2(1-q) = 0.$$

Using gaussian integration by parts and dividing by β^2 yields:

$$q = \mathbb{E}_z\left[\tanh^2\left(\beta\sqrt{2q}\,z\right)\right].$$

Thus we recovered the original Sherrington–Kirkpatrick limiting free energy formula. Now we shall show that the Parisi formula yields to the same expression when the infimum is taken w.r.t. to the set

$$D := \left\{ \zeta : [0,1] \to [0,1] \;\middle|\; \exists\, q \in [0,1] \text{ such that } \zeta(s) = \begin{cases} 0 & \text{if } s < q, \\ 1 & \text{if } s \geq q \end{cases} \right\},$$

which as hinted before is proved to be the set where to look for when the overlap is concentrated. Now to compute the initial condition in this framework we can exploit the structure of $\zeta(t) = \chi_{[q,1]}(t)$

$$\begin{cases} \dfrac{\partial f}{\partial t}(t,x) = -\beta^2 \left(\dfrac{\partial^2 f}{\partial x^2}(t,x) + \zeta(t) \left(\dfrac{\partial f}{\partial x}(t,x) \right)^2 \right), & (t,x) \in [0,1] \times \mathbb{R}, \\ f(1,x) = \log\cosh(x), & x \in \mathbb{R}. \end{cases} \tag{15}$$

so that (15) splits in two regimes, if $t \in [0,q]$, $\zeta(t) = 0$, $f(t,x)$ solves a standard backward heat equation with diffusion coefficient β, while for $t \in [q,1]$, $\zeta(t) = 1$, $f(t,x)$ solves a non-linear parabolic PDE with the same final condition of before.
The idea is the following: we solve this latter PDE to find the initial condition $f(q,x)$ and then we use this as the final condition of the backward heat equation to get to $f(0,0)$. The first PDE is just (15) restricted on the interval $[q,1]$. We make the change of variable $f(t,x) = \log u(t,x)$ so that $u(t,x)$ solves a backward heat equation with final condition $u(1,x) = \cosh(x)$. By Feynman-Kac we find that the solution is:

$$u(t,x) = \mathbb{E}_z^x \left[\cosh\left(x + \beta\sqrt{2(1-t)}\, z \right) \right] \quad \text{with } z \sim \mathcal{N}(0,1)$$

so that

$$f(q,x) = \log(\mathbb{E}_z^x \left[\cosh\left(x + \beta\sqrt{2(1-q)}\, z \right) \right]).$$

Now we solve the following:

$$\begin{cases} \dfrac{\partial f}{\partial t}(t,x) = -\beta^2 \dfrac{\partial^2 f}{\partial x^2}(t,x), & (t,x) \in [0,q] \times \mathbb{R}, \\ f(q,x) = \log(\mathbb{E}_z^x \left[\cosh\left(x + \beta\sqrt{2(1-q)}\, z \right) \right]), & x \in \mathbb{R}. \end{cases}$$

Proceding as before, using Feynman-Kac we get the solution to this PDE, which is:

$$f(t,x) = \mathbb{E}_{z'} \left[\log\left(\mathbb{E}_z^x \left[\cosh\left(x + \beta\sqrt{2(q-t)}\, z' + \beta\sqrt{2(1-q)}\, z \right) \right] \right) \right],$$

$$f(0,0) = \mathbb{E}_{z'} \left[\log\left(\mathbb{E}_z \left[\cosh\left(\beta\sqrt{2q}\, z' + \beta\sqrt{2(1-q)}\, z \right) \right] \right) \right].$$

Now the first expectation can be computed noting that fixed z' we are computing the quantity:

$$\mathbb{E}\left[\cosh(Z)\right] = e^{\sigma^2/2} \cosh(\mu), \quad \text{where } Z \sim \mathcal{N}(\mu, \sigma^2),\ \mu = \beta\sqrt{2q}\, z',\ \sigma^2 = 2\beta^2(1-q),$$

so that we get

$$f(0,0) = \mathbb{E}_z \left[\log\left(e^{\beta^2(1-q)} \cosh\left(\beta\sqrt{2q}\, z \right) \right) \right] = \beta^2(1-q) + \mathbb{E}_z \left[\log\cosh\left(\beta\sqrt{2q}\, z \right) \right].$$

The only thing we still have to compute is the following term:

$$\beta^2 \int_0^1 \zeta(t) t \, dt = \beta^2 \int_q^1 t \, dt = \frac{\beta^2}{2}(1-q^2),$$

and finally:

$$\lim_{N\to\infty} \overline{F_N}(\beta) = \inf_q \left(\log(2) + \beta^2(1-q) + \mathbb{E}_z \left[\log\cosh\left(\beta\sqrt{2q}\, z\right)\right] - \frac{\beta^2}{2}(1-q^2) \right),$$

$$\lim_{N\to\infty} \overline{F_N}(\beta) = \inf_q \left(\log(2) + \mathbb{E}_z \left[\log\cosh\left(\beta\sqrt{2q}\, z\right)\right] + \frac{\beta^2}{2}(1-q)^2 \right).$$

Thus yielding the original solution. To recap, we have shown that this naïf enriching works when the system is in a low-t phase and the variance of the overlap vanishes. In the next paragraph we tackle the full problem, which needs a more refined approach, as it has to deal with the fact that the variance of the overlap does not goes to zero in the limit anymore.

3.4 Consistency of HJB and Parisi for all t

In this final paragraph we describe on a high-level how one should enrich properly the free energy of SK to get the consistency with the Parisi formula for all values of t. This can be made by generating a gaussian process on a random environment which encodes a hierarchical structure that mimics better the behaviour of the gibbs measure at low temperature.
Given $K \in \mathbb{N}$ we store two sequences of parameters:

$$0 \;=\; \zeta_0 < \zeta_1 < \cdots < \zeta_K < \zeta_{K+1} = 1, \qquad 0 \;=\; q_0 < q_1 < \cdots < q_K < q_{K+1} = +\infty.$$

and define the simple function:

$$q \;=\; \sum_{k=0}^{K} q_{k+1}\, \mathbf{1}_{(\zeta_k,\, \zeta_{k+1}]}.$$

We now define a gaussian process $Z_q(\alpha)$, where $\alpha = (\alpha_1, \ldots, \alpha_K) \in \mathbb{N}^K$ is a multi-index representing a point in a K-level random environment called Poisson-Dirichlet cascade, see [2]. Each component α_k encodes a choice made at level k of a nested structure, and the process accumulates independent gaussian increments across levels.

$$Z_q(\alpha) \;=\; \sum_{k=0}^{K} \left(2q_k - 2q_{k-1}\right)^{1/2} z_{\alpha|k},$$

where $z_{\alpha|k} \sim \mathcal{N}(0,1)$ are independent standard gaussian random variables and $\alpha|k := (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k, k \le K$ denotes the vector of the first k element of α.
This construction induces a hierarchical correlation structure: two indices α^1 and α^2 that share a longer common path will have more terms in common in the sum defining $Z_q(\alpha)$, and therefore correspond to more strongly correlated values of the process.
Now defining ν_α as the weights of an appropriate random probability measure on $\mathbb{N}^K$

independent of all other sources of randomness, we can finally define our hamiltonian and enriched free energy:

$$H_N(t,q,\sigma,\alpha) \;=\; \sqrt{2t}\,H_N(\sigma) \;-\; Nt \;+\; Z_q(\alpha)\cdot\sigma \;-\; Nq_K,$$

$$\overline{F}_N(t,q) \;=\; -\frac{1}{N}\mathbb{E}\log 2^{-N}\sum_{\sigma\in\{\pm1\}^N}\sum_{\alpha\in\mathbb{N}^K}\exp\big(H_N(t,q,\sigma,\alpha)\big)\,\nu_\alpha\,.$$

By gaussian integration by parts we get the following approximate HJB-equation

$$\partial_t\,\overline{F}_N(t,q)-\sum_{k=0}^{K}(\zeta_{k+1}-\zeta_k)\left(\frac{\partial_{q_k}\overline{F}_N(t,q)}{\zeta_{k+1}-\zeta_k}\right)^2=\mathbb{E}\Big\langle\big(R_{1,2}-\mathbb{E}\langle R_{1,2}\mid\alpha^1\wedge\alpha^2\rangle\big)^2\Big\rangle.$$

Now we exploit a synchronization theorem which was proved by Panchenko in [5] [6] and [7] which let us recover the σ overlap $R_{1,2}$ from the hierarchical proximity of the associated indices α ($\alpha^1\wedge\alpha^2=k$ indicates that the two share the first k indexes):

Theorem 9 (Synchronization theorem)**.** *In the setting described above, it holds:*

$$\limsup_{N\to\infty}\mathbb{E}\Big\langle\big(R_{1,2}-\mathbb{E}\langle R_{1,2}\mid\alpha^1\wedge\alpha^2\rangle\big)^2\Big\rangle\;\le\;\frac{12}{K}.$$

This means that the hierarchical index α captures how and where pairs of replicas begin to differ. If two indices α^1 and α^2 agree up to k, their overlap has to lie in the interval between q_k and q_{k+1}. Given this, as K increases, these intervals become narrower, and in the limit $K\to\infty$, the overlap becomes almost completely determined by the depth of agreement between α^1 and α^2.
We say that the overlap $R_{1,2}$ *synchronizes* with the hierarchical structure of the random environment and this synchronization is what allows us to derive a sharp Hamilton–Jacobi-type equation for the limiting free energy.
So, by defining a functional derivative with respect to the path q, we can write, up to an error term that goes to zero as $K\to\infty$:

$$\sum_{k=0}^{K}(\zeta_{k+1}-\zeta_k)\left(\frac{\partial_{q_k}\overline{F}_N(t,q)}{\zeta_{k+1}-\zeta_k}\right)^2\;\approx\;\int_0^1\big(\partial_q\overline{F}_N(t,q,u)\big)^2\,du$$

$$\partial_t\overline{F}_N(t,q)\;-\;\int_0^1\big(\partial_q\overline{F}_N(t,q,u)\big)^2\,du\;\approx\;\mathbb{E}\Big\langle R_{1,2}-\mathbb{E}\langle R_{1,2}\mid\alpha^1\wedge\alpha^2\rangle\Big\rangle,$$

and taking the $K\to\infty$ we get

$$\partial_t f(t,q)\;-\;\int_0^1\big(\partial_q f(t,q,u)\big)^2\,du\;=0. \tag{16}$$

We note that this is an infinite-dimensional PDE since q lives in the function space $\mathcal{Q}$:

$$\mathcal{Q}(\mathbb{R}_{\ge0}):=\{q:[0,1)\to\mathbb{R}_{\ge0}\,|\,q\text{ is right-continuous and non-decreasing}\}\,.$$

This is a crucial shift: in the Curie–Weiss model, we had a classical PDE and could apply standard viscosity solution theory. But here, since the equation is infinite-dimensional, we no longer have an established theory of viscosity solutions and we do not even know if viscosity solutions are the right tool, since it is not even clear if the limit should be a

viscosity solution.
However, Mourrat argues that whatever notion of solution one could come up with, it will admit a variational formulation, as the hamiltonian is convex. Thus, he defines the following Hopf-Lax formulation of the problem as the solution to (16)

$$f(t,q) = \sup_{q' \in \mathcal{Q}} \left(\psi(q+q') - \frac{1}{4t} \int_0^1 (q'(u))^2 \, \mathrm{d}u \right).$$

Even if we do not prove this here, Mourrat concludes the argument by showing that this last formulation is equivalent to the Parisi formula, since this last is itself a variational principle over the space of cumulative distribution functions, which can be thought of as a reparametrisation of $\mathcal{Q}$.

3.5 Random energy model

Looking back at (7) we recognize that we could just define the Sherrington–Kirkpatrick model as a family of gaussian random variables indexed by σ and with covariance structure given by (9). Moved by this consideration, it is quite natural to define a broader class of gaussian processes by generalizing the covariance structure in the following way:

$$\mathbb{E} H_N(\sigma^1) H_N(\sigma^2) = N \, \xi(R_{1,2}).$$

A particularly important and studied model for which many result proved in the SK framework still hold (most notably an analogous of the Parisi formula) is the *pure p-spin* model, which arise by taking $\xi(x) = x^p$.
We do not delve deeper in the study of p–spin models but we shift our attention to the Random Energy Model (REM) which can be seen as a degenerate case of a p–spin model when p is sent to ∞. In this case $\xi(x) = \mathbf{1}_{x=1}$, so that the covariance between two configuration σ_1, σ_2 is zero unless they are the same, thus the gaussian random variables have become independent.

Definition 5. *(Random Energy Model) Given a family $(E_\sigma)_{\sigma \in \{\pm 1\}^N}$ of independent standard gaussian random variables, $t \geq 0$ and $\sigma \in \{\pm 1\}^N$, the hamiltonian for the random energy model is:*

$$H_N^{\mathrm{REM}}(t,\sigma) := \sqrt{2tN} \, E_\sigma.$$

The REM can be seen as another toy model of the SK model, since in this setting the covariance between the Hamiltonians has been set to zero, and as a result the behavior of the associated Gibbs measure is greatly simplified. On the other hand, it still encodes something more than the Curie–Weiss model, since the energy levels are random, and the system can therefore be regarded as a disordered system, albeit a trivial one.
The covariance simplification led us to hope that the enrichment required to describe the free energy via a Hamilton–Jacobi approach could significantly simplify the random environment needed to enrich the free energy.
However, this intuition turned out to be misleading and it seems that the only way to build an HJB equation for the enriched free energy of the REM is to use the same (infinite-dimensional) enrichment of SK. But doing this, we are immediately faced with a fundamental problem: in the REM, the covariance structure corresponds to the choice $\xi(x) = \mathbf{1}_{x=1}$ and if we plug this discontinuous function into the HJB equation

$$\partial_t f - \xi(\partial_h f) = 0,$$

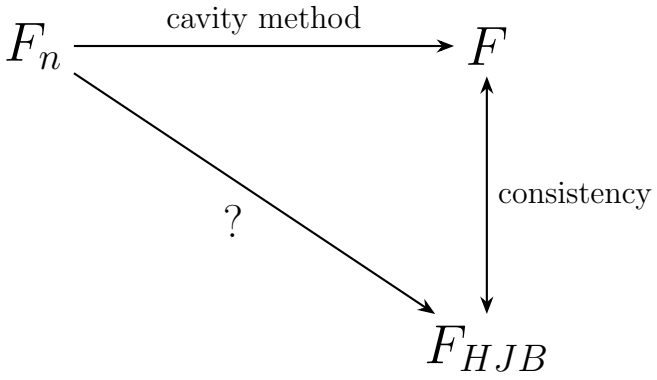
we obtain the following:

$$\partial_t f - \mathbf{1}_{\partial_h f = 1} = 0,$$

which is not well-posed in any standard sense because of the discontinuity of ξ.
This is interesting because it suggests that, even in a simplified setting, the complexity of the enrichment of SK cannot be reduced.

4 Conclusions

We have seen how, in the case of the Curie–Weiss model, the Hamilton–Jacobi–Bellman (HJB) framework proved to be both natural and effective, as the introduction of an external magnetic field enriched the model in a straightforward way and the limiting free energy turned out to be characterized uniquely via viscosity solutions of the HJB equation.
In this setting, Mourrat's approach provides a fully generative derivation of the free energy limit, avoiding any logical dependence on classical tools such as large deviation principles.
On the other hand, in the Sherrington–Kirkpatrick model, after a significantly more intricate enrichment of the free energy, Mourrat has been able to show that the Parisi formula is consistent with an *a priori* definition of solution via a variational representation. Indeed, the HJB equation for SK is infinite-dimensional, and we do not yet know how to properly define solutions in this context or whether viscosity solutions can be extended to this setting or even if such an extension would be appropriate.
This marks a substantial difference in the nature of the result: despite the conceptual appeal of this new generative approach to deriving limiting free energies, we still lack a fully rigorous and logically independent characterization of the result in the case of the SK model. At this point it is not even clear if the derivation could not rely on the prior knowledge of the Parisi formula.



In conclusion, the HJB approach offers a new appealing and potentially unifying framework to compute the limiting free energy of mean–field disordered systems. Although its current formulation is not yet fully developed in settings that give rise to infinite-dimensional Hamilton–Jacobi–Bellman equations, such as the Sherrington–Kirkpatrick model, the main obstacle seems to be of technical nature, i.e. identifying the appropriate notion of weak solution and establishing the corresponding existence and uniqueness results.

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