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Modeling and Forecasting Volatility of Financial Returns: A GARCH Analysis of Starbucks and Walmart

Metodi Statistici per la Finanza

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1 Introduction

Volatility modeling and forecasting are central topics in financial econometrics, given their importance for risk management, asset pricing, and portfolio allocation. Since Engle's (1982) ARCH model and Bollerslev's (1986) GARCH extension, conditional heteroskedasticity models have become essential for capturing volatility clustering in financial returns. Subsequent variants, such as the EGARCH (Nelson, 1991), GJR-GARCH (Glosten, Jagannathan, and Runkle, 1993), and APARCH (Ding, Granger, and Engle, 1993), were developed to address stylized features of financial time series, including leverage effects and asymmetric volatility responses. This paper provides both a theoretical overview and an empirical evaluation of ARCH-type models. As a case study, we analyze daily stock returns of Starbucks Corporation and Walmart Inc., two major consumer-oriented firms operating in distinct market segments. The objective is to compare the ability of ARCH, GARCH, EGARCH, GJR-GARCH, and APARCH models to capture and forecast volatility dynamics, offering insights into their practical relevance for equity market analysis.

2 Returns and Volatility

The returns of a financial instrument represent the most relevant data for financial time series analysis. Returns, in fact, are dimensionless and present statistical properties much more interesting than prices. They are the instrument that allow us to measure both the profitability and the risk of the considered financial instrument.
The most used formulation of such measure is the logarithmic returns

$$r_t = \log(P_t) - \log(P_{t-1})$$

where $P_t \in \mathbb{R}^+$ represents the price at time t of the considered financial instrument.
The volatility of the returns time series is a measure of their fluctuations and, as such, can be considered as a measure of the risk the agent faces when investing on such instruments.
In general, the volatility cannot be observed but it can be defined as the variance of the returns conditioned to the available information up to time t

$$V(r_t|I_{t-1})$$

Financial instruments' returns are usually uncorrelated, but rarely independent. Volatility tends to be persistent: we can observe periods of time of high volatility and periods of time of low volatility.
In financial markets, the dependence between returns manifests primarily in their second moments, which, conditional on the available information, vary over time. So, the hypothesis of constant variance $V(r_t|I_{t-1}) = \sigma^2$ is not coherent with the characteristic of a financial time series.
A more appropriate formulation is $V(r_t|I_{t-1}) = \sigma_t^2$.

2.1 Volatility Clustering

Volatility clustering is associated with serial dependencies in quantities that act as proxies for variance, such as squared returns r_t^2 (a proxy for variance) or absolute returns $|r_t|$ (a proxy for volatility).
The presence of volatility clustering is associated with the so-called power correlation effect or persistence. Usually, r_t is not autocorrelated, but $|r_t|^p$ is, particularly for $p = 1, 2$.
The correlations

$$\rho(|r_t|, |r_{t-h}|) \text{ or } \rho(|r_t|^2, |r_{t-h}|^2)$$

remain high even for large lags h, which explains their association with the concept of persistence.

2.2 Leverage and Asymmetry

Past positive and negative shocks (or returns) have an asymmetric impact on the current conditional variance. This stylized fact stems from an economic interpretation of market functioning, based on the idea that news released on the market triggers different reactions depending on its content. In particular, positive news has a smaller impact than negative news of the same magnitude.
The leverage effect suggests that the impact of a positive and negative shock differs not only in intensity but also in sign: positive shocks reduce risk.

3 ARCH Model

The Autoregressive Conditional Heteroskedasticity (ARCH) model, introduced by Engle (1982) [1], is designed to capture the phenomenon of time-varying conditional variance in time series data.
Formally, considering a univariate stochastic process

$$r_t = \mu_t + \epsilon_t$$

where μ_t is the conditional mean (possiblt constant or modelled separately), and the innovation ϵ_t is assumed to be a conditionally heteroskedastic disturbance with

$$\epsilon_t|I_{t-1} \sim (0, \sigma_t^2)$$

the $ARCH(q)$ model allows the conditional variance to evolve as a function of past squared shocks

$$\sigma_t^2 = \gamma + \sum_{j=1}^{q} \alpha_j \epsilon_{t-j}^2$$

with $\gamma > 0$ and $\alpha_j \geq 0 \forall i = 1, \ldots, q$ to ensure the positivity of the variance.

4 GARCH Model

The Generalized ARCH (GARCH) model, introduced by Bollerslev (1986) [2] as a generalization of ARCH model, provides a flexible framework for modelling time-varying conditional variances in time series data. These models are especially popular in financial econometrics due to their ability to capture volatility clustering, leptokurtosis and other stylized facts of asset returns.
Formally, considering a univariate stochastic process

$$r_t = \mu_t + \epsilon_t$$

where μ_t is the conditional mean (possiblt constant or modelled separately), and the innovation ϵ_t is assumed to be a conditionally heteroskedastic disturbance with

$$\epsilon_t | I_{t-1} \sim (0, \sigma_t^2)$$

the $GARCH(p,q)$ model assumes that the conditional variance depends both on past squared innovations and on its own past values

$$\sigma_t^2 = \gamma + \sum_{j=1}^{q} \alpha_j \epsilon_{t-j}^2 + \sum_{l=1}^{p} \beta_l \sigma_{t-l}^2$$

with $\gamma > 0$, $\alpha_j \geq 0 \forall i = 1, \ldots, q$ and $\beta_l \geq 0 \forall l = 1, \ldots, p$ to ensure the positivity of the variance.
Several extensions of this model have been developed to capture additional empirical features of financial time series:

4.1 GJR-GARCH Model

The GJR-GARCH model, introduces by Glosten, Jagannathan and Runkle (1993) [3], is an extension of the standard GARCH model designed to capture the leverage effect.
The $GJR - GARCH(p,q)$ model extends the $GARCH(p,q)$ model by allowing conditional variance to react asymmetrically to past shocks

$$\sigma_t^2 = \gamma + \sum_{j=1}^{q} \alpha_j \epsilon_{t-j}^2 + \sum_{j=1}^{q} \delta_j D_{t-j}^- \epsilon_{t-j}^2 + \sum_{l=1}^{p} \beta_l \sigma_{t-l}^2$$

with $\gamma > 0$, $\alpha_j \geq 0 \forall i = 1, \ldots, q$ and $\beta_l \geq 0 \forall l = 1, \ldots, p$ to ensure the positivity of the variance, $\delta_j \in \mathbb{R} \forall j = 1, \ldots, q$ and $D_{t-j}^- = 1$ if $\epsilon_{t-j} < 0$, 0 otherwise.
So, the impact of ϵ_{t-1}^2 on the volatility σ_t^2 depends on whether ϵ_{t-1} is positive or negative.
If $\epsilon_{t-j} \geq 0$ the impact on the volatility is $\alpha_j \epsilon_{t-j}^2$, while if $\epsilon_{t-j} < 0$ the impact is given by $(\alpha_j + \delta_j)\epsilon_{t-j}^2$.
If the parameters δ_j are statistically significant, that means there is a leverage effect.

4.2 E-GARCH Model

The Exponential GARCH (EGARCH) model, introduced by Nelson (1991) [4], is an asymmetric extension of the GARCH family designed to account for the asymmetry. The key innovation of EGARCH is that the logarithm of the conditional variance is modelled, which guarantees positivity of the variance without requiring non-negativity constraints on coefficients. Formally, considering a univariate stochastic process

$$r_t = \mu_t + \epsilon_t$$

where μ_t is the conditional mean (possiblt constant or modelled separately), and the innovation ϵ_t is assumed to be a conditionally heteroskedastic disturbance with

$$\epsilon_t | I_{t-1} \sim (0, \sigma_t^2)$$

The $E - GARCH(p,q)$ model model aims at modelling the logarithm of the conditional variance, that is specified as

$$\epsilon_t = e^{\frac{1}{2}h_t} z_t$$

$$h_t = \gamma + \sum_{j=1}^{q} \alpha_j g(z_{t-j}) + \sum_{l=1}^{p} \beta_l h_{t-l}$$

where $g()$ is a function of standardized innovations z_{t-j} defined as

$$g(z_{t-j}) = \theta z_{t-j} + \delta(|z_{t-j}| - E[|z_{t-j}|])$$

4.3 A-PARCH Model

The Asymmetric Power GARCH (APARCH) model, introduced by Ding, Granger and Engle (1993) [5], is a generalization of GARCH models that incorporates asymmetry and power transformations of the volatility, allowing richer modeling beyond the squared variance.
The $APARCH(p,q)$ conditional variance equation is given by

$$\sigma_t^\delta = \gamma + \sum_{j=1}^{q} \alpha_j (|\epsilon_{t-j}| - \theta_j \epsilon_{t-j})^\delta + sum_{l=1}^{p} \beta_l \sigma_{t-l}^\delta$$

with $\gamma > 0$, $\alpha_j \geq 0 \forall i = 1, \ldots, q$ and $\beta_l \geq 0 \forall l = 1, \ldots, p$ to ensure the positivity of the variance and $\delta > 0$ the power parameter.

5 Starbucks and Walmart

5.1 Modeling

The preliminary analysis of both Starbucks and Walmart return's time series exhibits characteristics typical of financial returns time series: non-normality, volatility clustering and serial correlations in squared returns. The results of the graphical analysis displayed on **Figures 1 and 2** indicate strong evidence of conditional heteroskedasticity, justifying the use of GARCH models. A GARCH(1,1) model with normally distributed innovations was estimated using data up to March 2023. For both assets, the estimated models, displayed on **Tables 1 and 2**, showed the sum of α_1 and β_1 parameters to be less than 1, indicating stationarity of the time series and the non-significance of the parameter μ.

Table 1: GARCH(1,1) with normal innovations fitted on Starbucks' returns

Parameter	Estimate	Std. Error	t value	Pr $(>\lvert t\rvert)$
gamma	0.075257	0.050955	1.4769	0.139694
alpha1	0.133736	0.030724	4.3528	0.000013
beta1	0.782599	0.048583	16.1086	0.000000

Table 2: GARCH(1,1) with normal innovations fitted on Walmart's returns

Parameter	Estimate	Std. Error	t value	Pr $(>\lvert t\rvert)$
gamma	0.044673	0.037708	1.1847	0.236130
alpha1	0.104143	0.019421	5.3624	0.000000
beta1	0.843522	0.030703	27.4739	0.000000

Moreover, the sign-bias tests did not provide any evidence of asymmetry in the residuals distribution (**Tables 3 and 4**).

Table 3: Sign Bias Test on GARCH(1,1) with normal innovations residuals (SBUX)

Test	t-value	prob	sig
Sign Bias	0.2576	0.7968	
Negative Sign Bias	0.6123	0.5405	
Positive Sign Bias	0.0892	0.9289	
Joint Effect	0.8195	0.8448	

Table 4: Sign Bias Test on GARCH(1,1) with normal innovations residuals (WMT)

Test	t-value	prob	sig
Sign Bias	0.7074	0.4795	
Negative Sign Bias	1.1891	0.2347	
Positive Sign Bias	1.0851	0.2782	
Joint Effect	2.7723	0.4281	

However, the standardized residuals exhibited non-normality, suggesting the normal distribution is not adequate for the innovations (**Figures 3 and 4**). To improve the fit, alternative GARCH-type models were estimated, including EGARCH, GJR-GARCH and APARCH, with both normal and Student-t innovations. Model comparisons was based on information criteria (AIC, BIC, etc.) (results displayed on **Tables 7 and 8**) and graphical diagnostics, that led to the choice of the $APARCH(1,1)$ model with Student-t innovations for the Starbucks' time series and the $APARCH(1,1)$ with normal innovations for the Walmart's. For both model the estimates are reported in **Tables 5 and 6**.

Table 5: APARCH(1,1) with Student-t innovations fitted on Starbucks' returns

Parameter	Estimate	Std. Error	t value	Pr $(>\lvert t\rvert)$
gamma	0.053965	0.031877	1.6929	0.090467
alpha1	0.106747	0.026937	3.9628	0.000074
beta1	0.891359	0.029256	30.4672	0.000000
theta1	0.447616	0.184708	2.4234	0.015377

Table 6: APARCH(1,1) with normal innovations fitted on Walmart's returns

Parameter	Estimate	Std. Error	t value	Pr $(>\lvert t\rvert)$
gamma	0.004156	0.036086	0.11517	0.908311
alpha1	0.072901	0.011454	6.36494	0.000000
beta1	0.940112	0.010795	87.08805	0.000000
theta1	0.774675	0.197087	3.93062	0.000085

In both cases the parameter γ is the only not significant parameter, giving evidence of strong significance on both ARCH and GARCH components, but also on the parameters that model the leverage phenomenon. In both cases the θ_1 parameter is positive and significant, suggesting that negative innovations on average have a stronger and positive influence on the volatility with respect to the positive innovations.

5.2 Forecasting

Static one-step-ahead volatility forecasts from April 2023 to March 2024 were generated using all fitted models (**Figure 5**) and the performance of the models was assessed using the Diebold-Mariano test, which compares predictive accuracy between model pairs. From **Tables 9 and 10** we can observe that for the Starbucks' experiment, the $APARCH(1,1)$ model with Student-t innovations significantly outperformed all models with normal innovations and the $GJR-GARCH(1,1)$ with Student-t innovations. While for Walmart's, the $APARCH(1,1)$ with normal innovations showed competitive predictive accuracy, although no clear dominance was observed across all model pairs.

References

[1] Engle, R. F. (1982). Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica*

[2] Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*

[3] Glosten, L. R., Jagannathan, R., & Runkle, D. E. (1993). On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks. *The Journal of Finance*

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6 Appendix

Table 7: Information Criteria of the fitted GARCH models (SBUX)

Criteria	**GARCH-norm**	**EGARCH-norm**	**GJR-norm**	**APARCH-norm**	**GARCH-std**	**EGARCH-std**	**GJR-std**	**APARCH-std**
Akaike	4973.143	4966.522	4955.575	4948.558	4822.854	4815.803	4815.828	4809.253
Bayes	4997.664	4997.172	4986.225	4985.338	4853.504	4852.583	4852.608	4852.163
Shibata	4973.104	4966.461	4955.514	4948.469	4822.792	4815.715	4815.739	4809.133
Hannan-Quinn	4982.459	4978.167	4967.220	4962.531	4834.498	4829.777	4829.801	4825.556

Table 8: Information Criteria of the fitted GARCH models (WMT)

Criteria	**GARCH-norm**	**EGARCH-norm**	**GJR-norm**	**APARCH-norm**	**GARCH-std**	**EGARCH-std**	**GJR-std**	**APARCH-std**
Akaike	4255.324	4209.156	4229.174	4206.767	3955.648	3950.022	3952.253	3949.952
Bayes	4279.844	4239.806	4259.825	4243.547	3986.298	3986.802	3989.034	3992.862
Shibata	4255.284	4209.094	4229.113	4206.678	3955.586	3949.934	3952.165	3949.832
Hannan-Quinn	4264.640	4220.801	4240.819	4220.740	3967.292	3963.996	3966.227	3966.254

The following tables are built so that each cell contains the Diebold-Mariano test statistic, calculated as the difference between the loss function of the model in the corresponding row and that of the model in the corresponding column. Therefore, a positive value indicates that the loss function of the model in the row is greater than that of the model in the column.
Since the Diebold-Mariano statistics asymptotically follows a $\mathcal{N}(0, 1)$ distribution, we consider values with an absolute magnitude greater than 1.64 to be significant.
If the value is less than -1.64 the model in the row has significantly better predictive capabilities than the model in the column. Conversely, if the value is greater than 1.64, the model in the column has significantly better predictive capabilities than the model in the row.

Table 9: Diebold Mariano test results on Starbucks' returns forecasts by GARCH models

	GARCH(1,1)-norm	**EGARC(1,1)-norm**	**GJR-GARCH(1,1)-norm**	**APARCH(1,1)-norm**	**GARCH(1,1)-std**	**EGARC(1,1)-std**	**GJR-GARCH(1,1)-std**	**APARCH(1,1)-std**
GARCH(1,1)-norm	NA	1.03250040	-0.8731069	0.7157661	1.19261809	1.5707007	0.008068564	2.216233
EGARCH(1,1)-norm	-1.032500402	NA	-1.7770057	-0.1647667	-0.06151333	1.8087997	-0.693751431	2.618747
GJR-GARCH(1,1)-norm	0.873106935	1.77700571	NA	1.3323392	1.36712752	2.4372529	1.375200991	2.55507
APARCH(1,1)-norm	-0.715766051	0.16476668	-1.3323392	NA	0.01269230	1.6299310	-0.512276595	4.354742
GARCH(1,1)-std	-1.192618091	0.06151333	-1.3671275	-0.0126923	NA	0.9191129	-0.740730614	1.513298
EGARCH(1,1)-std	-1.570700693	-1.80879971	-2.4372529	-1.6299310	-0.91911289	NA	-1.740890667	1.570203
GJR-GARCH(1,1)-std	-0.008068564	0.69375143	-1.3752010	0.5122766	0.74073061	1.7408907	NA	1.692789
APARCH(1,1)-std	-2.216233336	-2.61874738	-2.2555074	-4.3547416	-1.51329842	-1.5702026	-1.692788831	NA

Table 10: Diebold-Mariano test results on Walmart's returns forecasts by GARCH models

	GARCH(1,1)-norm	**EGARC(1,1)-norm**	**GJR-GARCH(1,1)-norm**	**APARCH(1,1)-norm**	**GARCH(1,1)-std**	**EGARC(1,1)-std**	**GJR-GARCH(1,1)-std**	**APARCH(1,1)-std**
GARCH(1,1)-norm	NA	3.1719349	0.2020811	-4.4798410	-2.1837610	-3.2917272	-0.5466890	-6.3946241
EGARCH(1,1)-norm	-3.1719349	NA	1.9188373	-2.2212515	0.5098833	-0.0194948	1.0308645	-1.4545331
GJR-GARCH(1,1)-norm	-0.2020811	-1.9188373	NA	2.0824751	-1.0312835	-1.6689511	-0.8695902	-1.9424531
APARCH(1,1)-norm	4.4798410	2.2212515	-2.0824751	NA	2.1799710	1.8583820	1.8237200	1.4781110
GARCH(1,1)-std	2.1837610	-0.5098833	1.0312835	-2.1799710	NA	-1.1952174	0.8449644	-2.9244411
EGARCH(1,1)-std	3.2917272	0.0194948	1.6689511	-1.8583824	1.1952174	NA	1.6832784	-2.0983942
GJR-GARCH(1,1)-std	0.5466890	-1.0308645	0.8695902	-1.8237196	-0.8449644	-1.6832784	NA	-1.9330280
APARCH(1,1)-std	6.3946241	1.4545331	1.9424531	-1.4781110	2.9244411	2.0983942	1.9330280	NA

Figure 1: Preliminary Analysis of Starbucks' returns

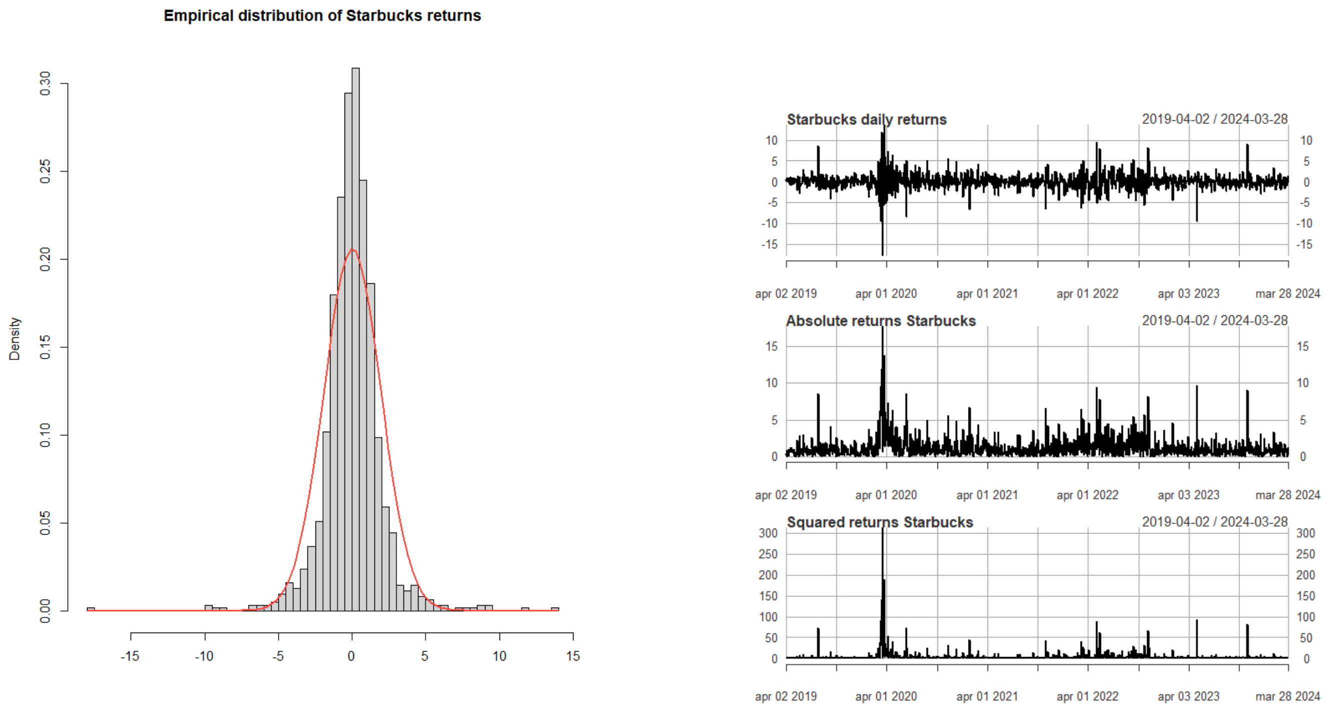

Figure 2: Preliminary Analysis of Walmart's' returns

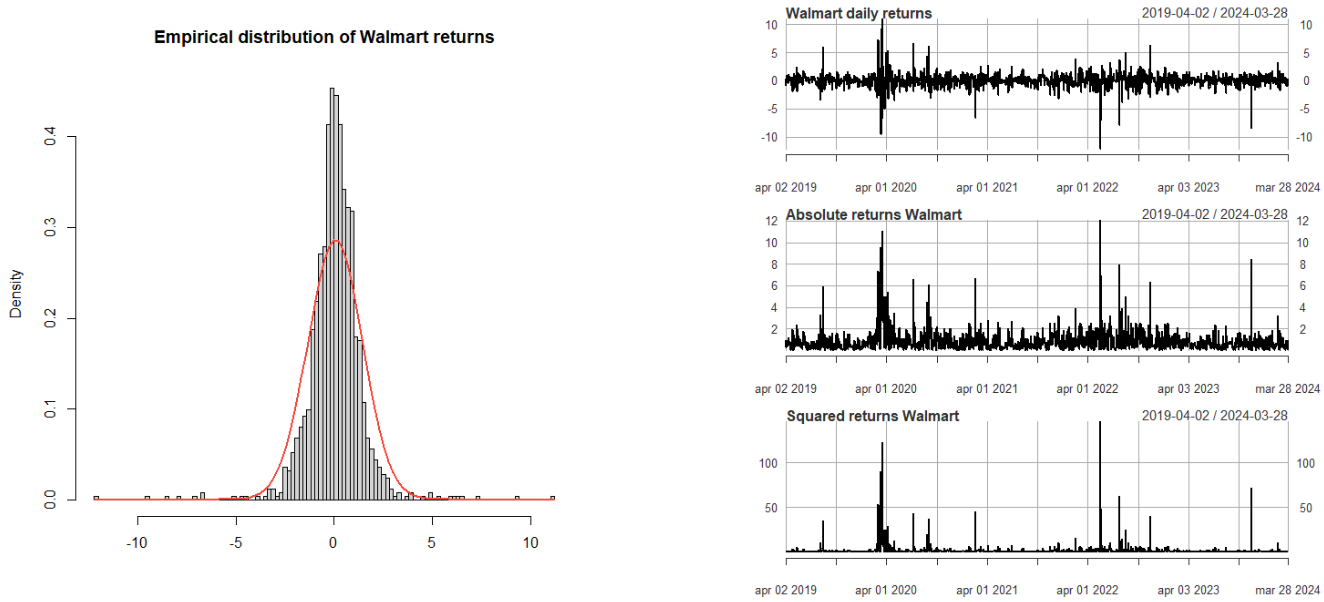

Figure 3: GARCH(1,1) with normal innovations standard residuals (SBUX)

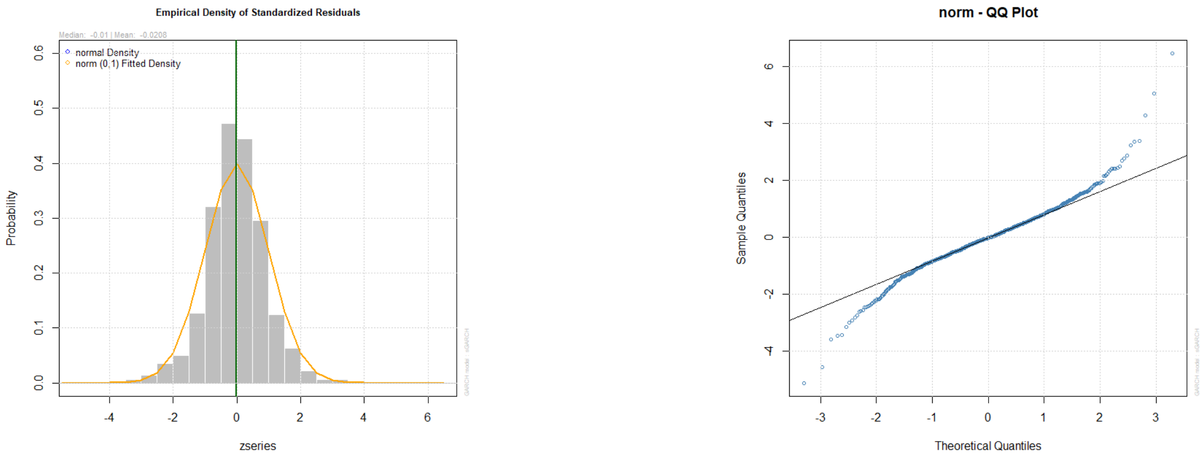

Figure 4: GARCH(1,1) with normal innovations standard residuals (WMT)

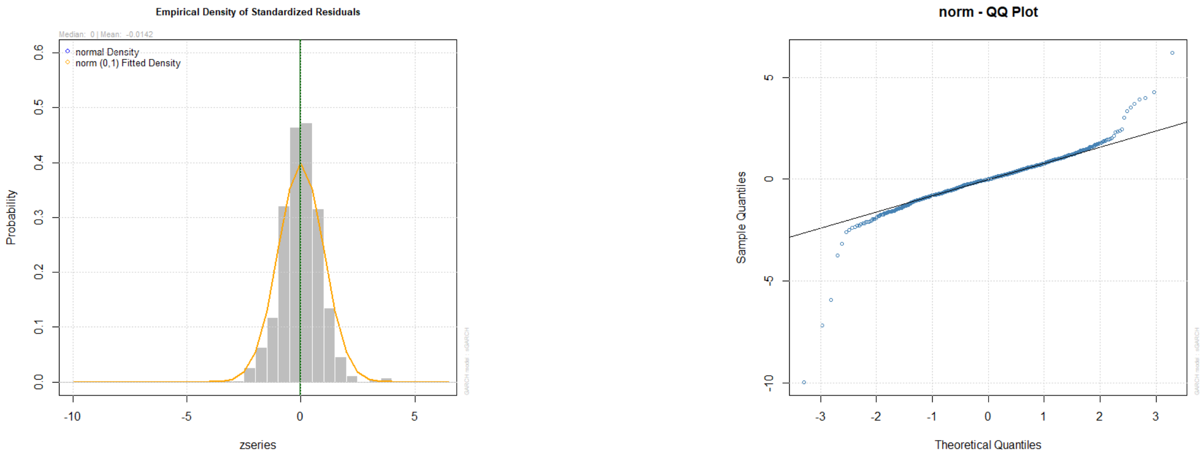

Figure 5: Volatility Forecasting