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Portfolio Optimization and Efficiency Testing: An Empirical Study on EuroStoxx 600 Stocks

Metodi Statistici per la Finanza

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1 Introduction

The construction and evaluation of investment portfolios is a central problem in modern finance, where statistical and optimization techniques provide rigorous tools to balance risk and return. This report presents both the theoretical background and empirical analysis of portfolio selection methods, with particular attention to the efficient frontier, the role of the risk-free asset, and the implications of short-selling constraints. Key concepts such as the minimum variance portfolio, the maximum trade-off portfolio, and the tangency portfolio are introduced, together with statistical procedures for assessing portfolio performance, including significance tests for the Sharpe ratio, equality tests, and exclusion tests for portfolio efficiency.

The experimental analysis is conducted on a dataset of twenty arbitrarily selected stocks from the EuroStoxx 600 index, representing four major economic sectors: Energy, Technology, Financials, and Real Estate. The study aims to illustrate how theoretical instruments translate into practical portfolio construction, and to provide a critical assessment of their performance under different assumptions and market conditions.

2 Efficient Frontier

The concept of the efficient frontier originates from the seminal work of Harry Markowitz (1952) [1] and constitutes a cornerstone of Modern Portfolio Theory. It provides a formal framework for the rational allocation of wealth across risky assets by explicitly considering the trade-off between expected return and risk.

In Markowitz's formulation, each portfolio is characterized by two fundamental quantities:

- the expected return, defined as the mean of the portfolio's return distribution;
- the risk, typically quantified by the variance (or equivalently the standard deviation) of returns.

Given a universe of risky assets, any portfolio can be represented as a point in the risk–return plane. While the set of all possible portfolios spans a convex region of this plane, not all of these portfolios are equally desirable. The efficient frontier is defined as the locus of portfolios that achieve the maximum expected return for a given level of risk, or equivalently, the minimum risk for a given expected return. Mathematically, it corresponds to the boundary of the feasible set obtained by solving the quadratic optimization problem:

$$\min_{\omega} \omega^T \Sigma \omega \text{ subject to } \mathbf{1}^T \omega = 1,\ \mu^T \omega = \mu_p$$

where ω denotes the vector of portfolio weights, Σ is the covariance matrix of asset returns, μ is the vector of expected asset returns, and μ_p is the desired portfolio return. By varying μ_p, one traces the entire efficient frontier. Portfolios that lie below the frontier are suboptimal, as there exists at least one alternative with higher return for the same risk or lower risk for the same return.

2.1 Capital Market Line

The introduction of a risk-free asset fundamentally alters the structure of the efficient set. According to the Capital Market Line (CML) theory, any combination of the risk-free asset and a single tangency portfolio of risky assets dominates the purely risky efficient frontier. In this setting, investors select portfolios along a straight line extending from the risk-free rate through the tangency portfolio. This line represents the optimal trade-off between risk and return in the presence of borrowing and lending at the risk-free rate.

2.2 Short Selling Constraints

In the classical Markowitz formulation, portfolio weights ω_i are allowed to take negative values, which corresponds to taking short positions in assets. While this assumption ensures convexity and leads to a frontier with an elegant parabolic shape in the risk–return plane, it is often unrealistic in practice due to regulatory restrictions, margin requirements, or institutional investment mandates that prohibit short sales.

When a no-short-selling constraint ($\omega_i \geq 0\ \forall i$) is imposed, the feasible set of portfolios becomes a simplex rather than the entire affine subspace defined by $\mathbf{1}^T \omega = 1$.

Despite these drawbacks, the short-selling constrained efficient frontier is considered a more realistic representation of the investment opportunities faced by many institutional investors, such as pension funds or mutual funds, which are legally or operationally restricted from shorting securities.

2.3 Relevant Portfolios

In this framework, it is often crucial to identify a few benchmark portfolios that play a central role in the analysis of risk–return trade-offs. These portfolios provide reference points along the efficient frontier and serve as foundations for both theoretical developments and practical asset allocation strategies:

- **Minimum Variance Portfolio**: The portfolio with the lowest variance among all feasible portfolios of risky assets. It represents the global leftmost point of the efficient frontier. Formally,

$$\omega_{VMG} = \arg\min_{\omega} \omega^T \Sigma \omega \text{ subject to } \mathbf{1}^T \omega = 1$$

- **Maximum Trade-Off Portfolio**: Also called the maximum Sharpe portfolio in the absence of a risk-free asset, it maximizes the ratio of excess expected return to risk, i.e., the slope of the efficient frontier. Formally,

$$\omega_{TO} = \arg\max_{\omega} \frac{\mu^T\omega}{\sqrt{\omega^T\Sigma\omega}} \text{ subject to } \mathbf{1}^T\omega = 1$$

where μ is the vector of expected asset returns.

- **Tangency Portfolio**: When a risk-free asset with return r_f is available, the tangency portfolio is the unique risky portfolio that maximizes the Sharpe ratio relative to the risk-free rate. It corresponds to the point of tangency between the efficient frontier and the Capital Market Line. Formally,

$$\omega_T = \arg\max_{\omega} \frac{\mu^T\omega - r_f}{\sqrt{\omega^T\Sigma\omega}} \text{ subject to } \mathbf{1}^T\omega = 1$$

3 Efficient Frontier Inference

In the presence of a risk-free asset, the slope of the efficient frontier corresponds to the Sharpe performance measure, denoted by Sh_{FE}. This index is defined as the ratio between the excess return of the tangency portfolio over the risk-free rate and its volatility, namely

$$Sh_{FE} = \frac{\mu_T - r_f}{\sigma_T}$$

where μ_T and σ_T denote the expected return and standard deviation of the tangency portfolio, respectively. The statistical analysis of Sh_{FE} allows for hypothesis testing and the construction of confidence intervals regarding the efficiency of the frontier. In, particular, under standard asymptotic conditions, the estimator

$$\hat{Sh}_{FE} = \frac{\hat{\mu}_T - r_f}{\hat{\sigma}_T}$$

is asymptotically distributed as

$$\hat{Sh}_{FE} \sim \mathcal{N}(Sh_{FE}, \frac{1}{T}(1 + \frac{Sh_{FE}^2}{2}))$$

This asymptotic variance provides the basis for significance testing of the efficient frontier. Specifically, one can test the null hypothesis $H_0 : Sh_{FE} = 0$, that is, the absence of a statistically significant slope, by using the standardized test statistic

$$\frac{\sqrt{T}(\hat{Sh}_{FE} - Sh_{FE})}{\sqrt{1 + \frac{Sh_{FE}^2}{2}}} \sim \mathcal{N}(0, 1)$$

Although this formulation is commonly used to evaluate the slope of the efficient frontier, the same inferential procedure can be applied to any portfolio. In practice, one can compute the Sharpe ratio of a given portfolio and test its significance using the same asymptotic distribution, thereby extending the methodology from the tangency portfolio to the entire set of feasible investment strategies.

3.1 Sharpe Ratio Equality Test

A natural extension of inference on the Sharpe ratio consists in testing the equality of performance across two portfolios. Specifically, one may consider the null hypothesis $H_0 : Sh_{p_1} = Sh_{p_2}$, where Sh_{p_1} and Sh_{p_2} denote the Sharpe ratios of two distinct portfolios. Under standard asymptotic assumptions, the joint distribution of the centered estimators converges in distribution to a bivariate normal law,

$$\sqrt{T} \begin{pmatrix} \hat{Sh}_{p_1} - Sh_{p_1} \\ \hat{Sh}_{p_2} - Sh_{p_2} \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, V)$$

where the covariance matrix V depends on the Sharpe ratios and the correlation ρ_{12} between the two portfolio returns. From this result, it follows that the difference in estimated Sharpe ratios satisfies

$$\sqrt{T}(\hat{Sh}_{p_1} - \hat{Sh}_{p_2}) \xrightarrow{d} \mathcal{N}(0, a)$$

with asymptotic variance

$$a = 2(1 - \rho_{12}) + \frac{Sh_{p_1}^2}{2} + \frac{Sh_{p_2}^2}{2} - Sh_{p_1}Sh_{p_2}\rho_{12}^2$$

Consequently, the null hypothesis of equal Sharpe ratio can be tested using the standardized statistic

$$\frac{\sqrt{T}(\hat{Sh}_{p_1} - \hat{Sh}_{p_2})}{\sqrt{a}} \sim \mathcal{N}(0, 1)$$

This methodology enables a rigorous statistical comparison of portfolio performance, beyond simple point estimates of Sharpe ratios.

3.2 Exclusion Test for Portfolio Efficiency

Another relevant inference problem concerns exclusion tests, which evaluate whether the efficient tangency portfolio derived from a restricted investment universe is statistically equivalent to the tangency portfolio obtained from the full set of assets. Suppose the initial universe contains $n = n_1 + n_2$ securities, and we exclude n_2 assets. The question is whether the efficient portfolio based on the reduced set of n_1 assets maintains the same efficiency as the original tangency portfolio in terms of its Sharpe ratio. This type of test has clear economic motivations: in practice, analysts or institutional investors may exclude certain categories of assets, such as small-cap stocks, highly risky securities, or those not meeting specific ethical or sustainability criteria (e.g., socially responsible investing).
Formally, let $\hat{Sh}_n$ and $\hat{Sh}_{n_1}$ denote the estimated Sharpe ratios of the tangency portfolios constructed from the

full and reduced universes, respectively. The null hypothesis is defined as

$$H_0 : Sh_n = Sh_{n_1}$$

The corresponding test statistic is

$$T\frac{Sh_n^2 - Sh_{n_1}^2}{1 + Sh_{n_1}^2} \sim \chi^2(n_2)$$

which is always nonnegative, since excluding assets cannot lead to higher attainable Sharpe performance.

A particular case of the exclusion test arises when assessing the efficiency of a specific portfolio p. Consider the asset set $s = [r_p, r_2, \ldots, r_n]$, where the first element corresponds to the portfolio p. The performance of portfolio p is then compared with that of the efficient tangency portfolio based on all n assets. Under the null hypothesis

$$H_0 : Sh_n = Sh_p$$

The test statistic is

$$T\frac{Sh_n^2 - Sh_p^2}{1 + Sh_p^2} \sim \chi^2(n-1)$$

which represents an exclusion test on the $n-1$ remaining assets. This formulation provides a rigorous statistical tool to determine whether a given portfolio can be considered efficient relative to the broader investment universe.

4 Eurostoxx 600: Case Study

The empirical analysis is based on a dataset of twenty arbitrarily selected stocks from the EuroStoxx 600 index, with each stock representing one of four economic sectors: Energy (SHELL, OMV, ORLEN, SUBSEA 7, REPSOL YPF), Technology (AIXTRON, ALTEN, SAGE GROUP, ASM INTERNATIONAL, SOITEC), Financials (SWEDBANK A, AEGON, ALLIANZ, ASSICURAZIONI GENERALI, AVIVA), and Real Estate (BRITISH LAND, CASTELLUM, COFINIMMO, COVIVIO, DERWENT LONDON).

To allow for a comprehensive assessment of portfolio performance under varying market conditions, the analyses were carried out over three distinct time horizons: S1 (31/12/1999 – 30/04/2024), S2 (31/12/1999 – 31/12/2016), and S3 (31/12/2016 – 31/12/2023).

4.1 Preliminary Analysis

The preliminary analysis revealed comparable average returns (**Figure 1**) across assets in the first two periods, with the same securities exhibiting negative performance. These negative returns were generally larger in absolute value during S2, while a consistent improvement in returns was observed in S3. The monthly variance (**Figure 2**) reached its maximum in the second period and its minimum in the third, indicating heightened volatility during S2 and more stable dynamics in S3. Overall, the results suggest a context of financial uncertainty and instability in S1 and S2, followed by a phase of sustained market growth in S3. The examination of correlations and covariances (**Figure 3**) confirmed, as expected, strong intra-sector dependencies; additionally, a notable degree of cross-sector correlation was detected between the Energy and Financial sectors.

4.2 Benchmark Portfolios

The computation of the efficient frontier (**Figure 4**) without the inclusion of a risk-free asset highlighted a markedly higher return-to-risk ratio for the benchmark portfolios in the third period compared to the first two. Across all time horizons, the market index was consistently positioned well below the efficient frontier, confirming its suboptimal performance relative to the constructed portfolios.

The portfolio weights (**Figure 5**) revealed distinctive asset contributions: ALLIANZ exhibited particularly strong performance, achieving a weight greater than one in the maximum trade-off portfolio during S3; conversely, SHELL, SAGE GROUP, and COFINIMMO displayed low-risk profiles, dominating the minimum variance portfolios. By contrast, AEGON and COVIVIO frequently received negative weights, in line with expectations derived from the preliminary analysis.

4.3 Long-Only Constraints

The introduction of weight constraints, prohibiting short selling, resulted in a rightward shift of the efficient frontier, displayed in **Figure 6** implying higher levels of risk for a given return compared to the unconstrained case. Despite this shift, the composition of the Minimum Variance portfolios (**Figure 7**) remained relatively stable, being consistently dominated by SHELL, SAGE GROUP, and COFINIMMO. In contrast, the Maximum Trade-Off portfolios (**Figure 7**) exhibited substantial variability across the three periods: S1 was characterized by a predominance of CASTELLUM, S2 was largely driven by Real Estate stocks, and S3 showed a strong concentration in Technology. Notably, Financial stocks were almost entirely absent from the optimal portfolios under these conditions.

4.4 Capital Market Line

The introduction of a risk-free asset, represented in this study by the most recent Euribor rate corresponding to each period, led to the construction of the Capital Market Line (CML), which is tangent to the efficient frontier at the tangency portfolio, in accordance with theoretical definitions (**Figure 8**). To examine the effect of investor preferences, three synthetic risk-aversion profiles were considered: conservative (risk aversion = 3), moderate (risk aversion = 1), and aggressive (risk aversion = 0.1). The analysis showed that, as risk aversion decreases, the optimal portfolio allocation shifts progressively from the minimum variance portfolio (which, in the presence of a risk-free asset, coincides with the risk-free investment itself) toward the maximum trade-off portfolio located on the CML (**Figure 9**).

The statistical assessment of the Sharpe ratios confirmed these findings. Specifically, the tangency portfolio significantly outperformed the GMV portfolio in all periods, with test statistics well above the critical threshold.

Table 1: Efficiency test on Tangency Portfolios

	S1	S2	S3
Test Statistics	4.582275	4.990978	4.096729

Conversely, the GMV portfolio only showed a statistically significant Sharpe ratio in S2, while in S1 its performance was not significantly different from the risk-free rate. These results highlight the superior risk-adjusted returns of the tangency portfolio relative to the GMV portfolio across different market conditions.

Table 2: Efficiency test on Minimum Variance Portfolios

	S1	S2	S3
Test Statistics	1.503899	2.413446	1.887736

4.5 Sector Performance Analysis

Given the strong intra-sector correlations observed among assets, an additional line of analysis concerns the relative performance of individual sectors across the three periods. To this end, we compared the Sharpe ratios of the Maximum Trade-Off portfolios constructed separately for each economic sector. This criterion was adopted as the Maximum Trade-Off portfolio provides the most efficient balance between risk and expected return, thereby offering a meaningful benchmark for sector-level performance.

Table 3: Sharpe Ratios divided by Economic Sector and Period

	Energy	Technology	Financials	Real Estate
S1	0.01300250	0.0043552922	0.011020112	0.010962155
S2	0.01435118	0.0003189084	0.006623746	0.037194734
S3	0.01147331	0.0403392360	0.020643656	-0.004435582

In addition, it may be relevant in practice to assess whether excluding a specific sector or group of assets significantly deteriorates overall portfolio performance. For instance, investors may wish to avoid securities from certain industries, such as defense. To address this issue, exclusion tests were carried out by removing the worst-performing sector in each of the three periods.

Table 4: Exclusion test on the Worst Performing Sector

	S1-Technology	S2-Technology	S3-Real Estate
Test Statistics	0.0139	0.0171	0.0504
p **value**	1	1	1

The results showed no statistically significant deterioration in performance following such exclusions ($p > 0.9999$), indicating that underperforming sectors exerted only a negligible impact on overall portfolio efficiency.

4.6 Inverse-Variance Weighted Portfolio

Another interesting application is to consider some basic custom mathematical rule, that set the ratio between risk and expected return, to widen the portfolio offer among which a customer could choose.

A portfolio with weights inversely proportional to variance, thus assigning lower weights to higher risk assets, was tested for efficiency. The test compares the Sharpe ratio of the constructed portfolio to that of the efficient frontier.

Table 5: Efficiency test on Inverse-Variance Weighted Portfolios

	S1	S2	S3
Test Statistics	0.06666921	0.1173026	0.1546972
p **value**	1	1	1

The Sharpe ratio test fails to reject the null hypothesis of zero Sharpe ratio for all portfolios. This indicates that the inverse-variance weighted portfolios do not deliver statistically significant risk-adjusted performance and therefore do not represent a practical construction method.

References

[1] H. Markowitz. *Portfolio Selection*. The Journal of Finance.

[2] J. Tobin. *Liquidity Preference as Behavior Towards Risk*. The Review of Economic Studies.

[3] W. F. Sharpe. *Mutual Fund Performance*. Journal of Business.

[4] J. D. Jobson and B. M. Korkie. *Performance Hypothesis Testing with the Sharpe and Treynor Measures*. Journal of Finance.

[5] O. Ledoit and M. Wolf. *Robust Performance Hypothesis Testing with the Sharpe Ratio*. Journal of Empirical Finance.

5 Appendix

Figure 1: Average Monthly Returns

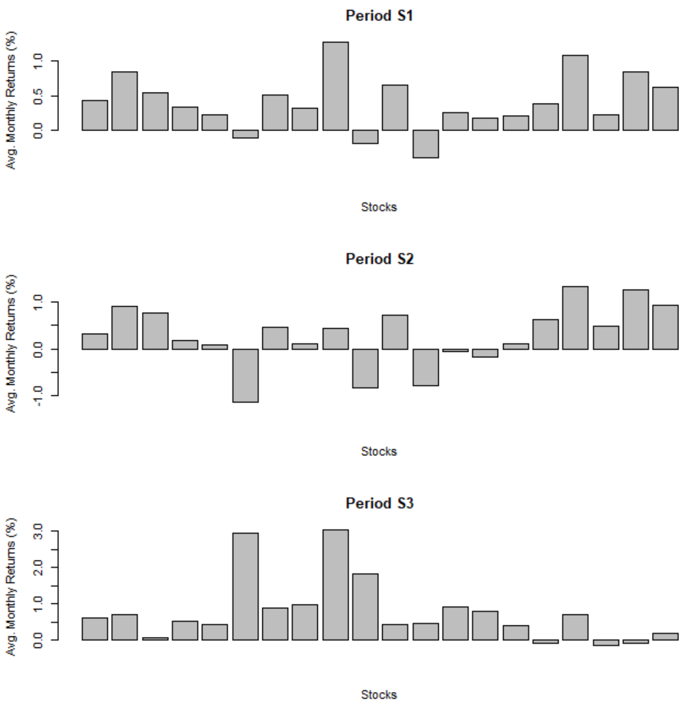

Figure 2: Monthly Variance

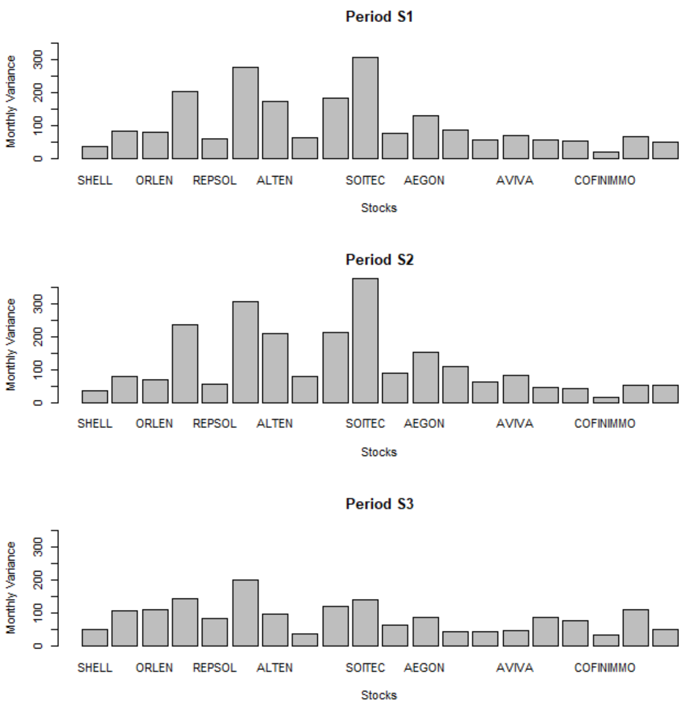

Figure 3: Covariance and Correlation Analysis

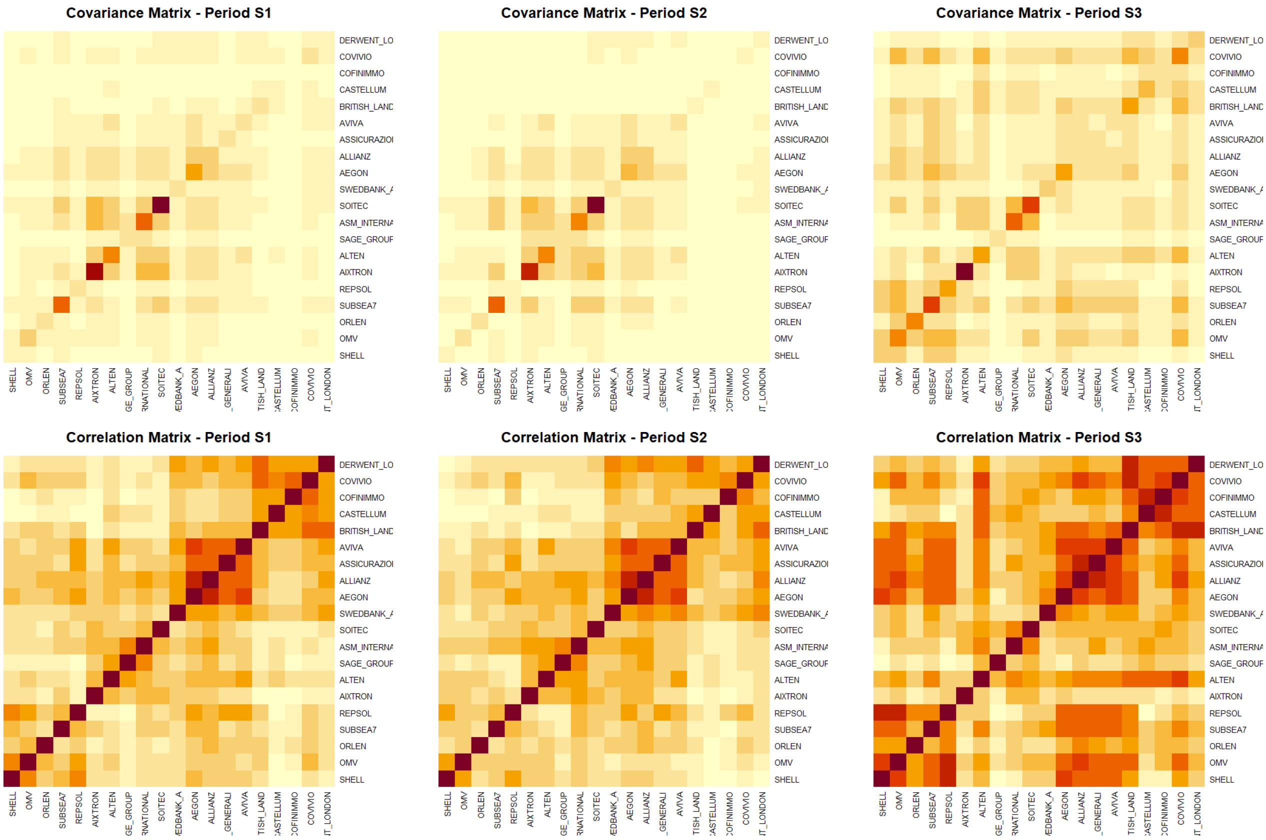

Figure 4: Efficient Frontier

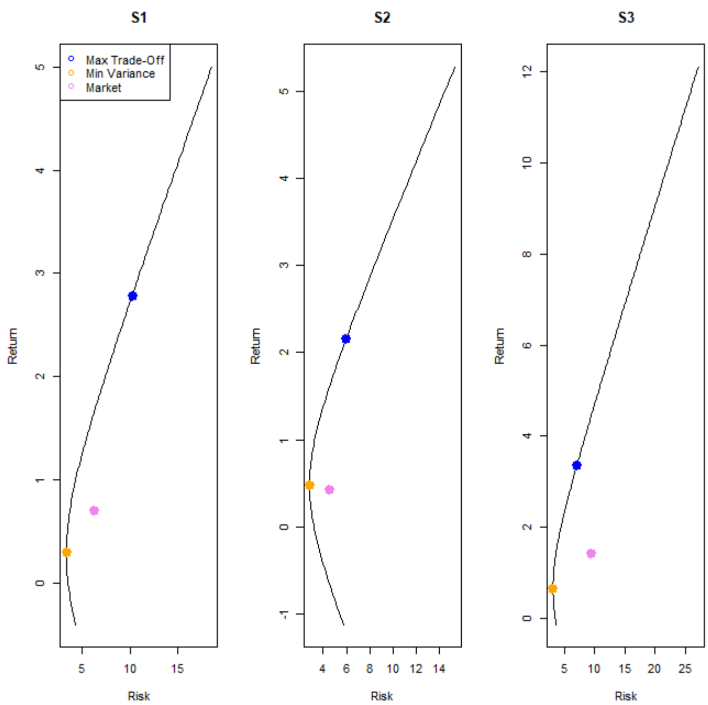

Figure 5: Maximum Trade-Off and Minimum Variance portfolios' weights across the different periods

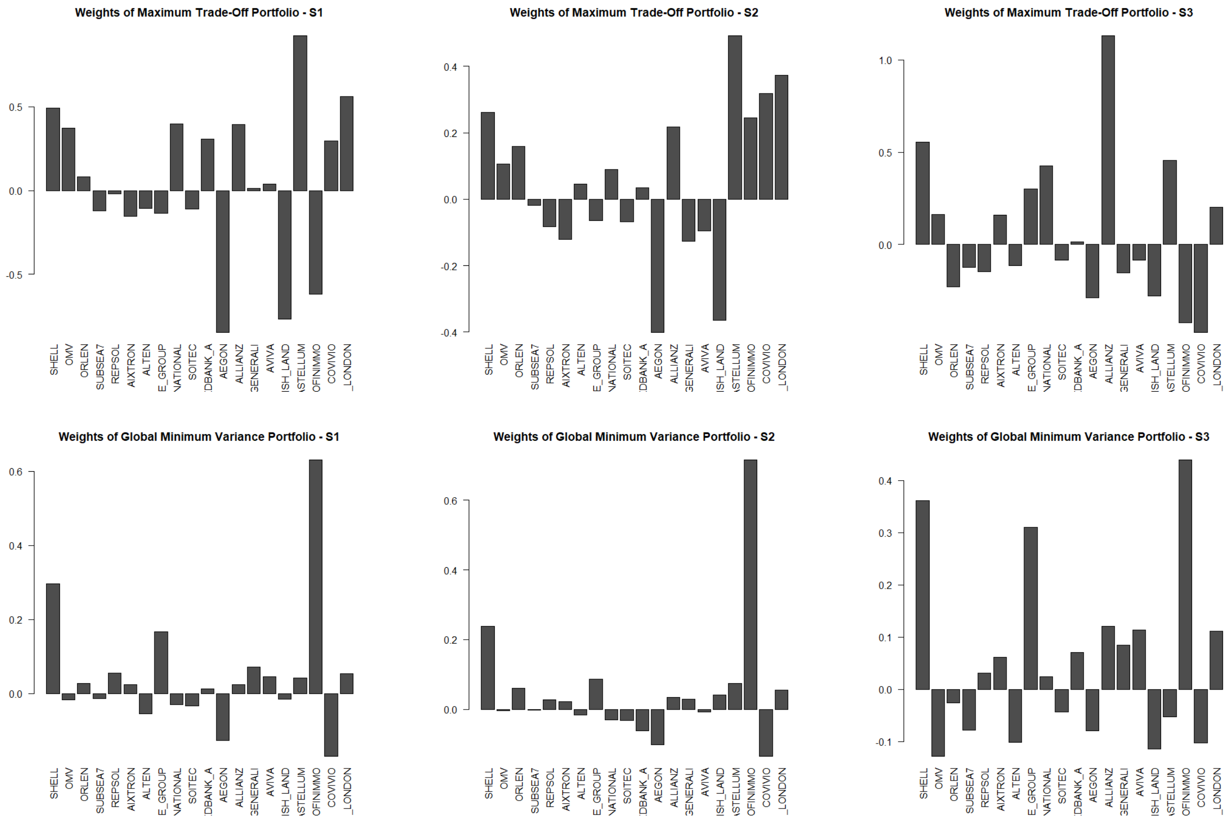

Figure 6: Long-Only Efficient Frontier

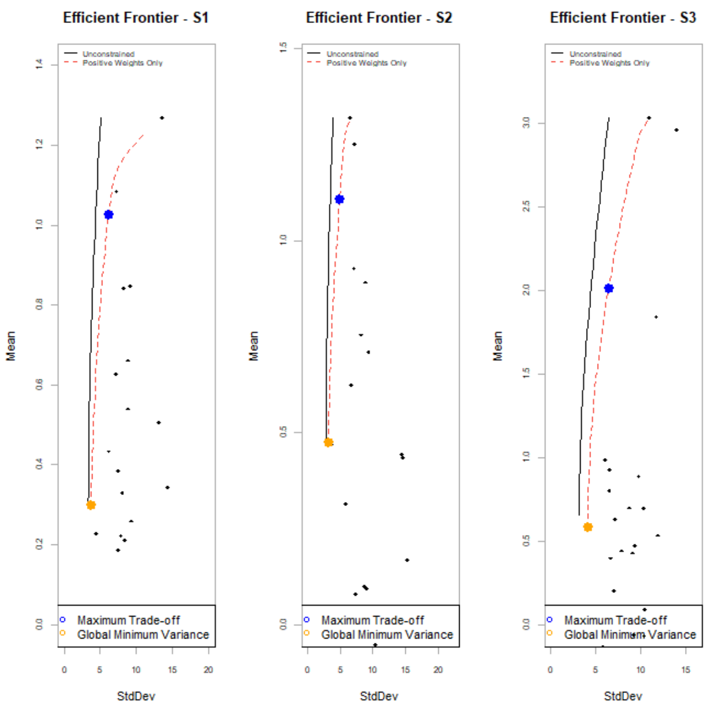

Figure 7: Long-Only Maximum Trade-Off and Minimum Variance portfolios' weights across the different periods

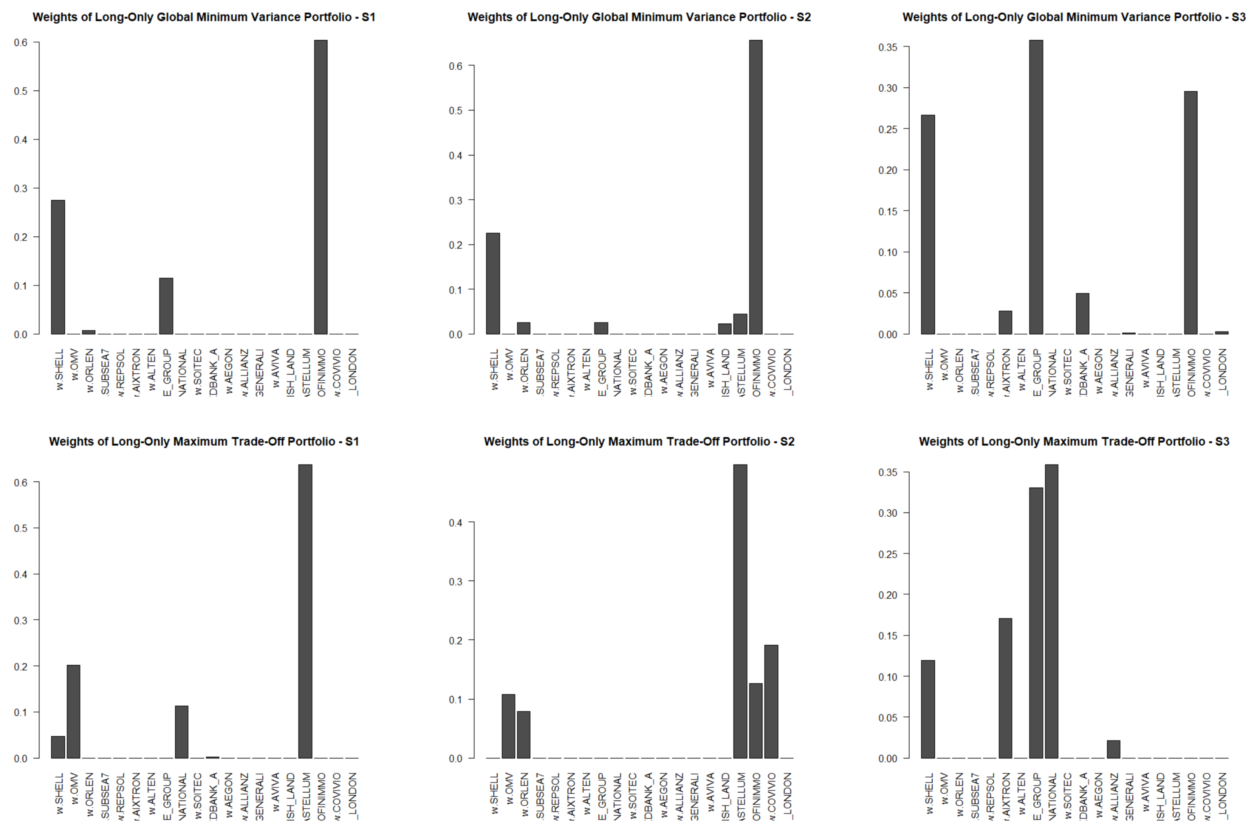

Figure 8: Capital Market Line

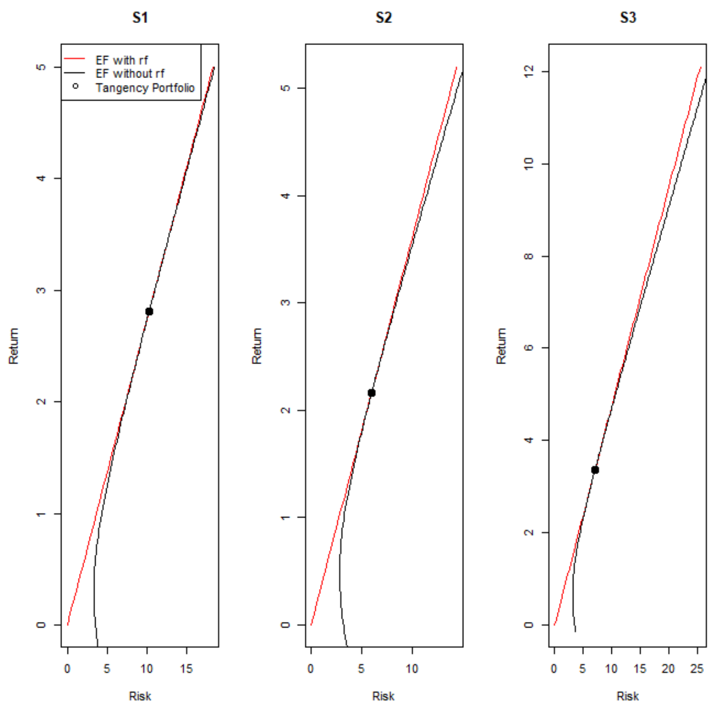

Figure 9: Optimal Portfolios with different Risk Aversions