Numerical solution of the wave equation using the Lattice Boltzmann methods

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1 Introduction: lattice Boltzmann models

Lattice Boltzmann models are a family of numerical schemes that consider the time and space evolution of quantities f_i, representing the density distribution of various possible physical quantities. These quantities move and collide on a regular lattice in 1D, 2D or 3D. The evolution is discrete in time and space. The time step is denoted δ_t and the lattice spacing δ_x.

A lattice Boltzmann (LB) equation is characterized by the following equations, for $i = 0, \dots, q$, where q is the number of lattice directions along which the population f_i can move:

$$f_i^{out}(\mathbf{r}, t) = f_i^{in}(\mathbf{r}, t) + \Omega_i \left(f^{in}(\mathbf{r}, t) \right) \tag{1}$$

and

$$f_i^{in}(\mathbf{r} + \delta_t \mathbf{v}_i, t + \delta_t) = f_i^{out}(\mathbf{r}, t) \tag{2}$$

The first equation is called the *collision* phase and the second one is called the *propagation* phase. The quantity Ω_i is called the collision term.

Figure 1 illustrates the collision-propagation process in a case where $q = 8$ directions are considered. The populations f_i^{in} are indicated as arrows entering a lattice site, and the populations f_i^{out} correspond to arrows exiting the site.

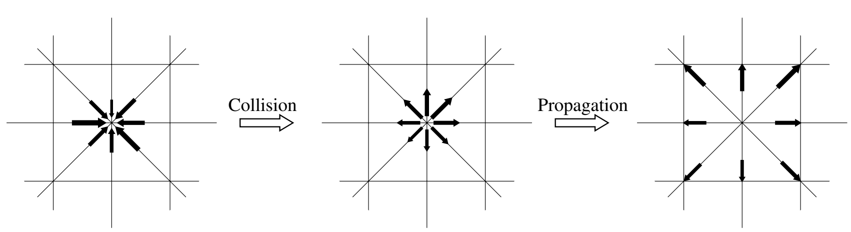

Figure 1: Illustration of the collision and propagation phases in a LB model defined on a 2D square lattice with 8 velocities.

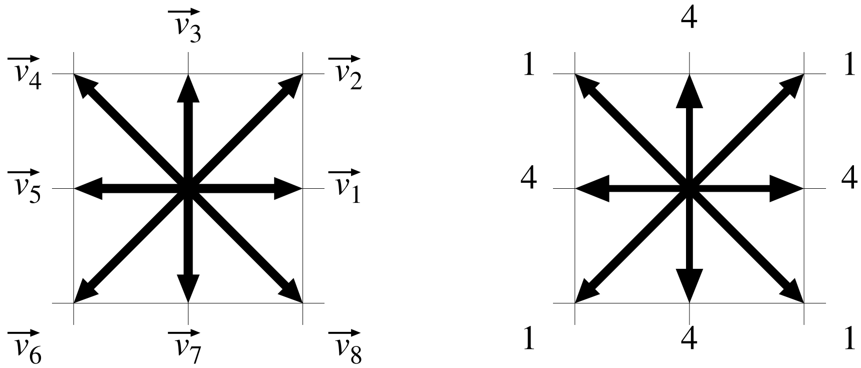

Figure 2: The D2Q8 lattice with 8 velocities, corresponding to a square lattice plus diagonals. The right panel shows the ratio of the weights associated with every direction: the diagonal directions should have a weight four times smaller than the main directions in order to ensure the isotropy, for instance when modeling hydrodynamics.

Figure 2 shows the velocity vectors $\mathbf{v}_i$, also in the case of a 2D lattice with 8 directions. Note that such a lattice is referred to as a **D2Q8 lattice**.

Note that $\mathbf{v}_i$ should be chosen so that $\mathbf{r} + \delta_t \mathbf{v}_i$ correspond to an existing lattice site, for any lattice site $\mathbf{r}$ and velocity vector $\mathbf{v}_i$.

2 Wave equation

The wave equation can be solved within the Lattice Boltzmann (LB) method. The resulting numerical scheme turns out to be the same as the so-called TLM method (Transmission Line Method).

In 2D one considers a $D2Q5$ lattice whose velocities are

$$\mathbf{v}_0 = (0,0) \quad \mathbf{v}_1 = v(1,0) \quad \mathbf{v}_2 = v(0,1) \quad \mathbf{v}_3 = v(-1,0) \quad \mathbf{v}_4 = v(0,-1)$$

where $v = \delta_x/dt$, δ_x being the lattice spacing and δ_t the time step. The 4+1 directions are illustrated in Fig. 3.

In 3D one considers a D3Q7 lattice, with $\mathbf{v}_0 = (0,0,0)$ and the other 6 velocity vectors corresponding to the 6 main lattice directions in 3D.

The field $\rho(\mathbf{r}, t)$ that obeys the wave equation is related to the f_i as

$$\rho(\mathbf{r}, t) = \sum_{i=0}^{q} f_i^{in}(\mathbf{r}, t) \equiv \sum_{i=0}^{q} f_i(\mathbf{r}, t) \tag{3}$$

where we defined $f_i \equiv f_i^{in}$ to simplify the notation. We also defined the current

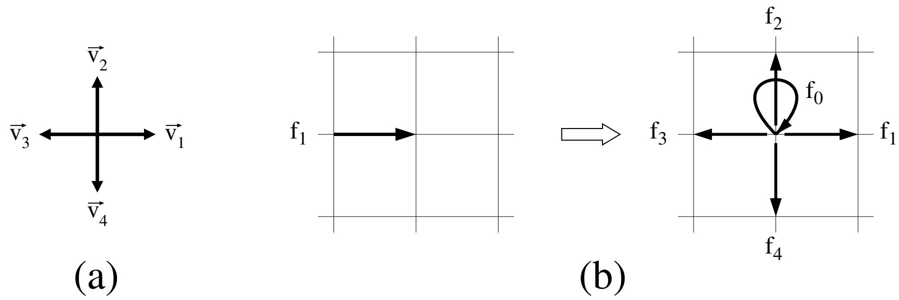

Figure 3: Sketch of the 2D LB wave model. The population f_i move along the four main lattice direction, and a population f_0 is at rest. When a f_i enters a lattice sites, it creates populations in all five directions, as specified by the evolution equations.

$\mathbf{j}$ associated to ρ as

$$\mathbf{j}(\mathbf{r},t) = \sum_{i=0}^{q} \mathbf{v}_i f_i(\mathbf{r},t) = \sum_{i=1}^{q} \mathbf{v}_i f_i(\mathbf{r},t) \tag{4}$$

The LB equation are then defined as

$$\begin{aligned} f_i^{out}(\mathbf{r},t) &= f_i^{in}(\mathbf{r},t) + \frac{1}{\tau}\left(f_i^{eq} - f^{in}(\mathbf{r},t)\right) \\ f_i^{in}(\mathbf{r}+\delta_t\mathbf{v}_i, t+\delta_t) &= f_i^{out}(\mathbf{r},t) \end{aligned} \tag{5}$$

with

$$\tau = \frac{1}{2}$$

and

$$\begin{aligned} f_i^{eq} &= w_i\rho + w_i\frac{\mathbf{j}\cdot\mathbf{v}_i}{c_s^2} \\ f_0^{eq} &= w_0\rho \end{aligned} \tag{6}$$

where w_i are called *weight factors* and c_s is the speed of the wave. There values will be given below. The weight w_i should add up to 1

$$\sum_{i=0}^{q} w_i = 1$$

In the continuous limit, eq. (5) with eq. (6) yield

$$\begin{aligned} \partial_t\rho + \partial_\beta j_\beta &= 0 \\ \partial_t j_\alpha - c_s^2\partial_\alpha\rho &= 0 \end{aligned} \tag{7}$$

When combined, these two equations give the wave equation

$$\partial_t^2\rho - c_s^2\nabla^2\rho = 0 \tag{8}$$

The rest population f_0 allows us to adjust the speed of the wave from place to place by having the value of w_0 depend of spatial location $\mathbf{r}$ or time t. For all non-zero velocities, it is important to use the same weight $w_i = w$. When $w_i = w$, we conclude form the condition

$$1 = \sum_{i=0}^{q} w_i = w_0 + qw$$

that

$$w = \frac{1 - w_0}{q} \tag{9}$$

In the LB approach, one can show that c_s is defined through the relation

$$\sum_{i=1}^{q} w_i v_{i\alpha} v_{i\beta} = c_s^2 \delta_{\alpha\beta} \tag{10}$$

For D2Q5 and D3Q7, it easy to show that

$$\sum_{i=1}^{q} v_{i\alpha} v_{i\beta} = 2v^2 \delta_{\alpha\beta} \tag{11}$$

where $v = \delta_x/dt$.

Therefore

$$\sum_{i=1}^{q} w_i v_{i\alpha} v_{i\beta} = 2\frac{(1 - w_0)}{q} v^2 \delta_{\alpha\beta} \tag{12}$$

and we obtain

$$c_s^2 = (1 - w_0)c_{max}^2 \qquad \text{with} \qquad c_{max}^2 \equiv 2\frac{v^2}{q}. \tag{13}$$

A consequence of this relation is that we must impose $0 \le w_0 < 1$. Then we obtain from (13) that the speed of the wave is such that

$$0 < c_s \le c_{max}$$

the maximum speed being achieved with $w_0 = 0$ i.e. without a rest population f_0. The refraction index is defined as

$$n = \frac{c_{max}}{c_s}$$

Using eqs. (9) and (13) when then obtain

$$w = \frac{1}{q}\frac{c_s^2}{c_{max}^2} = \frac{1}{n^2 q} \qquad w_0 = 1 - \frac{c_s^2}{c_{max}^2} = \frac{n^2 - 1}{n^2} \tag{14}$$

With the above value of w_i and the fact that $\tau = 1/2$, the LB wave model can also be written as

$$\boxed{\begin{aligned} f_i^{out} &= \frac{2}{n^2 q}\rho + \frac{1}{v}\mathbf{v}_i \cdot \mathbf{j} - f_i \qquad \text{for } i = 1 \text{ to } q \\ f_0^{out} &= 2\frac{n^2-1}{n^2}\rho - f_0 \end{aligned}} \tag{15}$$

Stability and accuracy: The above numerical scheme solves the wave equation for $\rho = \sum f_i$. It is unconditionally stable even in a very complex geometry. The reason is that, in addition to $\sum f_i$, and $\sum \mathbf{v}_i f_i$ which are conserved by the collision phase, $\sum f_i^2$ is also conserved.

The accuracy of the scheme is second order in δ_t and δ_x. It is however necessary that the wave length λ be large enough with respect to the lattice spacing δ_x. Empirically, it is found that $\lambda/\delta_x \approx 10$ is acceptable.

3 Boundary Conditions

3.1 Reflection

We also have to specify how to treat boundary conditions. On a reflexive wall (e.g. a mirror), the collision is replaced by the so-called *bounce-back* rule

$$f_i^{out}(\mathbf{r}, t) = f_{\mathrm{opp(i)}}^{in}(\mathbf{r}, t) \tag{16}$$

where opp(i) is the direction opposite to i. For instance on D2Q4, $\mathrm{opp}(1) = 3$, $\mathrm{opp}(2) = 4$, $\mathrm{opp}(0) = 0$.

3.2 Point source

A point source at a given location $\mathbf{r}$ for a wave can easily implemented by changing the collision phase as

$$f_i^{out}(\mathbf{r}, t) = A \sin(2\pi\nu t) \qquad \text{for } i = 1, \ldots, q \tag{17}$$

where ν is the desired frequency of the wave. Using the standard relation $c_s = \lambda\nu$, one creates a wave with a wave length

$$\lambda = \frac{c_s}{\nu}$$

3.3 Plan waves

To create a plan wave, equation (17) should be implemented on a line (2D case) or a plan (3D case). Note that it is then desirable to implement periodic boundary conditions on the domain, for the directions aligned with the wave plan.

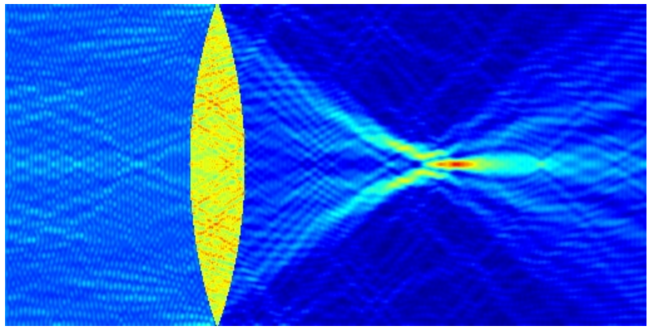

Figure 4: Simulation of a plan wave crossing a jens-shaped region with refraction index larger than 1.

Figure 5: Simulation of a the wave produced by a point source located at the focal point of a parabolic mirror. It results in a plan wave.

3.4 Attenuation

An attenuation of the wave can also be implemented on some desired lattice nodes. This can be useful to model an obstacle in which a part of the wave energy is dissipated. But is is especially useful to create a boundary layer around the computational domain to absorb the wave progressively and prevent artificial reflections at the limit of the domain.

An attenuation can be obtained my multiplying the f^{out} after collision and before propagation:

$$f_i^{in}(\mathbf{r} + \mathbf{v}_i\delta_t, t + \delta_t) = \beta f_i^{out} \tag{18}$$

for $\beta \in]0, 1[$.

To progressively absorb a wave that would otherwise continue to travel beyond the limit of the computational domain, one introduces a layer of lattice sites surrounding the domain boundary, with β decreasing slowly from 1 to a small value.

Figures 4 and 5 show two examples a the simulation of a wave, using the LB method.