Homework 1

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Exercise 1
Being the probability mass function constant and equal to $\frac{1}{N}$, we have:

$$E\left[Y\right] = \sum_{j=1}^{N}\left[jP\left(Y=j\right)\right] = \sum_{j=1}^{N}\left(j\frac{1}{N}\right) = \frac{1}{N}\sum_{j=1}^{N}j = \frac{N+1}{2}$$

Exercise 2
y is a function of x, so we can compute the expectation $E\left[y\right]$ using the PDF of x:

$$E\left[y\right] = \int_{-\infty}^{+\infty}(8+12x)f(x)dx = \int_{0}^{2}(8+12x)(1-\frac{x}{2})dx = \left[8x+6x^2-2x^2-2x^3\right]_0^2 = 16$$

Exercise 3
To compute the marginal PDFs we have to integrate the joint PDF on its domain along each axis.

We see that, for $X=x$, Y is between 0 and $1-x^2$, so we can compute the marginal of X as:

$$f_X(x) = \int_0^{1-x^2}\frac{15}{4}x^2dy = \frac{15}{4}x^2\int_0^{1-x^2}dy = \frac{15}{4}x^2\left(1-x^2\right),$$

for $-1<x<1$.

In the same way, we can see that for $Y=y$ we can compute the marginal of Y as:

$$f_Y(y) = \int_{-\sqrt{1-y}}^{+\sqrt{1-y}}\frac{15}{4}x^2dx = \frac{5}{2}\left(1-y\right)^{\frac{3}{2}},$$

for $0<y<1$.

Exercise 4

1. The marginal PDFs of X and Y are:

$$f_X(x) = \int_0^{\infty}6e^{-(2x+3y)}dy = 6e^{-2x}\int_0^{\infty}e^{-3y}dy = 2e^{-2x},$$

$$f_Y(y) = \int_0^{\infty}6e^{-(2x+3y)}dx = ... = 3e^{-3y}.$$

   As we can see, $f(x,y) = f_X(x)f_Y(y)$ (i.e. the total PDF is equal to the product of the two marginal PDFs) which is a sufficient condition for independency, so X and Y are independent.

2. Since X and Y are independent, the value of X is irrelevant w.r.t. the probability of having a given value of Y. Mathematically this can be expressed as $f(y|x) = f_Y(y)$. Therefore the expectation value $E\left[Y|\,X>2\right]$ is exactly equal to:

$$E\left[Y\right] = \int_0^{\infty}yf_Y\left(y\right)dy = \left[ye^{-3y}\right]_0^{\infty} - \int_0^{\infty}e^{-3y}dy = \frac{1}{3}$$

3. We want to compute the probability of getting a pair $(X = x, Y = y)$ such that $x > y$. To do that we can integrate the total PDF on the portion of its domain that satisfies this condition (i.e. on the portion of $R_+ \mathrm{x} R_+$ below the line $y = x$):

$$P(X > Y) = \int_0^\infty \left[\int_y^\infty f(x,y)dx\right] dy = 6\int_0^\infty e^{-3y}\left[e^{-2x}\right]_y^\infty dy = 3\int_0^\infty e^{-5y}dy = \frac{3}{5}$$

Exercise 5
We know that the probability for the laptop to last at least t years is:

$$P(T \geq t) = e^{-\frac{t}{5}},$$

therefore, the probability for the laptop to last less than t years (i.e. of braking within the t-th year) is:

$$P(T < t) = 1 - e^{-\frac{t}{5}}.$$

We want to compute $P(T < 3|T \geq 2)$, which can be written as:

$$P(T < 3|T \geq 2) = \frac{P(2 \leq T < 3)}{P(T \geq 2)}.$$

In the previous fraction the denominator is trivially $P(T \geq 2) = e^{-\frac{2}{5}}$. To compute the numerator we can notice that $P(T < t)$ is basically the CDF (if we consider t to be a continuous variable) for the event “laptop breaking”. Therefore, we can write the probability for the laptop to break during the third year as:

$$P(2 \leq T < 3) = \int_2^3 \frac{d}{dt}\left(1 - e^{-\frac{t}{5}}\right) dt = e^{-\frac{2}{5}} - e^{-\frac{3}{5}}.$$

Substituting the result in the previous expression, we get:

$$P(T < 3|T \geq 2) = 1 - e^{-\frac{1}{5}}.$$

Notice that also considering t as discrete we get the same result, since:

$$P(2 \leq T < 3) = P(T < 3) - P(T < 2) = 1 - e^{-\frac{3}{5}} - \left(1 - e^{-\frac{2}{5}}\right).$$