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EXOTIC DERIVATIVES - GROUP PROJECT

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1 Introduction

The purpose of this assignment is the analysis and evaluation of an investment certificate issued by Vontobel Financial Products Ltd.. Specifically, the financial instrument under analysis is a Capped Bonus Certificate (ISIN: CH1339135852) issued on April 5, 2024 with October 14, 2025 as the expiration date. The certificate is denominated in U.S. dollars (USD) and its payoff is linked to the performance of the S&P MidCap 400 Index. This index is composed of a pool of mid-cap companies of the U.S. stock market, with higher growth potential and higher volatility than large-cap companies.

Capped Bonus Certificates are investment instruments that offer a predefined return profile based on specific market conditions. They allow participation in the (positive) performance of the underlying, up to a cap level, while also offering a predefined return, called 'bonus', provided that the underlying never reaches, during the lifetime of the certificate, the protection barrier. Should the underlying, on the other hand, reach this barrier level, the value redeemed at maturity will be totally linked to the performance of the underlying, again up to the cap level, implying that repayment of the invested capital is not guaranteed. The investor is expected to hold this certificate until maturity, given the absence of early redemption features by either the issuer or the investor (callability and/or autocallability).

The issuer, Vontobel Financial Products Ltd. is based in Dubai and is part of the Swiss Vontobel Group. The product benefits from a guarantee issued by Vontobel Holding AG, with additional credit support through a Keep Well agreement from Bank Vontobel AG. These elements boost investor confidence, but do not completely eliminate the risk of financial losses, as the certificate does not provide capital protection. The reference exchange on which the certificate is listed is the SIX Swiss Exchange, which allows the instrument to be traded on the secondary market.

Given the presence of derivative instruments embedded in the certificate, it has a risk rating of 5 out of 7, indicating a relatively high level of complexity and risk. This is reflected in the intended investor of this instrument, namely a financially informed retail investor who possesses the experience and knowledge to be able to assess the risks associated with such a product. It is therefore not suitable for conservative or risk-averse investors, nor for those who do not possess an adequate understanding of the underlying dynamics.

2 Payoff Analysis

The Capped Bonus Certificate under analysis presents a payoff mechanism that depends on the performance of the S&P MidCap 400 Index, and on whether or not a predefined barrier will be breached during the life of the product.

The payoff structure revolves around three key components: the bonus level, the barrier level, and the cap. At issuance (April 5, 2024), the initial level of the underlying $S_0 = 2989.16$ is recorded and used as a reference point. A barrier level, set at 2391.33 (approximately 70% of the initial level), is continuously observed throughout the duration of the certificate, until the final Valuation Date (October 6, 2025). This barrier is crucial, as it determines whether or not the bonus mechanism will be activated at maturity.

If the price of the underlying remains strictly above the barrier throughout the entire observation period (April 5, 2024 – October 6, 2025), the investor will receive, at maturity, a cash amount equal to the reference price of the underlying on the Valuation Date (S_T) multiplied by the Ratio ($1000/S_0 = 0.33454$). However, this amount will be no less than the Bonus Amount (B.A. = USD 1070) and no greater than the Maximum Amount (M.A. = USD 1144.99). This structure provides conditional capital enhancement if the barrier is never breached, while also allowing participation in the underlying's performance up to a capped maximum.

Conversely, if the barrier is breached even once during the product's life, the bonus mechanism is deactivated. In this case, the certificate loses its conditional protection, and the final redemption amount is entirely linked to the actual performance of the underlying at maturity. If the index ends below its initial level, the investor incurs a proportional capital loss, just as if they had invested directly in the index. On the other hand, if the index ends above the initial level, the investor still benefits from the positive return, but again only up to the cap.

The whole payoff structure can be easily summarized by the following formula:

$$\text{Payoff}(T) = \text{M.A.} - 1000\left(K_1 - \frac{S_T}{S_0}\right)^+ + 1000\left(K_2 - \frac{S_T}{S_0}\right)^+ \cdot \mathbb{1}_{\left\{\min\limits_{t\in(0,T]} \frac{S_t}{S_0} > B\right\}}$$

where $K_1 = \text{M.A.}/1000$, $K_2 = \text{B.A.}/1000$ and $B = 2391.33/S_0$.

In summary, the Capped Bonus Certificate offers an appealing risk-return trade-off in markets that are expected to remain stable or rise moderately, while exposing the investor to potentially significant losses if the index suffers a severe decline during the life of the product.

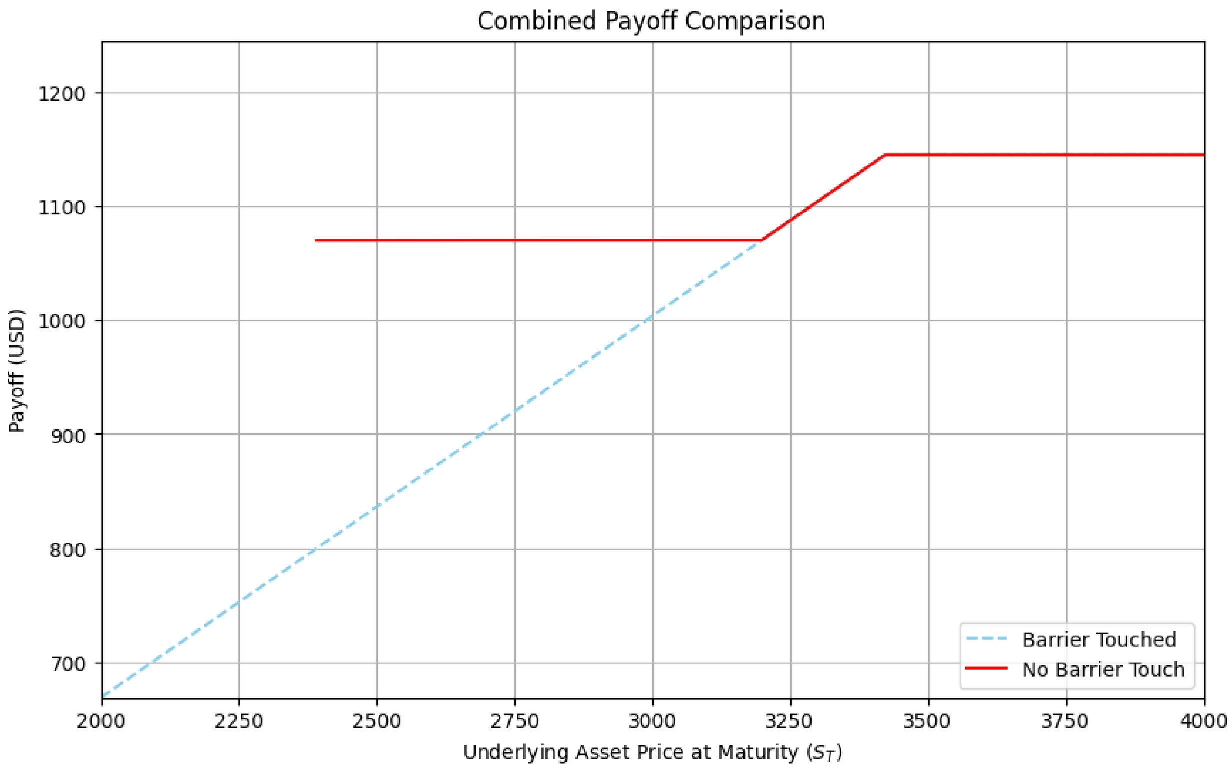

Figure 1: Combined graph of the payoff of the Certificate, as a function of S_T.

3 Replicating Strategy

One of the most effective tools for understanding and managing structured products is the construction of a replicating strategy. From a practical standpoint, such a replication serves as a theoretical hedging portfolio that a financial institution (e.g., the issuer) could implement to neutralize its exposure to the product. In other words, replicating the payoff with standard or exotic instruments allows for dynamic risk management and sensitivity analysis with respect to market movements.
Moreover, viewing the certificate through the lens of replication provides a deeper insight into its internal mechanics, revealing the financial instruments that implicitly lie behind the structured payoff. It makes transparent how the product reacts to various market conditions — such as volatility spikes — and clarifies the origin of its asymmetric risk-return profile.
The payoff of the Capped Bonus Certificate under analysis can be decomposed into three key components:

$$\text{Payoff}(T) = \underbrace{\text{M.A.}}_{\text{First Term}} \underbrace{- 1000 \left(K_1 - \frac{S_T}{S_0}\right)^+}_{\text{Second Term}} \underbrace{+ 1000 \left(K_2 - \frac{S_T}{S_0}\right)^+ \cdot \mathbb{1}_{\left\{\min\limits_{t \in (0,T]} \frac{S_t}{S_0} > B\right\}}}_{\text{Third Term}}$$

1. A **Zero-Coupon Bond (ZCB)** maturing at the certificate's expiry date with a notional amount equal to the Maximum Amount M.A. = USD 1144,99.
2. A **short position in 1000 European put options** written on the underlying index's performance S_T/S_0, with strike K_1 and maturity T.
3. A **long position in 1000 down-and-out put options** written on the underlying index's performance S_T/S_0, with strike K_2, maturity T and continuously monitored barrier level B.

Each component of the replicating strategy is influenced by different risk factors, such as interest rates, volatility, and the dynamics of the underlying. As a result, pricing the certificate effectively translates into pricing each individual component and aggregating their valuations under consistent market assumptions.

4 Risk Factors Analysis: Issuer's Point of View

Given the complex structure of the certificate under analysis, a number of risks emerge for the issuer, both financial and non-financial, that must be taken into account for proper management of this financial instrument. Indeed, the issuer, given its position as counterparty to the investor, must manage its exposure during the lifetime of the contract in order to ensure that the various risks are adequately hedged. The most important risk factors are described below.

4.1 Underlying Asset

In a Capped Bonus Certificate, the payoff mechanism involves both capped participation in the underlying asset's positive returns and conditional downside protection, provided the underlying never reaches the barrier during the certificate lifetime. Consequently, the management of underlying asset risk requires meticulous attention, as the exposure varies based on the asset's proximity to both the barrier and the cap levels.

Delta Risk: Delta measures the sensitivity of the certificate's value to small changes in the underlying asset's price. Initially, as the underlying trades significantly above the barrier, the certificate exhibits a moderate delta. However, as the underlying price approaches the barrier level, delta increases sharply. This intensified sensitivity arises because little downward movements may trigger the barrier, converting the product's payoff from conditionally protected to directly exposed to the underlying itself. Conversely, when the underlying approaches the cap, the delta sensitivity decreases due to limited upside gain. This delta exposure necessitates continuous dynamic hedging by the issuer to maintain a neutral position. The closer the underlying gets to the barrier or the cap, the more frequently and aggressively the issuer must rebalance its position, increasing both hedging complexity and cost.

Gamma Risk: Gamma measures the rate of change of delta as the underlying moves. For a Capped Bonus Certificate, gamma risk becomes particularly significant near the barrier level. This is due to the sudden transition in the payoff profile — from conditional downside protection to full participation in negative performance — if the barrier is breached. As the underlying approaches the barrier, the payoff's convexity increases sharply, leading to a spike in gamma and posing considerable challenges for effective hedging. In conclusion, from the issuer's perspective, upward movements in the underlying tend to be detrimental.

4.2 Volatility Risk

The value of the Capped Bonus Certificate is highly sensitive to changes in both implied volatility and the shape of the volatility surface (skew), due to the binary structure of its payoff.

When the implied volatility increases, the distribution of potential outcomes for the underlying index performance widens. As a result, the probability of breaching the barrier before maturity rises. Since the payoff is fixed and capped if the barrier is not breached, but becomes variable and potentially much lower if the barrier is hit, a higher volatility translates into a greater likelihood of entering in the riskier payoff regime. This shift decreases the expected value of the certificate from the investor's perspective and therefore represents a positive mark-to-market effect for the issuer.

In addition to the volatility level, the certificate is also sensitive to volatility skew — that is, the variation of implied volatility across strike prices. An increase in skew implies that lower strikes are assigned higher implied volatilities, effectively increasing the market-implied probability of downside scenarios. This has two consequences: it increases the likelihood of barrier breach and lowers the expected payoff when in the variable regime. Both effects, again, represents a positive mark-to-market effect for the issuer.

4.3 Time Decay

The options embedded in the certificate naturally lose time value as maturity approaches, a phenomenon called theta decay. For the issuer, who is long the vanilla put and short the down-and-out put, Theta is always detrimental, especially when the underlying approaches the critical barrier level. In particular, the short down-and-out put does not benefit from theta decay: ceteris paribus, as time passes, the probability of hitting the barrier falls, keeping the option alive. Likewise, the long vanilla put suffers from theta decay: long option positions typically carry negative theta and lose time value as the chance of finishing in-the-money diminishes, especially if the underlying remains significantly above the strike.

4.4 Interest Rate Risk

The certificate's replicating portfolio combines derivative positions with a zero-coupon bond, exposing it directly to interest rate fluctuations.
ZCB Discounting: The present value of the zero-coupon bond is inversely related to prevailing interest rates. When rates rise, the bond's price falls, and because the issuer is effectively short this bond, they stand to gain from an increase in rates.
Options Discounting: Interest rates also affect option values: higher rates tend to reduce the price of long put positions while increasing the value of short put positions. In this case, however, the decline in the long vanilla put's value typically outweighs the gain on the short down-and-out put, so overall the issuer does not materially benefit from rising rates within the equity option component.

4.5 Dividend Risk

The S&P MidCap 400 Index is a price-return index and does not reinvest dividends. Consequently, changes in expected dividends alter the forward trajectory of the index and affect the valuation of the embedded options.
Higher expected dividends lower the forward price of the index, increasing the value of both vanilla and barrier put options. Specifically, long vanilla-put positions gain value as dividend yields rise, while the short down-and-out put position loses value. Overall, from the issuer's perspective, an increase in dividend yield typically raises the certificate's fair value. Conversely, when expected dividends decline, the structure's value decreases, which benefits the investor.

4.6 Model Risk

The pricing and hedging of exotic derivatives, especially those involving barriers and path dependency, rely heavily on complex mathematical models. Inaccuracies in these models, due to incorrect assumptions, miscalibration, or oversimplification, can result in a mismatch between theoretical and realized hedging outcomes. The issuer must invest in robust modeling frameworks, regular validation procedures, and stress-testing environments to reduce the impact of model risk on the economic performance of the product.

5 Pricing the Certificate

5.1 A First Attempt: Heston Model

Volatility plays a central role in the pricing of equity-linked certificates, and given its major significance as a risk factor, a first attempt was made to calibrate a Heston model, which allows volatility to evolve stochastically over time and is often reflective of observed market behaviors such as volatility clustering and the volatility smile. The Heston model's structure, as reported in the formula below, includes a separate stochastic process for variance and theoretically offers a robust framework for capturing the nonlinearities inherent in option prices, especially when pricing path-dependent or volatility-sensitive instruments like the certificate under consideration.

$$\begin{cases} dS_t = (r - q)\, S_t\, dt + \sqrt{v_t}\, S_t\, dW_{1,t}^{\mathbb{Q}} \\ dv_t = \kappa(\theta - v_t)\, dt + \sigma\sqrt{v_t}\, dW_{2,t}^{\mathbb{Q}} \\ d\langle W_1^{\mathbb{Q}}, W_2^{\mathbb{Q}}\rangle_t = \rho\, dt \end{cases}$$

where the first equation represents the SDE for the underlying price process under the risk neutral measure $\mathbb{Q}$, the second equation shows a mean reverting process for the variance of S_t and the third introduces a correlation between the two Brownian motions.

However, the attempt to calibrate the Heston model was ultimately unsuccessful. The primary reason lies in the scarcity of liquid derivative data on the S&P 400, the index underlying the certificate. In contrast to large-cap indices like the S&P 500, the S&P 400 is less actively traded and lacks a broad ecosystem of listed options or volatility-linked instruments. The few options available - on ETFs that replicate the index, and not directly on the index - tend to be at-the-money and short-dated, with low open interest and trading volume. This absence of a deep and liquid options market undermines the reliability of implied volatility surfaces needed for robust Heston calibration. Moreover, the lack of traded options across a range of strikes and maturities prevents a meaningful fit of the model parameters, particularly those governing long-term mean reversion and volatility-of-volatility.

To overcome this information deficit and make the calibration lighter and more stable, we introduced two strategic simplifications. The first simplification concerns the correlation ρ between the Brownian motion driving the index price and the one driving its variance. Rather than treating ρ as a free parameter, we estimate it directly from historical data, computing the correlation between the daily log-returns of the S&P 400 and those of VIX (each series sampled over 252 trading days), and obtaining approximately –0.035. After having fixed ρ to this value, we removed one degree of freedom from our calibration problem. The second simplification addresses the initial variance. Instead of leaving v_0 free, we derived it from the historical volatility of the index, converting the standard deviation of the daily log-returns into annualized variance (the value obtained was approximately 0.16^2). With both ρ and v_0 pre-computed, only three parameters remained to be estimated: the mean-reversion speed κ, the long-term variance level θ, and the volatility-of-volatility σ. However, even with those simplifications, the minimum error remained significant, underscoring the inherent difficulty of fitting a pure Heston model when only few option quotes are available.

In light of these limitations, a more tractable alternative was adopted: the Black-Scholes model.

Although considerably simpler, the Black-Scholes framework is still widely used in practice and can serve as a reasonable approximation when richer models are impractical. The key assumptions and limitations embedded in this approach are fully acknowledged.

First and foremost, Black-Scholes assumes constant implied volatility throughout the life of the instrument. This is clearly an oversimplification, especially for a product with a maturity of over a year and embedded path-dependent features. To address this, a proxy for implied volatility was selected based on observable market data: specifically, the VIX index level as of the certificate's issuance date was used. While the VIX is constructed using S&P 500 options and does not directly reflect the volatility of the S&P 400, it nonetheless captures general investor expectations of market uncertainty under the risk-neutral measure. In the absence of a dedicated volatility index for the S&P 400, this choice represents a plausible approximation.

Second, the Black-Scholes model assumes a constant risk-free interest rate over the life of the contract. Given that the certificate is denominated in USD and has a time-to-maturity of approximately 1.5 years, the risk-free rate was proxied using the 1-year U.S. Treasury Bill yield at the issue date, as sourced from the Federal Reserve Economic Data (FRED). While this does not account for potential shifts in the yield curve or macroeconomic policy changes, it offers a market-consistent baseline rate for discounting cash flows under the risk-neutral measure.

A third assumption concerns the dividend yield on the underlying index. Since the S&P 400 does not publish an official dividend yield, and because dividend forecasts of the index's constituents were not readily accessible, this parameter was estimated from *SPDR S&P 400 U.S. Mid Cap UCITS ETF (Acc)*, an exchange-traded fund that tracks the S&P 400. This ETF-based proxy captures the aggregate dividend policy of the index constituents and offers a reasonable basis for modeling the cost-of-carry in the absence of direct index-level data.

Lastly, an important theoretical assumption of the Black-Scholes model is that log-returns are normally distributed, in accordance to the hypothesis that the underlying's time evolution is a geometric Brownian motion. However, empirical analysis of the S&P MidCap 400 index data reveals deviations from normality, with evidence of fat tails in the distribution of log-returns. This highlights a structural limitation of the model, which may understate the probability of extreme market moves, a critical consideration for pricing instruments with path-dependent features such as barrier options.

Taken together, these modeling choices reflect the necessary trade-off between theoretical rigor and practical feasibility. While the Black-Scholes model lacks the flexibility and realism of more advanced stochastic volatility frameworks, it remains a viable tool under data-constrained conditions, especially when pricing instruments tied to less liquid underlyings like the S&P 400. Should more robust data become available, future work may revisit the Heston model or alternative local volatility frameworks, but for the present application the Black-Scholes model provides a defensible and transparent basis for pricing the certificate.

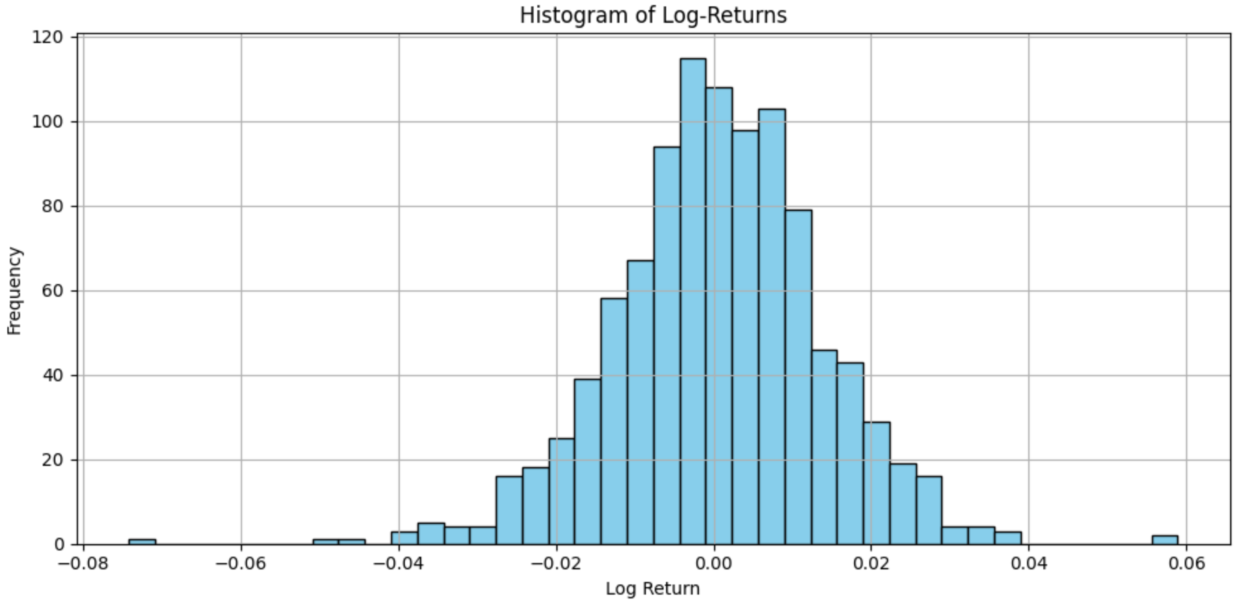

Figure 2: Histogram of frequency of log-returns of S&P 400 Index over the time period 1 May 2020 - 1 May 2024.

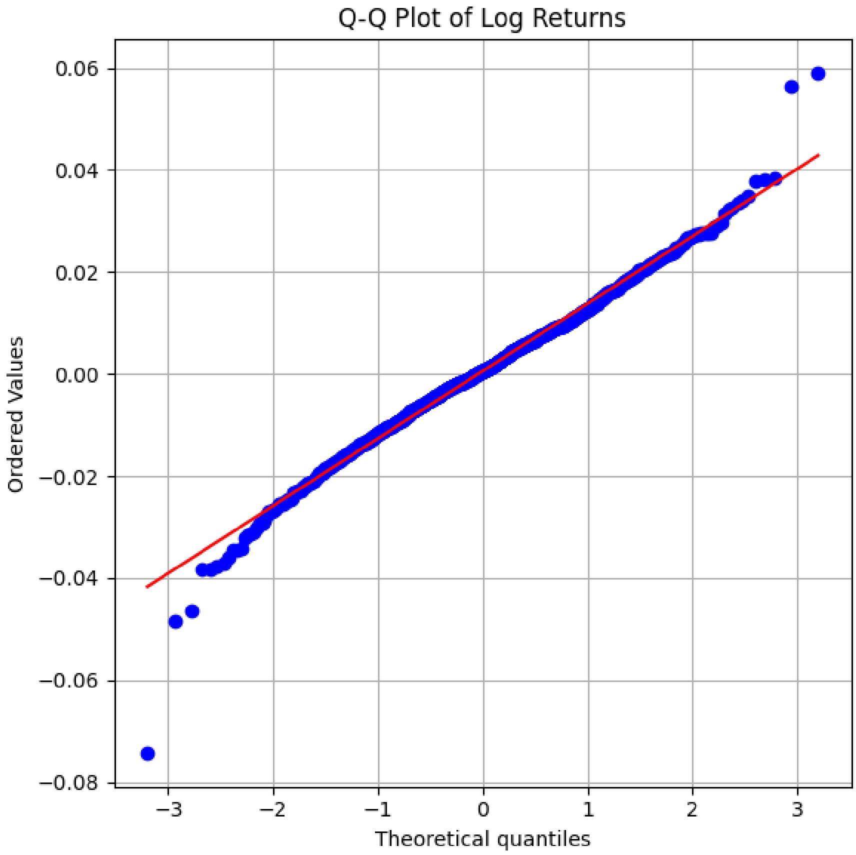

Figure 3: Q-Q plot of the log-returns of the S&P 400.

5.2 Pricing with Black-Scholes

In the Black-Scholes framework, the certificate can be priced directly using closed-form solutions for the components of the replicating strategy, eliminating the need for Monte Carlo simulations. According to the Black-Scholes model, the underlying asset S_t is assumed to follow a geometric Brownian motion under the risk-neutral measure $\mathbb{Q}$:

$$dS_t = (r - q)S_t\,dt + \sigma S_t\,dW_t^{\mathbb{Q}},$$

where r is the constant risk-free interest rate, q is the annualized dividend yield, σ is the volatility of the asset, and $W_t^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$.
A Zero Coupon Bond is priced simply discounting the face value, i.e. the amount of cash the holder will receive at maturity, hence:

$$ZCB(0) = FVe^{-rT}$$

The price of a European put option requires the computation of the expected value, under $\mathbb{Q}$, of the (discounted) payoff at maturity $(K - S_T)^+$. The well-known result is the following:

$$p(0) = Ke^{-rT}\Phi(-d_2) - S_0e^{-qT}\Phi(-d_1),$$

where

$$d_1 = \frac{\ln\bigl(S_0/K\bigr) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T},$$

and $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable.
For a down-and-out European put option with continuously monitored barrier level $B < K$, strike K, maturity T, and initial asset price $S_0 > B$, the result is more convoluted and it has been taken from "Option, Futures and Other Derivatives; John C. Hull; 11th Edition":

$$p_{\text{DO}}(0) = p(0) - p_{\text{DI}}(0)$$

where:

$$\begin{aligned} p_{\text{DI}}(0) = &- S_0e^{-qT}\Phi(-x_1) + Ke^{-rT}\Phi(-x_1 + \sigma\sqrt{T}) \\ &+ S_0e^{-qT}\left(\frac{H}{S_0}\right)^{2\ell}\bigl[\Phi(y) - \Phi(y_1)\bigr] \\ &- Ke^{-rT}\left(\frac{H}{S_0}\right)^{2\ell-2}\Bigl[\Phi(y - \sigma\sqrt{T}) - \Phi(y_1 - \sigma\sqrt{T})\Bigr] \end{aligned}$$

having previously defined:

$$\ell = \frac{r - q + \frac{1}{2}\sigma^2}{\sigma^2}, \quad x_1 = \frac{\ln\left(\frac{S_0}{H}\right)}{\sigma\sqrt{T}} + \ell\sigma\sqrt{T}, \quad y = \frac{\ln\left(\frac{H^2}{S_0K}\right)}{\sigma\sqrt{T}} + \ell\sigma\sqrt{T}, \quad y_1 = \frac{\ln\left(\frac{H}{S_0}\right)}{\sigma\sqrt{T}} + \ell\sigma\sqrt{T}.$$

These formulas enable analytical pricing of the certificate by evaluating each component and summing their values under the replication framework.
The theoretical value of the certificate, obtained by the replicating strategy, amounts to **965.25 USD**. This figure is lower than the market issue price of **1000 USD**: this discrepancy can be

attributed to several practical and structural considerations commonly observed in the pricing of structured financial products.

Firstly, the market price often includes a liquidity premium and a commercial spread added by the issuer. In practice, financial institutions do not sell structured products at their purely theoretical value. Instead, they incorporate a markup to account for market-making activities, balance sheet usage, and hedging costs. This results in a higher offer price for investors compared to the replicating strategy's valuation.

Secondly, the observed price difference may reflect transaction costs and embedded fees. The theoretical model typically excludes distribution fees, structuring costs, and other charges associated with issuing and managing the product. Instead, these costs are built into the market price, which represents the all-in cost to be borne by the investor.

Lastly, credit risk related to the issuer plays a significant role. Since the investor is exposed to the solvency of the issuer throughout the life of the product, the market price may also incorporate a discount linked to the issuer's perceived creditworthiness. This component, while absent in a purely risk-neutral valuation framework, can materially impact the price at which the certificate is sold to the public.

In conclusion, the 34.75 USD difference between the theoretical and market price reflects a combination of structural markups, operational costs, and credit risk considerations, all of which are inherent to the real-world issuance of structured financial instruments.

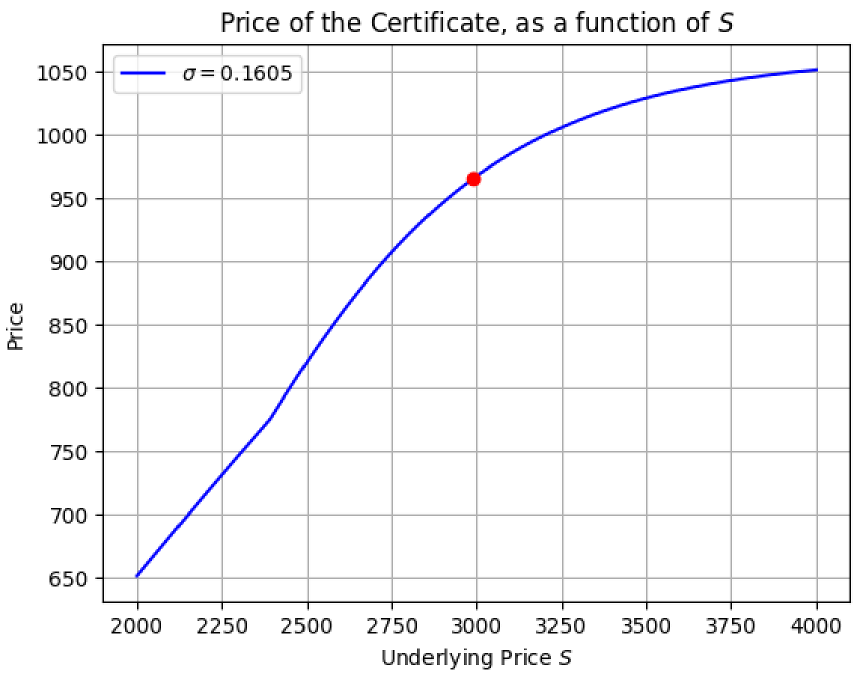

Figure 4: Price of the Certificate, as a function of S, computed with the Black-Scholes analytical formula. The red dot correspond to the price computed at $S = S_0$.

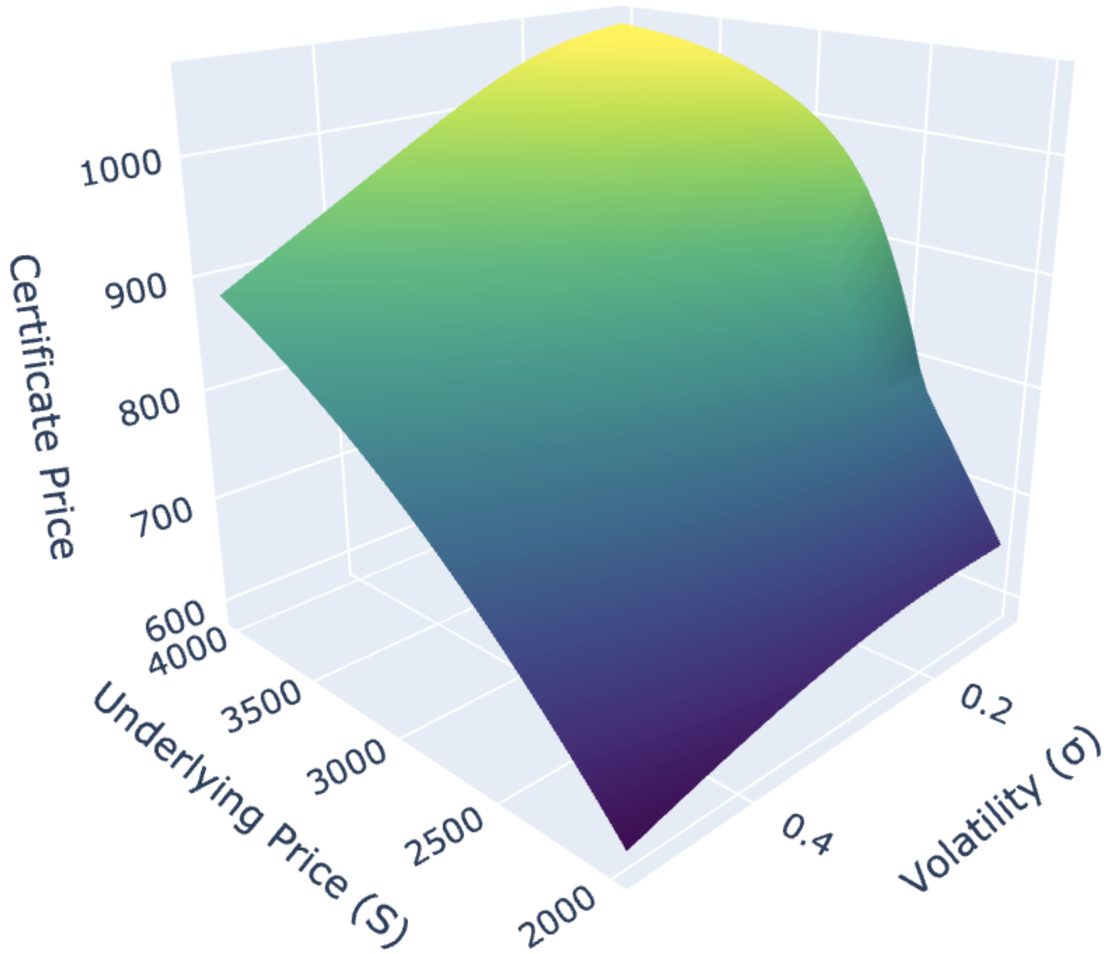

Figure 5: Price of the Certificate, as a function of S and σ, computed with the Black-Scholes analytical formula.

6 Sensitivities Analysis: Greeks

In this section, we define the main Greeks and present their values, calculated via numerical differentiation of the analytical pricing formula $V = V_{BS}(S, \sigma, r, t, T)$.

$$
\begin{aligned}
\Delta &= \frac{\partial V}{\partial S}, && \text{rate of change of price w.r.t. underlying spot} \\
\Gamma &= \frac{\partial^2 V}{\partial S^2}, && \text{rate of change of Delta w.r.t. underlying spot} \\
\nu \text{ (Vega)} &= \frac{\partial V}{\partial \sigma}, && \text{sensitivity to implied volatility} \\
\Theta &= \frac{\partial V}{\partial t}, && \text{time decay of the option value} \\
\rho &= \frac{\partial V}{\partial r}, && \text{sensitivity to interest rate} \\
\text{Volga} &= \frac{\partial^2 V}{\partial \sigma^2}, && \text{rate of change of Vega w.r.t. volatility} \\
\text{Vanna} &= \frac{\partial^2 V}{\partial S\, \partial \sigma}, && \text{rate of change of Vega w.r.t. underlying spot}
\end{aligned}
$$

Greek	**Value**
Delta (Δ)	0.195977
Gamma (Γ)	-0.000359
Vega (ν)	-756.624002
Theta (Θ)	69.342434
Rho (ρ)	-696.695618
Volga	480.738413
Vanna	0.382669

Table 1: Greeks at spot level $S = S_0 = 2989.16$

The sensitivities are consistent with the structure of a Capped Bonus Certificate and reflect the key features of its payoff: conditional capital protection (if the barrier is not breached), a bonus component, and a cap on upside participation.

- **Delta (Δ)** is approximately 0.196, reflecting limited directional exposure to the underlying. The product does not fully replicate movements in the underlying due to the cap. This low Delta arises from the net exposure of two options: one short vanilla put and one long barrier put.
- **Gamma (Γ)** is slightly negative and close to zero. This indicates nearly linear exposure in the central region of the payoff, far from the barrier and cap. The offsetting curvature of the long and short puts contributes to this small Gamma.
- **Vega (ν)** is strongly negative, indicating that the product loses value as implied volatility increases. This is due to the short vanilla put, which gains value (and thus produces a loss) when volatility rises, as the probability of ending in-the-money increases. Similarly, for the

long down-and-out put, increased volatility raises the probability of hitting the barrier, thus reducing its value.

- **Theta (Θ)** is positive, suggesting that the product benefits from time decay. The short vanilla put has positive Theta: as time passes, the likelihood of ending in-the-money decreases. Although the long down-and-out put has the opposite effect, overall, the vanilla put's time value decays faster, especially when the barrier is far from being triggered.
- **Rho (ρ)** is negative, as the product includes a zero-coupon bond that benefits from lower interest rates. The short vanilla put typically has positive Rho, meaning its value decreases as rates rise (beneficial in a short position). Conversely, the long down-and-out put has negative Rho. The net effect is that the product loses value when interest rates increase.
- **Volga** is positive, indicating convexity in the volatility exposure. As volatility rises, the negative Vega becomes more pronounced in magnitude, mainly due to the nonlinear behavior of the barrier option.
- **Vanna** is positive, indicating sensitivity of Vega to changes in spot. This is common in barrier structures, where the volatility exposure shifts significantly as the spot price approaches the barrier.

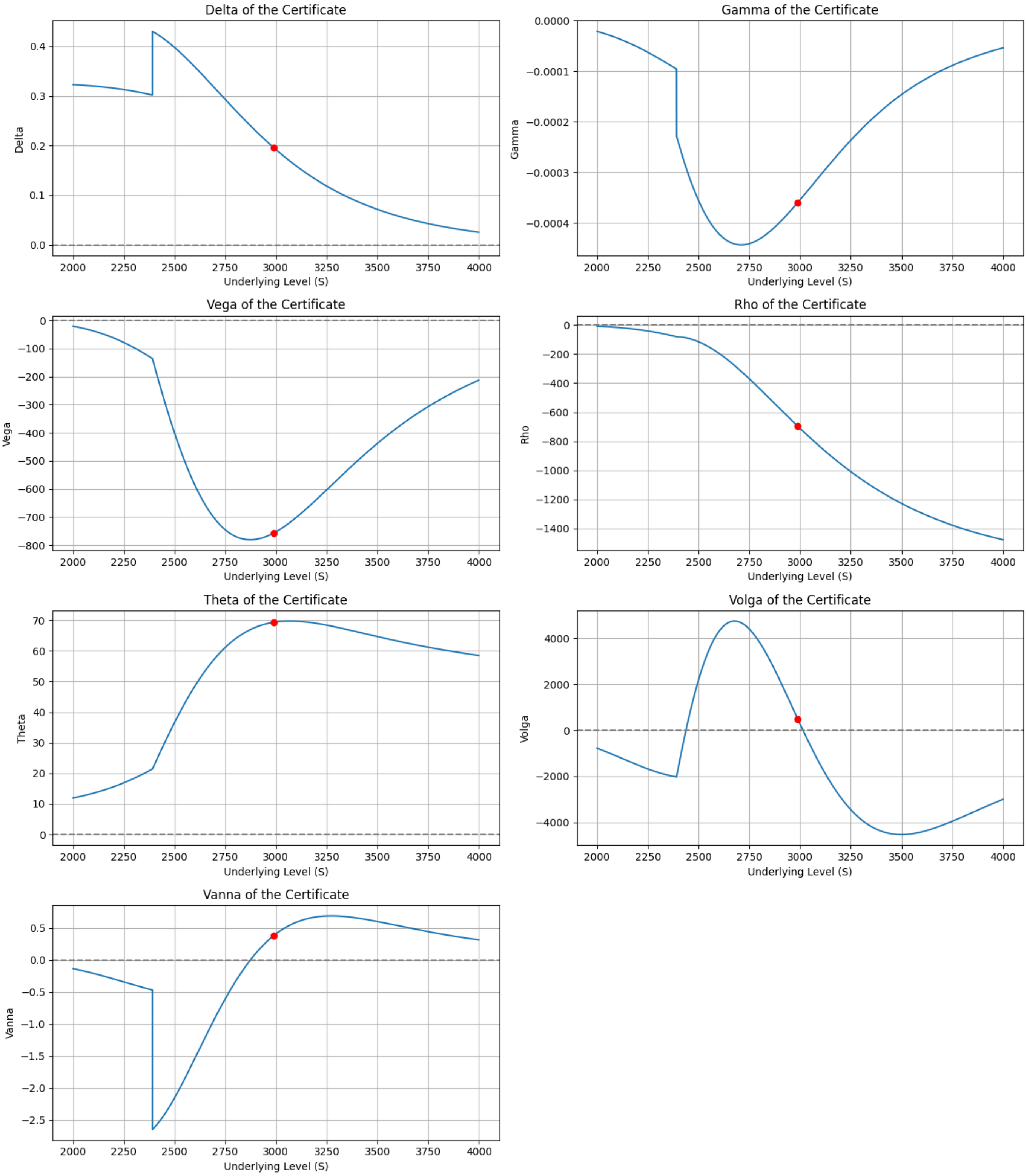

Figure 6: Greeks, as a function of the underlying level S. The red dot corresponds to $S = S_0$.

7 Summary Table for Risk Factors of the Option Components

The replicating strategy consists of two option components: a long down-and-out put and a short European put. The following tables show each component's individual parameter exposures.

	D-O Put Seller	**D-O Put Buyer**
Underlying	Depends	Depends
Interest rates	Depends	Depends
Dividends	Depends	Depends
Volatility	Long	Short
Skew	Long	Short
Barrier level	Long	Short

Motivation:

- **Underlying (S):** A decrease in the underlying's value increases the intrinsic value of the put. However, in a down-and-out structure, such a decrease also raises the likelihood of breaching the barrier, which would invalidate the option entirely. As a result, the buyer's exposure to the underlying is non-linear: initially negative (as in a vanilla put), but potentially turning neutral or even adverse as the spot approaches the barrier. The net sensitivity therefore depends on the spot level relative to the barrier.
- **Interest rates (r):** An increase in interest rates has two opposing effects on a down-and-out put. On one hand, higher rates reduce the present value of the strike price ($Ke^{-r(T-t)}$), leading to a lower value of the put: this is unfavorable for the buyer, who is effectively short interest rates. On the other hand, higher rates increase the forward price of the underlying ($F = Se^{(r-q)(T-t)}$), reducing the probability of the barrier being breached. This increases the likelihood that the option survives and pays off, which is favorable for the buyer, who is effectively long interest rates. The net effect depends on which of these two forces dominates, and is therefore context-dependent.
- **Dividends (q):** An increase in dividend yield tends to reduce the spot price, which would normally increase the value of a put by making it more in-the-money. However, for a down-and-out put, a lower spot also increases the likelihood of breaching the barrier and invalidating the option. As a result, the buyer's exposure to dividends depends on the spot level relative to the barrier: positive when the barrier is far, and potentially negative when the spot approaches the knock-out level.
- **Volatility (σ):** Greater volatility leads to larger and more probable barrier event (early knock-out), so the seller is long volatility exposure, and the buyer is short.
- **Skew:** A steeper skew (higher implied volatilities for lower strikes) increases the probability of the spot falling below the barrier. The seller is therefore long skew, and the buyer is short.
- **Barrier level (B):** A higher barrier level (closer to the spot price) raises the knock-out probability, increasing the seller's expected profit. Thus the seller is long the barrier parameter, and the buyer is short.

Parameter	**Put Seller**	**Put Buyer**
Underlying	Long	Short
Interest rates	Long	Short
Dividends	Short	Long
Volatility	Short	Long
Skew	Short	Long

Motivation:

- **Underlying (S):** A decrease in the value of the underlying makes the put more in-the-money, increasing its value for the buyer and decreasing it for the seller.
- **Interest rates (r):** An increase in r decreases the value of the vanilla put. Normally, the risk-free interest rate affects the price of an option in a less clear-cut way. In general, when interest rates rise, stock prices tend to fall (and viceversa), but it is important to emphasize that we are assuming that interest rates change while all the other variables stay constant. From the B&S formula for the vanilla put, one can appreciate that increasing interest rates lead to a lower value of the put itself. Finally, the seller is long interest rates, while the buyer is short.
- **Dividends (q):** A higher dividend yield reduces the expected future spot level (forward), increasing put value. The short-put seller therefore is short dividends exposure, while the buyer is long.
- **Volatility (σ):** Greater volatility increases the probability of the option finishing in-the-money, raising put value. The short-put seller is short vol, and the buyer is long vol.
- **Skew:** A steeper skew (higher implied vols at lower strikes) increases the price of out-of-the-money puts, again raising the value of the put. The seller is short skew, while the buyer is long.

8 Adding a Worst-Of Feature: Additional Risk Factors

The analysis conducted so far is based on a structured product whose risk-return profile depends on the performance of a single underlying asset (S&P 400). The hypothetical introduction of a basket of three underlyings and the resulting 'worst of' structure within the product's payoff significantly increase the complexity of managing the product. The payoff of an instrument with this feature is based on the worst performance among multiple underlyings. In the case of a basket containing three underlyings S_1, S_2, and S_3, the reference level at maturity T is:

$$\text{WorstOf}(T) = \min\left(\frac{S_1(T)}{S_1(0)}, \frac{S_2(T)}{S_2(0)}, \frac{S_3(T)}{S_3(0)}\right)$$

where $S_i(T)$ represents the price of the i-th stock at time T.

This feature introduces additional risk factors compared to the previously analyzed product. First, the certificate becomes highly sensitive to scenarios where even a single asset experiences a sharp drop, regardless of the performance of the other two. This can severely reduce or eliminate the product's payoff.

Valuation complexity increases as well, since it becomes necessary to consider the joint distribution of returns of the three assets, incorporating both their individual volatilities and the pairwise correlations. Correlation is a key driver of the product's risk profile: low correlation increases the probability that at least one asset underperforms significantly, reducing the expected value of the payoff for the investor. As a result, pricing such a product often requires Monte Carlo simulations or alternative advanced models, which heighten model dependency and raise the risk of mispricing due to incorrect assumptions or unstable calibrations.

From a hedging perspective, the complexity also increases substantially. The simple delta-gamma framework used for single-asset products becomes insufficient. It becomes necessary to account for higher-order sensitivities, such as cross-Gamma, to reflect interactions between assets. An additional layer of complexity arises from the dynamic nature of the "worst-performing" asset: the identity of the asset driving the payoff can change over time, resulting in discontinuous Greeks and abrupt shifts in the hedging strategy. This dynamic exposure demands sophisticated and frequently rebalanced hedging portfolios, which may incur substantial costs, especially under conditions of high volatility or low market liquidity.

In conclusion, replacing a single underlying with a worst-of structure across multiple assets leads to a significant increase in overall risk and introduces considerable complexity, both in the theoretical valuation of the product and in its active management. This trade-off between diversification potential and risk concentration must be carefully evaluated when designing or investing in such structures.

9 Conclusion

This report provided a comprehensive analysis of a capped bonus certificate, examining its payoff structure, pricing dynamics, replicating strategy, and risk profile.

The payoff was shown to be equivalent to a structure that pays a fixed bonus at maturity if the barrier is not breached, with the final redemption capped above a certain level. This asymmetry between downside protection and limited upside is central to the product's design.

Pricing was carried out using standard option pricing techniques under the Black-Scholes framework, and the results were coherent with market expectations. The replication using a long zero-coupon bond, a short European put, and a long down-and-out put effectively reproduced the product's payoff, offering transparency into both its construction and valuation.

The risk analysis confirmed that the product's sensitivities align with its payoff and replicating strategy. While the investor is protected from moderate downside moves (as long as the barrier holds), exposure to volatility, time, and rates plays a non-trivial role in the certificate's value evolution.

In summary, the capped bonus certificate delivers a structured investment profile balancing conditional protection and limited performance participation. The quantitative tools applied throughout this report provide a clear, coherent view of how the product behaves under different market conditions, supporting both valuation accuracy and informed investment decision-making.

10 Appendix: The Code

10.1 Black and Scholes Pricing

Contract Features

```
import yfinance as yf
from datetime import datetime
from datetime import date
import numpy as np
import matplotlib.pyplot as plt

# Define the date range
start_date = date(2024, 4, 5)
end_date = date(2025, 10, 6)
# Calculate the number of business days between the dates
total_days = np.busday_count(start_date, end_date)
# Convert days to years
T = total_days / 252
# Contract features
CAP = 3422.5882
BONUS = 3198.40
B = 2391.33
m = 0.33454
# Value of the underlying at the issuance date
S0 = 2989.16

# Vix Data
vix_data = yf.download("^VIX", start="2024-04-04", end="2024-04-06", interval="60m")
vix = vix_data['Close'].iloc[-1]

# Historical performance of the underlying: S&P 400 MidCap

# choose the dates
myStart = datetime(2020,5,1)
myEnd = datetime(2024,5,1)
data = yf.download("^MID", start = myStart, end = myEnd)
dates = data.index
prices = data["Close"].to_numpy()

plt.figure(figsize=(10, 5))
plt.plot(dates, prices, label="S&P 400 Close")
plt.plot([dates[0], dates[-1]], [B,B], color='red', linestyle='--', label='barrier level')
plt.title("Closing Prices of S&P 400")
plt.xlabel("Date")
plt.ylabel("Index Value")
plt.grid(True)
plt.xticks(rotation=45)
plt.legend()
plt.tight_layout()
plt.show()
```

Historical Performance and Normality Checks

```
# Daily log-returns
log_returns = np.log(prices[1:] / prices[:-1])
log_returns = log_returns.flatten()

# Histogram
plt.figure(figsize=(10, 5))
plt.hist(log_returns, bins=40, color='skyblue', edgecolor='black')
plt.title("Histogram of Log-Returns")
plt.xlabel("Log Return")
plt.ylabel("Frequency")
plt.grid(True)
plt.tight_layout()
plt.show()

from scipy.stats import shapiro, jarque_bera, probplot

# Q-Q plot
plt.figure(figsize=(6, 6))
probplot(log_returns, dist="norm", plot=plt)
plt.title("Q-Q Plot of Log Returns")
plt.grid(True)
plt.tight_layout()
plt.show()

# Shapiro-Wilk Test
shapiro_stat, shapiro_p = shapiro(log_returns)
print(f"Shapiro-Wilk Test:\n  Statistic = {shapiro_stat:.4f}, p-value = {shapiro_p:.4f}")

# Jarque-Bera Test
jb_stat, jb_p = jarque_bera(log_returns)
print(f"Jarque-Bera Test:\n  Statistic = {jb_stat:.4f}, p-value = {jb_p:.4f}")

# Asset price range
ST = np.linspace(2000, 4000, 20000)
# Payoff: barrier touched
payoff_barrier = m * CAP - m * np.maximum(CAP - ST, 0)
# Payoff: no barrier touched (only for ST > B)
ST_no_barrier = ST[ST > B]
payoff_no_barrier = (
    m * CAP - m * np.maximum(CAP - ST_no_barrier, 0) +
    m * np.maximum(BONUS - ST_no_barrier, 0)
)

# Common axis limits
x_min = ST[0]
x_max = ST[-1]
y_min = min(payoff_barrier.min(), payoff_no_barrier.min())
y_max = max(payoff_barrier.max(), payoff_no_barrier.max()) + 100
```

Payoff function

```python
# --- Create 1x2 grid plot (smaller) ---
fig, axs = plt.subplots(1, 2, figsize=(12, 5), sharey=True)

# Plot 1: Barrier touched
axs[0].plot(ST, payoff_barrier, color='skyblue', label='Barrier Touched')
axs[0].set_title('Payoff with Barrier Touch')
axs[0].set_ylabel('Payoff (USD)')
axs[0].set_xlim(x_min, x_max)
axs[0].set_ylim(y_min, y_max)
axs[0].grid(True)
axs[0].legend(loc='lower right')
# Plot 2: No barrier touched
axs[1].plot(ST_no_barrier, payoff_no_barrier, color='red', label='No Barrier Touch')
axs[1].set_title('Payoff without Barrier Touch')
axs[1].set_xlabel('Underlying Asset Price at Maturity ($S_T$)')
axs[1].set_ylabel('Payoff (USD)')
axs[1].set_xlim(x_min, x_max)
axs[1].set_ylim(y_min, y_max)
axs[1].grid(True)
axs[1].legend(loc='lower right')

plt.tight_layout()
plt.show()

# --- Combined plot (larger) ---
plt.figure(figsize=(10, 6))
plt.plot(ST, payoff_barrier, color='skyblue', linestyle='dashed', label='Barrier Touched')
plt.plot(ST_no_barrier, payoff_no_barrier, color='red', label='No Barrier Touch')
plt.title('Combined Payoff Comparison')
plt.xlabel('Underlying Asset Price at Maturity ($S_T$)')
plt.ylabel('Payoff (USD)')
plt.xlim(x_min, x_max)
plt.ylim(y_min, y_max)
plt.grid(True)
plt.legend(loc='lower right')
plt.show()
```

Black-Scholes Pricing Formula

```python
from scipy.stats import norm

r = 4.82/100          # assume a risk-free rate of 4.82%
sigma = 16.05/100     # assume a constant volatility of underlying of 16.05%
q = 1.64/100          # assume a dividend yield of underlying of 1.64%

def PriceBS(S0, sigma, r, T):

  # ---------------- The replicating portofolio is composed of: ---------------

  # 1. ZCB with face value of m*CAP (quantity = + 1)
  FV = m * CAP
  bond = FV*np.exp(-r*T)

  # 2. put option with strike = CAP (quantity = - m)
  K1 = CAP
  d1 = (np.log(S0/K1) + (r - q + 0.5*sigma**2)*T)/(sigma*np.sqrt(T))
  d2 = d1 - sigma*np.sqrt(T)
  put = K1 * np.exp(-r*T) * norm.cdf(-d2) - S0 * np.exp(-q*T) * norm.cdf(-d1)

  # 3. DO barrier put option with strike = BONUS and barrier level = B (quantity = + m)
  # (formula is taken from Hull, 11th edition, "Option, Futures and Other Derivatives")
  K2 = BONUS
  H = B
  D1 = (np.log(S0/K2) + (r - q + 0.5*sigma**2)*T)/(sigma*np.sqrt(T))
  D2 = D1 - sigma*np.sqrt(T)
  put_vanilla = K2 * np.exp(-r*T) * norm.cdf(-D2) - S0 * np.exp(-q*T) * norm.cdf(-D1)

  l = (r - q + 0.5*sigma**2)/(sigma**2)
  y = (np.log(H**2/(S0*K2)))/(sigma*np.sqrt(T)) + l*sigma*np.sqrt(T)
  x1 = (np.log(S0/H))/(sigma*np.sqrt(T)) + l*sigma*np.sqrt(T)
  y1 = (np.log(H/S0))/(sigma*np.sqrt(T)) + l*sigma*np.sqrt(T)

  DI_put = - S0 * np.exp(-q*T) * norm.cdf(-x1) + K2 * np.exp(-r*T) * norm.cdf(-x1 + \
        sigma*np.sqrt(T)) + S0 * np.exp(-q*T) * (H/S0)**(2*l) * (norm.cdf(y) - \
        norm.cdf(y1)) - K2 * np.exp(-r*T) * (H/S0)**(2*l - 2) * (norm.cdf(y - \
        sigma*np.sqrt(T)) - norm.cdf(y1 - sigma*np.sqrt(T)))

  DO_put = put_vanilla - DI_put

  # so the final price of the certificate is:
  price = bond - m * put + m * DO_put * (S0 > H)

  return price

print(f"The price of the Certificate is: {PriceBS(S0,sigma,r,T):.2f} USD")
```

Greeks

```
prices = np.array([PriceBS(S, sigma, r, T) for S in ST])
# Plot
plt.plot(ST, prices, label=f"$\\sigma = {sigma}$", color="blue")
plt.plot(S0, PriceBS(S0,sigma,r,T), 'ro')
plt.title("Price of the Certificate, as a function of $S$")
plt.xlabel("Underlying Price $S$")
plt.ylabel("Price")
plt.grid(True)
plt.legend()
plt.show()
# =========== Greeks ============
# Input values
h = 0.01
eps = 1e-4
# Delta
price_up = PriceBS(S0 + h, sigma, r, T)
price_down = PriceBS(S0 - h, sigma, r, T)
price = PriceBS(S0, sigma, r, T)
deltaC = (price_up - price_down) / (2 * h)
# Gamma
gamma_valC = (price_up - 2 * price + price_down) / h**2
# Vega
price_sigma_up = PriceBS(S0, sigma + eps, r, T)
price_sigma_down = PriceBS(S0, sigma - eps, r, T)
vega_valC = (price_sigma_up - price_sigma_down) / (2 * eps)
# Volga
volga_valC = (price_sigma_up - 2 * price + price_sigma_down) / eps**2
# Rho
price_r_up = PriceBS(S0, sigma, r + eps, T)
price_r_down = PriceBS(S0, sigma, r - eps, T)
rho_valC = (price_r_up - price_r_down) / (2 * eps)
# Theta (numerical derivative w.r.t. time passing)
price_T_down = PriceBS(S0, sigma, r, T - h)
theta_valC = - (price - price_T_down) / h
# Vanna
price_up_up = PriceBS(S0 + h, sigma + eps, r, T)
price_up_down = PriceBS(S0 + h, sigma - eps, r, T)
price_down_up = PriceBS(S0 - h, sigma + eps, r, T)
price_down_down = PriceBS(S0 - h, sigma - eps, r, T)
vanna_valC = (price_up_up - price_up_down - price_down_up + price_down_down)/(4*h*eps)
# Print the Greeks at S0
print(f"\nGreeks at S = S0 = {S0}:")
print(f"Delta  = {deltaC:.6f}")
print(f"Gamma  = {gamma_valC:.6f}")
print(f"Vega   = {vega_valC:.6f}")
print(f"Volga  = {volga_valC:.6f}")
print(f"Rho    = {rho_valC:.6f}")
print(f"Theta  = {theta_valC:.6f}")
print(f"Vanna  = {vanna_valC:.6f}")
```

Greeks II

```
h = 0.01
eps = 1e-4
deltas = []
gammas = []
vegas = []
rhos = []
thetas = []
volgas = []
vannas = []

for S in ST:
    price_up = PriceBS(S + h, sigma, r, T)
    price_down = PriceBS(S - h, sigma, r, T)
    price = PriceBS(S, sigma, r, T)
    delta = (price_up - price_down) / (2 * h)
    deltas.append(delta)
    gamma_val = (price_up - 2 * price + price_down) / h**2
    gammas.append(gamma_val)
    price_sigma_up = PriceBS(S, sigma + eps, r, T)
    price_sigma_down = PriceBS(S, sigma - eps, r, T)
    vega_val = (price_sigma_up - price_sigma_down) / (2 * eps)
    vegas.append(vega_val)
    volga_val = (price_sigma_up - 2 * price + price_sigma_down) / eps**2
    volgas.append(volga_val)
    price_r_up = PriceBS(S, sigma, r + eps, T)
    price_r_down = PriceBS(S, sigma, r - eps, T)
    rho_val = (price_r_up - price_r_down) / (2 * eps)
    rhos.append(rho_val)
    price_T_down = PriceBS(S, sigma, r, T - h)
    theta_val = - (price - price_T_down) / (h)
    thetas.append(theta_val)
    price_up_up = PriceBS(S + h, sigma + eps, r, T)
    price_up_down = PriceBS(S + h, sigma - eps, r, T)
    price_down_up = PriceBS(S - h, sigma + eps, r, T)
    price_down_down = PriceBS(S - h, sigma - eps, r, T)
    vanna_val = (price_up_up-price_up_down-price_down_up+price_down_down)/(4*h*eps)
    vannas.append(vanna_val)

fig, axes = plt.subplots(4, 2, figsize=(14, 16))
axes = axes.flatten()
greeks = [
    ("Delta", deltas, deltaC),
    ("Gamma", gammas, gamma_valC),
    ("Vega", vegas, vega_valC),
    ("Rho", rhos, rho_valC),
    ("Theta", thetas, theta_valC),
    ("Volga", volgas, volga_valC),
    ("Vanna", vannas, vanna_valC)
]
```

3D Plots

```
# Greeks Plot
for i, (name, values, value_at_S0) in enumerate(greeks):
    axes[i].plot(ST, values, label=name)
    axes[i].plot(S0, value_at_S0, 'ro')
    axes[i].set_title(f"{name} of the Certificate")
    axes[i].set_xlabel("Underlying Level (S)")
    axes[i].set_ylabel(name)
    axes[i].grid(True)
    axes[i].axhline(0, color="gray", linestyle="--")
fig.delaxes(axes[-1])
plt.tight_layout()
plt.show()
# 3D Plot
import plotly.graph_objects as go
S_vals = np.linspace(2000, 4000, 50)
sigma_vals = np.linspace(0.1, 0.5, 50)
S_grid, sigma_grid = np.meshgrid(S_vals, sigma_vals)
price_grid = np.array([[PriceBS(S, sig, r, T) for S in S_vals] for sig in sigma_vals])

from mpl_toolkits.mplot3d import Axes3D
fig = plt.figure(figsize=(12, 8))
ax = fig.add_subplot(111, projection='3d')
surf = ax.plot_surface(S_grid, sigma_grid, price_grid, cmap='viridis',
       edgecolor='k', alpha=0.9)

ax.set_title('Prezzo del certificato in funzione di S e sigma')
ax.set_xlabel('Underlying Price (S)')
ax.set_ylabel('Volatility')
ax.set_zlabel('Certificate Price')
fig.colorbar(surf, ax=ax, shrink=0.5, aspect=10)
plt.tight_layout()
plt.show()

fig = go.Figure(data=[go.Surface(z=price_grid, x=S_vals, y=sigma_vals,
      colorscale='Viridis')])
fig.update_layout(
    title='Prezzo del certificato in funzione di S e sigma',
    scene=dict(
        xaxis_title='Underlying Price (S)',
        yaxis_title='Volatility',
        zaxis_title='Certificate Price'
    ),
    autosize=True,
    margin=dict(l=40, r=40, b=40, t=40)
)

fig.show()
```

10.2 Attempted Heston Model Calibration

In the following boxes, the reader can find the implementation of the calibration scheme used for the Heston model.

Importing the Libraries

```
import pandas as pd
import numpy as np
from scipy.stats import norm
import matplotlib.pyplot as plt
from scipy.optimize import minimize
```

Reading the Data

```
vix_data = pd.read_csv("vix.csv")
sp400_data = pd.read_csv("sp400.csv")
vix_data.rename(columns={"Price":"VIXvalue"}, inplace=True)
sp400_data.rename(columns={"Price":"SPvalue"}, inplace=True)
merged = pd.merge(vix_data, sp400_data, left_index=True, right_index=True, how="inner")
merged["VIXvalue"]=pd.to_numeric(merged["VIXvalue"], errors="coerce")
merged["SPvalue"]=pd.to_numeric(merged["SPvalue"], errors="coerce")
VIX_r = np.log(merged['VIXvalue'] / merged['VIXvalue'].shift(-1))
SP_r = np.log(merged['SPvalue'] / merged['SPvalue'].shift(-1))
VIX_r = VIX_r.dropna()
SP_r = SP_r.dropna()
```

Converting the Data

```
prices = pd.read_excel("prices.xlsx", sheet_name=2)
mktprices = prices["media"].to_numpy()
maturities = prices["Days"].to_numpy()
strikes = prices["strike"].to_numpy()
tipi = prices["type"].to_numpy()
sp400_data
```

Parameters Estimation from Historical Data

```
correlation_matrix=np.corrcoef(VIX_r, SP_r)
v_0 = (np.std(SP_r))*np.sqrt(252)
```

Option Price Simulation Function

```
def simulazione(S0, K, r, V0, kappa, theta, sigma_vol, Nsim, T, tipo):
    rho = -0.03506433
    m = 200
    dt = T/m
    Vt = np.zeros(shape=(Nsim, m+1))
    Vt[:,0]=V0
    St = np.zeros(shape=(Nsim, m+1))
    St[:,0]=S0
    for j in range(1, m+1):
        Z1 = np.random.normal(size=Nsim)
        Z2 = np.random.normal(size=Nsim)
        St[:, j]=St[:, j-1] + r*St[:, j-1]*dt +
                np.sqrt(Vt[:, j-1])*St[:, j-1]*Z1*np.sqrt(dt)
        Vt[:, j]=np.maximum(0, Vt[:, j-1] + kappa*(theta - Vt[:, j-1])*dt +
                sigma_vol*np.sqrt(Vt[:, j-1])*np.sqrt(dt)*(rho*Z1 + np.sqrt(1
                -rho**2)*Z2))
    if tipo == "call":
        call_price_vector=np.exp(-r*T)*np.maximum(0, St[:,-1]-K)
        simulated_price=np.mean(call_price_vector)
    elif tipo == "put":
        put_price_vector=np.exp(-r*T)*np.maximum(0, K-St[:,-1])
        simulated_price=np.mean(put_price_vector)
    else:
        raise ValueError("deve essere c per la call o p per la put")
    return simulated_price
```

Prices Vector Function

```
def simulated_prices_vector(S0, strikes, r, V0, kappa, theta, sigma_vol,
Nsim, maturities, tipi):
    simulated_prices = np.zeros(len(maturities))
    for i in range(len(maturities)):
        simulated_prices[i] = simulazione(S0, strikes[i], r, V0, kappa,
        theta, sigma_vol, Nsim, maturities[i], tipi[i])
    return simulated_prices
```

Objective Function Definition

```
def objFunction(x, mktprices, S0, strikes, r, V0, Nsim, maturities, tipi):
    kappa = x[0]
    theta = x[1]
    sigma_vol = x[2]
    return np.sum(mktprices - simulated_prices_vector(S0, strikes, r, V0,
    kappa, theta, sigma_vol, Nsim, maturities, tipi))**2
```

Minimization

```
risk_free = 0.0399
VIX = 0.16015**2
simulazioni = 50000
S0 = 541.37
bounds = [(0.1, 1), (0.001, 1), (0.01, 0.1)]

N_opt = 100
results = []
for k in range(N_opt):
    ig = np.array([np.random.uniform(low,high) for low,high in bounds])
    parameters = minimize(objFunction, ig, args=(mktprices, S0, strikes,
    risk_free, VIX, simulazioni, maturities, tipi), bounds=bounds, method='Nelder-Mead')
    print("parametri: ", parameters.x)
    print("minimo: ", parameters.fun)
    results.append([parameters.x, parameters.fun])

df_out = pd.DataFrame(results)
df_out.to_excel("risultati_minimizzazione.xlsx", index = False)
```