BROWNIAN MOTION AND STOCHASTIC CALCULUS HW 3

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Discussed with classmates.

Exercise 1.

Proof.

By Taylor we can write

$$MV(f;0,\varepsilon) = \int_{B_\varepsilon} \left(f(0) + \sum \partial_i f(0)x_i + \frac{1}{2} \sum \partial_i^2 f(0)x_i^2 + \frac{1}{1!1!} \sum_{i \neq j} \partial_i \partial_j f(0)x_i x_j \right) ds + o(|x|^2)$$

and we note that the second and fourth term above is actually 0:

$$\int_{B_\varepsilon} \sum \partial_i f(0)x_i ds = \sum \partial_i f(0) \int_{B_\varepsilon} x_i ds = 0$$

by symmetry, and note that the operator

$$F_j : (\mathbb{R}^d)^* \to (\mathbb{R}^d)^*$$

where

$$F_j(f(x_1, x_2, \dots, x_n)) = f(x_1, \dots, -x_j, \dots, x_n)$$

is just flipping the space, thus is invariant under integral over ε-sphere for $f = \int_{B_\varepsilon} ds$, i.e.

$$\begin{aligned}\int_{B_\varepsilon} \sum_{i \neq j} \partial_i \partial_j f(0)x_i x_j ds &= \sum_{i \neq j} \partial_i \partial_j f(0) \int_{B_\varepsilon} ds = \sum_{i \neq j} \partial_i \partial_j f(0) F\left(\int_{B_\varepsilon} ds \right) \\ &= -\sum_{i \neq j} \partial_i \partial_j f(0) \int_{B_\varepsilon} ds\end{aligned}$$

and hence the value is 0. Now putting everything together we have (in definition of MV, the surface integral is normalized)

$$\begin{aligned}\lim_{\varepsilon \downarrow 0} \frac{MV(f;0,\varepsilon) - f(0)}{\varepsilon^2} &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \int_{B_\varepsilon} \frac{1}{2} \sum \partial_i^2 f(0)x_i^2 ds + o(\varepsilon^2) \\ &= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \frac{1}{2d} \Delta f(0) = \frac{1}{2d} \Delta f(0)\end{aligned}$$

□

Exercise 2.

Proof.

(1): First, WLOG let's shift the whole process down by $\pi/2$ so that we start now at 0. Note that $B_T \in \{-\pi/2, \pi/2\}$, and by Doob's stopping time we know

$$\mathbb{P}(B_T = \pi/2) = \mathbb{P}(B_T = -\pi/2) = \frac{1}{2}.$$

For $a \in \mathbb{R}$ we can have

$$\mathbb{P}(T < a, B_T = \pi/2) = \mathbb{P}(T < a, B_T = -\pi/2)$$

since $X \sim -X$. But note that

$$\mathbb{P}\left(\{T < a, B_T = \pi/2\} \cup \{T < a, B_T = -\pi/2\}\right) = \mathbb{P}(T < a)$$

by definition of B_T and thus

$$\mathbb{P}(T < a, B_T = \pi/2) = \frac{1}{2}\mathbb{P}(T < a) = \mathbb{P}(B_T = \pi/2)\mathbb{P}(T < a)$$

now for any $b \notin [-\pi/2, \pi/2)$ the relation

$$\mathbb{P}(T < a, B_T > b) = \mathbb{P}(B_T > b)\mathbb{P}(T < a)$$

trivially holds, and for any $b \in [-\pi/2, \pi/2)$ we know $\{B_T > b\} = \{B_T = \pi/2\}$ and thus

$$\mathbb{P}(T < a, B_T > b) = \mathbb{P}(B_T > b)\mathbb{P}(T < a)$$

also holds. So we know T and B_T are independent.

(2): WLOG we shift again to a Brownian motion starting at 0 and the stopping time T becomes $\min\{t : B_t = a \vee B_t = -b\}$ where $a + b = \pi$ and WLOG let $a < b$.

By Doob's stopping time we know B_T is a Martingale (same argument as in last time) and $\mathbb{E}[B_T] = \mathbb{E}[B_0] = 0$. Thus

$$\mathrm{Cov}(T, B_T) = \mathbb{E}[T B_T]$$

it suffices to show it nonzero.

Lemma 0.1. *$M_t := B_t^3 - 3tB_t$ is a Martingale.*

Proof. We omit the proof of L^1 and adapted since it's the same as we did in last homework, question 3. For Martingale property, we note that for $s < t$

$$B_t^3 = \left(B_s + (B_t - B_s)\right)^3 = B_s^3 + 3B_s^2(B_t - B_s) + 3B_s(B_t - B_s)^2 + (B_t - B_s)^3$$

and taking conditional expectation we have

$$\begin{aligned}
\mathbb{E}[B_t^3|\mathcal{F}_s] &= B_s^3 + 3B_s^2\mathbb{E}[B_t - B_s|\mathcal{F}_s] + 3B_s\mathbb{E}[(B_t - B_s)^2|\mathcal{F}_s] + \mathbb{E}[(B_t - B_s)^3|\mathcal{F}_s] \\
&= B_s^3 + 3B_s^2 \cdot 0 + 3B_s(t - s) + 0 = B_s^3 + 3(t - s)B_s
\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[3tB_t|\mathcal{F}_s] &= 3sB_s + 3s\mathbb{E}[B_t - B_s|\mathcal{F}_s] + 3(t-s)\mathbb{E}[B_s|\mathcal{F}_s] \\ &= 3sB_s + 3s \cdot 0 + 3B_s(t-s)\end{aligned}$$

and thus

$$\mathbb{E}[B_t^3 - 3tB_t|\mathcal{F}_s] = B_s^3 - 3sB_s = \mathbb{E}[B_s^3 - 3sB_s|\mathcal{F}_s]$$

so it is indeed a martingale. □

(A corollary of the above is that we can find Martingales of any degree using the coefficients of $(x-y)^k$ for even degrees on y, but we'll not need that here.)

Now Lemma + Doob's stopping time theorem says $B_T^3 - 3TB_T$ is a Martingale and thus we can compute

$$3\mathbb{E}[TB_T] = \mathbb{E}[B_T^3] = a^3\frac{b}{a+b} - b^3\frac{a}{a+b} = ab(a-b) \neq 0$$

thus

$$\mathrm{Cov}(T, B_T) = \mathbb{E}[TB_T] \neq 0$$

which concludes the proof.

□

Exercise 3.

Proof.

(1):

To solve:

$$\begin{cases} \partial_t p_t(y) = \frac{1}{2}\Delta_y p_t(y) \\ P_0(y) = \delta_x \end{cases}$$

Let's assume the solution is of the form

$$p_t(y) = e^{-\lambda t}\phi(y)$$

then the equation gives

$$\phi''(y) = -2\lambda\phi(y)$$

which yields only solution

$$\phi(y) = c_1 \sin(\sqrt{2\lambda}y) + c_2 \cos(\sqrt{2\lambda}y)$$

where imposing the boundary condition $p_t(0) = p_t(\pi)$ really gives us that $\lambda = \frac{k^2}{2}$ for $k \in \mathbb{Z}^*$ (since we want it to decay) and $p_t(y) = \sin(ky)$. Now linearity of solution yields formally

$$p_t(y) = \sum_{k=1}^{\infty} a_k e^{-k^2 t/2} \sin(ky)$$

where it's only defined if it converges. But plugging this in initial condition gives a_k to be the Fourier coefficients of $\delta_x(y)$ which in turn gives

$$a_k = \frac{2}{\pi}\int_0^{\pi} \delta_x(y)\sin(ky)dy = \frac{2}{\pi}\sin(kx)$$

and thus simply plugging in to our assumed function we get

$$p_t(y) = \frac{2}{\pi}\sum_{k=1}^{\infty} e^{-k^2 t/2} \sin(kx)\sin(ky)$$

which is indeed convergent since $e^{-k^2} \ll o(k^{-2})$ as $k \to \infty$ (this is even a computable geometric series).

(2):

When $t \to \infty$ the leading term is just $k = 1$ and thus

$$p_t(x, y) \sim \frac{2}{\pi}e^{-t/2}\sin(x)\sin(y)$$

in particular, to fit the form of question we have

$$\lambda = -\frac{1}{2}, \quad c(x) = \frac{2}{\pi}\sin(x), \quad \tilde{c}(y) = \sin(x).$$

(3):

We can of course give a full characterization of things using the formula

$$p_t(y) = \frac{2}{\pi}\sum_{k=1}^{\infty} e^{-k^2 t/2}\sin(kx)\sin(ky)$$

and integrate against y, but since we are eventually taking limit $t \to \infty$ and t and y are separable so eventually only the leading term matters. But to be less sloppy let's say that for any ε there exists large enough t such that (by (2))

$$\frac{1-\varepsilon}{1+\varepsilon}\frac{\int_a^b \frac{2}{\pi}e^{-t/2}\sin(x)\sin(y)dy}{\int_0^\pi \frac{2}{\pi}e^{-t/2}\sin(x)\sin(y)dy} \le \frac{\int_a^b p_t(y)dy}{\int_0^\pi p_t(y)dy} \le \frac{1+\varepsilon}{1-\varepsilon}\frac{\int_a^b \frac{2}{\pi}e^{-t/2}\sin(x)\sin(y)dy}{\int_0^\pi \frac{2}{\pi}e^{-t/2}\sin(x)\sin(y)dy}$$

and since ε is arbitrary we see that after limit the terms are really the same, thus plugging in we get

$$\begin{aligned}\lim_{t\to\infty}\mathbb{P}^x\left\{B_t \in I | t < T\right\} &= \lim_{t\to\infty}\frac{\int_a^b p_t(y)dy}{\int_0^\pi p_t(y)dy} = \lim_{t\to\infty}\frac{\int_a^b \frac{2}{\pi}e^{-t/2}\sin(x)\sin(y)dy}{\int_0^\pi \frac{2}{\pi}e^{-t/2}\sin(x)\sin(y)dy} \\ &= \frac{\cos a - \cos b}{2}.\end{aligned}$$

(4):

To not stop at $2t$ is to arrive at a certain point z at time t then go to y after another t.

To be explicit, we have that as $t \to \infty$ the leading term behaves like

$$\begin{aligned}p_{2t}(x,y) &= \int_0^\pi p_t(x,z)p_t(z,y)dz = \int_0^\pi \frac{2}{\pi}e^{-t/2}\sin(x)\sin(z)\frac{2}{\pi}e^{-t/2}\sin(z)\sin(y)dz \\ &= \frac{4}{\pi^2}e^{-t}\sin(x)\sin(y)\int_0^\pi \sin^2(z) = \frac{2}{\pi}e^{-t}\sin(x)\sin(y)\end{aligned}$$

and to compute the nominator we use a similar integral which states: density such that $B_t \in L$ conditioned on $T > 2t$:

$$\begin{aligned}p_{2t,a,b}(x,y) &= \int_a^b p_t(x,z)p_t(z,y)dz = \int_a^b \frac{2}{\pi}e^{-t/2}\sin(x)\sin(z)\frac{2}{\pi}e^{-t/2}\sin(z)\sin(y)dz \\ &= \frac{4}{\pi^2}e^{-t}\sin(x)\sin(y)\int_a^b \sin^2(z) = \frac{(2b-2a+\sin(2a)-\sin(2b))}{4}\frac{4}{\pi^2}e^{-t}\sin(x)\sin(y)\end{aligned}$$

and thus by putting those together (skipping the exact same approximating argument as above):

$$\begin{aligned}\lim_{t\to\infty}\mathbb{P}^x\left\{B_t \in I | 2t < T\right\} &= \lim_{t\to\infty}\frac{\int_0^\pi p_{2t,a,b}(x,y)dy}{\int_0^\pi p_{2t}(x,y)dy} = \lim_{t\to\infty}\frac{\frac{(2b-2a+\sin(2a)-\sin(2b))}{4}\frac{4}{\pi^2}e^{-t}\sin(x)\sin(y)}{\frac{2}{\pi}e^{-t}\sin(x)\sin(y)} \\ &= \frac{(2b-2a+\sin(2a)-\sin(2b))}{2\pi}.\end{aligned}$$

□

Exercise 4.

Proof.

(1): By Markov property, a Brownian motion starting at a new time is just a shifted Brownian motion, hence by independent increment

$$\sup_{x\in D}\mathbb{P}^x\{T>s+t\}=\sup_{x\in D}\mathbb{P}^x\{T>s\}\mathbb{P}^{B_s^x}\{T>t\}\le\sup_{x\in D}\mathbb{P}^x\{T>s\}\sup_{x\in D}\mathbb{P}^x\{T>t\}$$

in other words

$$q_{s+t}\le q_s q_t.$$

(2):

Notice that $f(t):=\log q_t$ is subadditive since $\log q_{t+s}\le\log q_t q_s=\log q_t+\log q_s$. Denote $s:=\inf_{t>0}\frac{f(t)}{t}$ (note this can be $-\infty$). We want to show that

$$\lim_{t\to\infty}\frac{f(t)}{t}=\inf_{t>0}\frac{f(t)}{t}.$$

Suppose contrary, then there exists a sequence t_n such that $\frac{f(t_n)}{t_n}\ge s+\varepsilon$ for some ε.

By definition of inf there exist $T\in(0,\infty)$ such that $\frac{f(T)}{T}<s+\frac{\varepsilon}{2}$. Since q_t is a probability, we have $f(t)\le 0$, and thus if we denote $t_n\in(mT,(m+1)T)$ then

$$\frac{f(t_n)}{t_n}\le\frac{f(mT)+f(t_n-mT)}{t_n}\le\frac{f(T)}{t_n/m}+\frac{f(t_n-mT)}{t_n}$$

where as $t_n\to\infty$ we know $\frac{t_n}{m}\le T+\frac{T}{m}\to T$ and hence

$$\lim_{n\to\infty}\frac{f(t_n)}{t_n}\le\frac{f(T)}{T}<s+\varepsilon/2$$

is a contradiction. Hence

$$\lim_{t\to\infty}\frac{\log q_t}{t}=\inf_{t>0}\frac{\log q_t}{t}$$

and we only need to show

$$\inf_{t>0}\frac{\log q_t}{t}\in(-\infty,0).$$

The <0 direction is clear, so we only need to convince ourselves that $\lim_{t\to\infty}\log q_t=O(t)$ for which we use the later bound.

Now, let $r:=r_x:=\operatorname{dist}(x,D)$ and $R:=R_x:=\max_{t\in\partial D}\operatorname{dist}(x,t)$ which are just the smallest and largest distance of x to the boundary. Now define $T_R^x:=\min\{t:|B_t-x|=R\}$ and $T_r^x:=\min\{t:|B_t-x|=r\}$, also, denote

$$T_R^{x,1}=\min\{t:B_t^1-x_1\ge R\},\quad T_r^{x,1}=\min\{t:B_t^1-x_1\ge r/\sqrt{d}\}$$

then

$$\mathbb{P}(T_R^{x,1} > t) \le \mathbb{P}^x\{T_R^x > t\} \le \mathbb{P}^x\{T > t\} \le \mathbb{P}\{T_r^x > t\} \le 2\mathbb{P}(T_r^{x,1} > t)$$

since to reach outside of B_R contains (as an event) going above $+R$ in one dimension, and to reach B_r at least one dimension has to go beyond $r/\sqrt{d}$, which again we use union bound to say that (probability of either reaching from below and from above) $\le$ (probability of reaching from below) + (probability of reaching from above).

So we can get our result if we can find a lower bound exponential in t of $\mathbb{P}(T_r^{x,1} > t)$. To be exponential in t roughly suggests that we should cut time in $\lceil t \rceil$ pieces, so we only need to show that there's positive probability that $\mathbb{P}(T_r^{x,1} > 1,$ start and ends in a same situation).

From discussion in class, we can find a neighborhood N_x of x, contained in ball of radius r, such that $\forall y \in N_x$, $\mathbb{P}(T_r^{y,1} > 1, B_1 \in N_x) > c_1$ for some constant c_1. In particular $x \in N_x$ so we get a bound. Thus

$$\mathbb{P}(T_r^{x,1} > t) \ge c_1^{\lceil t \rceil \frac{t}{\lceil t \rceil}} \ge c_1^{2\lceil t \rceil}$$

hence taking log

$$\log \mathbb{P}(T_r^{x,1} > t) \lessapprox O(t)$$

hence we are done for approximation of one point. But since all inequalities holds for all point (well, we can bound $r_x \le \mathrm{diam}(D)$), we get the bound that

$$\lim_{t\to\infty} \frac{\log q_t}{t} = C > -\infty$$

hence $\lambda \in (0, \infty)$.

(3):

First, we show that the sup is taken at $x = 0$. Thus, now shifting x to $a/2$ the boundary condition implies λ (from Q3 (1)) is $\dfrac{k^2\pi^2}{8a^2}$ which shows

$$p_t(y) = \sum_{k=1}^{\infty} a_k e^{-k^2\pi^2 t/8a^2} \sin\left(\frac{k\pi}{2a} y\right)$$

and by letting $t \to \infty$ we get

$$p_t(x, y) \sim \frac{2}{\pi} e^{-\pi^2 t/8a^2} \sin\left(\frac{\pi}{2a} x\right) \sin\left(\frac{\pi}{2a} y\right)$$

and hence

$$\begin{aligned}
\lim_{t\to\infty} \frac{\log q_t}{t} &= \lim_{t\to\infty} \frac{\log \mathbb{P}(T > t)}{t} = \lim_{t\to\infty} \frac{\log \int_0^{2a} \frac{2}{\pi} e^{-\pi^2 t/8a^2} \sin\left(\frac{\pi}{2a} x\right) \sin\left(\frac{\pi}{2a} y\right) dy}{t} \\
&= \lim_{t\to\infty} \frac{\log \frac{4}{\pi} - \frac{t\pi^2}{8a^2}}{t} \overset{x=a}{=} -\frac{\pi^2}{8a^2} + \lim_{t\to\infty} O(t^{-1}) = -\frac{\pi^2}{8a^2} = -\lambda_D.
\end{aligned}$$

$\square$

Exercise 5.

Proof.

(1): By DCT (compact support) and simple function approximation we can exchange the limit

$$\mathbb{E}^x\left[\int_0^\infty f(B_t)dt\right] = \int_0^\infty \mathbb{E}^x[f(B_t)]dt = \int_{\mathbb{R}^d} f(y)P_t(x,y)dy$$

for $P_t(x,y) = \frac{1}{(2\pi t)^{d/2}}e^{-|y-x|^2/2t}$ as defined in class. Plugging in we have

$$\begin{aligned}\mathbb{E}^x\left[\int_0^\infty f(B_t)dt\right] &= \int_0^\infty \mathbb{E}^x[f(B_t)]dt = \int_0^\infty \int_{\mathbb{R}^d} f(y)\frac{1}{(2\pi t)^{d/2}}e^{-|y-x|^2/2t}dydt \\ &= \int_{\mathbb{R}^d}\int_0^\infty \frac{1}{(2\pi t)^{d/2}}e^{-|y-x|^2/2t}dtf(y)dy = \int_{\mathbb{R}^d} G(x,y)f(y)dy\end{aligned}$$

where we've changed integral again by f compact support.

(2):

(Idea by Zihao He)

For $\varepsilon > 0$ define stopping time $T_\varepsilon := \inf\{t \geq 0 : |B_t - x| \geq \varepsilon\}$. And we separate the integral to get

$$\phi(x) = \mathbb{E}^x\left[\int_0^{T_\varepsilon} f(B_t)dt\right] + \mathbb{E}^x\left[\int_{T_\varepsilon}^\infty f(B_t)dt\right]$$

and by Strong Markov property we just shift by B_{T_ε} which is the same thing as uniformly shift to the ε ball:

$$\mathbb{E}^x\left[\int_{T_\varepsilon}^\infty f(B_t)dt\right] = \mathbb{E}^x\left[\mathbb{E}^{B_{T_\varepsilon}}\left[\int_0^\infty f(B_{t+T_\varepsilon})dt\right]\right] = \mathbb{E}^x[\phi(B_{T_\varepsilon})] = MV(\phi;x,\varepsilon)$$

but now miraculously we plug in question 1 to get

$$\frac{1}{2}\Delta\phi(x) = d\lim_{\varepsilon\to 0}\frac{MV(\phi;x,\varepsilon)-\phi(x)}{\varepsilon^2} = d\lim_{\varepsilon\to 0}\frac{-\mathbb{E}^x\left[\int_0^{T_\varepsilon} f(B_t)dt\right]}{\varepsilon^2}$$

and as $\varepsilon \to 0$ by Taylor (compact so $||f'|| < \infty$) on f we have $\int_0^{T_\varepsilon} f(B_t)dt = f(x)\cdot T_\varepsilon + o(\varepsilon)$ and from last homework we know the expectation of stopping time is $\mathbb{E}^x[T_\varepsilon] = \frac{\varepsilon^2}{d}$ and thus

$$\frac{-\mathbb{E}^x\left[\int_0^{T_\varepsilon} f(B_t)dt\right]}{\varepsilon^2} = \frac{-d\frac{\varepsilon^2}{d}f(x) + o(\varepsilon^3)}{\varepsilon^2}$$

taking the limit it goes to $-f(x)$.

$\square$