APPLIED DYNAMICAL SYSTEM HOMEWORK 6

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STAT 31410
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General ideas were discussed with many classmates in casual talks.

Exercise 1. *(5.10)*

The ODE system is

$$\begin{cases} \dot{x} = y + 2z + (x+z)^2 + xy - y^2 \\ \dot{y} = (x+z)^2 \\ \dot{z} = -2z - (x+z)^2 + y^2 \end{cases}$$

So to form it into the form of (5.34), we need to first identify its linearized reduction, i.e. E^s, E^c, E^u. This can be done via Jordan elimination of the linearized matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

for which by computing the eigenvalues we get that there's 2 center eigenspace dimensions and on stable, since the eigenvalues are $0, 0, -2$, therefore we will only have two equations of the form 5.34.

Row reducing the last term in the first row yields the following Jordan Ruduction:

$$A = PJP^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which means

$$P^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = JP^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{pmatrix} \dot{\alpha_1} \\ \dot{\alpha_2} \\ \dot{\beta} \end{pmatrix} = J \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ 0 \\ -2\beta \end{pmatrix}$$

and the exact form is (since $P^{-1}(x, y, z) = (\alpha, \beta)$)

$$\begin{cases} \dot{\alpha} = C\alpha + f(\alpha, \beta) \\ \dot{\beta} = S\beta + g(\alpha, \beta) \end{cases}$$

where

$$C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad S = (-2); \quad f(\alpha, \beta) = \begin{pmatrix} \alpha_1 \alpha_2 \\ \alpha_1^2 \end{pmatrix}; \quad g(\alpha, \beta) = -\alpha_1^2 + \alpha_2^2$$

(b): Let the equation of the center manifold be $(\alpha_1, \alpha_2, h(\alpha))$, and on $W^c_{loc}(0,0,0)$ we have

$$\frac{d\beta}{dt} = \frac{dh(\alpha)}{dt} = \frac{dh(\alpha)}{d\alpha_1}\frac{d\alpha_1}{dt} + \frac{dh(\alpha)}{d\alpha_2}\frac{d\alpha_2}{dt}$$

$$\Rightarrow -2h(\alpha) - \alpha_1^2 + \alpha_2^2 = \frac{dh(\alpha)}{d\alpha_1}(\alpha_2 + \alpha_1\alpha_2) + \frac{dh(\alpha)}{d\alpha_2}\alpha_1^2$$

but by Taylor we have that for α small enough (where h_{ij} refers to the Hessian at the origin)

$$h(\alpha) = h(0,0) + Dh(0,0)\alpha + \alpha^T \frac{1}{2} D^2 h(0,0)\alpha + \cdots = \frac{1}{2}h_{11}\alpha_1^2 + h_{12}\alpha_1\alpha_2 + \frac{1}{2}h_{22}\alpha_2^2 + o(\alpha^3)$$

So we can plug back to get

$$-2h(\alpha) - \alpha_1^2 + \alpha_2^2 = (h_{11}\alpha_1 + h_{12}\alpha_2)(\alpha_2 + \alpha_1\alpha_2) + (h_{22}\alpha_2 + h_{12}\alpha_1)\alpha_1^2 + o(\alpha^3)$$

and we can further plug $h(\alpha)$ into the left hand side and ge the following system if we neglect higher order terms:

$$\begin{cases} -(h_{11}+1)\alpha_1^2 & = & 0 \\ -2h_{12}\alpha_1\alpha_2 & = & h_{11}\alpha_1\alpha_2 \\ (1-h_{22})\alpha_2^2 & = & h_{12}\alpha_2^2 \end{cases} \quad \Rightarrow \quad \begin{cases} h_{11} & = & -1 \\ h_{12} & = & 1/2 \\ h_{22} & = & 1/2 \end{cases}$$

so the quadratic approximation is

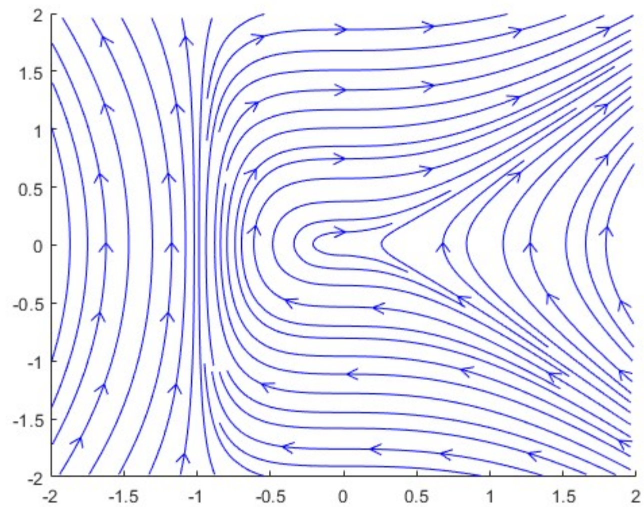
$$W^c_{loc}(0,0,0) = \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ -\alpha_1^2 + \frac{1}{2}\alpha_1\alpha_2 + \frac{1}{2}\alpha_2^2 \end{pmatrix} : \ (\alpha) \in \mathbb{R}^2 \right\}$$

(c):

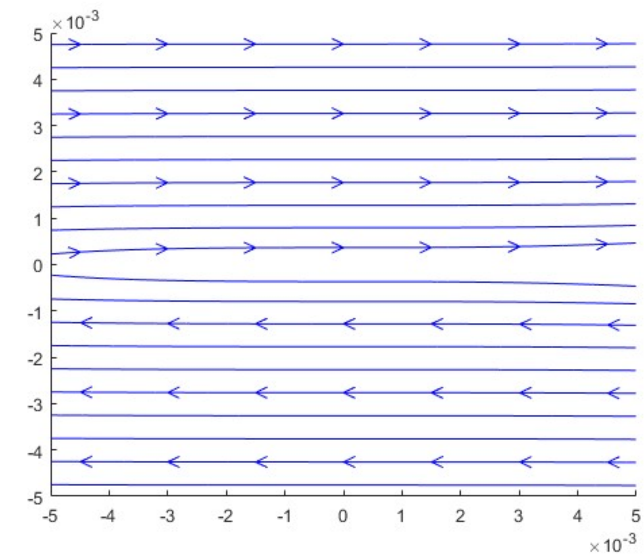
We have

$$\dot{\alpha} = C\alpha + f\left(\alpha, -\alpha_1^2 + \frac{1}{2}\alpha_1\alpha_2 + \frac{1}{2}\alpha_2^2\right) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\alpha + \begin{pmatrix} \alpha_1\alpha_2 \\ \alpha_1^2 \end{pmatrix} = \begin{pmatrix} \alpha_2 + \alpha_1\alpha_2 \\ \alpha_1^2 \end{pmatrix}$$

which yields the following graph:



which doesn't reveal much local behavior. Yet a closer look:



reveals that the origin is not stable.

Exercise 2. *(8.8)*

So we follow the exact step as in the $\mathbb{H}_2^2$ case to let (after reducion the quadratic term) the system in $\mathbb{H}_3^2$ be

$$\dot{X} = L + g_3(X) + o(X^3)$$

where we take

$$L = AX = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

and of course

$$DL = A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now, we try to eliminate the cubic term with the change of variable $X = Y + P_3(Y)$ for which we have

$$\dot{X} = \dot{Y} + DP_3\dot{Y} = [I + DP_3]\dot{Y}$$

and basic linear algebra tells us

$$(I + DP_3)^{-1} = I - DP_3 - DP_3^2 - \cdots = I - DP_3 + O(Y^4)$$

and by the original assumption we have

$$[I + DP_3]\dot{Y} = \dot{X} = A(Y + P_3(Y)) + g_3(Y + P_3(Y)) + O(Y^4)$$
$$\Rightarrow \dot{Y} = (I - DP_3 + O(Y^4)) \left(A(Y + P_3(Y)) + g_3(Y + P_3(Y)) + O(Y^4)\right)$$

where we note that $g_3(Y + P_3(Y)) = g_3(Y) + O(Y^2 \cdot P_3(Y)) = g_3(Y) + O(Y^5)$ to simplify:

$$\dot{Y} = AY + \left(AP_3(Y) - DP_3AY\right) + g_3(Y) + O(Y^4)$$

where using the Lie bracket $[v, u] = Duv - Dvu$ we try to minimize

$$[L, P_3] \approx g_3$$

Now we label the 8 cubic terms as

$$Y_1 = \begin{pmatrix} x^3 \\ 0 \end{pmatrix};\ Y_2 = \begin{pmatrix} x^2y \\ 0 \end{pmatrix};\ Y_3 = \begin{pmatrix} xy^2 \\ 0 \end{pmatrix};\ Y_4 = \begin{pmatrix} y^3 \\ 0 \end{pmatrix};$$
$$Y_5 = \begin{pmatrix} 0 \\ x^3 \end{pmatrix};\ Y_6 = \begin{pmatrix} 0 \\ x^2y \end{pmatrix};\ Y_7 = \begin{pmatrix} 0 \\ xy^2 \end{pmatrix};\ Y_8 = \begin{pmatrix} 0 \\ y^3 \end{pmatrix};$$

and then

$$\sum_{i=1}^{8} b_i[L, Y_i] = [L, P_3] \approx g_3 = \sum_{i=1}^{8} a_iY_i$$

for which we compute the following

$$[L, Y_1] = DY_1L - DLY_1 = \begin{pmatrix} 3x^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -y \\ x \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x^3 \\ 0 \end{pmatrix} = \begin{pmatrix} -3x^2y \\ -x^3 \end{pmatrix}.$$

The rest 7 are all calculated in the same way and the result is

$$[L,Y_1]=\begin{pmatrix}-3x^2y\\-x^3\end{pmatrix};\ [L,Y_2]=\begin{pmatrix}x^3-2xy^2\\-x^2y\end{pmatrix};\ [L,Y_3]=\begin{pmatrix}-y^3+2x^2y\\-xy^2\end{pmatrix};$$

$$[L,Y_4]=\begin{pmatrix}3xy^2\\-y^3\end{pmatrix};\ [L,Y_5]=\begin{pmatrix}x^3\\-3x^2y\end{pmatrix};\ [L,Y_6]=\begin{pmatrix}x^2y\\x^3-2xy^2\end{pmatrix};$$

$$[L,Y_7]=\begin{pmatrix}xy^2\\-y^3+2x^2y\end{pmatrix};\ [L,Y_8]=\begin{pmatrix}y^3\\3xy^2\end{pmatrix};$$

and

$$L=\begin{pmatrix}0&1&0&0&1&0&0&0\\-3&0&2&0&0&1&0&0\\0&-2&0&3&0&0&1&0\\0&0&-1&0&0&0&0&1\\-1&0&0&0&0&1&0&0\\0&-1&0&0&-3&0&2&0\\0&0&-1&0&0&-2&0&3\\0&0&0&-1&0&0&-1&0\end{pmatrix}$$

and the eigenvector and eigenvalues are encoded the following matlab result, where V contains eigenvalues as its columns and D has eigenvalues as its diagonal:

```
>> q88

V =

  -0.0000 - 0.1581i  -0.0000 + 0.1581i  -0.0038 - 0.3869i  -0.0038 + 0.3869i  -0.0357 - 0.0164i  -0.0357 + 0.0164i   0.1853 + 0.2871i   0.1853 - 0.2871i
   0.4743 + 0.0000i   0.4743 + 0.0000i   0.5268 - 0.0000i   0.5268 + 0.0000i   0.0246 + 0.2396i   0.0246 - 0.2396i   0.3650 + 0.0000i   0.3650 + 0.0000i
   0.0000 + 0.4743i   0.0000 - 0.4743i  -0.0115 - 0.1071i  -0.0115 + 0.1071i  -0.5862 + 0.0000i  -0.5862 + 0.0000i   0.1853 + 0.2871i   0.1853 - 0.2871i
  -0.1581 + 0.0000i  -0.1581 - 0.0000i   0.2470 - 0.0077i   0.2470 + 0.0077i   0.0082 - 0.3110i   0.0082 + 0.3110i   0.3650 - 0.0000i   0.3650 + 0.0000i
   0.1581 - 0.0000i   0.1581 + 0.0000i   0.2470 - 0.0077i   0.2470 + 0.0077i   0.0082 - 0.3110i   0.0082 + 0.3110i  -0.3650 - 0.0000i  -0.3650 + 0.0000i
   0.0000 + 0.4743i   0.0000 - 0.4743i   0.0115 + 0.1071i   0.0115 - 0.1071i   0.5862 - 0.0000i   0.5862 + 0.0000i   0.1853 + 0.2871i   0.1853 - 0.2871i
  -0.4743 + 0.0000i  -0.4743 - 0.0000i   0.5268 + 0.0000i   0.5268 + 0.0000i   0.0246 + 0.2396i   0.0246 - 0.2396i  -0.3650 + 0.0000i  -0.3650 - 0.0000i
  -0.0000 - 0.1581i  -0.0000 + 0.1581i   0.0038 + 0.3869i   0.0038 - 0.3869i   0.0357 + 0.0164i   0.0357 - 0.0164i   0.1853 + 0.2871i   0.1853 - 0.2871i


D =

  -0.0000 + 4.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.0000 - 4.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 + 2.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 - 2.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 2.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 2.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 - 0.0000i
```

and we're glad to see that there's 2 eigenvalues that are 0. Rather than checking that the two identified eigenvectors really is a combination of the last two rows, we just check them by computation. I did it by manually entering the representations of v_1 and v_2 in (8.42) and check that $L^T v_i = 0$:

```
>> q88
v1 in 8.42 is

u1 =

     3
     0
     1
     0
     0
     1
     0
     3

The transpose of L times v1 in 8.42 is

x =

     0
     0
     0
     0
     0
     0
     0
     0
```

```
>> q88
v2 in 8.42 is

u2 =

     0
     1
     0
     3
    -3
     0
    -1
     0

The transpose of L times v2 in 8.42 is

x2 =

     0
     0
     0
     0
     0
     0
     0
     0
```

Since there are only 2 dimension of the kernel and we've spotted two linearly independent eigenvectors, we are good. (Thank god I'm right!)

Also, since there are only 8 dimensions, and the matrix is diagonalizable, the column space does span the rest 6 dimensions, and together with the two we've shown above, they must span $\mathbb{R}^8$.

Exercise 3. *(8.13)*

The ODE system is

$$\begin{cases} \dot{x} = \mu x - y + a y^2 + x^3 \\ \dot{y} = x + \mu y + x y^2 + y^2 \end{cases}$$

(a):

If we do the complex transformation we get

$$\dot{z} = \lambda(\mu) z + \dots$$

where $\lambda(\mu) = \mu + i$, thus $\mathrm{Re}(\lambda) = \mu$.

$p = a y^2 + x^3$, $q = x y^2 + y^2$ and thus

$$\alpha(a) = \frac{1}{16}(6) - \frac{1}{16 \cdot 1}(-2a(2)) = \frac{3 + 2a}{8}.$$

(b):

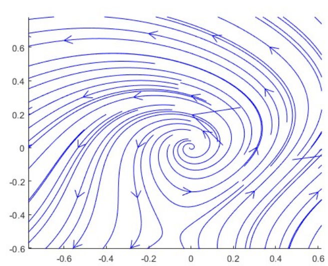
If $a \geq -\frac{3}{2}$ then $\alpha > 0$, which means that there is a bifurcation for when $\mu < 0$, which means that the limit cycle is unstable.

So in the (μ, a) plane, for $a > -\dfrac{3}{2}, \mu < 0$, there exists a subcritical bifurcation with an unstable limit cycle;

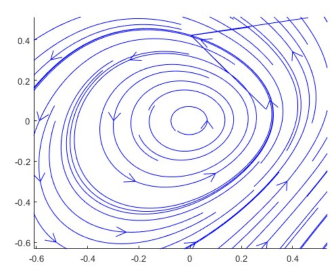
and for $a < -\dfrac{3}{2}, \mu > 0$, there exists a supercritical bifurcation with stable limit cycle.

(c):

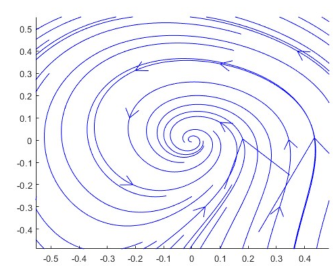
$\alpha > 0, \mu < 0$



$\alpha = 0, \mu = 0$



$\alpha < 0, \mu > 0$



and the stability is as expected.

Exercise 4. *(8.21)*

The ODE system is

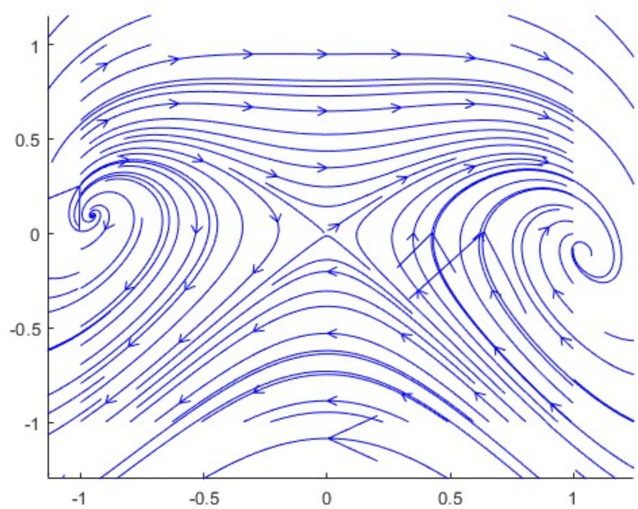
$$\begin{cases} \dot{x} = y + \varepsilon x \\ \dot{y} = x - xy - x^3 \end{cases}$$

(a):

The equilibria satisfies $\dot{x} = \dot{y} = 0$ which yields 3 pairs:

$$e_1 = (0,0), e_2, e_3 = \left(\frac{\varepsilon \mp \sqrt{\varepsilon^2 + 4}}{2}, \frac{-\varepsilon^2 \pm \varepsilon\sqrt{\varepsilon^2 + 4}}{2} \right)$$

And it's clear by the following graph that e_1 is a saddle, e_2(with minus on x-coord) is a source and e_3 is a sink:



(b):

According to (6.27) on page 203 of textbook, simply let $S(x, y) = (-x, y)$ then we have

$$-f(S(x,y)^T) = \begin{pmatrix} -y \\ x - xy - x^3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ x - xy - x^3 \end{pmatrix} = DS(z)f(z)$$

so indeed it is time-reversible.

As for it's not Hamiltonian, well, assume it is then

$$\frac{dH}{dy} = y \Rightarrow H = \frac{1}{2}y^2 + c(x)$$

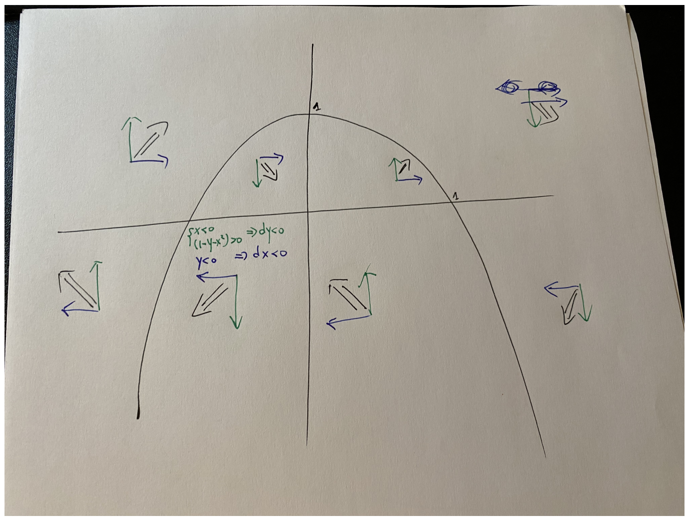
which means there's no cross terms, but

$$\frac{dH}{dx} = x - xy - x^3$$

means there is a term with x^2y, therefore its not Hamiltonian.

(c):

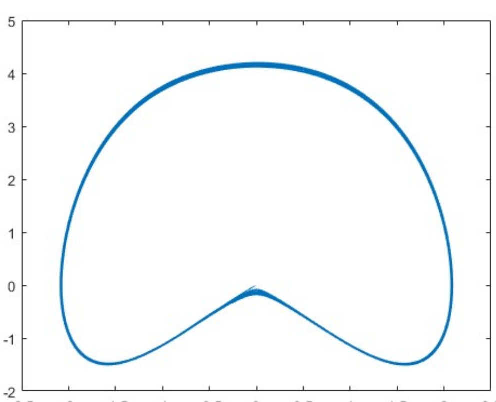
At the origin, the linearization is $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is a flip with respect to $x = y$, so the eigenvectors are $(1, -1)$ and $(1, 1)$. Now we draw the direction in each section by its system:



and we can probably guess that if there's a homoclinic orbit it starts in direction $(1, 1)$ or $(-1, -1)$.

If we can show that an orbit starting in the direction $(-1, -1)$ will eventually came back from either left up or crosses the y-axis (in which case we use the duality in x and claim that it will go back to the origin in the right half plane).

By a numerical depiction it's actually already certain that there's a homoclinic orbit like this if we start at $(-\varepsilon, -\varepsilon)$:



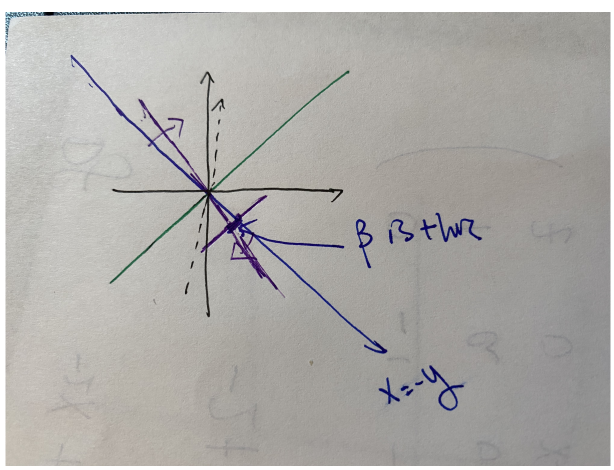
so the only thing is to prove it is. If we look at the time reversal symmetry, we get that if there is a limit of the orbit (we include ∞) then it must have $x = -x$, hence it's on the y axis. But there's no stable point on y-axis except the origin (even $0, \infty$ is not stable), so we must have that the curve goes back to the origin, and thus we are done.

(d):

We know that

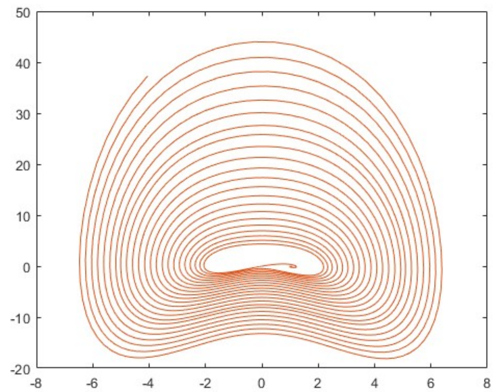
$$\sigma_0 = \lambda_1 + \lambda_2 = \frac{\varepsilon \mp \sqrt{\varepsilon^2 + 4}}{2} + \frac{\varepsilon \pm \sqrt{\varepsilon^2 + 4}}{2} = \varepsilon \neq 0$$

and we know that the matrix $A = \begin{pmatrix} \varepsilon & 1 \\ 1 & 0 \end{pmatrix}$ shifts the two eigenspaces clockwisely when $\varepsilon > 0$ and counter-clockwisely when $\varepsilon < 0$. But this also means that the original stable direction from right bottom has corresponding coordinate β in the Poincare surface that is larger than 0 for $\sigma_0 > 0$. This is best illustrated by the below graph I think:

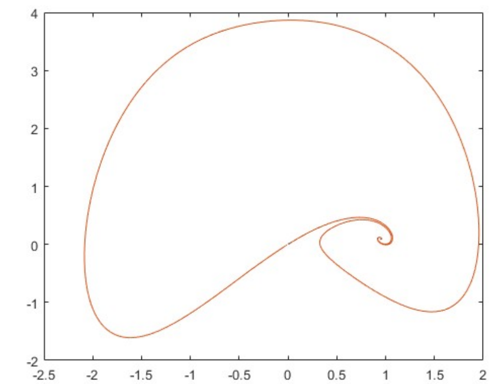


Therefore, $\beta \cdot \sigma_0 > 0$ always. But this means that there's a homoclinic bifurcation with no limit cycle created, which is well shown in the below graphs. Here, I did the curve with starting points at $0.01(\cos\theta, \sin\theta)$ for $\theta =$linspace$(0, 2\pi)$:

$\varepsilon = 0.1$



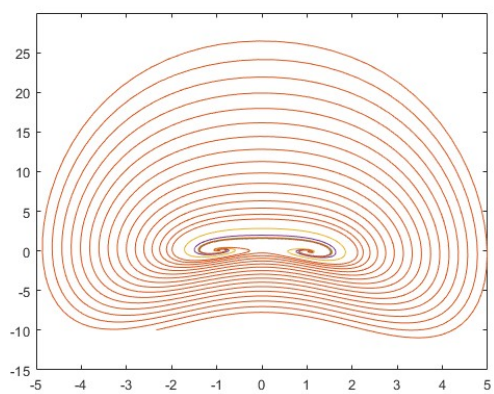
$\varepsilon = -0.1$



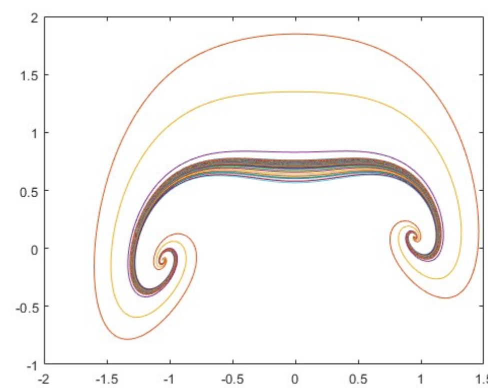
and it's clear that neither has a limit cycle as the inside points are all attracted by the sink and the outside curves are all diverging.

To make sure, I used the points around the source to see that there really is no limit cycle:

$\varepsilon = 0.1$



$\varepsilon = -0.1$



Indeed there's none.

Exercise 5. *(Substantial question)*

The ODE system is

$$\begin{cases} \frac{\partial w}{\partial t} = p - w - wb^2 + v\frac{\partial w}{\partial x} \\ \frac{\partial b}{\partial t} = -mb + wb^2 + \frac{\partial^2 b}{\partial x^2} \end{cases}$$

(1): We study the v parameter. The term $v\dfrac{\partial w}{\partial x}$ represents the loss of water due to slope of ground (since which ever way the slope is, the loss is certain), we know that $v\dfrac{\partial w}{\partial x} < 0$.

When the hill is going upwards to the right, as we go to the right there's less water, which means $\dfrac{\partial w}{\partial x} < 0$, in which case $v > 0$.

In the other case where the hill is downward, $\dfrac{\partial w}{\partial x} > 0$ and so $v < 0$.

So this means that $v > 0$ corresponds to higher on the right.

(2): Since the solution is spatial uniform and stationary, we get

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} = \frac{\partial b}{\partial t} = \frac{\partial b}{\partial x} = \frac{\partial^2 b}{\partial x^2} = 0$$

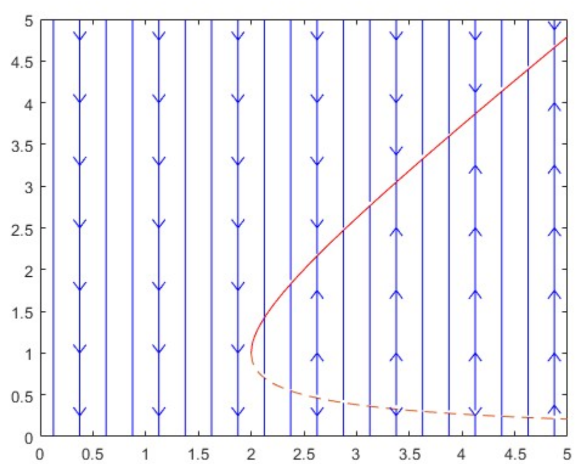
the system becomes

$$\begin{cases} p - w - wb^2 = 0 \\ -mb + wb^2 = 0 \end{cases}$$

which means

$$b = 0 \quad \text{or} \quad \frac{p \pm \sqrt{p^2 - 4m^2}}{2m}$$

which corresponds to a saddle node bifurcation exactly. If we were to plot it it will look like: (note $p > 0$)



As for the equilibrium at 0 that's also stable by the graph. (but that just means there's no plant anytime.)

(c):

By a change of variable $z = x - ct$ we get

$$\frac{\partial w}{\partial t} = \frac{dw}{dz}\frac{dz}{dt} = (-c)\frac{dw}{dz} \quad \text{and} \quad \frac{\partial w}{\partial x} = \frac{dw}{dz}\frac{dz}{dx} = \frac{dw}{dz}$$

and the same applies to b, hence

$$\frac{dw}{dz} = -\frac{1}{c}\left(p - w - wb^2 + v\frac{dw}{dz}\right) \Rightarrow \frac{dw}{dz} = \frac{wb^2 + w - p}{c + v}$$

and a similar computation yields

$$c\frac{db}{dz} = mb - wb^2 + \frac{d^2 b}{dz^2}$$

and by introducing $u = \frac{db}{dz}$ we get the new ODE system

$$\begin{cases} \dot{w} = \frac{w-p+wb^2}{c+v} \\ \dot{b} = u \\ \dot{u} = -cu + mb - wb^2 \end{cases}$$

(d):

So we treat the above new system as an ODE system and tries to determine how the parameters c and p affect the behavior. In particular, we examine its bifurcation points, if there is any. I follow instruction on class and divide this into 4 steps.

Step 1:

The first thing to do is to "move the fixed point" to the origin of the new coordinate system such that at the Hopf-bifurcation point we have

$$\begin{pmatrix} \dot{b} \\ \dot{u} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \mu & -\omega & 0 \\ \omega & -\mu & 0 \\ 0 & 0 & -\lambda_3 \end{pmatrix}\begin{pmatrix} b \\ u \\ w \end{pmatrix} + F\begin{pmatrix} b \\ u \\ w \end{pmatrix}$$

Note that there is a change of order of the system, because we can easily spot that w is not entangled with the other two (which can either be seen from the expression or just by their construction), and the other two are in fact in the center manifold.

The method to achieve this goal is similar to the method in problem 5.10 in textbook, whose solution is above. A Jordan diagonalization is required and I don't fancy that this is going to be concise due to the 3 parameters (though we can say that v is fixed, it is still there to be carried all through the computations.) Also, note that in the change of variables we also have constant term due to that $\dfrac{-p}{c+v}$ term.

Step 2:

Now we want to find the approximate center manifold $W^c_{loc}(0,0,0)$. This is done several times already in the homework from last to this. The reference for the step is very similar to (5.10) (b) above, as we are solving for the second order terms with 2 dimensions.

To be more precise, we will get something like

$$W^c_{loc}(0,0,0) = (b, u, \alpha b^2 + \beta bu + \gamma v^2 + \dots)$$

where our convention is to write

$$h(b,u) = \alpha b^2 + \beta bu + \gamma v^2 + \dots$$

and put it into the last coordinate. This must exist due to implicit function theorem.

Step 3:

Then we restrict to $W^c_{loc}(0,0,0)$ at the bifurcation point and get

$$\begin{cases} \dot{b} = -\omega u + F_1(b,u,h(b,u)) \\ \dot{u} = \omega b + F_2(b,u,h(b,u)) \end{cases}$$

where F_1 and F_2 is the first 2 terms induced from the expression in step 1.

Step 4:

Now we can follow the steps done in problem 8.13 and compute the value of $\alpha = \mathrm{Re}(c)$ and $\mathrm{Re}(\lambda)$. What we just did above is to find $p = F_1$ and $q = F_2$, and the clever thing to do is clearly apply the formula (8.59) to find α in terms of derivatives of p and q.

This will gives us α and $\mathrm{Re}(\lambda)$ in terms of μ and ω. But back to step 1, we've used a change of variables in the beginning, so we can do the inverse of that and get α and $\mathrm{Re}(\lambda)$ in terms of p, m and c.

Anyway, there must be some part of the parameter space c, m, p that corresponds to different signs of α and $\mathrm{Re}(\lambda)$. We divide the space into 9 parts and some are more interesting than others:

$\mu = 0$	$\alpha > 0$	$\alpha = 0$	$\alpha < 0$
$\mathrm{Re}(\lambda) > 0$	No Bifurcation.	$\rightarrow$	Hopf Bifurcation.
$\mathrm{Re}(\lambda) = 0$	$\downarrow$		$\uparrow$
$\mathrm{Re}(\lambda) < 0$	Hopf Bifurcation.	$\leftarrow$	No Bifurcation.

where we need $\mu = 0$ since that's where the bifurcation happens. And the boundaries are also important so I left that on.

Interpretations:

We note that p is related only to w, which means that a change in p would be like shifting the origin in the direction of w.

As for w itself, this is not in the center manifold. I think that the scale of it will affect how fast particles go in the center manifold if it proves that the change of variables does not separate this. If it does then this will just affect the unstable manifold.