APPLIED FUNCTIONAL ANALYSIS HOMEWORK 2

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Discussed with classmates.

Exercise 1. *(5.3) in book*

Proof.

With the sup-norm, the operator δ is bounded, and it's value is 1:

The norm in $\mathbb{R}$ is just the absolute value of the function, so

$$\frac{|\delta(f)|}{||f||_\infty} \leq 1$$

since $\delta(f) = f(0)$ and $|f(0)| \leq ||f||_\infty$ by definition of sup-norm. Thus $||\delta|| \leq 1$ is bounded by 1.

Now we just pick $f = 1$ and then

$$||\delta|| \geq \frac{|\delta(f)|}{||f||_\infty} = 1$$

which, combined with the above result, shows that $||\delta|| = 1$.

With the L^1-norm, the operator δ is unbounded:

Let $f_n(x) = (1-x)^n$, then

$$\frac{|\delta(f_n)|}{||f||_1} = \frac{1}{1/(n+1)} = n+1$$

which is unbounded since we can pick any n.

$\square$

Exercise 2. *(5.7) in book.*

Proof. (Discussed with Tim)

Using triangle equalities we get

$$\sin(a-b)=\sin(a)\cos(b)-\cos(a)\sin(b)$$

which gives

$$Kf=\int_0^1 \sin(\pi(x-y))f(y)dy=\sin(\pi x)\int_0^1 \cos(\pi y)f(y)dy-\cos(\pi x)\int_0^1 \sin(\pi y)f(y)dy$$

and since $\sin(\pi x)$ and $-\cos(\pi x)$ are not 0 function, and they are a pair of independent base (they are not multiple of each other!) in the function space, so we need the coordinate of these to be 0 in order for their combination to be 0. More explicitly

$$Kf=0\Rightarrow \int_0^1 \cos(\pi y)f(y)dy=0 \;\; and \;\; \int_0^1 \sin(\pi y)f(y)dy=0$$

which means

$$\ker(K)=\left\{ f\in C[0,1] \middle| \int_0^1 \cos(\pi y)f(y)dy=0 \;\; and \;\; \int_0^1 \sin(\pi y)f(y)dy=0 \right\}.$$

But just as we've argued above, sin and cos are independent, and since we can manipulate f so that the coordinates $\int_0^1 \cos(\pi y)f(y)dy$ and $\int_0^1 \sin(\pi y)f(y)dy$ can be what ever we want (as a production onto the base directions), so the range is just the span of the two bases:

$$\mathrm{Ran}(K)=\mathrm{span}\{\sin(\pi x),\cos(\pi x)\}.$$

□

Exercise 3. *(5.8) in book.*

Proof.

Equivalent norms on X leads to equivalent norms on $\mathcal{B}(X)$:

Use $||\cdot||_1, ||\cdot||_2$ to denote the two equivalent norms with

$$c||\cdot||_1 \leq ||\cdot||_2 \leq C||\cdot||_1.$$

For $T \in \mathcal{B}(X)$, T is bounded so we have

$$||T||_1 = \sup_x \frac{||Tx||_1}{||x||_1} \text{ and } ||T||_2 = \sup_x \frac{||Tx||_2}{||x||_2}$$

exist.

But we know

$$||T||_1 = \sup_x \frac{||Tx||_1}{||x||_1} \leq \sup_x \left(\frac{||Tx||_1}{||Tx||_2}\right) \sup_x \left(\frac{||Tx||_2}{||x||_2}\right) \sup_x \left(\frac{||x||_2}{||x||_1}\right) = \frac{c}{C}||T||_2$$

which in the same way we know that

$$||T||_2 \leq \frac{C}{c}||T||_1$$

hence the norms are equivalent on $\mathcal{B}(X)$ due to the fact that T is arbitrary.

$\square$

Exercise 4. *(5.14) in book.*

Proof.

(a): Note that if λ_i are eigenvalues of A, then $e^{\lambda_i t}$ are the eigenvalues of e^{tA} since

$$e^{tA}x = x + tAx + \frac{t^2A^2}{2}x + \cdots = x + t\lambda_1 x + \cdots = e^{t\lambda I}x$$

which means that the determinant of e^{tA} is the product of $e^{\lambda_i t}$. Then plugging in we have

$$\lim_{t\to 0}\frac{f(t)-1}{t} = \lim_{t\to 0}\frac{e^{\sum \lambda_i t}-1}{t} = \lim_{t\to 0}\sum \lambda_i + t(\sum \lambda_i)^2/2 + \cdots = \sum \lambda_i = \operatorname{tr}(A).$$

(b): Now since the derivative at s is given by

$$f'(s) = \lim_{t\to\infty}\frac{f(s+t)-f(s)}{t}$$

and we can compute (using result from next problem) since A and itself commutes that

$$f(s+t) = \det(e^{(s+t)A}) = \det(e^{sA}e^{tA}) = f(s)f(t)$$

thus

$$f'(s) = \lim_{t\to\infty}\frac{f(s+t)-f(s)}{t} = \lim_{t\to\infty}\frac{f(s)f(t)-f(s)}{t} = f(s)\lim_{t\to\infty}\frac{f(t)-1}{t} = \operatorname{tr}(A)f(s)$$

which is exactly the ODE.

(c): But since it's nothing but an ODE from $\mathbb{R}\to\mathbb{R}$, we know that by solutions to ODEs that

$$f' = \operatorname{tr}(A)f \quad \Rightarrow \quad f(s) = e^{\operatorname{tr}(A)s}$$

and by plugging in $f(s) = \det(e^{sA})$ we get

$$\det(e^{sA}) = e^{s\operatorname{tr}(A)}$$

which if we let $s = 1$ we have

$$\det(e^{A}) = e^{\operatorname{tr}(A)}.$$

$\square$

Exercise 5. *(5.15) in book.*

Proof. (Proof from Dynamical system textbook)

(a): This is just algebra computation. By timing the expansion and rearranging terms (we can do this since bounded operators to the k-th power over $k!$ is a convergent series) we get

$$\begin{aligned} e^A e^B &= \left(I + A + \frac{A^2}{2} + \dots\right)\left(I + B + \frac{B^2}{2} + \dots\right) \\ &= I + A + B + \frac{1}{2}\left(A^2 + 2AB + B^2\right) + \frac{1}{3!}\left(A^3 + 3A^2B + 3AB^2 + B^3\right) + \dots \end{aligned}$$

and for which we see that if A, B commutes, we have that each term is exactly the expansion of $(A+B)^n/n!$ and hence we have

$$e^A e^B = \sum_{k=1}^{\infty} \frac{(A+B)^k}{k!} = e^{A+B}$$

where since summation of bounded operators is still bounded, the last term makes sense.

(b):

Let's assume that there is a matrix C such that

$$e^A e^B = e^C$$

then it must satisfy that

$$\sum_{k=1}^{\infty} \frac{C^k}{k!} = \left(I + A + \frac{A^2}{2} + \dots\right)\left(I + B + \frac{B^2}{2} + \dots\right)$$

then both series has the lowest order term I and first order terms $A+B$. Let's assume that D is quadratic and E is cubic, and $C = A + B + D + E + \dots$ since the first order of C is fixed. Then we expand to get

$$\begin{aligned} e^c = I + (A+B) + \left(D + \frac{(A+B)^2}{2}\right) \\ + \left[E + \frac{1}{2}((A+B)D + D(A+B)) + \frac{1}{6}(A+B)^3\right] \end{aligned}$$

And if we were to compare the second order term with the expansion of $e^A e^B$ we get

$$\left(D + \frac{(A+B)^2}{2}\right) = \frac{1}{2}\left(A^2 + 2AB + B^2\right)$$

which gives

$$D = \frac{1}{2}[A, B]$$

where we use the square bracket as in the problem:

$$[A, B] = AB - BA.$$

Now let's compute E in a similar fashion and get

$$
\begin{aligned}
E &= \frac{1}{3!}\left(A^3+3A^2B+3AB^2+B^3\right)-\left[\frac{1}{2}\left((A+B)D+D(A+B)\right)+\frac{1}{6}(A+B)^3\right] \\
&= \frac{1}{3!}\left(A^3+3A^2B+3AB^2+B^3\right)-\left[\frac{1}{2}\left((A+B)[A,B]+[A,B](A+B)\right)+\frac{1}{6}(A+B)^3\right] \\
&= \frac{1}{12}\left[\left(AB^2-2BAB+B^2A\right)+\left(A^2B-2ABA+BA^2\right)\right] \\
&= \frac{1}{12}\left([A,[A,B]]-[B,[A,B]]\right)=0
\end{aligned}
$$

And we see that if we assume for higher order terms, say $F, G, H, \dots$, then they are terms that is made up of Lie brackets with $[A,[A,B]]$ and $[B,[A,B]]$, just the way $[A,B]$ appears in the third order brackets.

Thus, $E = F = \cdots = 0$ and $C = A + B + D = A + B + [A,B]$.

$\square$