Tommenix Yu HW2 (Discussed with classmates)

1. Show properties one by one:

- $Y_0=0$ by definition ✓

- Independence: We use the fact that Multivariate-Gaussian + (Cov = 0) ⇒ Independency

for $r<s<t$, we have (Y_t-Y_s, Y_r) is ~~iid~~ Multivariate Gaussian because that is $(tB_{\frac{1}{t}}-sB_{\frac{1}{s}}, rB_{\frac{1}{r}})$ & by def

$$tB_{\frac{1}{t}}-sB_{\frac{1}{s}}=(t-s)B_{\frac{1}{t}}+s(B_{\frac{1}{t}}-B_{\frac{1}{s}})=(t-s)B_{\frac{1}{t}}-s(B_{\frac{1}{s}}-B_{\frac{1}{t}})$$

is sum of ind Gaussian ⇒ Still Gaussian. Now we only have to show Cov = 0. to get $\underset{\text{independent}}{Y_t-Y_s \perp \sigma\{Y_r\}}$.

~~But that's~~

Here we use $\mathrm{Cov}(B_t,B_s)=\min\{t,s\}$ (alternative definition discussed in class)

to get $\mathrm{cov}(Y_t-Y_s, Y_r)=\mathrm{cov}(tB_{\frac{1}{t}}-sB_{\frac{1}{s}}, rB_{\frac{1}{r}})$

$$=tr\,\mathrm{cov}(B_{\frac{1}{t}},B_{\frac{1}{r}})-sr\,\underset{\mathrm{cov}}{(}B_{\frac{1}{s}},B_{\frac{1}{r}})$$

$$=tr\cdot\frac{1}{t}-sr\cdot\frac{1}{s}=r-r=0$$

Hence $Y_t-Y_s\perp\sigma\{Y_r\}$. But this argument works for all $r\leq s$

$\Rightarrow Y_t-Y_s\perp\sigma\{Y_r: r<s\}$

Hence Independence ✓

- Normal increments:

$$\forall t\geq s,\quad Y_t-Y_s=(t-s)B_{\frac{1}{t}}-s(B_{\frac{1}{s}}-B_{\frac{1}{t}})=:X_1+X_2$$

Note: sign ~~do~~ is ignored by ~~dis~~ symmetric distribution

& $X_1\perp X_2$ & $\begin{cases}X_1\sim N\left(0,\frac{(t-s)^2}{t}\right)\\ X_2\sim N\left(0,\frac{(t-s)\cdot s}{t}\right)\end{cases}$

$$\overset{\text{by independence}}{\Rightarrow} X_1+X_2\sim N\left(0,\frac{(t-s)^2}{t}+\frac{(t-s)\cdot s}{t}\right)=N(0,t-s)\ ✓$$

- Continuity:

  - for $t>0$, w.p.1 it's cts since $B_{\frac{1}{t}}$ is cts on $(0,\infty)$.
  - for $t=0$. $Y_t=0$ and we want to check w.p.1, $\lim_{t\to 0} Y_t=0$.

$$\text{But } \lim_{t\to 0} Y_t = \lim_{t\to 0} t\cdot B_{\frac{1}{t}} = \lim_{t\to\infty} \frac{1}{t}\cdot B_t = 0$$

from last homework.

Thus. Y_t is cts at 0 with .p. 1.

Now ~~combine~~ take union the 2 prob 1 set $\Rightarrow$ w.p.1. Y_t has cts path.

Tommenix Yu:

Q2: Just check: on $(\Omega, \mathcal{F}_t, \mathbb{P})$.

- $\Omega \in \mathcal{F}_T$ ✓
- complement:

  if $A \cap \{T \le t\} \in \mathcal{F}_t$, then $A^c \cap \{T \le t\} = \{T \le t\} \setminus (A \cap \{T \le t\}) \in \mathcal{F}_t$

  since $\mathcal{F}_t$ is σ-alg. $\Rightarrow A^c \in \mathcal{F}_T$. ✓
- countable union:

  If A_i, $i \in \omega$ is a ctb set of events ~~ϕ~~ in $\mathcal{F}_t$, then

  $$\left(\bigcup_{i\in\omega} A_i\right) \cap \{T \le t\} = \bigcup_{i\in\omega}\left(A_i \cap \{T \le t\}\right) \in \mathcal{F}_t$$

  $$\Rightarrow \bigcup_{i\in\omega} A_i \in \mathcal{F}_T. \quad ✓$$

So $\mathcal{F}_T$ is indeed a σ-alg.

Q3: for $r = \sqrt{2(1+\varepsilon)}$, we have by reflection principle that, if we define

$U_n := \{\exists t \in [n-1, n] \text{ with } |B_t - B_{n-1}| \geq \sqrt{2(1+\varepsilon)\log n}\} = \{M_n \geq \sqrt{2(1+\varepsilon)\log n}\}$

then $\cancel{\mathbb{P}(U_n)}$ since $B_{n-1+s} - B_{n-1} \sim N(0, s)$, we can pass this by taking prob.

i.e.

$$\mathbb{P}(U_n) = \mathbb{P}\{\exists t \in [0,1] \text{ with } |B_t| \geq \sqrt{2(1+\varepsilon)\log n}\} \leq 4\,\mathbb{P}\{B_1 \geq \sqrt{2(1+\varepsilon)\log n}\}$$

$$\lesssim e^{-(1+\varepsilon)\log n} \cdot \left(\frac{4}{\sqrt{2(1+\varepsilon)\log n}}\right) = c \cdot \frac{1}{n^{(1+\varepsilon)}\sqrt{\log n}}$$ is summable for all $\varepsilon > 0$.

Now, since ~~An~~ For n sufficiently large, $M_n \leq \sqrt{2(1+\varepsilon)\log n}$ for all ε

$\Rightarrow M_n \leq \sqrt{2\log n}$ as it's closed.

$\Rightarrow r \geq \sqrt{2}$ is all fine.

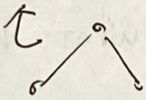
Tommenix Yu

Q4

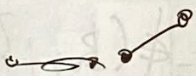
1) This is direct from definition: if $\Delta(2j-1,n+1)+\Delta(2j,n+1)) \geq \Delta(j,n)$

i.e.  2^{nd} variation is larger, we include the middle pt. in

the partition Π_n. So $\sum_{j=1}^{k_n}\left[B(t_{j,n})-B(t_{j-1,n})\right]^2 = \sum_{j=1}^{2^n} Y(j,n)$

 all pt in Π_n. corresponds to term here.

if not, then same argument.

 only 2 endpt is in Π_n. so is the value on right.

Q. (I realized that for a_1, a_2, a if $a = |a_1|+|a_2|$, then always $\Delta(j,n)$ is larger. ~~i.e.~~ since $(a+b)^2 \geq a^2+b^2$ for $a,b \geq 0$.

So if $B_{\frac{2j-1}{2^{n+1}}} \notin \left(B_{\frac{j-1}{2^n}}, B_{\frac{j}{2^n}}\right)$, then there's no choice of pt in middle.

WLOG assume it is ~~well order~~ an interval, if not switch

the $\Leftarrow$ direction of this statement is obvious.

in other words, we're choosing the "finest" choice of path with maximal variation, given the specification of information (only know information about $t = \frac{k}{2^{n+1}}$))

actually used in later parts!

2) First, $\sigma\left\{B_t : t \in \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)\right\}_{i=0,1..2^n}$ is a collection of independent σ-alg.

But really, given observation above (and it's shown by $(a+b)^2 \geq a^2+b^2$)

$Y(k,n)$ only depends on the path of B_t for $t \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)$, where pass to $t \in \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right]$ by continuity.

So $Y_{(k,n)}$ is msrb w.r.t $\sigma\{B_t : t\in[\frac{k-1}{2^n},\frac{k}{2^n})\}$

So $Y_{(k,n)}$ are independent of each other.

they are iid because $B_{\frac{1}{2^n}} - B_{\frac{0}{2^n}} \sim B_{\frac{2}{2^n}} - B_{\frac{1}{2^n}} \sim \dots \sim B_{\frac{2^n}{2^n}} - B_{\frac{2^n-1}{2^n}}$.

& $B_{\frac{1}{2^n}} - B_0 \sim 2^{-n/2} B_1$.

So there's a 1-1 correspondence from what's happening in $\underset{0 \quad \frac{1}{2^n}}{\llcorner\!\lrcorner}$ and $\underset{0 \quad 1}{\llcorner\!\lrcorner}$

by $f_n : x \rightsquigarrow 2^{-n/2} x$. where $f_n(B_t) = B_{(\frac{t}{2^n})}$, f_n acts on appropriate function space

Now ~~让~~ denote $E_n(1) := \{B_{\frac{1}{2^{n+1}}}$ is such that $|B_{\frac{1}{2^{n+1}}}| \geq |B_{\frac{1}{2^n}}|\}$

then $\exists\ \tilde{E}(1) := \{|B_{\frac{1}{2}}| \geq |B_1|\}$ from which $\tilde{f}_n(\tilde{E}(1)) = E_n(1)$

is 1-1 onto. actually generated function by f_n, $\tilde{f}_n : \Omega \to \Omega$ is an 1-1 onto map

Now, if $E_n(1) \ni \omega$, $\Rightarrow Y_{(1,n)} = \Delta(2j-1, n+1) + \Delta(2j, n+1) = B^2_{\frac{1}{2^{n+1}}} + (B_{\frac{1}{2^n}} - B_{\frac{1}{2^{n+1}}})^2$

& $\tilde{f}_n^{-1}(\omega)$ is s.t. $\tilde{E}(1)$ holds true

$$\Rightarrow Y = B^2_{\frac{1}{2}} + (B_1^2 - B^2_{\frac{1}{2}}) = 2^{-n}\left(B^2_{\frac{1}{2^{n+1}}} + \left(B_{\frac{1}{2^n}} - B_{\frac{1}{2^{n+1}}}\right)^2\right)$$

if $\omega \notin E_n(1)$. $\Rightarrow Y(1,n) = B^2_{\frac{1}{2^{n}}}$

& for $\tilde{f}_n^{-1}(\omega)$, $Y = 2^{-n} B^2_{(\frac{1}{2^{n}})}$

So for each $\omega\in\Omega$, $\exists\ \tilde{f}_n^{-1}(\omega)\in\Omega$ s.t. $Y = 2^{n}\, Y(1,n)$.

Thus, $Y(1,n) \overset{d}{\sim} Y\cdot 2^{-n}$

(after discussion with classmates:
a much easier way is just to find a copy of $\overset{iid}{B_t}$, then
show both are $\overset{d}{\sim}$ to new B_t's Y.) ~~But Quantile method above is correct so.~~

3) A ~~fact~~ fact we use is $\mathrm{Var}(\max(X,Y)) \le \mathrm{Var}(X)+\mathrm{Var}(Y)$

for $X,Y \ge 0$.

This is proven by indicator function + MCT

indicator is obvious : $\overbrace{\underbrace{(a\,(b}_{A}\,)\,c)}^{B}$ i.e. $\mathbb{P}(A)=a+b$

$\mathbb{P}(B)=b+c$.

then $\mathrm{Var}(\max\{\mathbb{1}_A,\mathbb{1}_B\}) = (a+b+c)^2 \le (a+b)^2+(b+c)^2 = \mathrm{Var}(\mathbb{1}_A)+\mathrm{Var}(\mathbb{1}_B)$.

Now MCT gives the result as $\mathrm{Var}(X)=\mathbb{E}[X^2]-\mathbb{E}[X]^2$ and limit passes into both.

$$\text{So } \mathrm{Var}[Y] \le \mathrm{Var}(B_1^2) + \mathrm{Var}\left(B_{\frac12}^2 + (B_{\frac12}-B_1)^2\right)$$

$$= \mathbb{E}[B_1^4] + \mathrm{Var}[B_{\frac12}^2] + \mathrm{Var}\left((B_{\frac12})^2\right]$$

$$= (3)!! + \left(\tfrac12\right)^4 (3!!)\times 2 = 3 + \tfrac18 3 = \tfrac{c}{8} < \infty$$

for $\mathbb{E}[Y]$, we note $\mathbb{E}[B_1^2]=1$

and by def of Y. $\mathbb{E}[Y] \ge \mathbb{E}[B_1^2]$

and we only need to check $\cancel{\mathbb{P}(\tfrac12 \in T_0) \ge 0}$

$\mathbb{P}(A) > 0$ where $A := \{|B_{\frac12}| > |B_1| \text{ and } \mathrm{sign}(B_{\frac12}) = \mathrm{sign}(B_1)\}$

But that's obviously true since

$\mathbb{P}(A) \ge \cancel{\int_{-\infty}^{\infty} \mathbb{P}\,\mathbb{1}_{\{|B_1|<|B_{\frac12}|,\ \mathrm{sign}(B_1)=\mathrm{sign}(B_{\frac12})\}}\, d\mu_{B_{\frac12}=x} \ge 0}$

$$\overset{\ge}{=} \frac12 \int_\Omega \mathbb{P}(B_{\frac12}=x)\cdot \mathbb{1}_{\{|B_1|<x\}}\, d\mathbb{P} > 0.$$

sign issue

$\Rightarrow \mathbb{E}[Y] \ge P(A)\cdot(1+\varepsilon) + P(A^c)\cdot 1 > 1$

4) in 2) we've shown $Y(i,n)$ iid & $Y(i,n) \sim 2^{-n} Y$.

By strong LLN in triangular form:

$$
\begin{array}{llll}
Y(1,0) & & & \\
Y(1,1) & Y(1,2) & & \\
Y(1,2) & Y(2,2) & Y(3,2) & Y(4,2) \\
\vdots & & & \ddots
\end{array}
$$

we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{2^n} Y(i,n)}{2^n} - \mathbb{E}[Y(1,n)] \xrightarrow{a.s.} 0$$

and $\mathbb{E}[Y(1,n)] = 2^{-n} \cdot \mathbb{E}[Y]$

So w.p. 1,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{2^n} Y(i,n)}{2^n} - 2^{-n} \cdot \mathbb{E}[Y] = 0$$

and $\sum_{i=1}^{2^n} Y(i,n) = \underbrace{\sum_{j=1}^{k_n} [B(t_j,n) - B(t_{j-1},n)]}_{S_n}$ by definition of Π_n & $Y(i,n)$

$$\Rightarrow \lim_{n\to\infty} \frac{S_n - \mathbb{E}[Y]}{2^n} = 0$$

$$\Rightarrow \lim S_n = \mathbb{E}[Y] \quad \text{w.p. } 1.$$

Tommenix Yu HW2

Q5:

1) $\mathbb{P}\{K_n=0\} = 1-\mathbb{P}\{B_s=0 \text{ for some } 1\leq s\leq 2\} =: 1-\mathbb{P}(A)$.

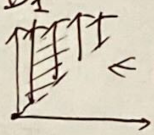
Computing $\mathbb{P}(A)$ we have:

$$\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A\mid B_1=x)\, \cancel{f_{B_1}}\, d\mu_{B_1} = \int_{-\infty}^{\infty} \mathbb{P}(A\mid B_1=x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

$$= 2\int_0^{\infty} \mathbb{P}(A\mid B_1=x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

and

$$\mathbb{P}(A\mid B_1=x) \overset{\text{Same distribution}}{=} \mathbb{P}\{\max_{0\leq s\leq 1} B_s \geq x\} \overset{\text{Reflection}}{=} 2\int_x^{\infty} 1\, d\mu_{B_1}$$



$$\Rightarrow \mathbb{P}(A) = 4\int_0^{\infty}\left[\int_x^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy\right]\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

$$= 4\frac{1}{2\pi}\cdot\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{\infty} e^{-\frac{r^2}{2}}r\,dr\,d\theta = 1-\frac{2}{\pi}\arctan 1.$$

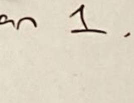
$$\Rightarrow \mathbb{P}\{K_n=0\} = 1-\mathbb{P}(A) = \frac{2}{\pi}\arctan 1 = \frac{1}{2}$$

2) Ok we've computed it above so I'll not really compute again for $\frac{j-1}{2^n}$ $\frac{j}{2^n}$, but as is shown in class, we have by scaling that

$$\mathbb{P}(E_{j,n}) = \mathbb{P}\left\{B_t=0 \text{ for } \exists\ 1\leq t\leq \frac{j}{j-1} = 1+\frac{1}{j-1}\right\}$$

$$= 1-\frac{2}{\pi}\arctan\left(\frac{1}{\sqrt{\frac{1}{j-1}}}\right) = 1-\frac{2}{\pi}\arctan\left(\sqrt{j-1}\right)$$

$$= \frac{2}{\pi}\arctan\left(\frac{1}{\sqrt{j-1}}\right)$$



~~arctan~~

$\tan\beta = \frac{1}{\tan\alpha}$

Now $\frac{1}{\sqrt{j-1}}$ is small we can just add things up using Taylor.

Say $j>5$, then $\frac{1}{\sqrt{j-1}}<\frac{1}{2}$ and taylor:

$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, But we actually need $j > 2^n$ which is convenient

we get

$$\cancel{\sum_{j>5}} \; 2^{-n/2} \sum_{j=2^n+1}^{2^{n+1}} \frac{2}{\pi}\arctan\left(\frac{1}{\sqrt{j-1}}\right)$$

$$= \frac{1}{2^{n/2}} \cdot \sum_{j=2^n+1}^{2^{n+1}} \cdot \frac{2}{\pi}\left(\frac{1}{\sqrt{j-1}} + O\left(\left(\frac{1}{\sqrt{j-1}}\right)^3\right)\right)$$

$$= \frac{2}{\pi}\cdot 2^{-n/2} \cdot \sum_{j=2^n+1}^{2^{n+1}} \left(\frac{1}{\sqrt{j-1}} + O\left(\frac{1}{2^{3n/2}}\right)\right)$$

$$= \left(\frac{2}{\pi}\cdot 2^{-n/2} \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{\sqrt{j-1}}\right) + \frac{2}{\pi}\cdot 2^{-n/2} \cdot O(2^{-3n/2}) \cdot \underbrace{2^n}_{\text{summation}}$$

$$= \frac{2}{\pi} \cdot \underbrace{2^{-n}}_{\Delta x} \cdot \sum_{j=2^n+1}^{j=2^{n+1}} \frac{1}{\sqrt{\frac{j-1}{2^n}}} + o(1)_{n\to\infty}$$

So $\lim_{n\to\infty} 2^{-n/2}\,\mathbb{E}[K_n] = \lim_{n\to\infty} \frac{2}{\pi} 2^{-n} \sum_{j=2^n+1}^{2^{n+1}} \frac{1}{\sqrt{\frac{j-1}{2^n}}}$

$= \frac{2}{\pi}\int_1^2 \frac{1}{\sqrt{x}}\,dx = \frac{2}{\pi}(2\sqrt{2}-2)$ by Riemann integration.

3). $\mathbb{P}(E_{j,n} \cap E_{k,n}) = \mathbb{P}(E_{k,n} \mid E_{j,n}) \cdot \mathbb{P}(E_{j,n})$

we have bound for $\mathbb{P}(E_{j,n})$ so let's look at $\mathbb{P}(E_{k,n} \mid E_{j,n})$

~~Say~~ ~~b~~ define $\sigma_j := \max\{s < \frac{j}{2^n} ; B_s = 0\}$

Since condition on $E_{j,n}$, $\sigma_j \in [\frac{j-1}{2^n}, \frac{j}{2^n}]$

Then problem becomes $\mathbb{P}\{B_t = 0 \text{ for } t \in (\frac{k-1}{2^n} - \sigma_j, \frac{k}{2^n} - \sigma_j)\}$

And now by same argument

$$\mathbb{P}\left\{B_t=0 \text{ for some } t\in\left(\frac{k-1}{2^n}-\sigma_j,\frac{k}{2^n}-\sigma_j\right)\right\} \text{ ; here scale: } \frac{k-1}{2^n}-\sigma_j \rightsquigarrow 1.$$

$$= 1-\frac{2}{\pi}\arctan\left(\frac{1}{\sqrt{\frac{1}{k-1-\sigma_j 2^n}}}\right) = \frac{2}{\pi}\arctan\left(\frac{1}{\sqrt{k-1-2^n\sigma_j}}\right).$$

Now just bound issue

$$\mathbb{P}(E_{j,n}\cap E_{k,n}) = \frac{4}{\pi^2}\cdot\arctan\left(\frac{1}{\sqrt{k-1-2^n\sigma_j}}\right)\cdot\arctan\left(\frac{1}{\sqrt{j-1}}\right)$$

But $2^n\sigma_j \in \left[2^n\frac{j-1}{2^n},\, 2^n\frac{j}{2^n}\right] = [j-1, j]$

$\Rightarrow$ $k-1-2^n\sigma_j \in [k-j, k-j-1]$ ok we first forget when $k=j+1$

and $j-1 \in [2^n, 2^{n+1}]$

So taylor gives $\arctan\left(\frac{1}{\sqrt{k-1-2^n\sigma_j}}\right)\cdot\arctan\left(\frac{1}{\sqrt{j-1}}\right)$

$$\leq \frac{1}{\sqrt{k-1-2^n\sigma_j}}\cdot\frac{1}{\sqrt{j-1}} \leq \frac{1}{\sqrt{k-j}\cdot 2^{n/2}}$$

for the other direction, we have to keep cubic term but roughly the same:

$$\arctan\left(\frac{1}{\sqrt{k-1-2^n\sigma_j}}\right)\cdot\arctan\left(\frac{1}{\sqrt{j-1}}\right)$$

$$\geq\left(\frac{1}{\sqrt{k-1-2^n\sigma_j}}-\frac{1}{3(\sqrt{k-1-2^n\sigma_j})^3}\right)\left(\frac{1}{\sqrt{j-1}}-\frac{1}{3(\sqrt{j-1})^3}\right) \geq \left(\frac{1}{\sqrt{k-j-1}}-\frac{1}{3(k-j)^{3/2}}\right)\left(2^{-(n+1)/2}-\frac{1}{3}\cdot 2^{-\frac{3n}{2}}\right)$$

$$\geq \frac{1}{\sqrt{k-j-1}\cdot 2^{(n+1)/2}} - \frac{1}{3(k-j)^{3/2}\cdot 2^{(n+1)/2}} - \frac{1}{3\sqrt{k-j-1}\, 2^{3n/2}}$$

$$= \left(\frac{\sqrt{k-j}}{\sqrt{k-j-1}}\cdot\frac{1}{\sqrt{2}} - \frac{1}{3\cdot(k-j)\cdot\sqrt{2}} - \frac{\sqrt{k-j}}{3\sqrt{k-j-1}\cdot 2^n}\right)\cdot\frac{1}{\sqrt{k-j}\cdot 2^{n/2}}$$

$$= \left(\frac{1}{3\sqrt{2}}\left(\frac{\sqrt{k-j}}{\sqrt{k-j-1}}-\frac{1}{k-j}\right)+\frac{\sqrt{k-j}}{3\sqrt{k-j-1}}\left(\frac{1}{\sqrt{2}}-\frac{1}{2^n}\right)+\frac{\sqrt{k-j}}{3\sqrt{2}\sqrt{k-j-1}}\right)\cdot\frac{1}{(k-j)2^{n/2}}$$

Where $\frac{\sqrt{k-j}}{\sqrt{k-j-1}} - \frac{1}{k-j} = \frac{(k-j)^{3/2} - \sqrt{k-j-1}}{(k-j)\sqrt{k-j-1}} > 0$ since nominator > 0.

and we see we can bound below by 0 on this.

Same for the second term.

So for the third term $\frac{1}{3\sqrt{2}} \cdot \frac{\sqrt{k-j}}{\sqrt{k-j-1}} \geq \frac{1}{3\sqrt{2}}$ uniform in n.

Thus for $c_2 = \frac{2}{\pi}$, $c_1 = \frac{1}{3\sqrt{2}} \cdot \frac{2}{\pi}$

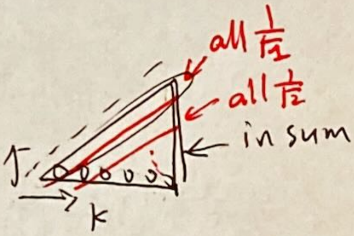
we have (for $k = j+1$, just adjust constant c_1, c_2 to suit that case (still uniform in n)). $c_1 \to \tilde{c}_1$, $c_2 \to \tilde{c}_2$

$$\frac{\tilde{c}_1}{2^{n/2}\sqrt{k-j}} \leq \mathbb{P}(E_{j,n} \cap E_{k,n}) \leq \frac{\tilde{c}_2}{2^{n/2}\sqrt{k-j}}.$$

4). $\mathbb{E}[K_n^2] = \mathbb{E}\Big[\sum_{\substack{j,k \in \mathbb{Z} \\ j,k \in [2^n+1, 2^{n+1}]}} \mathbb{1}_{\{E_{j,n} \cap E_{k,n}\}} \Big]$

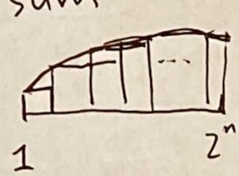
$$= \underbrace{\sum_{j=2^n+1}^{2^{n+1}} \mathbb{P}(E_{j,n})}_{\text{part (2)}} + 2 \underbrace{\sum_{j<k} \mathbb{P}(E_{j,n} \cap E_{k,n})}_{\text{part (3)}}$$

$$\leq \frac{2}{\pi}(2\sqrt{2}-2) \cdot 2^{n/2} + 2 \cdot c_2 \cdot \sum_{\substack{j<k \\ (\text{in range})}} \frac{1}{\sqrt{k-j}} \cdot 2^{-n/2}$$



$$\leq c_3 \cdot 2^{n/2} + \tilde{c}_2 \cdot 2^{-n/2} \cdot \sum_{i=1}^{2^n-1} \frac{2^n - i}{\sqrt{i}}$$

$$\leq c_3 \cdot 2^{n/2} + \tilde{c}_2 \cdot 2^{n/2} \cdot \sum_{i=1}^{2^n-1} \frac{1}{\sqrt{i}}$$ ← use left Riemann sum



$$\leq c_3 \cdot 2^{n/2} + \tilde{c}_2 2^{n/2} \cdot \left(\int_1^{2^n} \frac{1}{\sqrt{x}} \, dx \right)$$

$$\leq c_3 2^{n/2} + \tilde{c}_2 2^{\frac{n}{2}} \left(2 \cdot 2^{\frac{n}{2}} - 2 \right) = (c_3 - 2\tilde{c}_2) 2^{\frac{n}{2}} + 2 \cdot \tilde{c}_2 \cdot 2^n$$

$$\leq \left(|c_3 - 2\tilde{c}_2| + 2\tilde{c}_2 \right) \cdot 2^n =: c \cdot 2^n$$