PDE HOMEWORK 2

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Discussed with classmates.

Exercise 1.

Proof.

Notice that

$$\Delta\left(u+\max_{\overline{U}}|f|\frac{|x|^2}{2n}\right)=\Delta u+\max_{\overline{U}}|f|\sum_{i=1}^{n}\partial_{x_i}\frac{2x_i}{2n}=\Delta u+\max_{\overline{U}}|f|\geq 0$$

so $u+\max_{\overline{U}}|f|$ is subharmonic. By the next question (proven independently) we have

$$\max_{\overline{U}}\left[u+\max_{\overline{U}}|f|\frac{|x|^2}{2n}\right]=\max_{\partial U}\left[u+\max_{\bar{u}}|f|\frac{|x|^2}{2n}\right]\leq\max_{\partial U}|u|+\frac{|1|^2}{2n}\max_{\bar{u}}|f|$$
$$=\max_{\partial U}|g|+C'\max_{\bar{u}}|f|$$

and so for all x we know

$$u+\max_{\overline{U}}|f|\frac{|x|^2}{2n}\leq\max_{\partial U}|g|+C'\max_{\bar{u}}|f|$$

where we move term to right then take absolute value and then max over x we see that the result in question holds.

$\square$

Exercise 2.

Proof.

(a):

For v subharmonic, define

$$\phi(r) := \fint_{\partial B(x,r)} v(y)dS(y) = \fint_{\partial B(0,1)} v(x+rz)dS(z)$$

and thus we solve the derivative by directly putting things inside since $v \in C^2$ and hence the derivative is a continuous function on a compact set, hence bounded, then integrable, so DCT can be passed. Now we have

$$\begin{aligned}\phi'(r) &\overset{DCT}{=} \fint_{\partial B(0,1)} z \cdot \nabla v(x+rz)dS(z) = \fint_{\partial B(0,1)} \frac{\partial v}{\partial n}(x+rz)dS(z) \\ &= \fint_{\partial B(x,r)} \frac{\partial v}{\partial n}(y)dS(y) = \fint_{\partial B(x,r)} (n \cdot \nabla v)(y)dS(y) = \frac{1}{|\partial B(x,r)|}\int_{B(x,r)} (n \cdot \nabla v)(y)dS(y) \\ &= \frac{1}{|\partial B(x,r)|}\int_{B(x,r)} \Delta v(y)dS(y) \geq 0\end{aligned}$$

and thus

$$v(x) = \phi(0) \leq \phi(r) = \fint_{\partial B(x,r)} v(y)dS(y) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} v(y)dS(y)$$

for all r, then integrating over all r we get

$$v(x) = \phi(0) \leq \int_0^r \phi(t)dt = \int_0^r \fint_{\partial B(x,r)} v(y)dS(y)dt = \frac{1}{|B(x,r)|}\int_{B(x,r)} v(y)dS(y)$$

which is what we want.

(b):

We know v is continuous, then we know that if the max is attained at x in the interior of U, then at in neighborhood N around x everything is smaller or equal to $v(x)$, hence

$$v(x) \geq \fint_{B(x,r)} v(y)dS(y) \geq v(x)$$

so the equality holds. Now for every other point y in the interior of U, we know since U open and connected there is a path connecting x, y, which we can cover with metric balls contained inside U. This shows that everything in the interior is a constant, but then since v is continuous on $\overline{U}$ we get that it is a constant on the whole $\overline{U}$.

This means that if the maximum is in the interior it is also the maximum on the boundary. Thus we have

$$\max_{\overline{U}} v = \max_{\partial U} v.$$

(c):

Just take the derivative we get

$$\frac{\partial}{\partial x_i}\phi(u) = \phi'(u)\left(\frac{\partial}{\partial x_i}u\right)$$

and hence

$$\frac{\partial^2}{\partial x_i^2}\phi(u) = \phi''(u)\left(\frac{\partial}{\partial x_i}u\right)^2 + \phi'(u)\left(\frac{\partial^2}{\partial x_i^2}u\right)$$

summing up we have

$$\Delta\phi(u) = \phi''(u)\left(\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i}u\right)^2\right) + \phi'(u)\Delta u = \phi''(u)\left(\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i}u\right)^2\right) \geq 0$$

since $\phi'' \geq 0$.

This means $-\Delta\phi(u) \leq 0$ so subharmonic.

(d):

We just compute the Laplacian of v. It is

$$\Delta v = \sum_{j=1}^{n}\partial_j \sum_{i=1}^{n} 2u_{ij}\cdot u_i = \sum_{j=1}^{n}\sum_{i=1}^{n}\left(2u_{ijj}\cdot u_i + 2u_{ij}^2\right) = \left(\sum_{i=1}^{n} 2\Delta u_i\right) + 2\sum_{j=1}^{n}\sum_{i=1}^{n} u_{ij}^2 \geq 0$$

since u_i is harmonic and the first term goes away. Thus v is subharmonic.

(e):

The corresponding claims are: For $v \in C^2(\overline{U})$ superharmonic,

(a)' $v(x) \geq \frac{1}{|B(x,r)|}\int_{B(x,r)} v(y)dS(y)$.
(b)' $\min_{\overline{U}} v = \min_{\partial U} v$.
(c)' If u is harmonic, then $\phi(u)$ is superharmonic for ϕ concave.

And the only difference in the proofs are:

(a)' Sign flip in ϕ', as defined above;
(b)' Sign flip due to sign flip in (a)'.
(c)' $\phi''(u) \leq 0$ and the squares ≥ 0, so the product is less or equal to 0.

□

Exercise 3.

Proof.

Note that for $y \in \partial B(0,r)$, we know $|y| = r$ and for $|x| \leq r$ hence

$$\frac{r-|x|}{(|x|+r)^{n-1}} = \frac{(r+|x|)(r-|x|)}{(|x|+|y|)^n} \leq \frac{r^2-|x|^2}{|x-y|^n} \leq \frac{(r+|x|)(r-|x|)}{\Big||x|-|y|\Big|^n} = \frac{(r+|x|)}{(r-|x|)^{n-1}}$$

and thus we only need to show that the other terms line up. But since u harmonic we have

$$u(0) = \fint_{\partial B} u(y)dy = \fint_{\partial B} g(y)dy = \frac{1}{|\partial B|}\int_{\partial B} g(y)dy$$

where

$$|\partial B(0,r)| = \gamma(n)\cdot\frac{1^{n-1}}{r^{n-1}}$$

and hence

$$u(x) = \int_{\partial B} K(x,y)g(y)dy = \int_{\partial B}\frac{1}{\gamma(n)r}\frac{r^2-|x|^2}{|x-y|^n}g(y)dy$$

where we have

$$\int_{\partial B}\frac{1}{\gamma(n)r}g(y)dy = u(0)r^{n-2}$$

by harmonicity and the rest we can bound by the top most inequality, hence

$$u(0)r^{n-2}\frac{r-|x|}{(|x|+r)^{n-1}} \leq u(x) \leq u(0)r^{n-2}\frac{(r+|x|)}{(r-|x|)^{n-1}}.$$

$\square$

Exercise 4.

Proof.

Since n is finite we just estimate $\partial_n u$ at 0. That is, we have

$$\frac{u(he_n)-u(0)}{h} = \frac{1}{h}\left(\frac{2h}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+}\frac{g(y)}{|he_n-y|^n}dy - g(0)\right) = \frac{2}{n\alpha(n)}\int_{\partial\mathbb{R}^n_+}\frac{g(y)}{|he_n-y|^n}dy$$
$$= C_n\int_{S:=\left\{B(he_n,\varepsilon)\right\}}\frac{g(y)}{|he_n-y|^n}dy + C_n\int_{\partial\mathbb{R}^n_+\setminus S}\frac{g(y)}{|he_n-y|^n}dy$$
$$\geq C_n\int_S\frac{|y|}{|he_n-y|^n}dy$$

and now since we are confining ourselves on the $n-1$ dimensional subset

$$S := \left\{B(he_n,\varepsilon)\right\}$$

we know that for h small and ε smaller that $g(y) = h + O(\varepsilon)$ so it's bounded both above and below. Thus, for the rest we just have

$$C_n\int_S\frac{|y|}{|he_n-y|^n}dy \geq C_n'\int_S\frac{1}{|he_n-y|^n}dy = C_n'\cdot O\left(\int_0^r\frac{1}{|r|^n}r^{n-1}dr\right) = \infty$$

where the last step is just polar substitution plus rescaling to center at $y = he_n$. Thus

$$|\nabla u| \geq |\partial_n u| \geq \infty$$

so ∇u is not bounded around 0.

□