385. Tonimenix Yu HW 1. (Discussed with Classmates)

Ex 1:

- $Y_0 = 0$: $Y_0 = a^{-1} \cdot B_0 = 0$ ✓

- Independent increments:

$\sigma(Y_s - Y_r) = \sigma(B_s - B_r)$ since if $\mathcal{O}$ open then $a^{-1}\mathcal{O}$ is open, then by def of R.v. generated σ-alg they are the same since we're on $\mathbb{R}$

for the same reason, $\sigma\{Y_r : r \le s\} = \sigma\{B_r : r \le s\}$

So $\sigma(Y_s - Y_r)$ is independent to $\sigma\{Y_r : r \le s\}$ as $\sigma(B_s - B_r)$ is ind. to $\sigma\{B_r : r \le s\}$ ✓

- If $s < t$,

then $Y_s - Y_t = a^{-1} \cdot (B_{a^2 s} - B_{a^2 t}) \sim a^{-1} \cdot N(0, a^2 \cdot (s-t)) = N(0, s-t)$ ✓

- Continuous path.

If f cts then $a \cdot f(bx)$ is cts. so Y_t is cts path with $P = 1$ ✓

Hence, Y_t is a Brownian Motion. $\square$

Ex2:

- for $n \in \mathbb{N}$, by Law of Large Numbers we get $\lim_{n\to\infty} \frac{B_n}{n} = \mathbb{E} B_1 = 0$.

- for $\lim_{n\to\infty} \frac{M_n}{\log n}$, we note reflection principle yields

$$\mathbb{P}\left(\sup_{n\le t\le n+1} M_n \geq a \log n\right) \le 2\mathbb{P}\left(|B_{n+1} - B_n| \geq a \log n\right)$$

Where a is constant $a \neq 0$. Then use Exercise 5 { in fact, $e^{-\frac{x^2}{2}}\left(\frac{1}{x} - \frac{1}{x^3} + O\left(\frac{1}{x^5}\right)\right)$ by IBP, shown in ~~Ex~~ work for Ex5 (proven independently of this)

to get

$$2\mathbb{P}(|B_{n+1} - B_n| \geq a\log n) \le 2 \cdot e^{-\frac{a^2 \log^2 n}{2}}\left(\frac{1}{a\log n} - O\left(\frac{1}{\log^2 n}\right)\right)$$ where

$$\cancel{\le C \cdot \frac{1}{n^2 \log n} + O\left(\frac{1}{n^2 \log^3 n}\right)}$$

$e^{-C\cdot \log^2 n} = \left(\frac{1}{n}\right)^{C\cdot \log n} > n^{-1.5}$ when n large enough $\Rightarrow$ summable

which is summable.

Thus, $\forall a > 0$, $\mathbb{P}\left\{\limsup_{n\to\infty} \frac{M_n}{\log n} \geq a\right\} = 0$ with prob 1.

since $\sum_n^\infty \mathbb{P}(M_n \geq a \log n) < \infty$ ~~BB~~

& Borel Cantelli.

Now, $\forall t$, $B_t \le B_{\lfloor t \rfloor} + M_{\lfloor t \rfloor}$

Both exists $\Rightarrow$ well dfined.

So $\lim_{t\to\infty} \frac{B_t}{t} \le \lim_{t\to\infty} \frac{B_{\lfloor t \rfloor} + M_{\lfloor t \rfloor}}{t} \le \lim_{t\to\infty} \frac{B_{\lfloor t \rfloor}}{\lfloor t \rfloor} + \lim_{t\to\infty} \frac{M_{\lfloor t \rfloor}}{\lfloor t \rfloor} = 0$.

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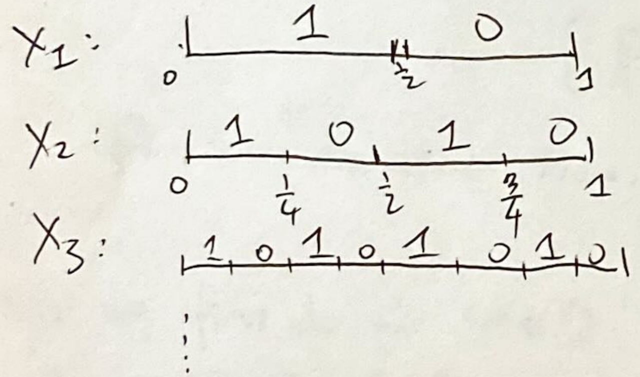
3. We construct a ${}^{k\text{th}}$ set of independent Bernoulli as usual (used a lot in Martingales).

- $X_1^{-1}(1) = [0, \frac{1}{2}]$; $X_1^{-1}(0) = [0,1] \setminus X_1^{-1}(1)$
- $X_2^{-1}(1) = [0, \frac{1}{4}] \cup [\frac{1}{2}, \frac{3}{4}]$;

⋮

- $X_n^{-1}(1) = \cancel{[0, \frac{1}{2^n}] \cup \ldots} \bigcup_{k=1}^{n} \left[\frac{2k-2}{2^k}, \frac{2k-1}{2^k}\right]$

Whose "graph" is this:



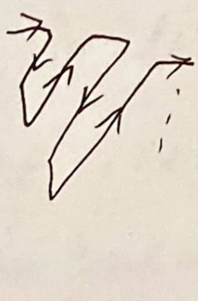
Then they are independent because they generate the σ-alg of a Martingale. (which, of course, is by computation of $P(A) \cdot P(B) = P(A \cap B)$.)

· Now we ~~construct~~ have $\mathbb{X} := \{X_1, X_2, X_3 \ldots\}$ which we ~~relable~~ ~~relable to~~ reshape to $(|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|)$.

form a 2-dim R.V. Table:

X_1	X_2	X_6
X_3	X_5	
X_4		

by



for which we relable as

$Y_{1,1} = X_1$,	$Y_{1,2} = X_2$	...
$Y_{2,1} = X_3$,	...	
$Y_{3,1} = X_4$ ...		

P3.

Now we use CLT to get normal distribution on each row:

Define: for each n

$$N_n = \lim_{m\to\infty} \sum_{i=1}^{m} \frac{(Y_{i,n}-\frac{1}{2})}{\sqrt{m}\cdot(\frac{1}{2})} \overset{CLT}{\sim} N(0,1).$$

Thus,

$N_1, N_2 \cdots$ are constructed Normal distributions, with independent σ-algebras since X_n has ind. σ-algebra.

$\square$.

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Q4. for fixed k, ε, (Method 1)

- $Y_n := \max\left\{ |B_{\frac{1}{2^n}} - B_0|, |B_{\frac{2}{2^n}} - B_{\frac{1}{2^n}}|, \dots, |B_{\frac{2^n}{2^n}} - B_{\frac{2^n-1}{2^n}}| \right\}$

  and denote $B_{\frac{k}{2^n}} - B_{\frac{k-1}{2^n}} =: X_n$

Then

$$\mathbb{P}\left(2^{n/2} Y_n < k\right) \overset{\text{independent}}{=} \left[\mathbb{P}\left(2^{n/2} \cdot X_1 < k\right)\right]^{2^n}$$

$$= \left[\mathbb{P}\left(2^{n/2} \cdot \frac{1}{2^{n/2}} |B_1| < k\right)\right]^{2^n}$$

$$= \left[\mathbb{P}(|B_1| < k)\right]^{2^n} = C_k^{(2^n)}$$

for some $C_k < 1$. Now, ~~idea~~ key is that $C_k^{(2^n)}$ is very very fast.

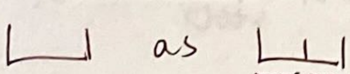
- Now, we first do for $s, t \in D$. Dyatic set.

  for $s, t \in D$, $\exists S \subseteq \mathbb{N}^+$ s.t. wlog $s \le t$, $t = s + \sum_{i \in S} \frac{1}{2^i}$

  & Since $t, s \in D$, S is actually a finite set with order less than N where $N := \max\{n_t, n_s\}$, where $t = \frac{k_t}{2^{n_t}}$; $s = \frac{k_s}{2^{n_s}}$, $k_t, k_s \equiv 1 \pmod 2$

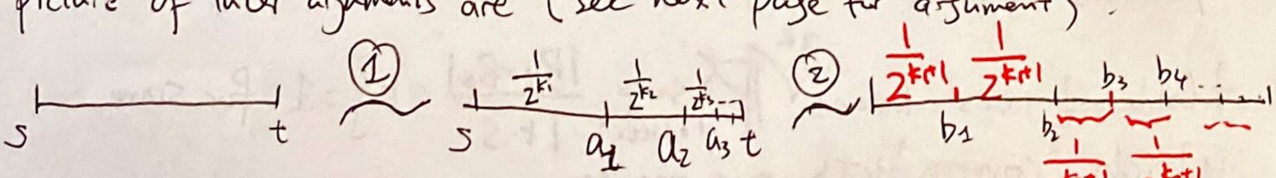
  Also, denote $M := \min\{n_t, n_s\}$.

  Now say $t - s = \frac{l}{2^n}$ for $l \in \mathbb{Z}$. the idea is we can always view



  , i.e. $\frac{1}{2^n} = \frac{2}{2 \cdot 2^n} = \frac{2}{2^{n+1}}$.

The picture of later arguments are (see next page for argument).



P5.

Using Trig inequality, we have (a_i, b_i explained in graph last page)

$\frac{1}{2^i}$, note $\sigma(i) \le N$

$i \mapsto n$ where a_{i-1} a_i

$$\frac{|B_t-B_s|}{\sqrt{s-t}} \le \frac{\sum_{i=1}^{l} |B_{a_i}-B_{a_{i-1}}|}{\sqrt{s-t}}$$

$2^{n+k} > 2^n$

to balance loss in $2^{-i/2}$

$$\Rightarrow 2^{-n/2}|B_t-B_s|\cdot\frac{1}{\sqrt{l}} \le \frac{1}{\sqrt{l}}\cdot\sum_{i=1}^{l}\left|2^{-i/2}\cdot Y_{\sigma(i)}\right|\cdot\left(2^{\frac{N-M}{2}}\right)$$

$$\Rightarrow \mathbb{P}\left(k \ge 2^{-n/2}|B_t-B_s|\right) \le C\cdot\sum_{i=1}^{l}(C_k)^{(2^{\sigma(i)})}$$ by linearity

Above is cut ①, for cut ② we get importantly

②

over and over m times

$$c\sum_{i=1}^{l}(C_k)^{(2^{\sigma(i)})} \le C\sum_{i=1}^{l} 2\,(C_k)^{(2^{\sigma(i)+1})} \le C\sum_{i=1}^{l} 2^m\,(C_k)^{(2^{\sigma(i)+m})}$$

$$= C\left[\sum_{i=1}^{l}(C_k)^{(2^{\sigma(i)})}\right] 2^m\cdot C_k^{2^m}$$

where as $m\to\infty$. $2^m C_k^{2^m} \to 0$ (take log to see)

Also, $C\cdot\sum_{i=1}^{l}(C_k)^{(2^{\sigma(i)})} \le C\cdot l\cdot C_k^{(2^{\sigma(1)})} \le \tilde{C} < \infty$

↑ constant.

Thus, $\frac{|B_t-B_s|}{\sqrt{s-t}} \le \tilde{C}\cdot\varepsilon$ by taking $m\to\infty$

- Now, Moving from D to $[0,1]$ we just use D dense in $[0,1]$ plus w.p. 1, B_t is cts. so $\sup_{t,s\in[0,1]}|B_t-B_s| = \sup_{s,t\in D}|B_t-B_s|$ w.p. 1.

So $\mathbb{P}\left\{\sup_{0<s<t<1}\frac{|B_t-B_s|}{\sqrt{t-s}} \le k\right\} \le \varepsilon$, for $\forall k, \forall \varepsilon$. w.p. 1

Yet Hölder-$\frac{1}{2}$ cts means $\exists k$ $\mathbb{P}\left\{\sup_{0<s<t<1}\frac{|B_t-B_s|}{\sqrt{t-s}} \le k\right\} = 1$ which contradicts our result.

$\Rightarrow$ W.P. 1, B_t is NOT Hölder-$\frac{1}{2}$ cts.

After discussion with classmates, realize that 0-1 law is much neater:
(Method 2):

By set inclusion

$$\mathbb{P}\left\{\sup_{0<s<t<1}\frac{|B_t-B_s|}{\sqrt{t-s}}\geq k\right\} \overset{\text{take } s\to 0 \text{ \& } s=0 \text{ passes by sup.}}{\geq} \mathbb{P}\left\{\sup_{0<r<1}\frac{|B_r|}{\sqrt{r}}\geq k\right\}$$

$$\geq \mathbb{P}\left\{\sup_{0<r<1}\frac{B_r}{\sqrt{r}}\geq k\right\}$$

Now, we construct a tail σ-algebra by

$$A_n := \left\{\sup_{0<r<\cancel{k_n}\, t_n}\frac{B_r}{\sqrt{r}}\geq k\right\}, \text{ then } A_1 \supseteq A_2 \supseteq \dots$$

any $k_n \to 0$

and $\mathbb{P}(A_n) \geq \mathbb{P}\left(\frac{B_{t_n}}{\sqrt{t_n}}\geq k\right) > 0$ by def of $N(0,1)$.

0-1 law implies ~~P(An)=p~~ $B := \bigcap_{n\to\infty} A_n$ has prob measure 1 or 0.

~~for $B = \lim_{n\to\infty} A_n$, $\mathbb{P}(B)$~~

But $\mathbb{P}\left(\frac{B_{t_n}}{\sqrt{t_n}}\geq k\right) = \mathbb{P}(B_1 \geq k) = C_k \leftarrow$ depend on k

So $\mathbb{P}(B) \geq C_k \Rightarrow \mathbb{P}(B) = 1$ and we are done.

Q5:

$$\int_x^\infty e^{-\frac{y^2}{2}}dy = \int_x^\infty \frac{1}{y}\cdot y e^{-\frac{y^2}{2}}dy \overset{\text{IBP}}{=} -\frac{e^{-\frac{y^2}{2}}}{y}\Big|_x^\infty - \int_x^\infty \frac{e^{-\frac{y^2}{2}}}{y^2}dy$$

$$= \frac{e^{-\frac{x^2}{2}}}{x} - \int_x^\infty \frac{1}{y^3} y e^{-\frac{y^2}{2}}dy$$

IBP again

$$= e^{-\frac{x^2}{2}}\left(\frac{1}{x}-\frac{1}{x^3}\right)+3\int_x^\infty \frac{e^{-\frac{y^2}{2}}}{y^4}dy$$

$$= e^{-\frac{x^2}{2}}\left(\frac{1}{x}-\frac{1}{x^3}\right)+o\left(\frac{1}{x^4}\cdot e^{-\frac{x^2}{2}}\right), \text{ by repeatedly IBP}$$

So

$$\mathbb{P}(N>x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\left(\frac{1}{x}-\frac{1}{x^3}+o\left(\frac{1}{x^4}\right)\right)$$

$$= e^{-\frac{x^2}{2}}\left[\frac{\left(\frac{1}{\sqrt{2\pi}}\right)}{x}+O(x^{-2})\right]$$

where $c_1 = \frac{1}{\sqrt{2\pi}}$.