PDE HOMEWORK 7

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Discussed with classmates.

Exercise 1.

Proof.

For $u \in C_0^2$ we have by FTC that

$$u(x)^2 = \left(\int_0^x u'(t)dt\right)^2 \leq \int_0^1 \int_0^1 (u'(x))^2 dxdy \leq \int_0^1 (u'(t))^2 dt$$

and thus putting yet another integral outside we have

$$\int_0^1 u(x)^2 dx \leq \int_0^1 \int_0^1 (u'(t))^2 dtdx \overset{Fubini\&t\leq 1}{\leq} \int_0^1 (u'(x))^2 dx.$$

For $u \in H_0^2$, we know that C_0^2 is dense in H_0^2 so we just pick $u_i \in C_0^2$ that goes to u, then pass the limit to get our result.

□

Proof. 2

$\Bigg($ Below is a proof 2 by Sobolev embedding and contradiction, this can be generated to any dimension.$\Bigg)$

Suppose it's not true then we can find $u_k \in H_0^1[0, 1]$ such that

$$\int_0^1 u_k^2 dx \geq k \int_0^1 (u')^2 dx$$

and now we use Sobolev embedding to get that H_0^1 is compact in L^2. Note that Theorem 1 in 5.8 in book is not enough since it omits the $n = 1$ case. But that is just because we have

$$||u||_{H_0^1} = ||u||_{L^2} + ||\nabla u||_{L^2} \geq ||\nabla u||_{L^2}$$

and thus any bounded space in H_0^1 (in the H_0^1 norm) we have it is Lipschitz (extended notion, as discussed briefly in class) thus we can apply the analogous version of Ascoli-Arzela (also talked in class) to know that the set

$$U := \{u \in H_0^1 | ||u||_{H_0^1} \leq 1\}$$

is a compact set. Thus H_0^1 is compact in L^2.

Now by compactness we define

$$v_k := \frac{u_k}{||u_k||_{L^2}}$$

to get

$$||v_k||_{L^2} = 1$$

and hence there exists limit of subsequence $v = \lim_{n\to\infty} v_{k_n}$ in L^2 sense. But then weak compactness means for $\phi \in C_c^\infty$ text functions

$$\langle v, \phi' \rangle = \lim_{n\to\infty} \langle v_{k_n}, \phi' \rangle = -\lim_{n\to\infty} \langle v'_{k_n}, \phi \rangle = 0$$

since

$$\int_0^1 (v'_{k_n})^2 dx \leq \frac{1}{k} ||v_{k_n}||_{L^2} \to 0.$$

But then

$$\langle v, \phi' \rangle = 0$$

for arbitrary ϕ so $v \equiv 0$.

Yet $v_{k_n} \xrightarrow{L^2} v$ and

$$\lim_{n\to\infty} ||v_{k_n}||_{L^2}^2 = 1 \neq 0 = ||v||_{L^2}^2$$

gives contradiction. So the Poincare inequality holds.

$\square$

Exercise 2.

Proof.

(i):

So we first give the intuition of this definition. This is just that

$$\int fv dx \approx \int \Delta^2 uv dx \overset{ibp\ twice}{\approx} \int \Delta u \Delta v dx$$

which is how we've defined it.

(ii):

The heuristics are just (below is just formal writing from (i))

$$\langle \Delta u, \Delta v \rangle \approx -\langle \nabla^3 u, \nabla v \rangle \approx \langle f, v \rangle$$

since by definition of $v \in H_0^2$ and boundary condition, we know the boundary term vanishes.

But that's nothing close to a proof. For a real proof we define

$$B[u, v] = \int_U \nabla u \nabla v dx$$

then

$$B[u, v] \leq c||\nabla u||_{L^2}||\nabla v||_{L^2} \leq c||u||_{H_0^2}||v||_{H_0^2}$$

thus if we define the functional

$$B[\cdot, v] : H_0^2(U) \to \mathbb{R}$$

then it is bounded, thus we apply Riesz on H_0^2 to define

$$B[u, v] =: \langle w, v \rangle$$

and hence we have an operator $A(u) := w$. Now we show that A is bounded above and below. We know

$$||Au||^2 = ||Au|| \cdot ||Au|| = B[u, Au] \leq \alpha||u|| \cdot ||Au||$$

which means $|A|$ is bounded above. Moreover, we have coercivity: Heuristic is that

$$L := \Delta^2 = \nabla \cdot I \nabla(\nabla \cdot I \nabla)$$

and to be precise we use Poincare:

$$B[u, u] = \int \Delta u \Delta u dx \geq ||\Delta u||_{L^2}^2$$

and now we use problem 1 (extended version of Poincare, where the 2 norm of the matrix is the same as the trace, that is the norm of Δ) to get that

$$(1 + C_1 + C_2)||\Delta u||_{L^2}^2 \geq ||\Delta u||_{L^2}^2 + ||\nabla u||_{L^2}^2 + ||u||_{L^2}^2 \geq ||u||_{H_0^2}^2$$

where C_1, C_2 is such that (by Poincare)

$$||\Delta u||_{L^2}^2 \geq C_1 ||\nabla u||_{L^2}^2; \quad ||\nabla u||_{L^2}^2 \geq \frac{C_2}{C_1} ||u||_{L^2}^2$$

thus bounded from below. So we let $(1 + C_1 + C_2)^{-1} =: \beta$ then

$$\beta ||u||_{H_0^2}^2 \leq B[u,u] = \langle Au, u \rangle \Rightarrow \beta ||u||_{H_0^2} \leq ||Au||$$

and so A is bounded from below. Thus by a priori estimate $\ker(A) = \{0\}$ and $\mathrm{Ran}(A)$ is closed. So we still need to show that the range of A is everything to get that A can be inverted. But since it's closed, if $u \perp \mathrm{Ran}\, A$ we know

$$0 = \langle Au, u \rangle = B[u,u] \geq \beta ||u||_{H_0^2}^2$$

and thus $u = 0$ (definiteness of norm). So the only vector orthogonal to $\mathrm{Ran}\, A$ is the null vector, hence it is the whole space H_0^2.

Thus we've shown that A is invertible. Now for the particular f in problem we just use Riesz again to get

$$\langle f, v \rangle =: \langle w, v \rangle$$

then we let $u = A^{-1} w$ to get

$$\langle f, v \rangle = \langle w, v \rangle = \langle Au, v \rangle = B[u, v]$$

and thus we've found a solution u.

Uniqueness:

Suppose there's 2 u_1, u_2 that satisfies the above condition, then

$$B[u_1 - u_2, v] = \langle f, v \rangle - \langle f, v \rangle = 0$$

so

$$\langle \Delta(u_1 - u_2), \Delta v \rangle = 0$$

for arbitrary $v \in H_0^2$. So $\Delta(u_1 - u_2) = 0$. Denote $s = u_1 - u_2$ then s is harmonic, but also to satisfy the condition we require $s = 0$ on ∂U. Now by maximum principle (and minimum) we know $s = 0$ everywhere on U. So $u_1 = u_2$ and thus the solution is unique.

(iii): From what I can think of as long as $\int_U f v dx$ is well defined, then everything above works through. And the largest space is $f \in \left(H_0^2(U)\right)^* = H_0^{-2}(U)$.

$\square$

Exercise 3.

Proof.

Just write out we have

$$\nabla \cdot A\nabla\phi(u) = \nabla\left(A\phi'(u)\nabla u\right) = (\nabla A)\phi'(u)\nabla u + \phi''(u)(\nabla u)^T A\nabla u + A\phi'(u)\Delta u$$

and since $A \geq 0$ so

$$\phi''(u)(\nabla u)^T A\nabla u \geq 0$$

by convexity of ϕ. Moreover, the other two terms can be written as

$$(\nabla A)\phi'(u)\nabla u + A\phi'(u)\Delta u = \phi'(u)\left(\nabla \cdot A\nabla u\right)$$

and thus

$$\begin{aligned} B[w,v] &= \int_U A\nabla w\nabla v dx \overset{v\in H_0^1}{=} -\int_U \nabla\cdot(A\nabla w)v dx \\ &= -\int_U \phi'(u)\nabla\cdot(A\nabla u)v dx - \int_U \phi''(u)(\nabla u)^T A\nabla u v dx \\ &\leq B[u,\phi'(u)v] + 0 \leq 0 \end{aligned}$$

where $\phi'(u)v \in H_0^1$ because u is bounded and $\phi'(u)$ is thus controlled both from above and below, and it is smooth.

□