INFLUENCE OF CORIOLIS FORCES ON BULK-EDGE CORRESPONDENCE IN GEOPHYSICAL WAVE MODELS

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ABSTRACT. In this paper, we focus on the 3 by 3 system in the geophysical wave transportation model in Fourier domain, with Coriolis force parameter $f(y)$. In particular we care about how the behavior of $f(y)$ affects the spectral flow for different energy levels. We can show that the spectral flow is always 2 if $f(y)$ is smooth ($||f'||_\infty < \infty$) and hence in this case the bulk edge correspondence holds. When $f(y)$ is non-smooth, in particular when it has a jump, we try to show that the spectral flow can be arbitrary, concluding the failure of bulk-edge correspondence.

CONTENTS

1. Introduction

Topology shed light to physics. In the geophysical fluid wave propagation model, we observed the Equatorial Kelvin and and Yanai waves that are propagating along the Earth's equator eastward. Remarkably, the existence of unidirectional Kelvin and Yanai waves has been predicted by the winding of the eigenmodes parameterized over a closed surface. This winding number is a topological invariant which we call the Chern number, parametrized by the Coriolis force f (Tauber, Delplace, Venaille, 2019). In particular, shown in (Delplace, Marston, Venaille, 2017, [1]), in the (k_x, k_y, f)-space where (k_x, k_y) are wavenumbers, the Chern number for the fiber bundle of eigenvectors with positive frequency eigenmodes has been explicitly computed to be 2, corresponding to the two equitorial unidirectional propagating waves, Kelvin and Yanai.

This kind of correspondence, generally speaking, is called bulk-edge correspondence, and appears in the study of quantum Hall effect, as well as that of topological insulators. Intuitively, think of two materials with different topological properties close to each other, and at their contact surface asymmetric transport of particles was observed. We call the materials as bulk, and the contact surface as edge, then the bulk-edge correspondence says that what is happening at the edge (asymmetric transport) is encoded by information at the bulk (topological properties of materials). In our geophysical wave model, we can think of the equator as the edge and the northern and southern hemispheres as bulk, and the unidirectional wave's appearance is analogous to the appearance of asymmetric transports. In this particular case with the earth's Coriolis force, we see from previous work that indeed the bulk-edge correspondence holds.

In the work below, we use the same system as in the geometric wave model, but we ask the question of what happens to the bulk-edge correspondence for a different Coriolis force parameter f. In particular, we investigate forces that are smooth ($\|f'(y)\|_\infty < \infty$) and those that have jumps. The result we have right now is for smooth forces the correspondence holds, while (hopefully) the appearance of jumps is the essential condition that breaks the correspondence.

2. THE QUESTION AND PREVIOUS WORKS

The major problem is to find the eigenvalue of the system (Vallis, [2])

$$H = \begin{pmatrix} 0 & D_x & D_y \\ D_x & 0 & if(y) \\ D_y & -if(y) & 0 \end{pmatrix}$$

where $D_x = \frac{1}{i}\partial_x; D_y = \frac{1}{i}\partial_y$.

The problem for $f = 0$ or $f = y$ or $f = f_0 \operatorname{sgn}(y)$ is solved, but it remains to solve for other cases, that is, we want to explore the eigenvalue E within the system

$$\begin{pmatrix} 0 & D_x & D_y \\ D_x & 0 & if(y) \\ D_y & -if(y) & 0 \end{pmatrix}\begin{pmatrix} \eta \\ u \\ v \end{pmatrix} = E\begin{pmatrix} \eta \\ u \\ v \end{pmatrix}$$

for which we can Fourier transform in $x \rightsquigarrow \xi$ such that the system becomes, since $\partial_x \rightsquigarrow i\xi$:

$$\begin{pmatrix} 0 & \xi & D_y \\ \xi & 0 & if(y) \\ D_y & -if(y) & 0 \end{pmatrix}\begin{pmatrix} \eta(y) \\ u(y) \\ v(y) \end{pmatrix} = E\begin{pmatrix} \eta(y) \\ u(y) \\ v(y) \end{pmatrix}$$

which we usually denote it just by H. We can also write explicitly as

$$\begin{cases} \xi u + Dv = E\eta \\ \xi\eta + ifv = Eu \\ D\eta - ifu = Ev \end{cases} \tag{2.1}$$

and to further simplify we eliminate η and u to get (Bal, page 29 [3]) the Sturm-Liouville equation

$$\left(D^2 + f^2 + \frac{\xi}{E}f'\right)v = \left(E^2 - \xi^2\right)v. \tag{2.2}$$

Here, we note that if we do Fouier transform in both x and y variables for the case f constant, then we get a definite matrix for which we can find three branches of eigenmodes in the $(\xi, \zeta) - E$ place where $x \rightsquigarrow \xi, y \rightsquigarrow \zeta$. We explicitly compute the eigenvalues to be $\xi = \pm\sqrt{\xi^2 + \zeta^2 + f^2}$ (class notes) and see that two of the three branches are similar to the two branches of the Dirac equation, and the one remaining is 0. But if f is non-constant then even Fourier transforming f would be hard in general and we shall not do that.

Previous work has done an explicit analysis of the discrete spectrum for the operator when $f(y) = y$ or $f(y) = \operatorname{sgn}(y)$:

In the case $f(y) = \lambda y$, we know $f' = \lambda$ is a constant and thus we can relate the problem to

$$a^*av = (D^2 + \lambda^2 y^2)v = \left(E^2 - \xi^2 - \lambda\left(1 - \frac{\xi}{E}\right)\right)v$$

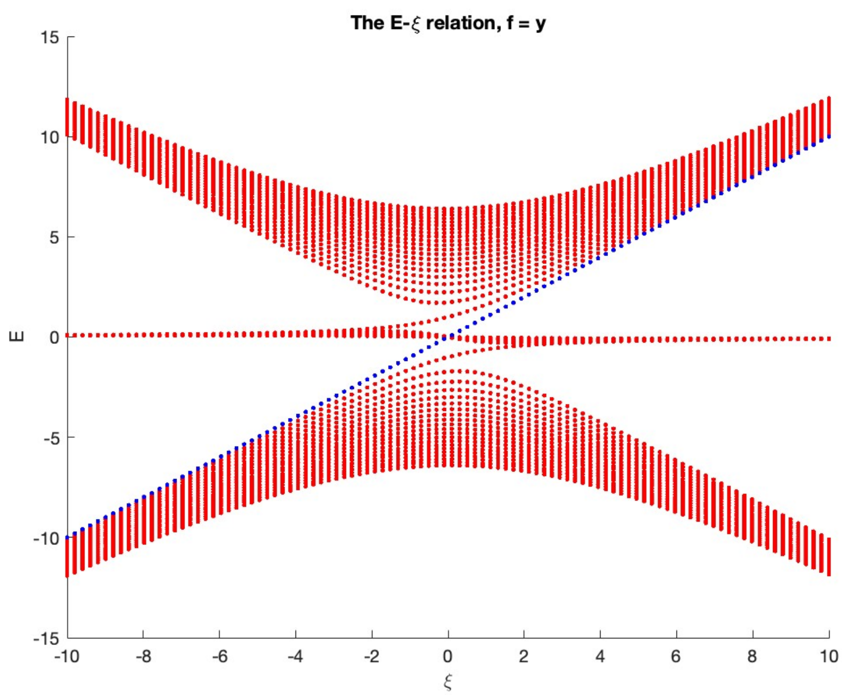
where $a := \partial_y + f(y)$, as is defined in last section. Thus, since we know that the branches of eigenvalues of a^*a is given by $2n\lambda$ we get by plugging in the corresponding eigenvector that

$$E_n^2 - \xi^2 - \lambda \frac{\xi}{E_n} = (2n+1)\lambda \quad \Rightarrow E_n^3 = (\lambda(2n+1) + \xi^2)E_n + \lambda\xi$$

and let's focus on the case $\lambda = 1$. We can factorize for $n = 0$ and get explicit expression

$$(E + \xi)(E^2 - \xi E - 1) = 0 \Rightarrow E = -\xi \text{ or } \frac{1}{2}\left(\xi \pm \sqrt{4 + \xi^2}\right)$$

but notice that we should discard all points here that satisfy $|E| = |\xi|$ as is discussed. Moreover, plotting all solutions for $n \in \mathbb{Z}^+$ we get the plot:



In particular, the blue mode is the Kelvin wave and the two waves that go from $-\infty$ to 0 and 0 to ∞ are Yanai waves.

Here the trivial case for $E = \frac{1}{2}\left(\xi \pm \sqrt{4+\xi^2}\right)$ is analytic in $\frac{1}{\xi}$ using binomial theorem: let $y = \frac{1}{\xi}, x = 4y^2$ then E is analytic in ξ because

$$\sqrt{4+\xi^2} = \frac{1}{y}(1+4y^2)^{\frac{1}{2}} = \frac{1}{y}\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}x^k = \frac{1}{y} + 2y - 2y^3 + 4y^5 + \dots$$

we'll see later that by Gato's work the branches are always analytic.

For the case $f(y) = \begin{cases} f_+ & y \geq 0 \\ f_- & y < 0 \end{cases}$, we obtain the boundary and jump conditions (deduction is in explained in section 6) (Bal, page 31)

$$-\partial_y^2 v + f^2 v = (E^2 - \xi^2)v, y \neq 0; \qquad -(v'(0^+) - v'(0^-)) + \frac{\xi}{E}(f_+ - f_-)v(0) = 0$$

and with some algebra we obtain

$$E = \frac{-\xi f_0}{\sqrt{f_e^2 + \xi^2}}$$

where $f_0 = \frac{1}{2}\left(f_+ - f_-\right), f_e = \frac{1}{2}\left(f_+ + f_-\right)$ where in particular

$$\lim_{\xi \to \infty} E = -\frac{1}{2}(f_+ - f_-)$$

which means there's nothing except the naive branch $E = \xi$ for $E < \left|\frac{1}{2}(f_+ - f_-)\right|$, hence the bulk-edge correspondence fails.

Note that here because the system is easy, we can get an explicit answer. For the more general case discussed in section 6, obtaining a precise solution very tedious and not necessarily meaningful. Rather, we will directly take asymptotics and see that the results generate.

3. Main results

In this work, for the eigenvalue problem

$$\begin{pmatrix} 0 & \xi & D_y \\ \xi & 0 & if(y) \\ D_y & -if(y) & 0 \end{pmatrix} \begin{pmatrix} \eta(y) \\ u(y) \\ v(y) \end{pmatrix} = E \begin{pmatrix} \eta(y) \\ u(y) \\ v(y) \end{pmatrix} \tag{3.1}$$

we characterized the behavior of eigenmodes of $E(\xi)$, especially how they behave when $\xi \to \pm\infty$. This will help us to compute the spectral flow of the problem, hence conclude whether edge-bulk correspondence hold.

Def 3.1. *Given the eigenmodes $E(\xi)$, the number of eigenmodes crossing the region $E = [a, a+\varepsilon]$ $\forall a > 0, \varepsilon > 0$ with direction, that is, upward crossings are counted as $+1$ while negative crossings counted -1, is called the spectral flow.*

Theorem 3.2. *If $f \in C^1(\mathbb{R}, \mathbb{R})$ is a function such that $||f'||_\infty$ is bounded, then for the eigenmodes of our system 3.1, when $E > 0$, one of the following must hold:*

- $E = \xi$;
- $E > |\xi|$ *and* $\lim_{\xi\to\infty} \dfrac{E}{\xi} = \pm 1$;
- $0 \le E < |\xi|$ *and* $\lim_{\xi\to\infty} E = 0$.

This, plus the fact that our system's eigenmodes E is odd in ξ, and a specification of one anomaly branch, we get the following result:

Theorem 3.3. *If $f \in C^1(\mathbb{R}, \mathbb{R})$ is a function such that $||f'||_\infty$ is bounded, the spectral flow is always 2.*

The fact that for smooth coriolis force we have spectral flow 2 means that the bulk-edge correspondence holds. However, as we'll see, when the coriolis force has jumps, we really expect the correspondence to fail.

Theorem 3.4. *For the same system 3.1, for $-\infty = y_0 < y_1 < \cdots < y_n < y_{n+1} = \infty$, let $f(y) = f_i$ for $y_i \le y \le y_{i+1}$ where $f_0 < 0$ and $f_n > 0$, and define $d_i := f_i - f_{i-1}$. Then, as $\xi \to \infty$, all E such that does not go to infinity or exactly equal to 0 goes to $-\dfrac{1}{2}d_i$, for $d_i > 0$, and as $\xi \to -\infty$, $E \to -\frac{1}{2}d_j$ for $d_j < 0$.*

This shows that in this particular case, the bulk-edge correspondence fails, since the spectral flow for a, ε small is 1 yet it is 2 for a big.

Our goal is to find an explicit eigenvector v of the Strum-Liouville problem 2.2 induced by 3.1 that goes to $\frac{1}{2}d$ for f with a jump of value d. If this can be done, we can create any eigenmodes as we want, and the buld-edge correspondence is of course false.

4. Numerical characterization

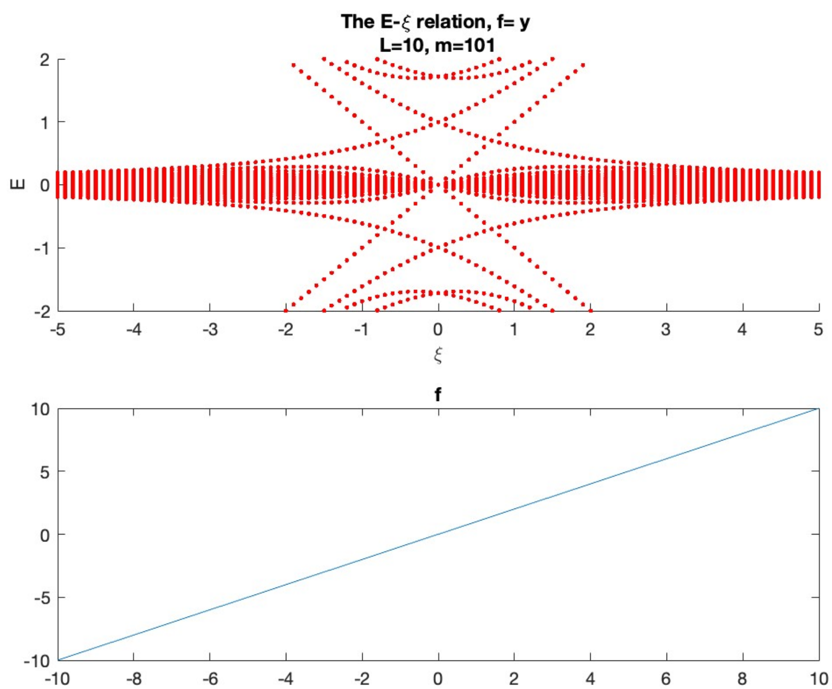
4.1. **Full system Neumann condition:** First we solve the original system 3.1 with Neumann boundary conditions: to get over the obstacle that the system is dependent on all of y. One way to implement is to assume that we are only dealing with the period $y \in [-10, 10]$, idea being that most interesting things of f happen when y is small. Now that we are on a compact domain we can approximate the differential operator and get: for $l = 10$, finite difference matrix gives (n is how many points we've chosen in the discretization)

$$D = \frac{n-1}{2l} \begin{pmatrix} -2 & 2 & & & & \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 \\ & & & & -2 & 2 \end{pmatrix}$$

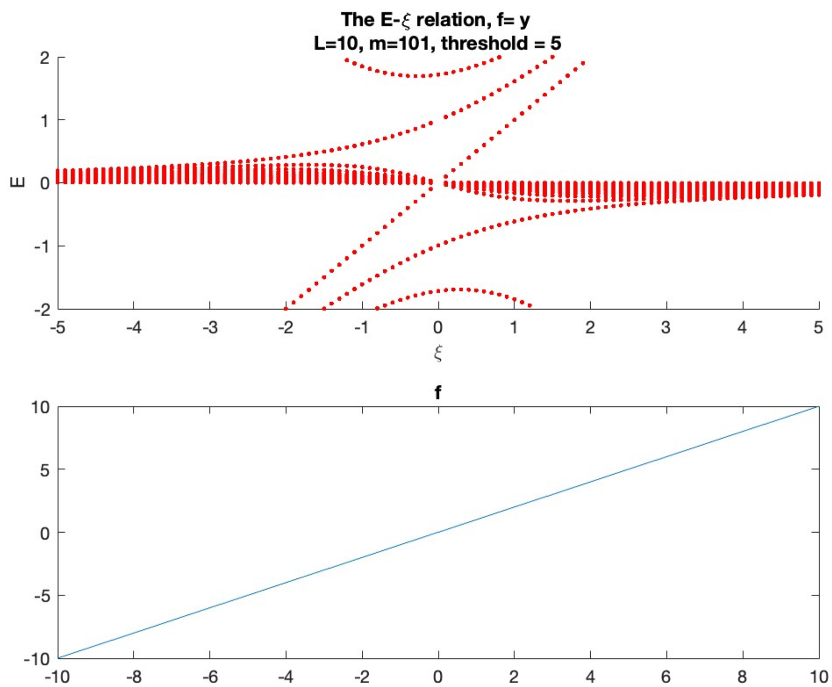
and the operator becomes

$$H = \begin{pmatrix} 0 & \xi * I & \frac{1}{i} D \\ \xi * I & 0 & \mathrm{diag}(if) \\ \frac{1}{i} D & -\mathrm{diag}(if) & 0 \end{pmatrix}$$

which we can find eigenvalues for fixed ξ value and known f. For the case $f = y$ we plot all corresponding E for each $\xi \in [-5, 5]$ and get the plot:



yet there are many artificial waves created by numerical impreciseness, most of which correspond to a highly oscillating eigenvector. To get rid of those modes we do Fourier transform on the eigenvectors and abandon those with high frequencies. In practice, I Fourier transformed the eigenvector and got rid of all those points with corresponding eigenvectors' 2-norm $\geq$ threshold$=5$, the result is:



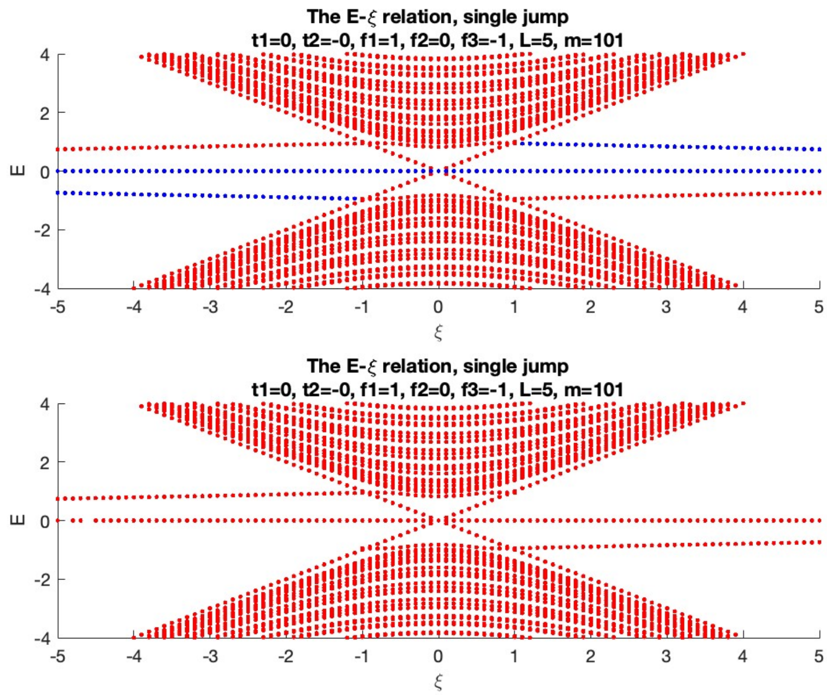
which is indeed very close to the real curves for this case.

With similar ideas to what we're about to do in later sections, it is shown in P31(Bal) that for a function of the form $f = f_+ \cdot \mathbb{1}_{y>0} + f_- \cdot \mathbb{1}_{y<0}$, the only admissible solution is of the form:

$$E = \frac{-\xi f_0}{\sqrt{f_e^2 + \xi^2}}$$

where $f_0 = \frac{1}{2}(f_+ - f_-)$ and $f_e = \frac{1}{2}(f_+ + f_-)$. In particular, we note that as $\xi \to \infty$ we have $E \to -1$.

This is also validated with our code:



where I colored the oscillating eigenvalues blue to discern. Here the Yanai-like mode expresses a clear gap from 0, and no modes around 0 appear. As we will show later, the asymptotic behavior is closely related to the jumps in the graph of f, as for a smooth f E cannot go to a constant at $\xi = \infty$.

4.2. **Strum-Liouville problem with Dirichlet conditions.** There are two problems with the above method: it is very slow and we have to manually delete points numerically generated. We use the Strum-Liouville problem 2.2 and discretize with Dirichlet condition

$$D = \frac{n-1}{2l}\begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}.$$

The idea is to let $\mu = \dfrac{\xi}{E}$ be a fixed constant and hence getting

$$\left(D^2 + f^2 + \frac{\xi}{E} f'\right) v = (1-\mu^2)E^2 v$$

For each eigenvalue S, we compute $E = \pm\sqrt{S/(1-\mu^2)}$ and $\xi = \mu E$. The problem with this method is that $1-\mu^2$ will blow up when $\mu \to \pm 1$, but we'd get a very precise characterization within the discretization range $[-L, L]$ for other angles. Since we are finding eigenvalue of a much smaller system, and there is nothing bad numerically generated, we overcame our obstacles. Below are the results:

The graph for $f = y$ gives:

and the graph for $f = \mathrm{sgn}(y)$ gives:

5. Smooth Coriolis force

Here we try to find the spectral flow of $\hat{H}$. The main method is to analysis the Asymptotic behavior of E as $\xi \to \infty$. But first let's throw away the continuous spectrum, which is irrevalent of our analysis of eigenmodes. Moreover, there's no worry for residue spectrum since our operator is self-adjoint. From 2.2 we see that the ξ-E relation is odd, so we only discuss when $\xi \to +\infty$.

5.1. **Absolute continuous spectrum estimation.**

Lemma 5.1. *(Weyl's Criterion) If A, B are self-adjoint operators such that $(A+i)^{-1}-(B+i)^{-1}$ is compact, then*

$$\sigma_c(A) = \sigma_c(B).$$

Lemma 5.2. *For an operator of the form $A = f(x)g(D)$, $x \in \mathbb{R}^d$ and $D = -i\Delta$. If f, g are bounded and $f, g \in L^p$, $2 \le p < \infty$, then $A \in \mathcal{I}_p$ which in particular A is compact.*

Theorem 5.3. *If $\eta, u, v \in L^2$ in system 2.1 are uniformly continuous, and if $|f'(y)| < C$ for constant C, then $E^2 < 1 + \xi^2$ for our equation.*

Proof.

Rewriting things we get

$$B := D^2 + f^2 + \frac{\xi}{E} f' = (D^2 + 1) + \left(f^2 - 1 + \frac{\xi}{E} f' \right) =: A + w$$

then, to fit into the standard of Weyl's Criterion, we try and get

$$B + i = A + i + w \quad \Rightarrow \quad (A+i)^{-1} = (B+i)^{-1}(I - w(A+i)^{-1})$$

which means that if $(B+i)^{-1}w(A+i)^{-1}$ is compact then we can claim that $\sigma_c(A) = \sigma_c(B)$.

And since ST is compact for T compact and S bounded, we only need to show $w(A+i)^{-1}$ is compact and $(B+i)^{-1}$ bounded.

To show $(B+i)^{-1}$ bounded we assume $(B+i)^{-1}w = h$ then we have

$$\begin{aligned}
&w = (D^2 + 1 + i + f^2 - 1 + \frac{\xi}{E} f')h \\
\Rightarrow &\langle w, h \rangle = ||h_y||^2 + (1 + f^2 - 1 + \frac{\xi}{E} f')||h||^2 + i||h||^2 \\
\Rightarrow &(1 + f^2 - 1 + \frac{\xi}{E} f')||h||^2 \le \mathrm{Re}(\langle w, h \rangle) - ||h_y||^2 \le |\langle w, h \rangle| \le ||w|| \cdot ||h|| \\
\Rightarrow &||h|| \le C||W||
\end{aligned}$$

for $C = \min_y (1 + f^2 - 1 + \frac{\xi}{E} f')^{-1}$ which is attained since f and f' is bounded.

To show $w(A+i)^{-1}$ is compact we use the fact that $w \in L^2$ since it's compact and $(A+i)^{-1} = g(D)$ for $g(x) = \dfrac{1}{x^2+1+i}$ where $g \in L^2$ since it is of order x^{-2} and is well behaved around 0. Hence, Lemma 2.2 shows that $w(A+i)^{-1}$ is compact.

Now Weyl's condition says

$$\sigma_c\left(\frac{\xi}{E}f' = (D^2+1) + \left(f^2 - 1 + \frac{\xi}{E}f'\right)\right) = \sigma_c(D^2+1) = [1,\infty)$$

and hence for all eigenvalue $(E^2-\xi^2)$ we can find, we must have $E^2-\xi^2 < 1$ i.e. $E^2 < 1+\xi^2$. ☐

Theorem 5.4. *If $\eta, u, v \in L^2$ in system 2.1 are uniformly continuous, and if $|f'(y)| < C$ for constant C, then for fixed y either $\lim\limits_{\xi\to\infty} E = \infty$ or $\lim\limits_{\xi\to\infty} E = 0$.*

Proof.

Assume $\lim_{\xi\to\infty} E = c$, then as $\xi \to \infty$, the equation 2.2 becomes

$$\begin{aligned}\lim_{\xi\to\infty}\left(D^2 + f^2 + \frac{\xi}{E}f'\right)v &= \lim_{\xi\to\infty}\left(E^2-\xi^2\right)v \\ \Rightarrow \lim_{\xi\to\infty}\left(D^2 + f^2 + \frac{\xi}{c}f'\right)v &= \lim_{\xi\to\infty}\left(c^2-\xi^2\right)v\end{aligned}$$

Since $|f| < C$ and $\dfrac{\xi}{E}f'v = O(\xi)$ thus $D^2v = O(\xi^2)$. But we also know that $\langle D^2v, v\rangle > 0 > -\xi^2||v||^2$, so

$$||(D^2+\xi^2)v|| > \xi^2||v|| = O(\xi^2) \gg O(\xi)$$

contradict to the equation.

Thus, $\lim\limits_{\xi\to\infty} E \neq c$ so either $\lim\limits_{\xi\to\infty} E = \infty$ or $\lim\limits_{\xi\to\infty} E = 0$. ☐

5.2. **Specifying the discrete spectrum.** We now consider the discrete spectrum, i.e. the eigenmodes. As is specified above, we only consider $E > 0$.

Lemma 5.5. $E^2 - \xi^2 < 1$.

This is directly from theorem 5.3 above. Now, we consider the case for $\xi^2 = E^2$:

Proposition 5.6. *$E = \xi$ is always a branch of eigenmode. There is exactly one point such that $E = -\xi$.*

Proof. First consider the case $v = 0$. Plugging in this into 3.1 we get

$$\begin{cases}\xi u = E\eta \\ \xi\eta = Eu \\ D\eta - ifu = 0\end{cases}$$

which means either $u = \eta = 0$ or $\xi^2 = E^2$. But $v = 0$ already so it cannot be the first case, so $E = \pm\xi$ and $\mu \pm \eta = 0$. We define operator

$$\mathfrak{a} := \partial_y + f(y); \qquad \mathfrak{a}^* = -\partial_y + f(y)$$

and we get the question

$$\mathfrak{a}\eta = \mathfrak{a}u = 0$$

for which we have that the only possible solution is (Bal, page 29-30)

$$\eta = u = ce^{-F(y)}$$

where $F(y) = \int_0^y f(s)ds$ and c is such that $||u|| = ||\eta|| = 1$.

Now for $v \neq 0$, let's discuss for $E^2 = \xi^2$. If $E = \xi$ then the system becomes $D_y v + ifv = 0$ which is equivalently $\mathfrak{a}^* v = 0$, which obtains no solution in L^2 because f is positive at ∞ and negative at $-\infty$. Hence we are left with $E = -\xi$.

In this case we need to solve $D_y v - ifv = 0$ which is $\mathfrak{a}v = 0$. The solution is $v = ce^{-F(y)}$ with normalization. To get u, η we use the original system to get

$$\xi(\eta + u) + D_y v = 0, \quad D_y \eta - ifu = Ev = -\xi v$$

which gives

$$(\partial_y - f)\eta = -\mathfrak{a}^*\eta = i\frac{f^2 - \xi^2}{\xi}v, \quad u = -\eta - \frac{ifv}{\xi}.$$

Using multiplier $e^{-F(y)}$ we obtain:

$$\eta = (c' - A(y))\frac{e^{F(y)}}{\xi}; \quad u = \frac{A(y)e^{F(y)} - ife^{-F(y)}}{\xi}$$

where

$$A(y) = i\int_0^y (f^2(s) - \xi^2)e^{-2F(s)}ds$$

and for it to be L^2 we need $c' = 0$ and still need to show that $A(y)e^{F(y)}$ is L^2.

But we also have the constraint

$$(\mathfrak{a}^*\eta, v) = (\eta, \mathfrak{a}v) = 0 = \frac{-i}{\xi}(\xi^2(v, v) - (fv, fv))$$

which means $-\xi = E = \frac{\|fv\|}{\|v\|} := \frac{\|v'\|}{\|v\|}$.

Thus, there is exactly one point with $E = -\xi$, this completes the proof. □

Lemma 5.7. *The eigenmodes are analytic for $E \neq 0$.*

This is shown in [Kato, section 7.1]. And this allows us to talk of the derivative E' in ξ.

Lemma 5.8. *The derivative in ξ is $|E'(\xi)| \leq 1$.*

Proof. First, by analyticity the derivative is well defined. Now we take derivative on the original system

$$\frac{d}{dy\xi}(\hat{H}(\xi)\phi(\xi)) = \frac{d}{dy\xi}(E(\xi)\phi(\xi)) \quad \Rightarrow \quad \hat{H}'\phi + \hat{H}\phi' = E'\phi + E\phi'$$

where we take inner product with ϕ and by the fact that we've normalized v so that $\|v^2\| = 1$ our equation $(\phi', \phi) = 0$ by differentiation. So we get

$$(H'\phi, \phi) = E'(\phi, \phi)$$

and explicitly

$$H' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with eigenvalues $\pm 1, 0$, which means $E' \in [-1, 1]$ since we can decompose ϕ into the orthogonal basis of eigenfunctions. □

For convenience, denote $\mu = \dfrac{\xi}{E}$. Now we split the space $(E > 0)$ into three pieces: $\mu > 1$, $\mu \in (-1, 1)$, and $\mu < -1$, and label them component $1, 2, 3$ respectively. Note that we've shown already for $\mu = \pm 1$.

Corollary 5.9. *Except the one branch that crosses the point on $\xi = -E$, all other branches that has a point in component 2 goes to $-\infty$ at $-\infty$ and ∞ at ∞.*

This is directly by the derivative analysis, and the fact that no branches touches the Kelvin mode. Now we'll show that the Yanai wave corresponds to the one that crosses $\xi = -E$.

Lemma 5.10. *The branch contains the only possible point on $E = -\xi$ goes to ∞ at ∞ and stays in component 2, and on the left it stays in component 3.*

Proof. The only thing we need to check is that the derivative at the point of crossing is not -1. But this means ϕ is exactly the eigenfunction that corresponds to the eigenvalue -1 of H', i.e. $u = -v$ and $v = 0$. But we've discussed that this point is for $v = 0$, so we can rule it out. □

Theorem 5.11. *If $||f'||_\infty < \infty$, then $E \to 0$ as $\xi \to \infty$.*

Proof. Denote the left hand side of the Sturm-Liouville problem 2.2 as

$$L_\mu v = \left(D^2 + f^2 + \frac{\xi}{E}f'\right) v = \left(E^2 - \xi^2\right) v = (1 - \mu^2)E^2 v$$

and we note

$$\mathfrak{a}^*\mathfrak{a} = (-\partial_y + f)(\partial_y + f) = -\partial_y^2 + f^2 - f', \quad \mathfrak{a}\mathfrak{a}^* = -\partial_y^2 + f^2 + f'$$

so

$$L_\mu = \mathfrak{a}^*\mathfrak{a} + (1+\mu)f' = \mathfrak{a}\mathfrak{a}^* + (\mu-1)f'.$$

Taking inner product with v we get

$$\|\mathfrak{a}v\|^2 + (\mu+1)(f'v,v) = (E^2-\xi^2)\|v\|^2$$

and in particular in component 1 we have $(f'v,v)<0$ and

$$(\xi^2-E^2)||v||^2 \le -(\mu+1)(f'v,v) = (\mu+1)|(f'v,v)| \le \frac{\xi+E}{E}||f'||_\infty||v||^2$$

which yields

$$E(\xi-E) \le ||f'||_\infty$$

and that, as we take $\xi\to\infty$, since $||f'||_\infty$ is bounded, we know $E\to 0$.

On the other hand with the same reason in component 3

$$(\xi^2-E^2)\|v\|^2 = (-\xi-E)(-\xi+E)\|v\|^2 \le \frac{-E-\xi}{E}(f'v,v) \le \frac{-E-\xi}{E}\|f'\|_\infty\|v\|^2$$

hence

$$E(-\xi+E) \le \|f'\|_\infty$$

and we are done. □

Corollary 5.12. *If $f'>0$ then there is nothing at all in component 1.*

Proof. Using the fact above

$$\|\mathfrak{a}v\|^2 + (\mu+1)(f'v,v) = (E^2-\xi^2)\|v\|^2$$

and similarly

$$\|\mathfrak{a}^*v\|^2 + (\mu-1)(f'v,v) = (E^2-\xi^2)\|v\|^2$$

we get that for $\mu>-1$ L_μ is a non-negative operator with no solution $v\neq 0$ for $\mu>1$ since here $E^2-\xi^2<0$. □

5.3. **Further progress.** Now we consider the case of $||f'||$ unbounded at a point, while f is bounded.

Here, let's assume that f' is undefined around a point t, and is well behaved anywhere else. Let $g_n' := f' \cdot \mathbb{1}_{\left[t-\frac{1}{n},t+\frac{1}{n}\right]}$, and define

$$g_n(x) := C + \int_{t-\frac{1}{n}}^{x} g_n'(y)dy$$

so that $f - g_n$ is f flattened in middle and squished so that it is continuous, and shifted so that it is negative at $-\infty$ and positive at ∞. To get an always valid C, we just find the zero point of f, then make it still 0. WLOG we let $C = 0$ in the following.

Since f is continuous and bounded, we have $||g_n||_\infty \to 0$ so that

$$h_n := f - g_n \to f$$

uniformly.

What we do know is that h_n all has a well behaved spectral flow. Define accordingly $\mathfrak{a}_n := \partial_y + f - g_n = \partial_y + h_n$ and E_n, ϕ_n is the corresponding eigenvector to ξ (this is not well-defined. Just vaguely thinking of this as the corresponding one to E, if indeed there is only one such...). Then we will have the following:

$$(\mathfrak{a}\mathfrak{a}^* + (\mu+1)f')v = (E^2 - \xi^2)v$$

$$\left(\mathfrak{a}\mathfrak{a}^* + g_n^2 - 2g_n f - g_n' + (\mu+1)(f' - g_n')\right) v_n = (E_n^2 - \xi^2)v_n$$

For convenience, we use g to denote g_n in below deduction.

And we apply dot product with v_n, v respectively to the two equations to get (here we assume $v_i, v \in \mathbb{R}$, so $\langle v, v_n\rangle = \langle v_n, v\rangle$):

$$\left\langle \left(-\partial_y^2 + f^2 + \frac{\xi}{E}f'\right) v, v_n \right\rangle = (E^2 - \xi^2)\langle v, v_n\rangle$$

$$\left\langle \left(-\partial_y^2 + (f-g)^2 + \frac{\xi}{E}(f' - g')\right) v_n, v \right\rangle = (E_n^2 - \xi^2)\langle v, v_n\rangle$$

and take their difference we get

$$(E_n^2 - E^2)\langle v, v_n\rangle = \langle g(g-2f)v, v_n\rangle + \xi \left\langle \left(\frac{f'-g'}{E_n} - \frac{f'}{E}\right) v, v_n \right\rangle$$

but $g \to 0$ uniformly so as $n \to \infty$ the term $\langle g(g-2f)v, v_n\rangle \to 0$.

Now, consider $f' - g' = \mathbb{1}_{y \notin [t-1/n, t+1/n]} f'$ so

$$\left\langle \frac{f'-g'}{E_n} v, v_n \right\rangle = \frac{1}{E_n}\int_{y\notin[t-1/n,t+1/n]} f' v v_n dy = \frac{1}{E_n}\langle f' v, v_n\rangle - \frac{1}{E_n}\int_{t-\frac{1}{n}}^{t+\frac{1}{n}} f' v v_n dy$$

and hence

$$\xi\left\langle\left(\frac{f'-g'}{E_n}-\frac{f'}{E}\right)v,v_n\right\rangle=\frac{\xi(E-E_n)}{E_nE}\langle f'v,v_n\rangle-\frac{\xi}{E_n}\int_{t-\frac{1}{n}}^{t+\frac{1}{n}}f'vv_ndy$$

and

$$(E_n^2-E^2)\langle v,v_n\rangle=\langle g(g-2f)v,v_n\rangle+\frac{\xi(E-E_n)}{E_nE}\langle f'v,v_n\rangle-\frac{\xi}{E_n}\int_{t-\frac{1}{n}}^{t+\frac{1}{n}}f'vv_ndy$$

$$\Rightarrow EE_n(E_n^2-E^2)\langle v,v_n\rangle=EE_n\langle g(g-2f)v,v_n\rangle+\xi(E-E_n)\langle f'v,v_n\rangle-E\xi\int_{t-\frac{1}{n}}^{t+\frac{1}{n}}f'vv_ndy$$

and we know from previous analysis that $E_n=O(\xi^{-1})$. And got stuck here since we get the same bound.

Re thinking it we note that the really important step is

$$(\xi^2-E^2)||v||^2\leq\frac{-E-\xi}{E}(f'v,v)$$

where we have

$$E(-\xi+E)\leq\frac{(f'v,v)}{(v,v)}$$

this gives us for all $||f'||_\infty<\infty$ the desired result. And even for continuous f whose derivative is not bounded at a point x we can still apply this since we have

$$(f'v,v)=\int_{B_\varepsilon(x)}f'|v|^2dy+\int_{R\setminus B_\varepsilon(x)}f'|v|^2dy$$

which is bounded if $f'\in L^{1+\varepsilon}_{loc}$ and $v\in L^{\frac{2\varepsilon}{1+\varepsilon}}_{loc}$.

6. WITH JUMPS

6.1. **Piecewise constant functions.** We see that the case for $f = y$, since $|f'(y)| \equiv 1 < C$ it should satisfy that as $\xi \to \infty$, E tends to either ∞ or 0. This is indeed the case, as shown in the plot.

We have relied on the fact that f' exists and is bounded above. since we also know that f has to be bounded at infinity, and

$$0 < \lim_{y\to\infty} f(y) < C; \quad -C < \lim_{y\to-\infty} f(y) < 0$$

the only savior could only be non-differentiable points. The case for one jump is well discussed in the paper, with results mensioned above:

Proposition 6.1. *For* $f(y) = \begin{cases} f_+ & y \geq 0 \\ f_- & y < 0 \end{cases}$

the only admissible solution with $0 < \lim_{\xi\to\infty} E < \infty$ *has expression*

$$E = \frac{-\xi f_0}{\sqrt{f_e^2 + \xi^2}}$$

where $f_0 = \frac{1}{2}\left(f_+ - f_-\right), f_e = \frac{1}{2}\left(f_+ + f_-\right)$.

Corollary 6.2. *For the expression above,* $\lim_{\xi\to\infty} E = -f_0$.

The domain wall is $\frac{1}{2}\left(f_+ - f_-\right)$, in other words, one half the size of the jump. The one half is because we still need to consider the fact that $E \notin (-f_0, f_0)$ automatically times 2 to the length.

Now we try to find similar results for functions with 2 jumps.

Proposition 6.3. *For* v *differentiable almost everywhere and* $f(y) = \begin{cases} f_1 & y \geq t_1 > 0 \\ f_2 & y \in [t_2, t_1] \\ f_3 & y < t_2 < 0 \end{cases}$ *where* $f_3 = -f_1 < 0$ *and* $f_2 = 0$, $t_1 = -t_2 = 1$, *then the only non-trivial* E *is* $\lim_{\xi\to\infty} E = \frac{1}{2}\left(f_1 - f_2\right)$.

Proof.

Finding the system to solve:

Denote $d := \frac{1}{2}\left(f_1 - f_2\right) = \frac{1}{2}\left(f_2 - f_3\right)$.

First we dealt with the jump conditions. Integrating and taking limit to equation 2.2 we have (since v is continuous)

$$\lim_{\varepsilon\to 0}\int_{t_1-\varepsilon}^{t_1+\varepsilon}\left(D^2+f^2+\frac{\xi}{E}f'\right)v\,dy=\lim_{\varepsilon\to 0}\int_{t_1-\varepsilon}^{t_1+\varepsilon}\left(E^2-\xi^2\right)v\,dy$$

$$\Rightarrow \lim_{\varepsilon\to 0}\int_{t_1-\varepsilon}^{t_1+\varepsilon}\left(D^2+\frac{\xi}{E}f'\right)v\,dy=0$$

$$\Rightarrow \lim_{\varepsilon\to 0}-v'(y)\Big|_{t_1-\varepsilon}^{t_1+\varepsilon}+\frac{\xi}{E}(f_1-f_2)v(t_1)=0$$

$$\Rightarrow v'(1^-)-v'(1^+)+\frac{\xi d}{E}v(1)=0$$

and similarly for the jump at $t_2=-1$ we have

$$v'(-1^-)-v'(-1^+)+\frac{\xi d}{E}v(-1)=0$$

So the system we have to solve is the following:

$$\begin{cases}\left(E^2-\xi^2\right)v=\left(D^2+f^2\right)v & y\notin\{t_1,t_2\}\\ v'(1^-)-v'(1^+)+\frac{\xi d}{E}v(1)=0 & \\ v'(-1^-)-v'(-1^+)+\frac{\xi d}{E}v(-1)=0 & \end{cases}$$

This is a second order ODE we know solution to, and since v is bounded, we know the sign on the exponential of v. To be explicit we know the solution is of the following form:

$$v(y)=\begin{cases}c_1e^{-\mu_1 y} & y>1\\ ae^{\mu_2 y}+be^{-\mu_2 y} & y\in[-1,1]\\ c_2e^{\mu_3 y} & y<-1\end{cases}$$

where $\mu_1,\mu_2,\mu_3>0$ and a,b,c_1,c_2 are constants with degree one freedom.

Using this v, we can further plug in to modify our system as:

$$\begin{cases}E^2-\xi^2=f_1^2-\mu_1^2=f_2^2-\mu_2^2=f_3^2-\mu_3^2\\ v'(1^-)-v'(1^+)+\frac{\xi d}{E}v(1)=0\\ v'(-1^-)-v'(-1^+)+\frac{\xi d}{E}v(-1)=0\end{cases}$$

<u>Getting implicit function of E and ξ:</u>

Now we start to solve this system.

First, note that $\mu_1^2-\mu_3^2=f_1^2-f_3^2=0$ by our assumption, so $\mu_1=\mu_3$ since both are greater than 0 by how they are defined.

Then, since v is continuous at ± 1 we have

$$\begin{cases} ae^{\mu_2} + be^{-\mu_2} = c_1 e^{-\mu_1} \\ ae^{-\mu_2} + be^{\mu_2} = c_2 e^{-\mu_3} = c_2 e^{-\mu_1} \end{cases} \Rightarrow \begin{pmatrix} e^{\mu_2} & e^{-\mu_2} \\ e^{-\mu_2} & e^{\mu_2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c_1 e^{-\mu_1} \\ c_2 e^{-\mu_1} \end{pmatrix}$$

which gives

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} e^{\mu_2} & e^{-\mu_2} \\ e^{-\mu_2} & e^{\mu_2} \end{pmatrix}^{-1} \begin{pmatrix} c_1 e^{-\mu_1} \\ c_2 e^{-\mu_1} \end{pmatrix} = \frac{e^{-\mu_1}}{e^{2\mu_2} - e^{-2\mu_2}} \begin{pmatrix} c_1 e^{\mu_2} - c_2 e^{-\mu_2} \\ -c_1 e^{-\mu_2} + c_2 e^{\mu_2} \end{pmatrix}.$$

For convenience we denote $\beta = e^{\mu_2}$, $\alpha = \beta^2 = e^{\mu_2}$.

Now plugging into first boundary condition we have

$$\begin{aligned} & v'(1^-) - v'(1^+) + \frac{\xi d}{E} v(1) = 0 \\ \Rightarrow & \mu_2 (ae^{\mu_2} - be^{-\mu_2}) + c_1 \mu_1 e^{-\mu_1} + \frac{\xi d}{E} c_1 e^{-\mu_1} = 0 \\ \Rightarrow & c_1 \left[\mu_2 \left(\alpha + \frac{1}{\alpha} \right) + \left(\alpha - \frac{1}{\alpha} \right) \left(\mu_1 + \frac{\xi d}{E} \right) \right] = c_2 \cdot 2\mu_2 \end{aligned}$$

and for the second boundary condition we get

$$\begin{aligned} & v'(-1^-) - v'(-1^+) + \frac{\xi d}{E} v(-1) = 0 \\ \Rightarrow & c_2 \mu_3 e^{-\mu_3} - \mu_2 (ae^{-\mu_2} - be^{\mu_2}) + \frac{\xi d}{E} c_2 e^{-\mu_3} = 0 \\ \Rightarrow & c_2 \left[\mu_2 \left(\alpha + \frac{1}{\alpha} \right) + \left(\alpha - \frac{1}{\alpha} \right) \left(\mu_1 + \frac{\xi d}{E} \right) \right] = c_1 \cdot 2\mu_2 \end{aligned}$$

and dividing the two equations yields

$$\frac{c_2}{c_1} = \frac{c_1}{c_2}$$

which means $c_2 = \pm c_1$.

But we can rule out $c_2 = c_1$ since in that case

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{e^{-\mu_1}}{e^{2\mu_2} - e^{-2\mu_2}} \begin{pmatrix} c_1 e^{\mu_2} - c_2 e^{-\mu_2} \\ -c_1 e^{-\mu_2} + c_2 e^{\mu_2} \end{pmatrix} = \frac{c_1 e^{-\mu_1}}{\beta + \beta^{-1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

which means $a = -b$.

Now plugging into the boundary condition we see that

$$v'(1^-) = \mu_2 \left(ae^{\mu_2} - be^{-\mu_2} \right) = \mu_2 a(\beta + \beta^{-1}) = \mu_2 c_1 e^{-\mu_1}$$

and the boundary condition becomes

$$\mu_1 + \mu_2 + \frac{\xi d}{E} = 0$$

whereas the second boundary condition gives (the change of sign is because taking derivative on v has different sign for $y > 1$ and $y < 1$)

$$\mu_1 - \mu_2 + \frac{\xi d}{E} = 0$$

together they mean $\mu_2 = 0$, which is impossible since then $v = a + b = 0$ in middle but $v = c_1 e^{-\mu_1 y}$ is never 0 except when $c_1 = c_2 = 0$, which means $v = 0$ and hence not an eigenvector.

Thus, we know $c_1 = -c_2$.

Now we do the same as above to get a neater expression of a and b:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{e^{-\mu_1}}{e^{2\mu_2} - e^{-2\mu_2}} \begin{pmatrix} c_1 e^{\mu_2} - c_2 e^{-\mu_2} \\ -c_1 e^{-\mu_2} + c_2 e^{\mu_2} \end{pmatrix} = \frac{c_1 e^{-\mu_1}}{\beta - \beta^{-1}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

note that even though $a = -b$ the denominator does not cancel with the term in boundary condition, and we get

$$\begin{aligned} &v'(1^-) - v'(1^+) + \frac{\xi d}{E} v(1) = 0 \\ \Rightarrow &\mu_2 \left(\beta + \frac{1}{\beta} \right) + \left(\mu_1 + \frac{\xi d}{E} \right) \left(\beta - \frac{1}{\beta} \right) = 0 \end{aligned}$$

and here since we've canceled c_2 already the second boundary condition should yield the same equation.

A bit of arrangement gives

$$\frac{\xi d}{E} = \frac{(\mu_1 + \mu_2)\beta + (\mu_1 - \mu_2)\beta^{-1}}{\beta^{-1} - \beta}.$$

Taking asymptotics:

Since $f_2 = 0$ the ODE tells us $\beta = e^{\mu_2} = e^{\sqrt{\xi^2 - E^2}} = O_{\xi \to \infty}\left(e^{\xi}\right)$ and hence directly taking the limit we have

$$\begin{aligned} \lim_{\xi \to \infty} \frac{\xi d}{E} &= \lim_{\xi \to \infty} \frac{e^{\sqrt{\xi^2 - E^2}} \left(\sqrt{\xi^2 - E^2 + d^2} + \sqrt{\xi^2 - E^2} \right) + O(\xi^{-1} e^{-\xi})}{e^{-\sqrt{\xi^2 - E^2}} - e^{\sqrt{\xi^2 - E^2}}} \\ &= \lim_{\xi \to \infty} \frac{e^{\sqrt{\xi^2 - E^2}}(2\xi)}{-e^{\sqrt{\xi^2 - E^2}}} = -2\xi \end{aligned}$$

which yields $\lim_{\xi \to \infty} E = -\frac{d}{2}$. □

Corollary 6.4. *For the same setting as above, except that* $f_2 \neq 0$ *and* $f_2 \in (f_3, f_1)$*, we expect* $E =\in \left\{ \frac{1}{2}|f_1 - f_2|, \frac{1}{2}|f_2 - f_3| \right\}$.

Proof. Denote $d_1 = f_1 - f_2, d_2 = f_2 - f_3$.

Most is the same as in the above computation till the steps:

$$c_1\left[\mu_2\left(\alpha+\frac{1}{\alpha}\right)+\left(\alpha-\frac{1}{\alpha}\right)\left(\mu_1+\frac{\xi d_1}{E}\right)\right]=c_2\cdot 2\mu_2$$

$$c_2\left[\mu_2\left(\alpha+\frac{1}{\alpha}\right)+\left(\alpha-\frac{1}{\alpha}\right)\left(\mu_1+\frac{\xi d_2}{E}\right)\right]=c_1\cdot 2\mu_2$$

where we see we can't cancel everything.

So we denote $S_1 := \mu_1 + \frac{\xi d_1}{E}$, $S_2 := \mu_1 + \frac{\xi d_2}{E}$ and cancel c_1 by plugging in to get

$$\left[\mu_2\left(\alpha+\frac{1}{\alpha}\right)+\left(\alpha-\frac{1}{\alpha}\right)\left(\mu_1+\frac{\xi d_1}{E}\right)\right]\cdot\left[\mu_2\left(\alpha+\frac{1}{\alpha}\right)+\left(\alpha-\frac{1}{\alpha}\right)\left(\mu_1+\frac{\xi d_2}{E}\right)\right]=4\mu_2^2$$

$$\Rightarrow\mu_2^2\left(\alpha-\frac{1}{\alpha}\right)^2+\left(\alpha-\frac{1}{\alpha}\right)^2S_1S_2+\mu_2\left(\alpha^2-\frac{1}{\alpha^2}\right)(S_1+S_2)=0$$

$$\Rightarrow\mu_2^2\left(\alpha-\frac{1}{\alpha}\right)+\left(\alpha-\frac{1}{\alpha}\right)S_1S_2+\mu_2\left(\alpha+\frac{1}{\alpha}\right)(S_1+S_2)=0$$

$$\Rightarrow\alpha\left(\mu_2+S_1\right)\left(\mu_2+S_2\right)-\frac{1}{\alpha}\left(\mu_2-S_1\right)\left(\mu_2-S_2\right)=0$$

but now we note $\lim_{\xi\to\infty}\alpha = e^{\xi}$ and all the rest are at most $O(\xi)$, so by taking the limit on ξ we can throw away half of the equation and get

$$\left(\mu_2+S_1\right)\left(\mu_2+S_2\right)=0$$

which yields either $E = -\frac{1}{2}d_1$ or $E = -\frac{1}{2}d_2$.

Now WLOG assume $f_2 > 0$ so that $d_2 > d_1$. We take $E = \frac{1}{2}d_1$. □

Now we generate the above result to the general case for piecewise constant functions.

Lemma 6.5. *(Shift invariance) If $H(y,\xi)v(y,\xi) = E(\xi)v(y,\xi)$ is a solution, then so is $H(y+k,\xi)v(y+k,\xi) = E(\xi)v(y+k,\xi)$.*

The reason is just that $\partial_y v(y+k) = v'(y+k)$ and everything else is shifted.

Theorem 6.6. *For $-\infty = y_0 < y_1 < \cdots < y_n < y_{n+1} = \infty$, let $f(y) = f_i$ for $y_i \le y \le y_{i+1}$ where $f_0 < 0$ and $f_n > 0$, and define $d_i := f_i - f_{i-1}$. Then, as $\xi \to \infty$, all E such that does not go to infinity or exactly equal to 0 goes to $-\frac{1}{2}d_i$, for $d_i > 0$, and as $\xi \to -\infty$, $E \to -\frac{1}{2}d_j$ for $d_j < 0$.*

Proof. We do this by asymptotic behavior and computation.

From the same deduction, we know that the system we need to solve has jump conditions:

$$v'(y_i^-) - v'(y_i^+) + \frac{\xi d_i}{v} v(y_i) = 0$$

We also know that we can find eigenvalue v such that

$$v(y) := a_i e^{\mu_i y} + b_i e^{-\mu_i y}, \qquad y_i \le y \le y_{i+1}$$

where for integrable $b_0 = 0$ and $a_n = 0$.

With this particular eigenvector v we can write down the full system we need to solve, including the direct result of system, the jump conditions, and the continuity condition:

$$\begin{cases} E^2 - \xi^2 = f_i^2 - \mu_i^2 \\ a_{i-1} e^{\mu_{i-1} y_i} + b_{i-1} e^{-\mu_{i-1} y_i} = a_i e^{\mu_i y_i} + b_i e^{-\mu_i y_i} \\ \mu_{i-1} a_{i-1} e^{\mu_{i-1} y_i} - \mu_{i-1} b_{i-1} e^{-\mu_{i-1} y_i} - \mu_i a_i e^{\mu_i y_i} + \mu_i b_i e^{-\mu_i y_i} + \frac{\xi d_i}{v} a_i e^{\mu_i y_i} + b_i e^{-\mu_i y_i} = 0 \end{cases}$$

For ease we denote

$$A_{i-1}^i := a_{i-1} e^{\mu_{i-1} y_i}, \quad A_i^i := a_i e^{\mu_i y_i}; \quad B_{i-1}^i := b_{i-1} e^{-\mu_{i-1} y_i}, \quad B_i^{:} = b_i e^{-\mu_i y_i}$$

and thus using the continuity condition we can write

$$v(y_i) = \frac{\mu_i}{\mu_i + \mu_{i-1}} (A_{i-1}^i + B_{i-1}^i) + \frac{\mu_{i-1}}{\mu_i + \mu_{i-1}} (A_i^i + B_i^i)$$

and plugging into the jump condition we get

$$A_{i-1}^i \left(\mu_{i-1} + \frac{\mu_{i-1}}{\mu_i + \mu_{i-1}} \frac{\xi}{E} d_i \right) + B_{i-1}^i \left(-\mu_{i-1} + \frac{\mu_{i-1}}{\mu_i + \mu_{i-1}} \frac{\xi}{E} d_i \right)$$
$$+ A_i^i \left(-\mu_i + \frac{\mu_i}{\mu_i + \mu_{i-1}} \frac{\xi}{E} d_i \right) + B_i^i \left(\mu_i + \frac{\mu_i}{\mu_i + \mu_{i-1}} \frac{\xi}{E} d_i \right) = 0$$

and by asymptotics we obtain that if E does not go to ∞ as is stated in thoerem, then $\lim_{\xi \to \infty} \mu_i = \xi$ and hence the whole thing is simplified to

$$A_{i-1}^i \left(2 + \frac{d_i}{E} \right) + B_{i-1}^i \left(-2 + \frac{d_i}{E} \right) + A_i^i \left(-2 + \frac{d_i}{E} \right) + B_i^i \left(2 + \frac{d_i}{E} \right) = 0$$

which then after resembling gives (now, we denote $\alpha_i := e^{2\xi y_i}$)

$$\frac{a_i \alpha_i + b_{i-1}}{a_{i-1} \alpha_i + b_i} = \frac{A_i^1 + B_{i-1}^i}{A_{i-1}^1 + B_i^i} = \frac{2E + d_i}{2E - d_i} \tag{6.1}$$

and now let's further denote

$$D_i := \frac{2E + d_i}{2E - d_i}, \quad M_i := \prod_{i=1}^{i} D_i$$

and start from beginning: Since $b_0 = 0$ we get from 6.1 that

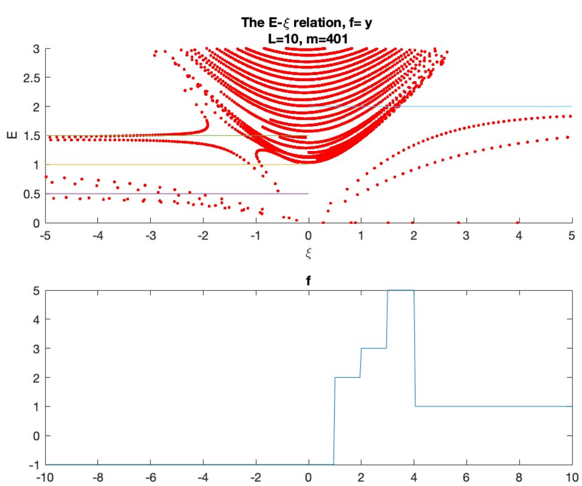
$$
\begin{aligned}
a_1 &= \frac{1}{\alpha} D_1(a_0\alpha_1 + b_1) = M_1 a_0 + D_1 \frac{b_1}{\alpha_1} \\
a_2 &= \frac{1}{\alpha_2}(-b_1 + D_2(a_1\alpha_2 + b_2)) = M_2 a_0 + D_2 D_1 \frac{b_1}{\alpha_1} + D_2 \frac{b_2}{\alpha_2} - \frac{b_1}{\alpha_2} \\
a_3 &= M_3 a_0 + M_3 \frac{b_1}{\alpha_1} + \frac{M_3}{D_1}\frac{b_2}{\alpha_2} + \frac{M_3}{D_1 D_2}\frac{b_3}{\alpha_3} - \frac{M_3}{D_1 D_2}\frac{b_1}{\alpha_2} - \frac{b_2}{\alpha_3} \\
&\vdots \\
a_n &= M_n a_0 + \left(\sum_{j=1}^{n} \frac{M_n}{\prod_{i=1}^{j-1} D_j} \frac{b_i}{\alpha_j} \right) - \left(\sum_{j=1}^{n} \frac{M_n}{\prod_{i=1}^{j+1} D_j} \frac{b_i}{\alpha_j + 1} \right) = 0
\end{aligned}
$$

But note that $\alpha_i = e^{\xi y_i} \to \infty$ if $y_i < 0$. So we WLOG assume $y_n < 0$, then it gives us $M_n a_0 = 0$, and $a_0 \neq 0$ otherwise $v \equiv 0$, which means $M_n = 0$ which means $E = -\frac{1}{2} d_i$ for some i. If $d_j < 0$, then by analysis above we know that E cannot reach 0 hence this cannot be an asymptotic, hence here E can converge to only those

For the case where $y_n \geq 0$ we just apply shift invariance to shift to left, then get the same result for E.

For $d_j < 0$, run the same computation for $\xi \to -\infty$, then use shift invariance to right and get the result. □

This is very well seen from the graph. For a curve with 4 jumps, we see that there's exactly 4 asymptotic corresponding to their signs:



Proposition 6.7. *Let f be a function that has a jump of value d at t and $||f'(y)||_\infty < \infty$ for $y \neq t$. Then for $\delta > 0$, $E = -\frac{d}{2}$, and $z \in [t, t+\delta]$ the vector valued function*

$$\phi = \begin{pmatrix} -\frac{i(f(z)+E)}{\xi}e^{-\xi z} + i\xi e^{-\xi z} \\ -ie^{-\xi z} \\ e^{-\xi z} \end{pmatrix}$$

is such that

$$H\phi - E\phi = O\left(\frac{e^{-\xi z}}{\xi}\right) \to 0$$

as $\xi \to \infty$.

What this tells us is that we have a localized approximate eigenvalue-eigenfunction pair for the operator. Note that the method we obtain this eigenfunction is nothing but taking the asymptotic behavior of the case when f has the same jump but is flat elsewhere.

Proof. First, WLOG assume $t = 0$, and denote $\phi = (\eta, u, v)^T$.

The fact that the following goes through only when $E = -\frac{d}{2}$ is due to the jump condition, which is shown in section 2.

We compute

$$D_y\eta = -\left[\partial_y\left(f(y)\frac{e^{-\xi y}}{\xi}\right) + E\partial_y\frac{e^{-\xi y}}{\xi}\right]\Bigg|_{y=z} = e^{-\xi z}\left(E + f(z) - \frac{f'(z)}{\xi}\right)$$

and thus

$$\begin{aligned} H\phi &= \begin{pmatrix} 0 & \xi & D_y \\ \xi & 0 & if(y) \\ D_y & -if(y) & 0 \end{pmatrix}\begin{pmatrix} -\frac{i(f(z)+E)}{\xi}e^{-\xi z} + i\xi e^{-\xi z} \\ -ie^{-\xi z} \\ e^{-\xi z} \end{pmatrix} \\ &= \begin{pmatrix} -i\xi e^{-\xi z} + i\xi e^{-\xi z} \\ -i(f(z)+E)e^{-\xi z} + if(z)e^{-\xi z} \\ D_y\eta - f(z)e^{-\xi z} \end{pmatrix} = \begin{pmatrix} 0 \\ -iEe^{-\xi z} \\ \left(E - \frac{f'(z)}{\xi}\right)e^{-\xi z} \end{pmatrix} \end{aligned}$$

and hence

$$H\phi - E\phi = \begin{pmatrix} \frac{iE(f(z)+E)}{\xi}e^{-\xi z} \\ 0 \\ -\frac{f'(z)}{\xi}e^{-\xi z} \end{pmatrix} = O\left(\frac{e^{-\xi z}}{\xi}\right) \to 0$$

which completes the proof. $\square$

7. Futher Works

7.1. To find in Kato's book.

We need two results in perturbation theory to conclude that the spectral flow can be anything for general functions, and we refer to Kato's textbook [4] in hope to find the following two results:

Theorem 7.1. *For operator $H(\xi)$, function $\phi(\xi)$ and $E(\xi)$, if as $\xi \to \infty$*

$$H(\xi)\phi(\xi) \to E\phi(\xi)$$

then there exists eigenvalue E' and eigenfunction $\phi'(\xi)$ with $|E'-E| < \delta$, $||\phi(\xi)-\phi'(\xi)|| < \delta$ for any δ, with $\xi(\delta)$ big enough.

Theorem 7.2. *For given ε, δ if $||H' - H|| < \varepsilon$ and $H'(\xi)\phi'(\xi) = E'(\xi)\phi'(\xi)$, then there exists exists eigenvalue E and eigenfunction $\phi(\xi)$ with $|E'-E| < \delta$, $||\phi(\xi)-\phi'(\xi)|| < \delta$ such that $H\phi = E\phi$. Also there is a one-one correspondence between the set of eigenfunctions of H' and the set of eigenfunctions of H.*

With the above two theorems, we can conclude that the bulk edge correspondence fails.

Theorem 7.3. *If f contains jumps, then the spectral flow is not always* 2*.*

Proof. Theorem 7.2 says that we can approximate the asymptotic behavior of the eigenmodes $E(\xi)$ within δ large gap, hence the eigenmodes tends to at most # jump values with size $-\frac{d}{2}$; Theorem 7.3 says that all these values actually has some eigenmodes that tends to it. Combined, it means the spectral flow is the same value as the corresponding step function, which is not 2 at all energy levels. □

This would end the discussion by concluding that the bulk-edge correspondence fails in this case.

7.2. Naive Try Outs.

Given this, I tried to use simple function approximation to do for general cases with jump. The setting is that if f satisfies that it is a constant $f_+ > 0$ for $y > y_+$ and is $f_- < 0$ for $y < y_-$, and is well behaved (say $||f'||_\infty < C$ except on a countable collection of points within $[y_-, y_+]$; Or measurable), then we know there exists a simple function approximation f_n of the function f that we can pick $||f - f_n||_\infty \to 0$. In particular we note that

Of the setting I have two approaches:

Approach 1: This requires more effort to show that v changes in H^2 norm continuously on f.

Lemma 7.4. *If v is a H^2 continuous function of f (given the equation), then we know that for fixed ξ, the change of ξ with f is small, in particular we have that, at ∞, the only asymptotics are characterized solely by the jump values in the same fashion as before mentioned.*

Proof. We have

$$\left\langle \left(D^2 + f^2 + \frac{\xi}{E} f' \right) v, v \right\rangle = (E^2 - \xi^2)||v||^2$$

$$\left\langle \left(D^2 + f_n^2 + \frac{\xi}{E_n} f_n' \right) v_n, v_n \right\rangle = (E_n^2 - \xi^2)||v_n||^2$$

Since $||v_n||_{H^2} \to ||v||_{H^2}$ and $||f_n - f||_\infty \to 0$, we know $||v_n|| \to ||v||$. We denote all the value of jumps caused by simple function approximation and is decaying to 0 by g_i, and those appear at points t_i and we denote the jumps originally in f by y_i. Then, subtracting what we have by one another, for large n we have

$$\sum_j \frac{\xi}{E_n} g_j v_n^2(t_j) + \sum_i \frac{\xi}{E_n} d_i v_n^2(y_i) - \sum_i \frac{\xi}{E} d_i v^2(y_i) = (E_n^2 - E^2)||v||^2 + O(\varepsilon)$$

which we note that the corresponding $g_i \to 0$ as well because we are approximating, also $v_n(y) \to v(y)$ hence reassemble the above yields

$$\xi d_i \sum_i (E v_n(y_i)^2 - E_n v(y_i)^2) = E E_n (E_n^2 - E^2)||v||^2$$

which for ξ arbitrarily large we know for all $n > N$ such that $v_n(y_i) \to v(y_i)$, hence

$$\sum_i (E v_n(y_i)^2 - E_n v(y_i)^2) \to 0$$

implies

$$\sum v(y_i) = 0 \quad \text{or} \quad E_n \to E$$

But if it is the first case, then the left hand side is exactly 0, and since we know right hand side cannot be 0 ($E_n \neq 0$ by above theorem) so we know $E_n \to E$. □

Approach 2: The perturbation to eigenvalues of a matrix is itself an object, and possibly could be done without analysing the eigenvector. The perturbation in our case is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i(f - f_n) \\ 0 & -i(f - f_n) & 0 \end{pmatrix}$$

which is very small. Maybe we can show with a method that generates the linear algebra problem $(A + \delta V)x = (\lambda + \delta\mu)x$?

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