APPROXIMATION THEORY HOMEWORK 6

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Discussed with classmates.

Exercise 1.

Proof.

(a): We use induction. For $j = 1$, we have (boundary condition vanish since periodic)

$$||f'||_2^2 = \int_{-\pi}^{\pi} f'f'dx \overset{ibp}{=} \int_{-\pi}^{\pi}(f'')(-f)dx + 0 \overset{Holder}{\leq} ||f''||_2 \cdot ||-f||_2 = ||f''||_2 \cdot ||f||_2$$

and take square root on both sides we have

$$||f'||_2 \leq ||f''||_2^{\frac{1}{2}} \cdot ||f||_2^{\frac{1}{2}}$$

which finishes the case for $j = 1$.

Now assume that the condition hold for $j \leq n-1$, now for $j = n$, we have

$$\begin{aligned}||f^{(n)}||^2 = \int_{-\pi}^{\pi} f^{(n)}f^{(n)}dx = \int_{-\pi}^{\pi} f^{(n+1)}(-f^{(n-1)})dx &\overset{Holder}{\leq} ||f^{(n+1)}||_2 \cdot ||f^{(n-1)}||_2 \\ &\overset{IH}{\leq} ||f^{(n+1)}|| \cdot ||f^{(n+1)}|| \cdot ||f||^{\frac{1}{n}}\end{aligned}$$

which after rearranging we get

$$\Rightarrow ||f^{(n)}||^{\frac{n+1}{n}} \leq ||f^{(n+1)}|| \cdot ||f||^{\frac{1}{n}} \Rightarrow ||f^{(n)}|| \leq ||f^{(n+1)}||^{\frac{n}{n+1}} \cdot ||f||^{\frac{1}{n+1}}$$

and thus we are done.

(b):

Just plug in result of (a) repeatedly we have

$$\begin{aligned}||f^{(j)}|| &\leq ||f^{(j+1)}||^{\frac{j}{j+1}} \cdot ||f||^{\frac{1}{j+1}} \leq ||f^{(j+2)}||^{\frac{j}{j+1}\cdot\frac{j+1}{j+2}} \cdot ||f||^{\frac{1}{j+1}+\frac{j}{j+1}\cdot\frac{1}{j+2}} \\ &\leq \cdots \leq ||f^{(k)}||^{M_k}||f||^{S_k}\end{aligned}$$

where the product

$$M_k := \frac{j}{j+1} \cdot \frac{j+1}{j+2} \cdots \frac{k-1}{k} = \frac{j}{k}$$

and the sum

$$\begin{aligned}
S_n &= \frac{1}{j+1} + \frac{1}{j+2} \cdot \frac{j}{j+1} + \frac{1}{j+3} \cdot \frac{j}{j+1}\frac{j+1}{j+2} + \cdots + \frac{1}{k-1}\frac{j}{j+1} \cdots \frac{k-1}{k} \\
&= j\left(\frac{1}{j(j+1)} + \frac{1}{(j+1)(j+2)} + \cdots + \frac{1}{(k-1)k}\right) \\
&= j\left(\frac{1}{j} - \frac{1}{j+1} + \frac{1}{j+1} - \frac{1}{j+2} + \cdots + \frac{1}{k-1} - \frac{1}{k}\right) \\
&= j\left(\frac{1}{j} - \frac{1}{k}\right) = \frac{k-j}{k}
\end{aligned}$$

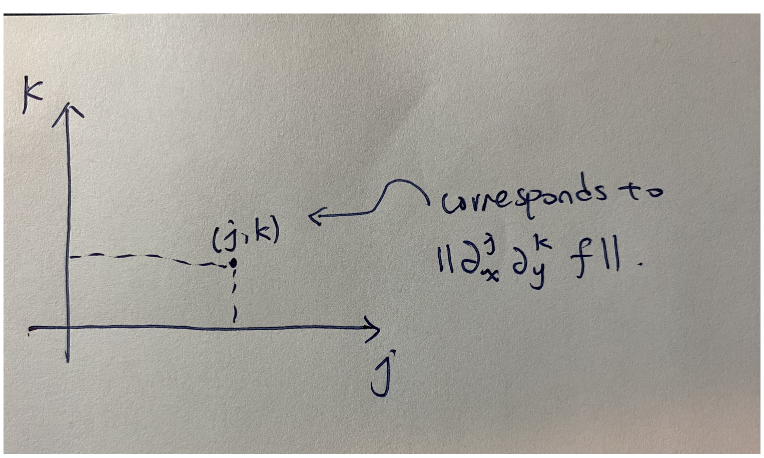
and plugging in the computation results we get

$$||f^{(j)}|| \leq ||f^{(k)}||^{\frac{j}{k}} \cdot ||f||^{\frac{k-j}{k}}$$

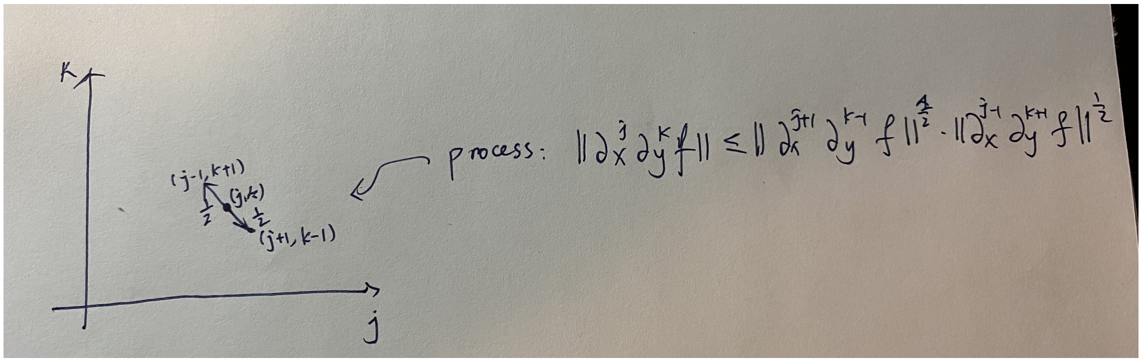
which is what we want.

(c):

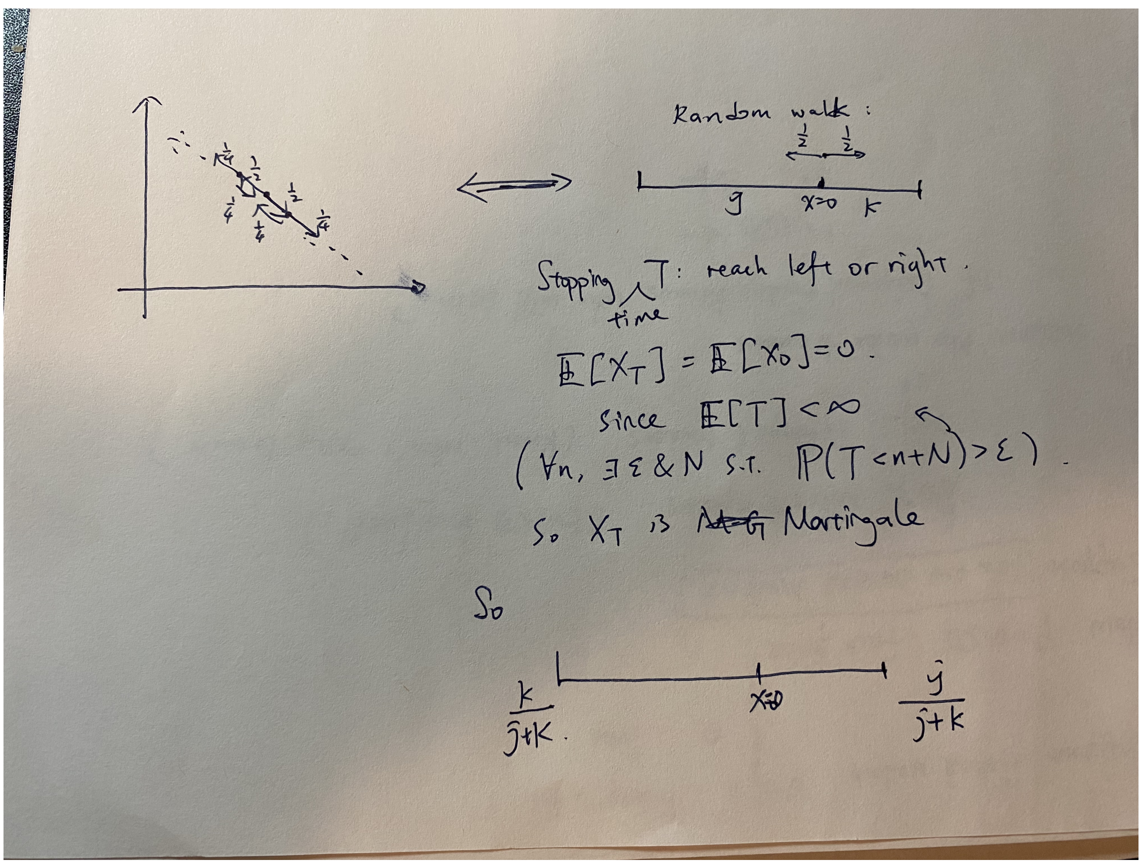
The key point is that we have to disentangle the cross terms, and the way we do it is to do it in the way as "one up one down": we have by Holder and integral by parts that $||\partial_x^j \partial_y^k f|| \leq ||\partial_x^{j+1}\partial_y^{k-1} f||^{\frac{1}{2}} ||\partial_x^{j-1}\partial_y^{k+1} f||^{\frac{1}{2}}$ and note also that we can do the same expansion for each individual term on the right. This leads to a huge number of computation and indexing if one do this directly. However, note that first we create the space of j and k, in which we express the order of derivatives as coordinates in space, like below:



and the inequality property and iteration can be vied as a process:



and since in part (b) we know already how to deal with both axis, so we only need to compute, after doing repeatedly this process, the degree of terms that are on the axis. But this is not so easy since the middle parts need to be eliminated by limits as degrees goes to 0, yet that is why we have random walk/Martingale theory. Note that we can view this as a simple random walk of a man starting at value 0 and ending either at k or $-j$, as illustrated:



now we know X_T is a Martingale by condition in picture. Thus

$$0 = \mathbb{E}[X_0] = \mathbb{E}[X_T] = -j\mathbb{P}(X_T = -j) + k\mathbb{P}(X_T = k)$$

which gives us

$$\mathbb{P}(X_T = -j) = \frac{k}{j+k}; \quad \mathbb{P}(X_T = k) = \frac{j}{j+k}$$

thus mapping back to derivatives we have

$$||\partial_x^j \partial_y^k f|| \le ||\partial_x^{j+k} f||^{\frac{j}{j+k}} ||\partial_y^{k+j} f||^{\frac{k}{j+k}}$$

where now we apply (b) to each term to get

$$||\partial_x^{j+k} f||^{\frac{j}{j+k}} \le ||\partial_x^m f||^{\frac{j}{j+k} \cdot \frac{j+k}{m}} ||f||^{\frac{j}{j+k} \cdot \frac{m-(j+k)}{m}}$$
$$||\partial_y^{j+k} f||^{\frac{k}{j+k}} \le ||\partial_y^m f||^{\frac{k}{j+k} \cdot \frac{j+k}{m}} ||f||^{\frac{k}{j+k} \cdot \frac{m-(j+k)}{m}}$$

and simply multiplying the above we have

$$||\partial_x^j \partial_y^k f|| \le ||\partial_x^m f||^{\frac{j}{m}} ||\partial_y^m f||^{\frac{k}{m}} ||f||^{1-\frac{j+k}{m}}$$

as desired.

$\square$

Exercise 2.

Proof.

(a):

Plug in and do integral by parts we have

$$
\begin{aligned}
|\hat{f}_k| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikx} f(x)dx\right| \overset{ibp}{=} \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}\frac{e^{-ikx}}{-ik}f'(x)dx\right| = \dots \\
&\leq \frac{1}{2\pi|k|^j}\left|\int_{-\pi}^{\pi} e^{-ikx} f^{(j)}(x)dx\right| \leq \frac{1}{2\pi|k|^j}||f^{(j)}||_1
\end{aligned}
$$

where the boundary term vanishes because both f and e^{-ikx} are 2π periodic functions.

(b):

Plug in we have that for j not a multiple of n:

$$
0 = \int_{-\pi}^{\pi} e_j(x)dx \approx \sum_{k=0}^{n-1}\frac{2\pi}{n}e^{2\pi ijk/n} = \frac{2\pi}{n}\frac{e^{2\pi ij}-1}{e^{2\pi ij/n}-1} = 0
$$

since the nominator is 0. As for $j = ln$ then the exponential is always 1 so the sum is 2π.

(c): plugging in the Fourier transform of f and then exchange sum and integration we get

$$
\hat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}}\hat{f}_m\int_0^{2\pi}e^{i(m-k)x}dx \approx \sum_{m\in\mathbb{Z}}\hat{f}_{mn+k}
$$

by part b (up to a sign, but that's easily dealt with by changing summation order). From part (a) we know that for $f \in C^j$, $\hat{f}_k = O(k^{-j})$ and thus the difference is

$$
\sum_{m\in\mathbb{Z};m\neq 0}\hat{f}_{mn+k} = \sum_{m\in\mathbb{Z};m\neq 0} O((mn+k)^{-j}) = O(m^{-j})
$$

and thus decays when $j \geq 2$, and diverges when j is only 1. And the bound is just (from (a))

$$
\left|\hat{f}_k - \sum_{m\in\mathbb{Z}}\hat{f}_{mn+k}\right| \leq \frac{1}{2\pi}||f^{(j)}||_1\sum_{m\in\mathbb{Z};m\neq 0}\frac{1}{(mn+k)^j} = O\left(\sum_{m\in\mathbb{Z};m\neq 0}\frac{1}{(mn+k)^j}\right).
$$

Ok maybe one want a more precise bound (to be honest the above is precise enough...) then we can get (heavily discussed with others)

- If $|k| < |n|$

$$
\sum_{l\in\mathbb{Z}\setminus\{0\}}\frac{1}{|l+\frac{k}{n}|^j} \leq 2\sum_{l=1}^{\infty}\frac{1}{l^j} + \frac{1}{|1-\frac{k}{n}|^j} \leq \frac{2j}{j-1} + \frac{1}{|1-\frac{k}{n}|^j}
$$

So

$$
\text{error} \leq \frac{\|f^{(j)}\|_1}{2\pi|n|^j}\left(\frac{2j}{j-1} + \frac{1}{|1-\frac{k}{n}|^j}\right)
$$

• If $|k| > |n|$, there exists $C \in \mathbb{Z}$, s.t. $\dfrac{k}{n} = C + \dfrac{k'}{n}$, $|k'| < |n|$, similarly we have

$$\sum_{l\in\mathbb{Z}\setminus\{0\}} \frac{1}{|l+\frac{k}{n}|^j} \leq \frac{2j}{j-1} + \frac{1}{|1-\frac{k}{n}|^j} + \frac{1}{|\frac{k'}{j}|^j} - \frac{1}{|C-\frac{k'}{n}|^j}$$

So

$$\text{error} \leq \frac{\|f^{(j)}\|_1}{2\pi|n|^j}\left(\frac{2j}{j-1} + \frac{1}{|1-\frac{k}{n}|^j} + \frac{1}{|\frac{k'}{j}|^j} - \frac{1}{|C-\frac{k'}{n}|^j}\right)$$

(d):

Compute the error we get ($\tilde{f}$ refers to the approximation above) by plugging in the Fourier series of f:

$$\left\|\sum_{k=-n+1}^{n-1} \tilde{f}_k e^{ikx} - f\right\|_2 \leq \left\|\sum_{k=-n+1}^{n-1} \left(\hat{f}_k - \tilde{f}_k\right) e^{ikx}\right\|_2 + \left\|\sum_{k\geq n} \hat{f}_k e^{ikx}\right\|_2$$
$$\leq \frac{1}{2\pi}\sum_{k=-n+1}^{n-1} \frac{1}{2\pi}||f^{(j)}||_1 \sum_{m\in\mathbb{Z};m\neq 0} \frac{1}{(mn+k)^j} + \frac{1}{2\pi}\sum_{k\geq n}|\hat{f}_k|$$

and to bound the first term note that a shift of each k from 0 to $n-1$ to nm is just summing on each term in $\mathbb{Z}$, and the for the other side we're left with one mn term, thus the first term (without constant) after reordering summation is

$$\sum_{k=-n+1}^{n-1}\sum_{m\in\mathbb{Z};m\neq 0} \frac{1}{(mn+k)^j} = 2\sum_{l\in\mathbb{Z},l\neq 0}\frac{1}{l^j} - \sum_{h\in\mathbb{Z};h\neq 0}\frac{1}{(hn)^j}$$

So let the usual zeta function denote what it is, i.e.

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

then the above is (for sanity, $j \geq 1$ is assumed)

$$2\sum_{l\in\mathbb{Z},l\neq 0}\frac{1}{l^j} - \sum_{h\in\mathbb{Z};h\neq 0}\frac{1}{(hn)^j} = 4\zeta(j) - \frac{2\zeta(j)}{n^j}.$$

Now for the second term, the tail of summation of coefficients. For this we just integrate and get

$$\frac{1}{2\pi}\sum_{k\geq n}|\hat{f}_k| \leq \frac{1}{\pi}\int_{n-1}^{\infty}\frac{1}{2\pi x^j}||f^{(j)}||_1 dx = \frac{||f^{(j)}||_1}{4\pi^2(j-1)}(n-1)^{-j+1}$$

and combining we have

$$\left\|\sum_{k=-n+1}^{n-1}\tilde{f}_k e^{ikx} - f\right\|_2 \leq \frac{||f^{(j)}||_1}{4\pi^2}\left(4\zeta(j) - \frac{2\zeta(j)}{n^j} + \frac{(n-1)^{-j+1}}{j-1}\right).$$

$\square$

Exercise 3.

Proof.

(a):

Analytic part: we have $\hat{f}_0 = 0$ and for $k \neq 0$ we note that the function is odd so we only need to compute the sin part of the integral to get the coefficients (and we can reduce to only positive part):

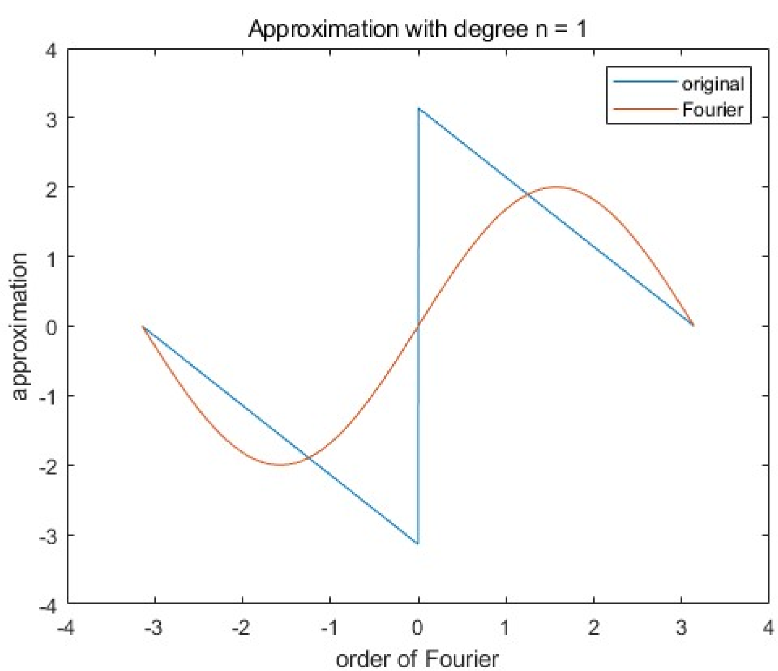
$$\hat{f}_k = \frac{1}{\pi} \int_0^\pi (\pi - x) \sin kx dx \overset{ibp}{=} \frac{2}{k}.$$

Because there is a jump at 0, so we can locally map it to sgn function with a homeomorphism, and we've shown that sgn function has Gibb's phenomenon, thus this function f, via transformation Φ (existence by Implicit function theorem) we have

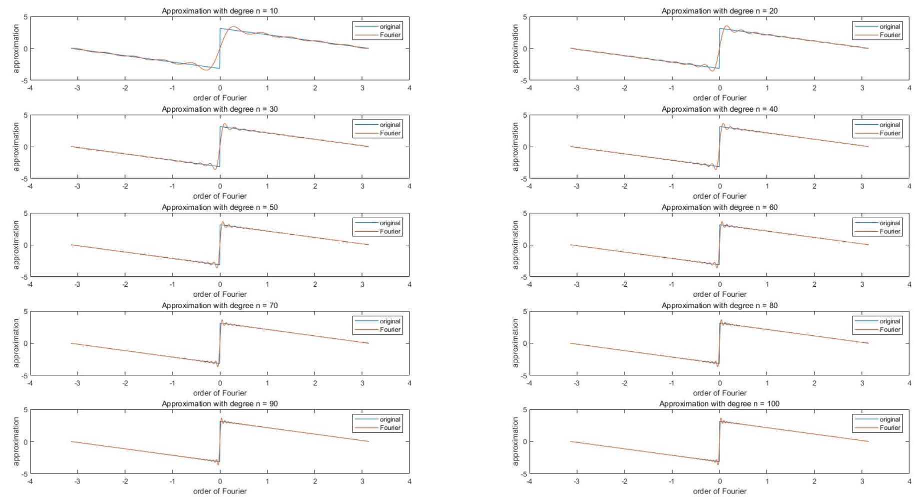
$$f = \pi \cdot \Phi(\text{sgn})$$

and since there's Gibb's phenomenon for sgn, there is one for f. And indeed we can check it below:

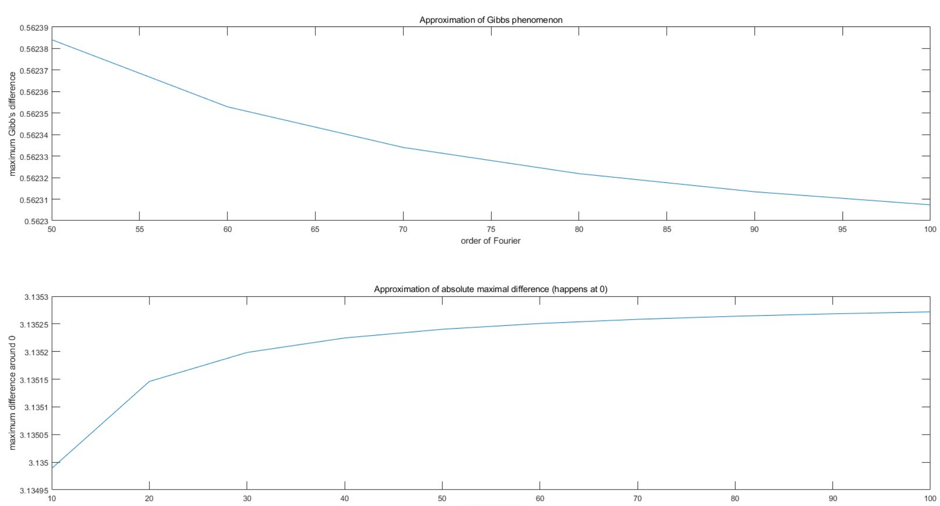
For $n = 1$ the result is (all generated by file "TommenixYu_q3a"):



and for other n the results are:



and we can see Gibb's phenomenon clearly. Now we compute and plot both the maximum L^∞ error, as well as the error caused by Gibb's phenomenon:



where the error of Gibb's phenomenon is a constant (precision's pretty hype) and the maximum error goes to π. But that is obvious since the maximum error occur at $x = \pm\varepsilon$ for ε really really really small, where the Fourier series goes to 0 but the original function is $\pm\pi$.

(b):

We note the function is even so we only do the cos part this time:

$$\hat{f}_k = \frac{2}{2\pi}\int_0^\pi x(1+\cos x)\cos(kx)dx = \frac{1}{\pi}\int_0^\pi x\cos kx dx + \frac{1}{\pi}\int_0^\pi x\cos(x)\cos(kx)dx$$
$$= \frac{1}{\pi}\int_0^\pi x\cos kx dx + \frac{1}{2\pi}\int_0^\pi x\cos((k-1)x)dx + \frac{1}{2\pi}\int_0^\pi x\cos((k+1)x)dx$$

Using integral by parts (same as in part (a)) we have

$$\frac{1}{\pi}\int_0^\pi x\cos kx dx = \begin{cases} \frac{\pi}{2} & k=0 \\ \frac{(-1)^k-1}{\pi k^2} & k\neq 0 \end{cases}$$

and the rest two terms are exactly the same except that the scaling and k is different. Anyway we plug in and get

$$\frac{1}{2\pi}\int_0^\pi x\cos((k-1)x)dx = \begin{cases} \frac{\pi}{4} & k=1 \\ \frac{(-1)^{k-1}-1}{\pi(k-1)^2} & k\neq 1 \end{cases}$$

$$\frac{1}{2\pi}\int_0^\pi x\cos((k+1)x)dx = \begin{cases} \frac{\pi}{4} & k=-1 \\ \frac{(-1)^{k+1}-1}{\pi(k+1)^2} & k\neq -1 \end{cases}$$

so the Fourier series is

$$S_n(f)(x) = \frac{\pi}{2} - \frac{2}{\pi} + \left(\frac{\pi}{2} - \frac{4}{\pi}\right)\cos x + 2\sum_{k=2}^{n}\left(\frac{(-1)^k-1}{\pi k^2} + \frac{(-1)^{k-1}-1}{\pi(k-1)^2} + \frac{(-1)^{k+1}-1}{\pi(k+1)^2}\right)\cos kx$$

As for is there Gibb's phenomenon, the answer is no because the function is Lipschitz so the Fourier series converge. For a proof of this fact we just note:

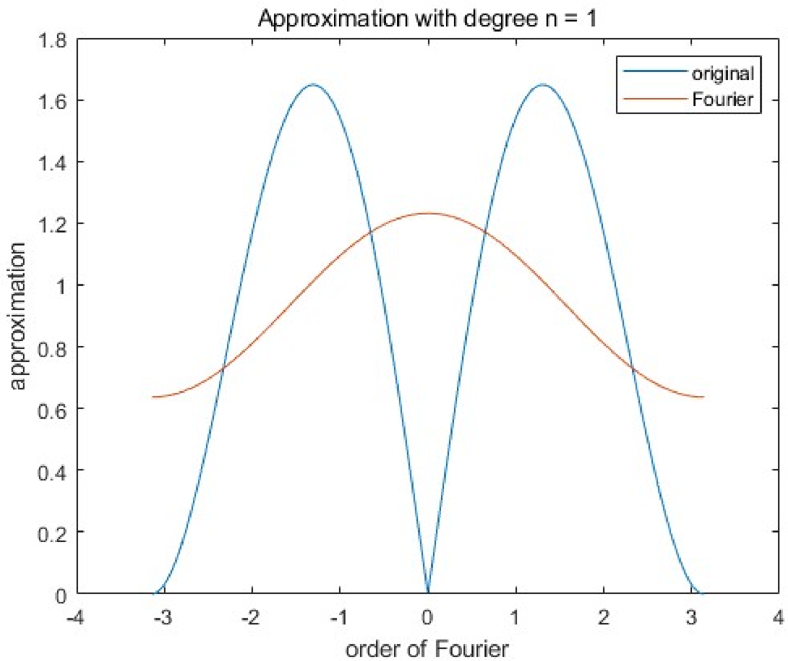
$$||f(x) - D*f(x)||_\infty \leq \int_{-\pi}^{\pi} |D(y)(f(x-y)-f(x))|\, dy$$
$$= \int_{-\pi}^{\pi} \left|\frac{\sin(x-y)(n+1/2)}{\sin((x-y)/2)}(f(x-y)-f(x))\right| dy$$

we note that D_n has the property that it decays far away from 0 (even though it's not an approximate identity, it does have the two properties of integrates to 1 and vanishes away from critical point, as seen in class). Thus, we first separate the integral into close enough to 0 and far away. For the faraway part it goes to 0 by above argument. For the close to 0 part, so since f is Lipschitz the quotient $\dfrac{f(x+h)-f(x)}{h}$ is bounded even when $h \to 0$, i.e. we can write out (continuing the above)

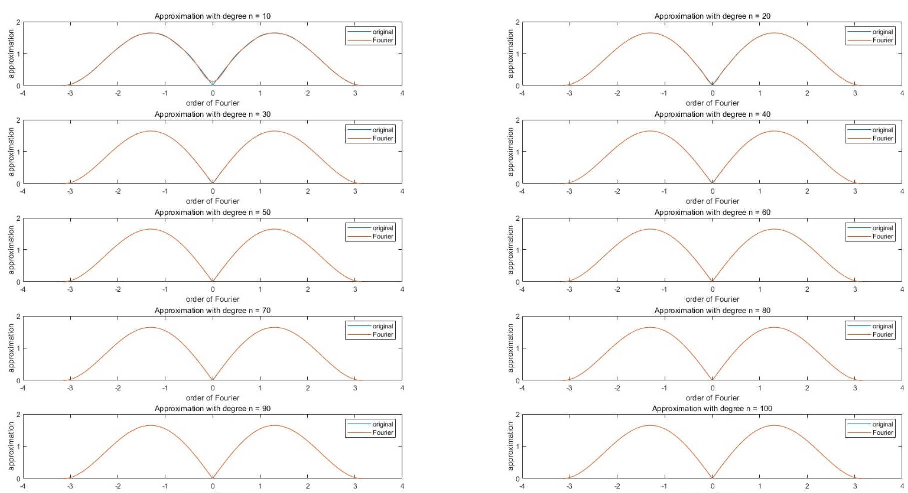
$$\lessapprox \int_{-\varepsilon}^{\varepsilon} \left|\sin(x-y)(n+1/2)\frac{y}{\sin((x-y)/2)}\frac{f(x-y)-f(x)}{y}\right| dx + O(1)$$

and we see that the three terms are all bounded, thus as $\varepsilon \to 0$ the whole thing tends to 0. In other words we have this result.

For $n = 1$ the result is (all generated by file "TommenixYu_q3b"):

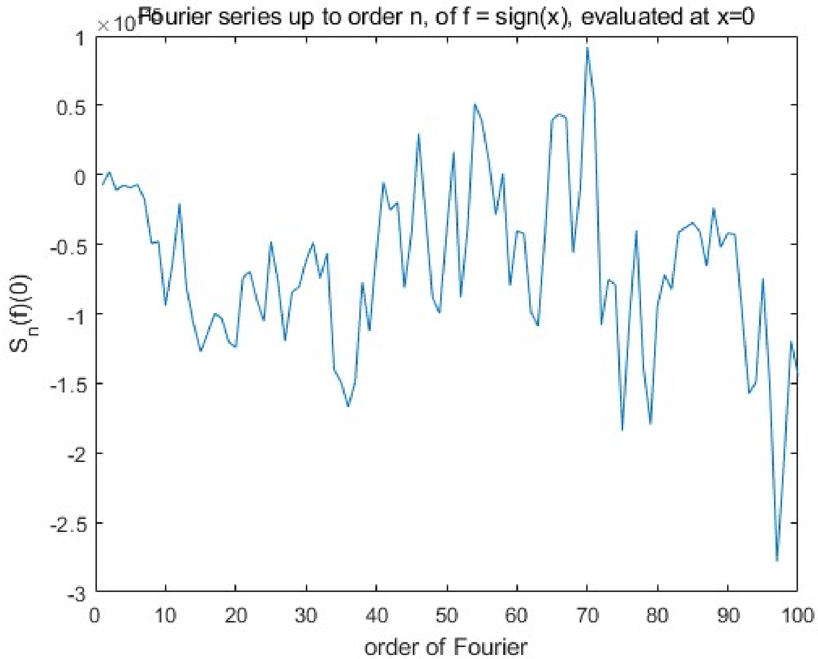


and for other n the results are:



and we see that the Fourier series converges and there's no Gibbs phenomenon. This makes sense because the function is Lipschitz.

(c): The Fourier series will converge to $\dfrac{\lim_{x\to x_*^+} f(x) + \lim_{x\to x_*^-} f(x)}{2}$. To verify this I just picked the sign function, and as we expected the result should be around 0 at around 0 (generated by "TommenixYu_q3c"):



(d):

Here we assign $f(0) = 0$.

First, we note that the function is odd, so it's even coefficients, a_k, are 0. And to compute we have

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(kt)\,\mathrm{sgn}(t)dt = \frac{2}{\pi k}\left(1-(-1)^k\right)$$

and let

$$S_m(f) := \sum_{k=1}^{m} b_k \sin(kx)$$

then we know

$$S_{2n}(f) := \sum_{k=1}^{n} \frac{4}{(2k-1)\pi}\sin((2k-1)x).$$

And notice that the closest zero point to 0 is $\pm\frac{\pi}{2n}$ since we directly take derivative

$$S_n(f)' = \frac{4}{\pi}\sum_{k=1}^{n}\cos(2k-1)x = \frac{2}{\pi}\left[\frac{e^{ix}(1-e^{i2nx})}{1-e^{i2x}} + \frac{e^{-ix}(1-e^{-i2nx})}{1-e^{-i2x}}\right]$$

and we plug in $x = \frac{\pi}{2n}$ to have

$$S_n(f)'\left(\frac{\pi}{2n}\right) = \frac{2}{\pi}\frac{e^{i\pi/(2n)} - e^{-i\pi/(2n)} + e^{-i\pi/(2n)} - e^{i\pi/(2n)}}{(1-e^{i\pi/n})(1-e^{-i\pi/n})} = 0.$$

Now to compute the exact value we have: (same as in (a))

$$\begin{aligned}
S_{2n}(f)\left(\frac{\pi}{2n}\right) &= \frac{4}{\pi}\left(\sin\left(\frac{\pi}{2n}\right) + \frac{\sin\left(\frac{3\pi}{2n}\right)}{3} + \cdots + \frac{\sin\left(\frac{(2n-1)\pi}{2n}\right)}{2n-1}\right) \\
&= \frac{\pi}{2n}\frac{4}{\pi}\left(\frac{\sin\left(\frac{\pi}{2n}\right)}{\frac{\pi}{2n}} + \frac{\sin\left(\frac{3\pi}{2n}\right)}{\frac{3\pi}{2n}} + \cdots + \frac{\sin\left(\frac{(2n-1)\pi}{2n}\right)}{\frac{(2n-1)\pi}{2n}}\right) \\
&\approx \frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt = 1 + 2\cdot 0.0894898722...
\end{aligned}$$

where the constant is called Wilbraham-Gibbs constant. So we can see that the constant at least does not grow with n.

Well, since Jeremy also asks for $x = \dfrac{j\pi}{2n}, -n+1 \le j \le n-1$, that is, he is curious about the decay rate of catears, we have to do this now:

$$\begin{aligned}
S_{2n}(f)\left(\frac{j\pi}{2n}\right) &= \frac{4}{\pi}\left(\sin\left(\frac{j\pi}{2n}\right) + \frac{\sin\left(\frac{3j\pi}{2n}\right)}{3} + \cdots + \frac{\sin\left(\frac{(2n-1)j\pi}{2n}\right)}{2n-1}\right) \\
&= \frac{\pi}{2n}\frac{4}{\pi}\left(\frac{\sin\left(\frac{j\pi}{2n}\right)}{\frac{\pi}{2n}} + \frac{\sin\left(\frac{3j\pi}{2n}\right)}{\frac{3\pi}{2n}} + \cdots + \frac{\sin\left(\frac{(2n-1)j\pi}{2n}\right)}{\frac{(2n-1)\pi}{2n}}\right) \\
&\approx \frac{2}{\pi}\int_0^{j\pi}\frac{\sin t}{t}dt
\end{aligned}$$

(those are the few times I'm glad I did Latex it...)

$\square$

<u>Total propagated threshold:</u>

For $\pm v_i > 0$, we have that the total propagated threshold T is

$$T = v_i \pm \left(\frac{1.75}{10000} + L \right) \cdot \frac{A}{10000 \cdot 2.33 \cdot \sigma_i} = v_i \pm \left(\frac{1.75}{10000} + w \cdot \beta_{\cdot,1} \right) \cdot A$$

and the factor exposure to hedge (FE) is

$$FE = \pm \left(\frac{1.75}{10000} \pm w \cdot \beta_{\cdot,1} \right) \cdot A$$

where v is the vector of exposure, L is total propagated loss, A is total AUM, and $\beta_{\cdot,1}$ is the first column of the β matrix, or just β for factor 1.