BROWNIAN MOTION AND STOCHASTIC CALCULUS HW 8

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Discussed with classmates.

Exercise 1.

Proof.

(1): No:

Suppose only finitely many, call it N, since non-constant it cannot be sum of constant iid. So let $X = \sum_{i=1}^{n} Y_i$. Since Y has range with order > 2 so when $n > N$, X would have more than N values since at least one value of Y has the largest non-zero absolute value. Contradiction.

(2): Yes.

Since sum of two Poisson distribution is still Poisson (with scaling), so it is infinitely divisible. Pick $X = -P$ for P Poisson then we are done by letting $K = 1$.

(3): Yes.

Poisson takes value in integers, and integers are closed under addition, so Poission distribution satisfies the condition. □

Exercise 2.

Proof.

(1)

For f cadlag, it has only finite jump points, since if infinite then we cut $[0,1]$ into decreasing closed sets that goes down to measure 0, then one of them has infinitely many jumps, which means there is a point for which left or right limit does not exist, contradiction. So let K be the jumps points, then f is near continuous with K since there's finitely many compact intervals and there's no explosion (left limit exists), so we can uniformly bound the derivative, i.e. such $\varepsilon - \delta$ pair can be found.

(2):

Denote the function by f. Near continuity means left limit and right limit exist for all $x \in [0,1]$ and f is continuous (with uniform δ for ε, this guarantees no explosion, i.e. left limit exists) on $[0,1]\backslash K$. Thus, pick the g cadlag with

$$g\Big|_{[0,1]\backslash K} = f\Big|_{[0,1]\backslash K}$$

and if $1 \in K$ then we pick $g(1) = 0$ then g would satisfy the condition.

$\square$

Exercise 3. *(6.6)*

Proof.

(1): (steps in book & discussion with Tong Xie)

A Poisson process, viewed as a compound Poisson process, have Levy measure $\mu^{\#}$ with $\mu^{\#}(\mathbb{R}) = \mu^{\#}(\{1\})$ just by definition. Thus, to compensate X_t we compute

$$\int_{-\infty}^{\infty} [e^y - 1]\mu^{\#}(dy) = e - 1$$

and hence $X_t - (e-1)t$ is compensated. If $r = e - 1$ then we know already that $\tilde{S}_t$ is a Martingale so we can pick $M_t \equiv 1$.

Now we discuss when $r \neq e - 1$. Follow the book and define $S_t := e^{X_t}$ and $h(y) := e^y - 1$ then note that

$$S_t - S_{t-} = S_{t-}h(X_t - X_{t-}) = S_{t-}h(X_t - X_{t-})$$

in particular we define

$$\hat{X}_t := \sum_{s \le t} h(X_s - X_{s-}) = (e-1)X_t$$

then this process has a Levy measure μ (also a probability measure) with $\mu(\{e-1\}) = 1$ and mean $m = e - 1$. For $\hat{Y}_t := \hat{X}_t - (e-1)t$, which is $\hat{X}_t$ compensated, we can write

$$dS_t = S_{t-}d\hat{X}_t = S_{t-}d\hat{Y}_t + (e-1)S_{t-}dt$$

now product rule says

$$d\tilde{S}_t = \tilde{S}_{t-}(-rt + d\hat{X}_t)$$

and our goal is to make $-rt + d\hat{X}_t$ a Martingale under the new measure. From section 6.5 in book it suffices to find a ν such that $\hat{X}_t\Big|_{\nu} \sim PP(r)$ and also equivalent.

There exists such ν because we only need

$$\int_{-\infty}^{\infty} e^{-x}\nu(dx) = r$$

and in particular that's not unique.

Now note that for it to truly be a Martingale instead of a local martingale we can just check square integrability:

$$\mathbb{E}[\tilde{S}_t^2] = \mathbb{E}[e^{2(X_t - rt)}] < \infty$$

since $X_t \sim PP(1)$.

To pick M_t we directly use result in section 6.5 that

$$M_t = \exp\left\{\sum_{s \le t} \frac{d\nu}{d\mu}(X_s - X_{s-}) - (\nu(\mathbb{R}) - \mu(\mathbb{R}))t\right\} = \exp\left\{\sum_{s \le t} \frac{d\nu}{d\mu}(X_s - X_{s-}) - \frac{r - e + 1}{e - 1}t\right\}$$

or for a more concrete expression just pick $\nu = \frac{r}{e-1}\mu$.

(2):

From above we know ν is not unique. Hence, by definition of Q (same in page 208)

$$Q\{X_{t+\Delta t} - X_t \in V\} = \nu(V)\Delta t + o(\Delta t)$$

that is not unique, and also M_t is not unique since it also depends on ν.

$\square$

Exercise 4.

Proof.

(1): Yes, since

$$\int_0^1 |x| \frac{dx}{x} = 1.$$

(2):

$$\mathbb{E}[X_t] = t \int_0^1 x \frac{dx}{x} = t;$$

$$\mathrm{Var}(X_t) = t \int_0^1 x^2 \frac{dx}{x} = \frac{t}{2}.$$

(3):

Since μ measures the Poisson rate of jumps of sizes, the rate for jump $> \frac{1}{2}$ is

$$\int_{\frac{1}{2}}^1 \mu(dy) = \log 2$$

and thus it's the same thing as computing the probability of a Poisson process with rate $\log 2$ that did not jump for $t \leq 2$. Plugging into Poisson process we get what we want is

$$P = e^{-2\log 2} = \frac{1}{4}.$$

(4):

Let $f(x) = x^4$ then g is by definition the generator

$$g = Lf(x) = \int_0^1 [f(x+y) - f(x)]\, \mu(dy) = \int_0^1 [(x+y)^4 - x^4] \frac{dy}{y} = 4x^3 + 3x^2 + \frac{4}{3}x + \frac{1}{4}$$

where the expression $Lf(x)$ is deduced in class.

$\square$

Exercise 5.

Proof.

(1):

$$|x|\mu(dx) \sim \frac{1}{x^{3/2}}$$

for $x \to 0$ so it is not integrable, hence not GCPP I. But it is GCPP II since $|x|^2\mu(dx)$ is integrable.

(2):

$$\mathbb{E}[X_t] = t \lim_{\varepsilon \to 0} \left(\int_{\varepsilon}^{1} + \int_{-1}^{\varepsilon} \right) x \frac{1}{|x|^{5/2}} dx = 0$$

$$\operatorname{Var}(X_t) = t \int_{-1}^{1} \frac{1}{|x|^{-\frac{1}{2}}} = 4t$$

where the integral is also principal value integral.

(3):

For the same reason the equivalent Poisson rate is

$$\int_{\frac{1}{2}}^{1} \frac{1}{|x|^{5/2}} dx = \frac{4\sqrt{2}-2}{3}$$

and what we want is

$$P = e^{-\frac{8\sqrt{2}-4}{3}}.$$

(4):

$$\begin{aligned} Lf(x) &= \lim_{\varepsilon \to 0} \left(\int_{\varepsilon}^{1} + \int_{-1}^{\varepsilon} \right) \left[(x+y)^4 - x^4 - y4x^3 \right] \frac{1}{|y|^{5/2}} dy \\ &= \lim_{\varepsilon \to 0} \left(\int_{\varepsilon}^{1} + \int_{-1}^{\varepsilon} \right) \left[\frac{6x^2}{|y|^{1/2}} + 4x \operatorname{sgn}(y)\sqrt{|y|} + |y|^{3/2} \right] dy \\ &= \lim_{\varepsilon \to 0} \left(\int_{\varepsilon}^{1} + \int_{-1}^{\varepsilon} \right) dy = 6x^2 \cdot 4 + \frac{4}{5} = 24x^2 + \frac{4}{5}. \end{aligned}$$

□

Exercise 6.

Proof.

(1): Write $f(X_t, t) = C_t[\sin X_t]^a$, and denote $C_t =: g(t)$ for clarity, then use Ito's formula to get (I interchangably used X_t and B_t at the convenient positions)

$$dM_t = \left(g'(t)[\sin X_t]^a - a(a-1)g(t)[\sin X_t]^{a-2}\right) dt + g(t)a\cos X_t[\sin(X_t)]^{a-1} dB_t$$

then for M_t to be a local Martingale we need it to be a stochastic integral, hence we solve ODE

$$g'(t)[\sin X_t]^a - a(a-1)g(t)[\sin X_t]^{a-2} = 0$$

which gives solution

$$C_t = g(t) = \exp\left\{at - \frac{1}{2}a(a-1)\int_0^t \cot^2 X_s ds\right\}$$

which is C_1 because it's smooth combined with integral of continuous function. Now to make our logic sound we redo from start to compute with Ito that

$$\begin{aligned} dM_t &= g(t)a\cos X_t[\sin(X_t)]^{a-1} dB_t = \left(\frac{g(t)a\cos X_t[\sin(X_t)]^{a-1}}{M_t}\right) M_t dB_t \\ &= a\cot X_t M_t dB_t =: A_t M_t dB_t. \end{aligned}$$

(2): If Girsanov can pass through, then the satisfied expression would be

$$dW_t = dB_t - A_t dt = dB_t - a\cot(B_t)dt$$

up till T_n for all n, where T_n is the stopping time that bounds both A_t and M_t.

(3): (Notification by Tim Su and Tong Xie) Note that if we view $\mathbb{P}^*$ as the original measure then we can express X_t as

$$dX_t = a\cot(X_t)dt + dW_t$$

which is exactly the expression in Hw6 prob 3! Using the result we get that $\mathbb{P}^*\{T = \infty\} = 1$ for $a \geq \frac{1}{2}$ (otherwise I'm so stuck).

□