PDE HOMEWORK 6

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Discussed with classmates.

Exercise 1.

Proof.

Here

$$F = \partial_t u + b \cdot \nabla_x u - f = \begin{pmatrix} 1 \\ b \end{pmatrix} \cdot \nabla u - f = \begin{pmatrix} 1 \\ b \end{pmatrix} \cdot p - f$$

and thus

$$\begin{cases} \dot{x} = \partial_p F = \begin{pmatrix} 1 \\ b \end{pmatrix} \\ \dot{z} = p \cdot \dot{x} = f \end{cases}$$

and thus we can write out

$$x(s) = (s, x_0 + sb), \qquad z = z_0 + \int_0^s f(x(\tau))d\tau$$

and for any given point (t, x) we backtrack to get

$$t = s; x_0 + sb = x \Rightarrow x_0 = x - tb$$

so $z_0 = u(x_0) = g(x - tb)$ and plugging in everything we have

$$\begin{aligned} u(t, x) = z(t) = z_0 + \int_0^t f(x(\tau))d\tau &= g(x - tb) + \int_0^t f(\tau, x_0 + \tau b)d\tau \\ &= g(x - tb) + \int_0^t f(\tau, x + (\tau - t)b)d\tau. \end{aligned}$$

□

Exercise 2.

Proof.

(1):

Writing out we have

$$\begin{cases} x_1u_1 + x_2u_2 = 2u \Rightarrow F = x \cdot p - 2z = 0 \\ u(x_1, 1) = g(x_1) \Rightarrow x_2(s) = 1 \end{cases}$$

and the Character system is

$$\begin{cases} \dot{x} = \partial_p F = x \\ \dot{z} = \partial_p F \cdot x = 2z \end{cases}$$

and hence

$$\begin{cases} x_1(s) = x_1^0 e^s \\ x_2(s) = x_2^0 e^s = e^s \\ z = z_0 e^{2s} = u(x(0))e^{2s} = u(x_1^0, 1)e^{2s} = g(x_1^0)e^{2s} \end{cases}$$

and for given point (x_1, x_2) we plug in to get

$$x_1 = x_1(s) = x_1^0 e^s; \quad x_2 = x_2(s) = e^s;$$

$$\Rightarrow x_1^0 = \frac{x_1}{e^s} = \frac{x_1}{x_2}$$

and thus

$$z_0 = g\left(\frac{x_1}{x_2}\right)$$

$$\Rightarrow u(x_1, x_2) = z(s) = e^{2s} g\left(\frac{x_1}{x_2}\right) = x_2^2 g\left(\frac{x_1}{x_2}\right).$$

To check this we note that the initial condition is easily checked, and the PDE we plug in to get indeed

$$\begin{aligned} x_1u_1 + x_2u_2 &= x_1\left(x_2^2 g'\left(\frac{x_1}{x_2}\right)\frac{1}{x_2}\right) + x_2\left(2x_2 g\left(\frac{x_1}{x_2}\right) - x_2^2 g'\left(\frac{x_1}{x_2}\right) x_1 \frac{1}{x_2^2}\right) \\ &= 2x_2^2 g\left(\frac{x_1}{x_2}\right) = 2u. \end{aligned}$$

(2):

Writing out we have

$$\begin{cases} uu_1 + u_2 = 1 \Rightarrow F = (z, 1) \cdot p - 1 = 0 \\ u(x_1, x_1) = \frac{1}{2}x_1 \Rightarrow x_2^0 = x_1^0 \end{cases}$$

and the Character system is

$$\begin{cases} \dot{x} = \partial_p F = (z, 1) \\ \dot{z} = \partial_p F \cdot p = 1 \end{cases}$$

and hence

$$\begin{cases} x_2(s) = x_2^0 + s = x_1^0 + s \\ z = z_0 + s = s + g(x_1^0) = s + \frac{1}{2}x_1^0 \\ x_1(s) = x_1^0 + \int_0^s z_0 + t dt = x_1^0 + \frac{s^2}{2} + \frac{s}{2}x_1^0 \end{cases}$$

where for any given point $x = (x_1, x_2)$ we have

$$x_2 = x_1^0 + s; \quad x_1 = x_1^0 + \int_0^s z_0 + t dt = x_1^0 + \frac{s^2}{2} + \frac{s}{2}(x_1^0)^2$$

so plugging in we have

$$x_1^0 = x_2 - s; \quad x_1 = x_2 - s + \frac{s^2}{2} + \frac{s}{2}(x_2 - s) = s\left(\frac{x_2}{2} - 1\right) + x_2$$

and thus

$$s = \frac{2x_1 - 2x_2}{x_2 - 2}$$

and so the solution is

$$u(x_1, x_2) = z(s) = s + \frac{1}{2}x_1^0 = \frac{2x_1 - 2x_2}{x_2 - 2} + \frac{1}{2}\left(x_2 - \frac{2x_1 - 2x_2}{x_2 - 2}\right) = \frac{x_2^2 - 6x_2 + 4x_1}{2x_2 - 4}$$

where the boundary condition and PDE are checked to satisfy.

□

Exercise 3.

Proof.

(1):

Write out

$$L(q) = \sup_p \left(p \cdot q - \frac{1}{r}|p|^r \right)$$

and if we just choose $p = q^{\frac{1}{r-1}}$ then the thing inside the sup is

$$|q|^{1+\frac{1}{r-1}} - \frac{1}{r}|q|^{\frac{r}{r-1}} = \left(1 - \frac{1}{r}\right)|q|^{\frac{r}{r-1}} = \frac{1}{s}|q|^s$$

and we just check that this indeed is the maximum. Note that $p \cdot q$ is concave (linear) in q and so is $-H(p)$ concave, so derivative $= 0$ means global maximum of the expression inside sup, and take derivative we have

$$\partial_p \left(p \cdot q - \frac{1}{r}|p|^r \right) = q - |p|^{r-2}p$$

and using $p = q^{\frac{1}{r-1}}$ we have

$$\partial_p \left(p \cdot q - \frac{1}{r}|p|^r \right)\Bigg|_{p=q^{\frac{1}{r-1}}} = q - q^{\frac{r-2}{r-1}+\frac{1}{r-1}} = q - q = 0$$

so it's indeed the maximum, thus the supremum. So

$$L(q) = \sup_p \left(p \cdot q - \frac{1}{r}|p|^r \right) = \frac{1}{s}|q|^s$$

due to computation above.

(2):

H is convex is just by taking derivatives:

$$H'' = (Ap + b)' = A \geq 0$$

since A positive definite, and hence H is convex.

Now writing out

$$L(q) = \sup_p \left(p \cdot q - \frac{1}{2}p^T Ap + b^T p \right)$$

note $p^T q + b^T p$ is linear in p and $-\frac{1}{2}p^T Ap$ is concave so finding the place where derivative is 0 is same as finding the sup. So just take derivative to get

$$\partial_p \left(p \cdot q - \frac{1}{2}p^T Ap - b^T p \right) = q - Ap - b = 0 \Rightarrow p = A^{-1}(q - b)$$

now plug in to compute we get

$$
\begin{aligned}
L(q) &= \left(p \cdot q - \frac{1}{2}p^T A p + b^T p\right)\Big|_{p=A^{-1}(q-b)} \\
&= q^T A^{-1}(q-b) - \frac{1}{2}(q-b)^T A^{-1}(q-b) - b^T A^{-1}(q-b) \\
&= \frac{1}{2}(q-b)^T A^{-1}(q-b)
\end{aligned}
$$

(where we note that the transform is convex, as expected).

$\square$

Exercise 4.

Proof.

(1):

Reduce to only proving $q \in \partial H(p) \iff p \cdot q = H(p) + L(q)$:

If we have $q \in \partial H(p) \iff p \cdot q = H(p) + L(q)$ where $L = H^*$ and H is arbitrary, we just let $H := L$ then by duality of Legendre transform we have

$$q \in \partial L(p) \iff p \cdot q = L(p) + L^*(q) = L(p) + H(q)$$

and with a change of variable names we get the other half of the problem:

$$p \in \partial L(q) \iff q \cdot p = L(q) + H(p)$$

$q \in \partial H(p) \Rightarrow p \cdot q = H(p) + L(q)$:

This mimics the proof in class. Assume $q \in \partial H(p)$ holds.

$p \cdot q \leq H(p) + L(q)$:

by definition we know

$$L(q) = \sup_s (p \cdot s - H(p)) \geq p \cdot q - H(q)$$

which gives us the result.

$p \cdot q \geq H(p) + L(q)$:

$q \in \partial H(p)$ by definition means that for all r we have

$$H(r) \geq H(p) + q \cdot (r - p) \iff p \cdot q \geq H(p) + (q \cdot r - H(r))$$

since this holds for all r we have

$$p \cdot q \geq H(p) + \sup_r (q \cdot r - H(r)) = H(p) + L(q).$$

$q \in \partial H(p) \Leftarrow p \cdot q = H(p) + L(q)$:

We want to show that $\forall r$ we have

$$p \cdot q \geq H(p) + (q \cdot r - H(r))$$

which is equivalent to the definition of $q \in \partial H(p)$. But

$$p \cdot q = H(p) + L(q) \geq H(p) + (q \cdot r - H(r))$$

for all r, so we are done.

(2):

What we need to do is to prove the second equality sign. For this we note that if y is the minimum of the function

$$f(y) := tL\left(\frac{x - y}{t}\right) + g(y)$$

then $f'(y) = 0$ hence
$$-t\frac{1}{t}DL\left(\frac{x-y}{t}\right) + Dg(y) = 0$$
so
$$DL\left(\frac{x-y}{t}\right) = Dg(y)$$
where we note that DL as a derivative might not even be defined since we know nothing about the smoothness L except that it is convex. But we can use the generalized subdifferential ∂L to proceed. Thus we know
$$Dg(y) \in \partial L\left(\frac{x-y}{t}\right)$$
which by (1) we know it's equivalent to
$$\frac{x-y}{t} \in \partial H\left(Dg(y)\right)$$
and when ∇H is indeed defined the right hand side is a set of one element and hence
$$\frac{x-y}{t} = \nabla H(\nabla g(y))$$
from which we know
$$|x-y| = t\nabla H(\nabla g(y)) \leq tR$$
since R is defined to be the sup of $|\nabla H(\nabla g)|$.

Thus any minimum, including the global one, is attained inside the ball $B(x, tR)$ and thus
$$\min_{y\in\mathbb{R}^n}\left\{tL\left(\frac{x-y}{t}\right) + g(y)\right\} = \min_{y\in B(x,tR)}\left\{tL\left(\frac{x-y}{t}\right) + g(y)\right\}.$$

$\square$

Exercise 5.

Proof.

Let's fit into the form of Hamilton-Jacobi:

$$\begin{cases} \partial_t u + H(\nabla u) = 0 & U \\ u(x,0) = g(x) & t = 0 \end{cases}$$

to get

$$H(x) = |x|^2; \quad g = \chi_E$$

and hence

$$L(q) = \sup_p (p \cdot q - H(p))$$

to find the maximum we take derivative to get

$$(p \cdot q - H(p))' = q - 2p = 0$$

so $p = \frac{1}{2}q$ and

$$L(q) = \frac{1}{4}q^2.$$

Now we use the Hopf-Lax formula to get

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) + \chi_E(y) \right\}$$

so in order for the minimum to be attained χ_E has to be 0 otherwise it's ∞, larger than any exact value when $y \in E$. So

$$u(x,t) = \min_{y \in \mathbb{R}^n} \left\{ tL\left(\frac{x-y}{t}\right) \right\} = \min_{y \in E} t\frac{1}{4t^2}(x-y)^2 = \frac{1}{4t} \operatorname{dist}(x,E)^2$$

where the last equality holds because a) E is closed and the minimum is attained and b) the minimum is just the distance by definition.

$\square$