BROWNIAN MOTION AND STOCHASTIC CALCULUS HW 5

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Discussed with classmates.

Exercise 1.

Proof.

For convenience, define mesh $\Pi^{(n)}$ on $[r,s]$ with $r = t_0 < t_1 < \cdots < t_n = s$ and $t_i := r + \frac{i}{n}(s-r)$. Then, using these nodes we can define

$$f^{(n)}(x) := \sum_{i=1}^{n} \mathbb{1}_{t_{i-1} \le x < t_i} f(t_{i-1})$$

which just means $f^{(n)}(x) = f(t_{i-1}) =: F_n^n$ for $t_{i-1} \le x < t_i$, and $\{F_j^i\}$ is a countable collection of sequences (F^i is the i-th sequence).

But then we have by sbp that for any $\omega \in \Omega$

$$\int_r^s f^{(n)}(t)dB_t = \sum_{i=1}^{n} f(t_{i-1})[B_{t_i} - B_{t_{i-1}}] = (F_{n-1}^n B_{t_n} - F_0^n B_{t_0}) - \sum_{i=1}^{n} B_{t_i}(F_{t_i}^n - F_{t_{i-1}}^n)$$

but then we take $n \to \infty$ and by continuity of f we know $F_{n-1}^n \to f(s)$ and since f differentiable and B_t continuous we have

$$\sum_{i=1}^{n} B_{t_i}(F_{t_i}^n - F_{t_{i-1}}^n) \to \int_r^s B_t df(t) = \int_r^s B_t f'(t)dt$$

whereas the left hand side converges because we can smoothly truncate f outside of $[0,t]$ (where it's defined) to make it compact support, then the limit converges as a stochastic integraion

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})[B_{t_i} - B_{t_{i-1}}] = \int_r^s f(t)dB_t$$

from discussion in class if we take $f^{(n)}$ as a constant random variable measurable with respect to $\mathcal{F}_t$ (actually this is much easier since f is deterministic so $f^{(n)} \in \mathcal{F}_0$). Hence

$$\int_r^s f(t)dB_t = \lim_{n\to\infty}(F_{n-1}^n B_{t_n} - F_0^n B_{t_0}) - \sum_{i=1}^{n} B_{t_i}(F_{t_i}^n - F_{t_{i-1}}^n) = f(s)B_s - f(r)B_r - \int_r^s B_t f'(t)dt.$$

We use Riemann sum see that $\int_r^s f(t)dB_t \sim N$:

Writing out Riemann sum we get

$$\int_r^s B_t f'(t)dt = \lim_{n\to\infty} \sum_{i=1}^n B_{t_{i-1}} \left(f(t_i) - f(t_{i-1}) \right).$$

Now we note that

$$f(s)B_s - f(r)B_r - \sum_{i=1}^n B_{t_{i-1}} \left(f(t_i) - f(t_{i-1}) \right)$$

$$= \left(f(s) - f(r) - \sum_{i=1}^n \left(f(t_i) - f(t_{i-1}) \right) \right) B_r + \sum_{i=2}^n \left(f(s) - \sum_{j=i}^n \left(f(t_i) - f(t_{i-1}) \right) \right) [B_{t_i} - B_{t_{i-1}}]$$

$$= 0 \cdot B_r + \sum_{i=2}^n \left(f(s) - f(s) + f(t_{i-1}) \right) [B_{t_i} - B_{t_{i-1}}] = \sum_{i=2}^n f(t_{i-1})[B_{t_i} - B_{t_{i-1}}]$$

which is a sum of independent normal distributions, thus normal.

But on the other hand we have pointwise limit by Riemann integral that

$$\lim_{n\to\infty} \left[f(s)B_s - f(r)B_r - \sum_{i=1}^n B_{t_{i-1}} \left(f(t_i) - f(t_{i-1}) \right) \right] = f(s)B_s - f(r)B_r - \int_r^s B_t f'(t)dt$$

which then implies weak converge, which means the limiting random variable has normal distribution, which by IBP is what we want.

$\square$

Exercise 2.

Proof.

The idea is that we first show the result for $t_1 \leq t \leq t_2$, then we inductively show the result for all $t_j \leq t \leq t_{j+1}$ holds.

For $t_1 \leq t \leq t_2$:

We explicitly compute

$$\begin{aligned}\left(\int_0^t A_s dB_s\right)^4 &= \left(\sum_{i=1}^{2} A_{t_{i-1}}[B_{t_i} - B_{t_{i-1}}]\right)^4 \\ &= \sum_{i=1}^{2} A_{t_{i-1}}^4 [B_{t_i} - B_{t_{i-1}}]^4 + 6A_{t_1}^2 A_{t_0}^2 [B_{t_2} - B_{t_1}]^2 [B_{t_1} - B_{t_0}]^2 \\ &\quad + 4\sum_{i \neq j} A_{t_{j-1}}^3 A_{t_{i-1}} [B_{t_j} - B_{t_{j-1}}]^3 [B_{t_i} - B_{t_{i-1}}]\end{aligned}$$

but for those with the largest index corresponding to an odd power, we know it vanishes after taking expectation, for instance if we let $i = 2, j = 1$ below then

$$\begin{aligned}&\mathbb{E}\left[A_{t_{i-1}} A_{t_{j-1}}^3 [B_{t_i} - B_{t_{i-1}}]^3 [B_{t_j} - B_{t_{j-1}}]\right] = \mathbb{E}\left[\mathbb{E}\left[A_{t_{i-1}} A_{t_{j-1}}^3 [B_{t_i} - B_{t_{i-1}}]^3 [B_{t_j} - B_{t_{j-1}}] \middle| \mathcal{F}_{t_{j-1}}\right]\right] \\ =&\mathbb{E}\left[A_{t_{i-1}} A_{t_{j-1}}^3 [B_{t_j} - B_{t_{j-1}}] \mathbb{E}\left[[B_{t_i} - B_{t_{i-1}}]^3 \middle| \mathcal{F}_{t_{i-1}}\right]\right] = \mathbb{E}\left[A_{t_{i-1}} A_{t_{j-1}}^3 [B_{t_j} - B_{t_{i-1}}] \cdot 0\right] = 0\end{aligned}$$

with the exact same manipulation we can throw away all terms with odd power largest index.

Now we compute the leftover terms one by one:

$$\begin{aligned}\mathbb{E}\left[\sum_{i=1}^{2} A_{t_{i-1}}^4 [B_{t_i} - B_{t_{i-1}}]^4\right] &= \mathbb{E}\left[\sum_{i=1}^{2} A_{t_{i-1}}^4 \mathbb{E}\left[[B_{t_i} - B_{t_{i-1}}]^4 \middle| \mathcal{F}_{t_{i-1}}\right]\right] = 3\mathbb{E}\left[\sum_{i=1}^{2} (t_i - t_{i-1})^2 A_{t_{i-1}}^4\right] \\ &\leq 3C^4 \sum_{i=1}^{n} (t_i - t_{i-1})^2\end{aligned}$$

and similarly (again, $i = 2, j = 1$ below)

$$\begin{aligned}&\mathbb{E}\left[A_{t_{i-1}}^2 A_{t_{j-1}}^2 [B_{t_i} - B_{t_{i-1}}]^2 [B_{t_j} - B_{t_{j-1}}]^2\right] \\ \leq&C^4 \mathbb{E}\left[[B_{t_j} - B_{t_{j-1}}]^2 \mathbb{E}\left[[B_{t_i} - B_{t_{i-1}}]^2 \middle| \mathcal{F}_{t_i}\right]\right] = C^4 (t_i - t_{i-1}) \mathbb{E}\left[[B_{t_j} - B_{t_{j-1}}]^2\right] \\ =&C^4 (t_i - t_{i-1})(t_j - t_{j-1})\end{aligned}$$

And thus adding all together

$$\mathbb{E}\left[\left(\int_0^t A_s dB_s\right)^4\right] \leq 3C^4 t^2 \leq 4C^4 t^2$$

for this case.

Now assume we've finished for $t_{k-1} \le t < t_k$ and we want to show for $t_k \le t < t_{k+1}$, we just compute (again we throw away odd terms by IH, and exact same computation as above (and a similar one for second order))

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t A_s dB_s\right)^4\right] &\le \mathbb{E}\left[|Z_{t_n} + A_{t_n}[B_{t-t_n}]|^4\right] \\ \le& \mathbb{E}[Z_{t_n}^4] + \mathbb{E}[|A_{t_n}[B_{t-t_n}]|^4] + \mathbb{E}[Z_{t_n}^2 \cdot |A_{t_n}[B_{t_{n+1}-t_n}]|^2] \\ \le& 3C^4 t_n^2 + 3C^4(t-t_n)^2 + 6C^4 t_n(t-t_n) \le 3C^4t^2 \le 4C^4t^2\end{aligned}$$

which inductively proves our result.

(2):

Define the approximating simple process $A_s^{(n)}$ to A_s in the same manner as in class

$$A_s^{(n)} := n\int_{\frac{k-1}{n}t}^{\frac{k}{n}t} A_w dw, \qquad \frac{k}{n}t \le s \le \frac{k+1}{n}t$$

thus $A_s^{(n)}$ is simple and we can see that $A_s^{(n)} \to A_s$ pointwise.

Now, for convenience denote $A := \int_0^t A_s dB_s$ and $B_n := \int_0^t A_s^{(n)} dB_s$, then we have

And we have

$$\mathbb{E}\left[A^4 - B_n^4\right] = \mathbb{E}\left[\left(A^2 + B_n^2\right)\left(A^2 - B_n^2\right)\right]$$

where $A_s^{(n)} \xrightarrow{L^2} A_s$. Moreover

$$A^2 + B_n^2 \le C^2 \int_0^t |dB_s| \le q^2(\omega)C^2$$

for some constant q dependent on ω and in the same manner we bound $A + B_n \le 2Cq(\omega)$.

Putting together we have

$$\lim_{n\to\infty}\left(A^2 + B_n^2\right)\left(A^2 - B_n^2\right) \le \lim_{n\to\infty} 2q^2C^2\left(A^2 - B_n^2\right) \to 0$$

so $\mathbb{E}\left[\lim_{n\to\infty}(A^4 - B_n^4)\right] = 0$ and by DCT we have

$$\lim_{n\to\infty}\mathbb{E}\left[(A^4 - B_n^4)\right] = \mathbb{E}\left[\lim_{n\to\infty}(A^4 - B_n^4)\right] = 0$$

which, together with part 1 we know

$$\mathbb{E}\left[\left(\int_0^t A_s dB_s\right)^4\right] \le 4C^4t^2.$$

$\square$

Exercise 3.

Proof.

Using the very same method as in part (1) of question 2 above, we have shown already that the terms with odd largest degree is 0, this gives us only one type 2-1 term left, i.e.:

$$
\begin{aligned}
\left(\int_0^t A_s dB_s\right)^3 &= \left(\sum_{i=1}^n A_{t_{i-1}}[B_{t_i} - B_{t_{i-1}}]\right)^3 \\
&= \sum_{i=1}^n A_{t_{i-1}}^3 [B_{t_i} - B_{t_{i-1}}]^3 + \sum_{i\neq j} A_{t_{i-1}}^2 A_{t_{j-1}} [B_{t_i} - B_{t_{i-1}}]^2 [B_{t_j} - B_{t_{j-1}}] \\
&\quad + \sum_{i,j,k \text{ distinct}} A_{t_{i-1}} A_{t_{j-1}} A_{t_{k-1}} [B_{t_i} - B_{t_{i-1}}][B_{t_j} - B_{t_{j-1}}][B_{t_k} - B_{t_{k-1}}] \\
&= \sum_{i>j} A_{t_{i-1}}^2 A_{t_{j-1}} [B_{t_i} - B_{t_{i-1}}]^2 [B_{t_j} - B_{t_{j-1}}]
\end{aligned}
$$

and taking expectation we have

$$
\mathbb{E}\left[\left(\int_0^t A_s dB_s\right)^3\right] = \sum_{i>j} (t_i - t_{i-1}) \mathbb{E}\left[A_{t_{j-1}} \mathbb{E}\left[A_{t_{i-1}}^2 [B_{t_j} - B_{t_{j-1}}] \middle| \mathcal{F}_{t_j - 1}\right]\right]
$$

where we just define $A_s := \mathbb{1}_{B_{t_j} - B_{t_{j-1}} > 0}$ for $s \in [t_{j-1}, t_j)$, which in other words we have

$$
\mathbb{E}\left[A_{t_{i-1}}^2 [B_{t_j} - B_{t_{j-1}}] \middle| \mathcal{F}_{t_j - 1}\right] = \varepsilon > 0
$$

and since $\mathbb{E}[\mathbb{1}_{B_{t_j} - B_{t_{j-1}} > 0}] = \dfrac{1}{2}$ we can compute

$$
\mathbb{E}\left[\left(\int_0^t A_s dB_s\right)^3\right] \geq \sum_{i>j} (t_i - t_{i-1}) \frac{1}{2} \varepsilon \geq 0.
$$

For the degree one claim it is correct. We know $B_t \overset{d}{\sim} -B_t$, and thus

$$
\mathbb{P}\left(\sum_{i=1}^n A_{t_{i-1}}[B_{t_i} - B_{t_{i-1}}] > 0\right) = \mathbb{P}\left(\sum_{i=1}^n A_{t_{i-1}}[-B_{t_i} + B_{t_{i-1}}] > 0\right) = \mathbb{P}\left(\sum_{i=1}^n A_{t_{i-1}}[B_{t_i} - B_{t_{i-1}}] < 0\right)
$$

and since $\mathbb{P}\left(\sum_{i=1}^n A_{t_{i-1}}[B_{t_i} - B_{t_{i-1}}] = 0\right) = 0$ we know

$$
\mathbb{P}\left(\sum_{i=1}^n A_{t_{i-1}}[B_{t_i} - B_{t_{i-1}}] > 0\right) = \frac{1}{2}.
$$

□

Exercise 4.

Proof.

(1):

By Markov's inequality we have for each $s \in \mathcal{D}_n$

$$\mathbb{P}\left\{|Z_{s+2^{-n}} - Z_s|^\alpha \geq 2^{-n\frac{\beta}{2}}\right\} \leq \frac{\mathbb{E}[|Z_{s+2^{-n}} - Z_s|^\alpha]}{2^{-n\frac{\beta}{2}}} \leq c2^{-n(1+\beta)}2^{n\frac{\beta}{2}} = c2^{-\frac{n\beta}{2}} \cdot 2^{-n}$$

and thus if we only want more than 1 s that satisfies it, we can bound it by union bound:

$$\mathbb{P}\left\{\exists s \in \mathcal{D}_n : |Z_{s+2^{-n}} - Z_s|^\alpha \geq 2^{-n\frac{\beta}{2}}\right\} \leq 2^n c 2^{-\frac{n\beta}{2}} \cdot 2^{-n} = c2^{-\frac{n\beta}{2}}.$$

(2):

Denote event

$$E_n := \mathbb{P}\left\{\exists s \in \mathcal{D}_n : |Z_{s+2^{-n}} - Z_s| \geq 2^{-n\varepsilon}\right\}$$

then

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \sum_{n=1}^{\infty} c2^{-\frac{n\beta}{2}} < \infty$$

and thus by Borel Cantelli with probability 1 the event is not infinitely recurring, hence w.p.1 for sufficiently large n and all $s \in \mathcal{D}_n$ we have

$$|Z_{s+2^{-n}} - Z_s| \leq 2^{-n\varepsilon}.$$

(3):

From last part for each ω there is a particular N such that for $n > N$ the bound holds. And because there is a unique binary representation of real number there is a $I \subset \mathbb{Z}^*$ such that (WLOG $t > s$):

$$t - s = \sum_{i \in I \subset \mathbb{Z}^*} \frac{1}{2^i}$$

and thus there exists an increasing sequence $\{t_n\}$ such that $s = t_0, t = \lim_{n\to\infty} t_n$ and $t_n - t_{n-1} = \dfrac{1}{2^{i_n}}$ where i_n is the nth smallest integer in I (if only finite step we make all remaining $t_m = t$). This is defined so we can do triangle inequality

$$|Z_t - Z_s| \leq \sum_{n=1}^{\infty} |Z_{t_n} - Z_{t_{n-1}}|$$

For $U := \max\left\{I \bigcap \{1, 2, \dots, N\}\right\}$ there is a unique constant $C' := C'(\omega)$ that bounds $\sum_{n=1}^{U} |Z_{t_n} - Z_{t_{n-1}}|$. For the remaining terms we can use part (2) and get bound

$$\sum_{n=U}^{\infty} |Z_{t_n} - Z_{t_{n-1}}| \leq \sum_{n=U}^{\infty} 2^{-n\varepsilon} < C''|t - s|^\varepsilon$$

and putting together we have

$$|Z_t - Z_s| \leq C' + C''|t-s|^{\varepsilon} \leq C|t-s|^{\varepsilon}$$

since $|t-s| > 2^{-N}$ on the sum of first finite terms.

Extension to a continuous function is just because we have a continuous control on a dense set, and we are on $\mathbb{R}$. $\square$

Exercise 5.

Proof.

(1):

The fact that $\hat{Z}_t$ is adapted follows from definition.

$\hat{Z}_t$ is L^1 because for any $0 \le t < 1$ we know f is compact therefore bounded on $[0, t \wedge T]$ and hence if we denote $E := \{P \le 1\}$ we have

$$\int_\Omega \hat{Z}_t d\mathbb{P} = \int_{\{\Omega - E\}} \int_0^{T\wedge t} f(s) dB_s d\mathbb{P} \in L^1.$$

Martingale property:

For any fixed $t < 1$, we know again that f is bounded on $[0, t]$, and hence we know from class that Z_s for $s \in [0, t]$ is a Martingale. But this argument extends to $s < 1$ since for each s we can pick $t \in (s, 1)$ and run the argument. Now, for $s < t$ and $\forall A \in \mathcal{F}_s$

$$\begin{aligned} \int_A Z_{t\wedge T} d\mathbb{P} &= \int_{A\cap\{T\le s\}} Z_T d\mathbb{P} + \int_{A\cap\{s<T<t\}} Z_t d\mathbb{P} + \int_{A\cap\{T\ge t\}} Z_t d\mathbb{P} \\ &= \int_{A\cap\{T\le s\}} Z_T d\mathbb{P} + \int_{A\cap\{s<T<t\}} Z_s d\mathbb{P} + \int_{A\cap\{T\ge t\}} Z_s d\mathbb{P} = \int_A Z_{s\wedge T} d\mathbb{P} \end{aligned}$$

and hence it $\hat{Z}_t$ is a Martingale.

(2):

Since $f \in L^1[0, 1)$ we have L^1. Adapted is from definition.

We know the improper integral

$$\int_0^1 f^2(s) ds < \infty$$

just by p-test, in particular this means we can exchange the limit and expectation on the simple process approximation, i.e. if we denote $\hat{Z}_t^{(n)}$ as a simple process approximation to $\hat{Z}_t$, then really

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^{t\wedge T} (f(s)^{(n)} - f(s))^2 ds\right] = 0$$

by above improper integral estimation. Also, by $L^2[0, 1)$ of f we know the variance rule holds so we can get the square out and get convergence of $\hat{Z}_t^{(n)}$ by Cauchy sequence in L^2, which cannot be other thing other than $\hat{Z}_t$. So

$$\lim_{n\to\infty} \hat{Z}_t^{(n)} = \hat{Z}_t$$

in L^2. This gives us Martingale property because we know (from the same argument for L^2 in class) $\hat{Z}_t^{(n)}$ is a Martingale, but note that they are also L^1 so we can exchange limit and

have
$$\mathbb{E}[\hat{Z}_t|\mathcal{F}_s] = \mathbb{E}[\lim_{n\to\infty} \hat{Z}_t^{(n)}|\mathcal{F}_t] \overset{DCT}{=} \lim_{n\to\infty} \hat{Z}_s^{(n)} = \hat{Z}_t.$$

(3): (idea by Liam)

The stopping time we pick is
$$T := \min\{t : \hat{Z}_t = 1\}$$
and it's easy to see then $\hat{Z}_t$ is not a martingale (expectation not 0) but hard to see that $\mathbb{P}(T < 1) = 1$. We show it now.

We want to make the vertical fluctuation horizontal, to do so we define $t_k = \sum_{j=0}^{k} \frac{1}{n-j}$ for integer $k \le n-1$. Then we define a left Riemann sum approximation to f, i.e.
$$A_s^{(n)} = \left(1 - \frac{j}{n}\right)^{-\frac{1}{2}} \qquad \frac{j}{n} \le s < \frac{j+1}{n}$$
which we can still conclude convergence because f is deterministic.

Then we define
$$Y(t_k) = Z^{(n)}\left(\frac{k}{n}\right)$$
for grid points (which by construction goes like $O(\log k)$) and for $0 < s < t_{k+1} - t_k$ we define
$$Y(t_k + s) = Z^{(n)}\left(\frac{k}{n} + s\frac{n-k}{n}\right)$$
with idea being for how long we've stretched B_s in the integral, we reduce the stretch to a horizontal one. So by the correspondence between Y and Z, if Y touches 1 with probability 1 before $\log n$ as $n \to \infty$, then the result will hold. So we only need to show Y is a Brownian motion.

For convenience, we denote the map $F : \mathbb{R} \to \mathbb{R}$ such that $F(x) = y$ is the element such that $Y(x) = Z^{(n)}(y)$, which is well-defined and 1-1 since that's how we've defined Y.

Lemma 0.1. *Y is a Brownian motion.*

Proof. $Y_0 = 0$ is obvious. The fact that it's continuous follows from the continuity of Z_n.

To show independent increment, note that $Y(t) - Y(s) \in \sigma\left(B_w : F(s) \le w \le F(t)\right)$ and $Y(s) \in \mathcal{F}_{F(s)}$ thus they are independent.

To show Gaussian increment, we compute (say $t_j \le s < t < t_i$)
$$Y(t) - Y(s) = \left(1 - \frac{j}{n}\right)^{-\frac{1}{2}} (B_{F(t_{j+1})} - B_{F(s)}) + \sum_{k=j+1}^{i-1} \left(1 - \frac{k}{n}\right)^{-\frac{1}{2}} (B_{F(t_{k+1})} - B_{F(t_k)})$$
$$+ \left(1 - \frac{i-1}{n}\right)^{-\frac{1}{2}} (B_{F(t)} - B_{F(t_{i-1})})$$

and we note that for $t_i \leq a < b < t_{i+1}$ we can compute

$$\left(1-\frac{i}{n}\right)^{-\frac{1}{2}}[B_{F(b)}-B_{F(a)}] = B\left(\frac{n}{n-i}\frac{b(n-i)+i}{n}\right)-B\left(\frac{n}{n-i}\frac{a(n-i)+i}{n}\right) = B(b)-B(a) \sim N(0,b-a)$$

and apply this repeatedly we get (by independent increment)

$$Y(t)-Y(s) \sim N(0,t_{j+1}-s)+\sum_{k=j+1}^{i-1} N(0,t_{i+1}-t_i)+N(0,t-t_{i-1}) \sim N(0,t-s).$$

So indeed Y is a Brownian motion. □

Thus, for each n there's a corresponding Y^n that is defined like above and is a Brownian motion. Thus

$$\mathbb{P}(T<1) = \mathbb{P}\left(\max_{0\leq s\leq \log n+O(1)} B_s \geq 1\right)$$

which goes to 1 as $n \to \infty$ since the stopping time $\tau := \min\{t : B_t = 1\}$ has

$$\mathbb{P}(\tau < \infty) = 1.$$

□