APPLIED FUNCTIONAL ANALYSIS HOMEWORK 7

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Discussed with classmates.

Exercise 1. *(8.12) in book*

Proof.

By a priori estimate (proposition 5.30) we know that if a bounded operator has norm expression also bounded from below, then it has closed range and trivial kernel. Thus, we already know offhand that the operator A is invertible on it's range Ran A, i.e. the equation $Ax = y$ has a unique solution for $y \in \text{Ran } A$.

So it suffices us to use the self adjoint condition to prove Ran $A = \mathcal{H}$, the whole space.

But we know $\mathcal{H} = \overline{\text{Ran } A} \oplus \ker A^* = \overline{\text{Ran } A} \oplus \{0\}$ since A is self adjoint, thus above implies $\overline{\text{Ran } A} = \mathcal{H}$, and using the fact that it's closed we're done.

$\square$

Exercise 2. *(8.13) in book.*

Proof.

By definition, $(u_\alpha \otimes u_\alpha)(x) = \langle u_\alpha, x\rangle u_\alpha$. Moreover, we know that the set is orthogonal.

$\underline{\Rightarrow:}$

If u_α are orthonormal basis, then by definition 6.27 and theorem 6.26 we have that

$$x = \sum_{\alpha\in\mathcal{A}} \langle u_\alpha, x\rangle u_\alpha = \sum_{\alpha\in\mathcal{A}} (u_\alpha \otimes u_\alpha)(x)$$

for any x, and hence

$$\sum_{\alpha\in\mathcal{A}} u_\alpha \otimes u_\alpha = I.$$

$\underline{\Leftarrow:}$

Again, if we know that $\sum_{\alpha\in\mathcal{A}} u_\alpha \otimes u_\alpha = I$ holds, then by theorem 6.26 again we know that u_α is a complete orthonormal set, hence an orthonormal basis.

$\square$

Exercise 3. *(8.14) in book. (Discussed with Tim)*

Proof.

By sesqui-linearity we get

$$\langle x, Ay\rangle - \langle x, By\rangle = 0 \Rightarrow \langle x, (A-B)y\rangle = 0$$

which is arbitrary in x so $(A-B)y = 0$. But y is also arbitrary, so $A - B = 0$, hence $A = B$.

Now, if (after shifting terms and combining using sesqui linearity)

$$\langle x, (A-B)x\rangle = 0$$

which means if we compute directly the inner product as in lemma 8.26, we get

$$\begin{aligned}
&\langle y, (A-B)x\rangle \\
=&\frac{1}{4}\big(\langle x+y, (A-B)(x+y)\rangle - \langle x-y, (A-B)(x-y)\rangle \\
&-i\langle x+iy, (A-B)(x+iy)\rangle + i\langle x-iy, (A-B)(x-iy)\rangle\big)
\end{aligned}$$

where since all terms of $\langle x, (A-B)x\rangle$ and for y are cancelled we get

$$\langle y, (A-B)x\rangle = 0$$

which by above means $A = B$.

For real space we just take $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ then we note

$$\langle x, Ax\rangle = \langle x, Bx\rangle = 0$$

yet $A \neq B$.

$\square$

Exercise 4. *(8.17) in book.*

Proof.

Let $\langle \cdot, \cdot \rangle_D$ denote the dual product (if space is Hilbert it's just inner product, but no such assumption here). Then we have

$$||x_n - x||_X \leq \delta \Rightarrow \left| \langle x_n, y \rangle_D - \langle x, y \rangle_D \right| \leq ||x_n - x||_X \cdot ||y||_{X^*} \leq c\delta$$

for any $y \in X^*$. Thus strong convergence implies weak convergence.

Now we show that in a finite dimensional space weak convergence implies strong convergence. But for that space we can just find finite orthonormal basis $e_1, \dots, e_n$, and every element in the sequence (thus in the space) can be written as

$$x_k = \sum_{i=1}^{n} a_i^k e_i.$$

Moreover, we denote

$$x = \sum_{i=1}^{n} a_i e_i.$$

Thus, if for all $y \in \mathcal{H}$ we have $\langle x_n, y \rangle \to \langle x, y \rangle$, then in particular taking $y = e_i$ for all $1 \leq i \leq n$ we have

$$a_i^k \to a_i$$

as $k \to \infty$. Thus, finding N such that the difference in each dimension is less than δ we get

$$||x_n - x|| \leq n \cdot \delta \leq \varepsilon$$

if for every ε we pick $\delta = \varepsilon / n$, since n is finite.

So we have weak convergence implies strong convergence in finite dimension Hilbert spaces. The other direction follows from the general statement in the beginning.

□

Exercise 5. *(8.18) in book.*

Proof.

Theorem in class claims that: Let $\{e_\alpha\}$ be a basis of $\mathcal{H}$ (not necessarily orthogonal), then we have that

$$x_n \rightharpoonup x \iff \begin{cases} ||x_n|| \leq M \\ \langle e_\alpha, x_n \rangle \to \langle e_\alpha, x \rangle, \forall \alpha \in I. \end{cases}$$

Thus, since the sequence of orthonormal vectors is bounded (has norm 1), we only need to check for any basis.

We define the basis generated by $\{u_n\}$ by letting $\mathcal{H} = \mathcal{M} \oplus [U]$, and if $\mathcal{M}$ is trivial we use the complete basis $\{u_n\}$; If $\mathcal{M}$ is non trivial we find an orthonormal basis of $\mathcal{M}$ (since it's still Hilbert), then we concatenate all the new basis to $\{u_n\}$.

Using this new basis, we compute

$$\langle u_\alpha, u_n \rangle \to 0 = \langle u_\alpha, 0 \rangle$$

since we'd go past the counting ordinal n eventually, then the rest is 0 due to orthogonality.

$\square$

Exercise 6. *(8.20) in book.*

Proof.

First, $\inf f(x) > -\infty$ since ϕ is a bounded function and hence
$$f(x) = \frac{1}{2}||x||^2 - \phi(x) \geq \frac{1}{2}||x||^2 - C||x||$$
where the sign is due to negative sign in front of ϕ, and the quadratic equation attains it's minimum.

We now show that the function f is strictly convex, which will imply that if the infimum is attained, it is attained at a unique point (otherwise the line segment between the 2 infimum points contradicts strict convexity).

To see that it's strictly convex, we first note that
$$-\phi\left(\theta x + (1-\theta)y\right) = -\theta\phi(x) - (1-\theta)\phi(y)$$
since it's linear, so it's convex. So we only have to show that $\frac{1}{2}||x||^2$ is strictly convex. But this is because it is the combination of a strictly convex function $h = x^2$ that is increasing on the domain $[0, \infty)$ and a convex function $g = ||x||$, reason:
$$g\left(\theta x + (1-\theta)y\right) = ||\theta x + (1-\theta)y|| \leq ||\theta x|| + ||(1-\theta)y|| \leq \theta||x|| + (1-\theta)||y||$$
so $f = h(g(x)) - \phi(x)$ is strictly convex since $h(g(x))$ is and $-\phi(x)$ is convex.

Now we show that the infimum is attained. Again we use the bound C in the definition of bounded function to get that for large enough R, for $\forall ||x|| \geq R$ we have
$$f(x) \geq \frac{1}{2}||x||^2 - C||x|| \geq \frac{1}{2}R^2 - CR > L$$
for some large L that is the minimum value on the circle of radius R. Hence if a sequence of $f(x_n)$ converges to the infimum, then for all large enough $n > N_1$ $||x_n|| \leq R$.

But then due to Banach Alaoglu we know that any bounded ball in a Hilbert space is weakly compact, and since f is a functional, there exists subsequence of x_n as defined above such that $x_{\phi(n)} \rightharpoonup x$, but we know that $f(x_{\phi(n)}) \to \inf f(x)$ and thus the limit can only be the infimum point. Thus such a point exists and we are done.

$\square$