PDE HOMEWORK 3

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Discussed with classmates.

Exercise 1.

Proof.

For an arbitrary $v \in C_c^\infty(U)$, we define

$$i(\tau) = I[u + \tau v] = \int_U \sqrt{1 + |\nabla u|^2 + 2\tau \nabla u \cdot \nabla v + \tau^2 |\nabla v|^2} dx$$

and note that we can exchange integral and differentiation since if we call the inside function f we have

- f is integrable because it's positive;
- f' exists and is continuous by computation;
- $\int f'$ exists and is continuous by computation.

So we can get

$$i'(\tau) = \int_U \frac{2\tau |\nabla v|^2 + 2\nabla u \nabla v}{2\sqrt{1 + |\nabla u|^2 + 2\tau \nabla u \cdot \nabla v + \tau^2 |\nabla v|^2}} dx$$

which we take $\tau = 0$ to get by the minimality of u

$$0 = i'(0) = \int_U \frac{\nabla u \nabla v}{\sqrt{1 + |\nabla u|^2}} dx \overset{IBP}{=} 0 - \int_U v \cdot \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) dx$$

where the boundary term vanishes because v vanishes on the boundary. Thus since v is arbitrary we get that

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0.$$

□

Exercise 2.

Proof.

Just do Fourier transform we get

$$\begin{cases} (\partial_t + (|\xi|^2 + c))\hat{u} = \hat{f} & \xi \in \mathbb{R}^n, t > 0 \\ \hat{u} = \hat{g} & t = 0 \end{cases}$$

and solving ODE we get

$$\left[e^{t(|\xi|^2+c)}\hat{u}\right]' = e^{t(|\xi|^2+c)}\hat{f}$$

which implies

$$e^{t(|\xi|^2+c)}\hat{u} = \hat{g}(\xi) + \int_0^t e^{s(|\xi|^2+c)}\hat{f}(\xi, s)ds$$

i.e.

$$\hat{u} = e^{-t(|\xi|^2+c)}\hat{g}(\xi) + \int_0^t e^{-(t-s)(|\xi|^2+c)}\hat{f}(\xi, s)ds$$

and inverse transform to get

$$u = \int_{\mathbb{R}^n} e^{-tc}\Phi(t, x-y)g(y)dy + \int_0^t \int_{\mathbb{R}^n} e^{sc-tc}\Phi(x-y, t-s)f(y, s)dyds$$

where

$$\Phi(x, t) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{|x|^2}{4t}}$$

and the only thing left is to verify this formula.

(Since not asked to prove so just roughly the idea:)

For $t > 0$ apply $(\partial_t - \Delta)$ to Φ by DCT we see that the g term cancels out and the f term becomes exactly what we need. So we have

$$(\partial_t - \Delta + c)u = f, \quad t > 0.$$

For $t \to 0$ we note that the integral $\int_{\mathbb{R}^n} e^{sc-tc}\Phi(x-y, t-s)f(y, s)dy \leq ||f||$ because Φ is an approximate identity, so the latter integral goes to 0 as $\int_0^t 1ds \to 0$. The first goes to what we want because e^{sc-tc} vanishes and the rest is the same as proven in class.

Thus u is indeed a solution of the system.

□

Exercise 3.

Proof.

We define

$$v(x,t) = u(x,t) - g(t).$$

First we extend $v(-x,t) = -v(x,t)$ for $x > 0$, which is still as smooth after extension because $u = g$ on $x = 0$. Taking derivatives we've transformed our problem into:

$$\begin{cases} (\partial_t - \partial_{xx})v = -\partial_t g & \mathbb{R}_+ \times (0,\infty) \\ (\partial_t - \partial_{xx})v = \partial_t g & \mathbb{R}_- \times (0,\infty) \\ v(0,t) = 0 \end{cases}$$

and we can just denote

$$f(x,t) = -\partial_t(g) + 2\mathbb{1}_{x<0}\partial_t g(t)$$

and get

$$(\partial_t + \xi^2)\hat{v} = \hat{f}$$

solving ODE yields (really the same as everything in last question, constant is 0 here)

$$\hat{v} = \int_0^t e^{-(t-s)\xi^2}\hat{f}(y,s)ds$$

where Duhamel's principle plus the convolution identity gives

$$\begin{aligned} v &= \int_0^t \int_{\mathbb{R}} \Phi(t-s,x-y)f(y,s)dyds \\ &= -\int_0^t \partial_s g(s)\int_{\mathbb{R}} \Phi(t-s,x-y)dyds + 2\int_0^t \partial_s g(s)\int_{-\infty}^0 \Phi(t-s,x-y)dyds \\ &= -g(t) + 2\int_0^t \partial_s g(s)\int_{-\infty}^0 \Phi(t-s,x-y)dyds \\ &= -g + 2\int_0^t g(s)\frac{\partial}{\partial s}\left(\int_0^\infty \Phi(t-s,x-y)dy\right)ds + g(s)\int_{-\infty}^0 \Phi(t-s,x-y)dy\Big|_{s=0}^t \\ &= -g + 2\int_0^t g(s)\frac{\partial}{\partial s}\left(\int_0^\infty \Phi(t-s,x-y)dy\right)ds \end{aligned}$$

since at $s = t$ the boundary exponential decay is way faster than the decay in denominator. The only toil left is to compute the derivative and we have

$$\frac{\partial}{\partial s}\left(\int_0^\infty \Phi(t-s,x-y)dy\right) = \frac{\partial}{\partial s}\int_0^\infty \frac{1}{\sqrt{4\pi(t-s)}}e^{-\frac{(x-y)^2}{4(t-s)}}dy$$

and we let $z = \dfrac{x-y}{\sqrt{4(t-s)}}$ to get the heat kernel and compute

$$
\begin{aligned}
&= -\frac{1}{\sqrt{\pi}}\frac{\partial}{\partial s}\int_{\frac{x}{\sqrt{4(t-s)}}}^{\infty} e^{-z^2}dz = -\frac{1}{\sqrt{\pi}}\frac{\partial \frac{x}{\sqrt{4(t-s)}}}{\partial s}\cdot\frac{\partial}{\partial \frac{x}{\sqrt{4(t-s)}}}\int_{\frac{x}{\sqrt{4(t-s)}}}^{\infty} e^{-z^2}dz \\
&= \frac{x}{4\sqrt{\pi}(t-s)^{3/2}}e^{-\frac{x^2}{4(t-s)}}
\end{aligned}
$$

And plugging back yields

$$v = -g + 2\int_0^t g(s)\frac{x}{4\sqrt{\pi}(t-s)^{3/2}}e^{-\frac{x^2}{4(t-s)}}ds$$

and using $v = u - g$ we get

$$u = \frac{x}{\sqrt{4\pi}}\int_0^t \frac{1}{(t-s)^{3/2}}e^{-\frac{x^2}{4(t-s)}}g(s)ds.$$

Verify this: The only problematic point is $(0,0)$ where there's a singularity. But we'll just focusing on taking derivatives away from that point, thus what we'll write down is well-defined. We directly compute

$$
\begin{aligned}
\partial_t u &= \frac{x}{\sqrt{4\pi}}\int_0^t \partial_t\left(\frac{1}{(t-s)^{3/2}}e^{-\frac{x^2}{4(t-s)}}\right)g(s)ds + \frac{x}{\sqrt{4\pi}}\frac{1}{(t-s)^{3/2}}e^{-\frac{x^2}{4(t-s)}}g(s)\Big|_{s=t} \\
&= \frac{x}{\sqrt{4\pi}}\int_0^t \partial_t\left(\frac{1}{(t-s)^{3/2}}e^{-\frac{x^2}{4(t-s)}}\right)g(s)ds + 0 = \frac{x}{\sqrt{4\pi}}\int_0^t\left(\frac{(-6t+x^2+6s)e^{-\frac{x^2}{4(t-s)}}}{4(t-s)^{7/2}}\right)g(s)ds
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta u &= 2\frac{1}{\sqrt{4\pi}}\int_0^t \partial_x\left(\frac{1}{(t-s)^{3/2}}e^{-\frac{x^2}{4(t-s)}}\right)g(s)ds + \frac{x}{\sqrt{4\pi}}\int_0^t \partial_x^2\left(\frac{1}{(t-s)^{3/2}}e^{-\frac{x^2}{4(t-s)}}\right)g(s)ds \\
&= \frac{2}{\sqrt{4\pi}}\int_0^t -\frac{xe^{-\frac{x^2}{4(t-s)}}}{2(t-s)^{5/2}}g(s)ds + \frac{x}{\sqrt{4\pi}}\int_0^t\left(\frac{(-2t+x^2+2s)e^{-\frac{x^2}{4(t-s)}}}{4(t-s)^{7/2}}\right)g(s)ds
\end{aligned}
$$

Thus brutal computation gives:

$$
\begin{aligned}
\partial_t - \Delta u &= \frac{x}{\sqrt{4\pi}}\int_0^t\left(\frac{(-4t+4s)e^{-\frac{x^2}{4(t-s)}}}{4(t-s)^{7/2}}\right)g(s)ds - \frac{1}{\sqrt{4\pi}}\int_0^t -\frac{xe^{-\frac{x^2}{4(t-s)}}}{(t-s)^{5/2}}g(s)ds \\
&= -\frac{1}{\sqrt{4\pi}}\int_0^t \frac{xe^{\frac{x^2}{4(t-s)}}}{(t-s)^{5/2}}g(s)ds + \frac{1}{\sqrt{4\pi}}\int_0^t \frac{xe^{\frac{x^2}{4(t-s)}}}{(t-s)^{5/2}}g(s)ds = 0.
\end{aligned}
$$

If $t = 0$ then since g is bounded around 0 we know the integrand is bounded and thus the integral goes to 0. So $u = 0$.

For $x \to 0$ we have

$$\lim_{x\to 0} u(x,t) \to \int_0^t \delta_{s-t} g(s) dt = g(t)$$

just by the definition of what is Φ inside.

So it indeed satisfies the conditions.

<u>Smoothness?</u>

Again, we don't need to worry about the singularity at $(0,0)$ so since Φ is smooth and bounded, and g is bounded for any given (x,t), so for any given point we can use DCT to pass the differential operator inside the integral. So u is infinitely differentiable away from the boundary.

$\square$

Exercise 4.

Proof.

Direct computation tells us

$$(\partial_t - \Delta)u_\lambda = \lambda^2 u_t(\lambda x, \lambda^2 t) - \lambda \cdot \lambda u_{xx}(\lambda x, \lambda^2 t) = \lambda(\partial_t - \Delta)u(\lambda x, \lambda^2 t) = 0.$$

Now for v since we can change order of differentiation as they are all defined, we have

$$\begin{aligned}(\partial_t - \Delta)v &= x\nabla u_t + 2u_t + 2tu_{tt} - 2\Delta u - x \cdot \nabla(\Delta u) - 2t\Delta u_t \\ &= x \cdot \nabla((\partial_t - \Delta)u) + 2(\partial_t - \Delta)u + 2t(\partial_t - \Delta)u_t = 0\end{aligned}$$

where everything is computed similarly so I just show detail of one of the linear algebra involved:

$$\Delta(x \cdot \nabla u) = \Delta\left(\sum_i x_i \partial_i u\right) = \sum_j 2\partial_j \partial_j u + \sum_i x_i \partial_i \sum_j \partial_j \partial_j u = 2\Delta u + x \cdot \nabla(\Delta u).$$

$\square$

Exercise 5.

Proof.

Usual trick has that

$$(i\partial_t + \Delta)u = 0 \quad \Rightarrow \quad (i\partial_t - |\xi|^2)\hat{u} = 0 \quad \Rightarrow \quad (\partial_t + i|\xi|^2)\hat{u} = 0$$

and by multiplying $e^{it|\xi|^2}$ and solving the ODE we get

$$e^{it|\xi|^2}\hat{u} = C$$

where plugging in $t = 0$ we have $C = \hat{g}$ and

$$\hat{u} = e^{-it|\xi|^2}\hat{g}.$$

Thus

$$u(x,t) = \int_{\mathbb{R}^n} \left(\mathcal{F}^{-1} e^{-it|\xi|^2}\right)(x-y,t)g(y)dy$$

and we compute the inverse of $e^{-it|\xi|^2}$:

$$\begin{aligned}
\mathcal{F}^{-1} e^{-it|\xi|^2}(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\xi}e^{-t|\xi|^2}d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\xi - t|\xi|^2}d\xi = \frac{1}{(2\pi)^n}\prod_{i=1}^n \int_{-\infty}^{\infty} e^{ix_i\xi_i - t\xi_i^2}d\xi_i \\
&= \frac{1}{(2\pi)^n}\prod_{i=1}^n \int_{-\infty}^{\infty} e^{-\left(\sqrt{it}\xi_i - \sqrt{it}\frac{x_i}{2t}\right)^2} e^{i\frac{x_i^2}{2t}}d\xi_i = \frac{1}{(2\pi)^n}\prod_{i=1}^n \frac{1}{\sqrt{it}}\int_{-\infty}^{\infty} e^{-z^2}e^{i\frac{x_i^2}{2t}}dz \\
&= \frac{1}{(2\pi)^n}\prod_{i=1}^n \frac{1}{\sqrt{it}} e^{i\frac{x_i^2}{4t}}\sqrt{\pi} = \frac{1}{(4\pi it)^{n/2}}e^{i\frac{|x|^2}{4t}}
\end{aligned}$$

and hence plugging $x - y$ and noting that $|x-y|^2 = |x|^2 - 2x\cdot y + |y|^2$ we get

$$\begin{aligned}
u(x,t) &= \int_{\mathbb{R}^n}\left(\frac{1}{(4\pi it)^{n/2}}e^{i\frac{|x-y|^2}{4t}}\right)g(y)dy \\
&= \frac{e^{i\frac{|x|^2}{4t}}}{(4\pi it)^{n/2}}\int_{\mathbb{R}^n} e^{-i\frac{x\cdot y}{2t}}e^{i\frac{|y|^2}{4t}}g(y)dy.
\end{aligned}$$

<u>If $|y|^2 g(y) \in L^1$ then it is the solution:</u>

Basically that condition just means the derivatives are defined. So we compute by chain rule and check that everything cancels:

$$u_t = -\frac{n}{2t}u - \frac{1}{(4\pi it)^{n/2}}\int_{\mathbb{R}^n}\frac{i|x-y|^2}{4t^2}e^{i\frac{|x-y|^2}{4t}}g(y)dy$$

and

$$\Delta u = -\frac{1}{(4\pi it)^{n/2}}\int_{\mathbb{R}^n}\frac{(|x|^2 - 2x\cdot y + |y|^2 - 2int)}{4t^2}e^{i\frac{|x-y|^2}{4t}}g(y)dy$$

where this is well defined because $|y|^2 g(y) \in L^1$. Their difference is

$$(i\partial_t - \Delta)u = \frac{1}{(4\pi i t)^{n/2}} \int_{\mathbb{R}^n} \left[-\frac{in}{2t} - \frac{|x-y|^2}{4t^2} + \frac{(|x|^2 - 2x\cdot y + |y|^2 - 2int)}{4t^2} \right] e^{i\frac{|x-y|^2}{4t}} g(y)dy = 0.$$

<u>If $g \in L^2 \cap L^1$, then L^2 energy is conserved:</u>

Using the convension of Fourier transform in class we have

$$||u||_2 = \frac{1}{(2\pi)^n} ||\hat{u}||_2 = \frac{1}{(2\pi)^n} ||e^{-it|\xi|^2} \hat{g}||_2$$

where as

$$||g||_2 = \frac{1}{(2\pi)^n} ||\hat{g}||_2$$

so it suffices us to prove that

$$||\hat{g}||_2 = ||e^{-it|\xi|^2} \hat{g}||_2$$

which is just by writing out

$$||e^{-it|\xi|^2} \hat{g}||_2 = \int_{\mathbb{R}^n} (e^{-it|\xi|^2} \hat{g}) \overline{(e^{-it|\xi|^2} \hat{g})} d\xi = \int_{\mathbb{R}^n} \hat{g} \overline{\hat{g}} d\xi = ||\hat{g}||_2.$$

$\square$