APPLIED DYNAMICAL SYSTEM HOMEWORK 2

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STAT 31410
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General ideas were discussed with many classmates in casual talks.

Exercise 1.

Solution:

If $\beta = 1$, then $x \sim e^t$ which is not finite blowup.

In other cases,

$$\frac{dx}{dt} = x^\beta \Rightarrow \int \frac{1}{x^\beta} dx = \int 1 dt \Rightarrow \frac{x^{1-\beta}}{1-\beta} = t + c$$

where by plugging in initial value we get $c = \dfrac{x_0^{1-\beta}}{1-\beta}$.

By a change of variables $\alpha = 1 - \beta$, we have $x = (\alpha t + x_0^\alpha)^{\frac{1}{\alpha}}$. So we see that if $\alpha > 0$, then we are fine.

If $\alpha < 0$, then $x = \dfrac{1}{(\alpha t + x_0^\alpha)^{\frac{-1}{\alpha}}}$, which means that when $(\alpha t + x_0^\alpha)^{\frac{-1}{\alpha}} = 0$ we have our blow up point.

By computation we get

$$\alpha t + x_0^\alpha = 0 \Rightarrow t = -\frac{x_0^\alpha}{\alpha}$$

In this case, as long as the right-hand side is real then there is a blow-up time. This means we are good when $x > 0$, and when $x < 0$ we are good if α is an integer. So the "finite time blow-up space" would be

$$\{(x_0, t) | x_0 > 0 \ \& \ \alpha \leq 0 \text{ or } x_0 < 0 \ \& \ \alpha \in \mathbb{Z}^-\}$$

where $\alpha = 1 - \beta$

Now, for $\beta = 2, x_0 = -1$, then $T = -\dfrac{1}{-1} = 1$ and

$$\tau = \int_0^t 1 + \frac{1}{(1-s)^2} ds = t + \frac{1}{1-s}\Big|_0^t = t + \frac{1}{1-t} - 1.$$

The above equation means that when $t \in (0, 1)$, τ is an increasing function in t and it range from 0 to ∞.

Since $y(\tau) = x(t(\tau))$,

$$\frac{dy}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = x^2 \cdot \left(1 - \frac{1}{(1-t)^2}\right)$$

where if we let $1 - t = z$, we have $\tau = \frac{1}{z} - z$, which yields

$$z = \frac{-t + \sqrt{\tau^2 + 4}}{2}.$$

In addition, $x = \frac{1}{1-t}$ here, so plugging back the solution is

$$\frac{dy}{d\tau} = \frac{1}{z^2} - \frac{1}{z^4}$$

for z defined above.

Exercise 2.

If $\beta \geq 1$, then the function f is differentiable in J and has its derivative $|-\beta x^{\beta}|$ bounded on an interval, so there is unique solution.

If $\beta < 1$, then there $f \to \infty$ around 0 so there are no Lipschitz conditions, then we might be able to find a family of solutions. So let's just try to solve it.

Note first that the derivative is even, so the function should be odd with respect to

$(0, x(0)) = (0, 0)$, so its odd.

Solving it only for the positive x we get:

$$\frac{dx}{dt} = -x^{\beta} \quad \Rightarrow \quad \int x^{\beta} = \int -dt \quad \Rightarrow \quad \frac{x^{1-\beta}}{1-\beta} = -t + C \quad \Rightarrow \quad x = [(1-\beta)(-t+C)]^{\frac{1}{1-\beta}}$$

where, at first glance we can only choose $C = 0$ due to initial values.

However, since when C is not zero there is really nothing contradicting the ODE function, we can do the change of variable $\tau_c = t - c$ where $dt = d\tau_c$.

Plugging in the solution of ODE above we get $x = \left[(1-\beta)(-\tau_c)\right]^{\frac{1}{1-\beta}}$ are all valid solutions (checked, it works).

Thus, the family of solution is (by plugging back)

$$x = [(1-\beta)(-t+c)]^{\frac{1}{1-\beta}}$$

Exercise 3.

In the proof of unique solution in class via the Banach Contraction theorem, we've used it to construct a contraction. Quoted below is from notes:

> Now note that x is the solution to the function $T : C^0(J, B_b(x_0)) \to C^0(J, B_b(x_0))$ where
> $$T(u) = x_0 + \int_0^t f(u(s))ds$$
> and the function space is complete. We will now find the contraction even when the Lipschitz constant k of f is not confined (but fixed).
> $$\begin{aligned} d(T(u_1), T(u_2)) &= \left| \int_0^t f(u_1(s))ds - \int_0^t f(u_2(s))ds \right| \\ &= \left| \int_0^t \left(f(u_1(s)) - f(u_2(s)) \right) ds \right| \\ &\le \left| \int_0^t \left(k|u_1(s) - u_2(s)| \right) ds \right| \\ &\le k * \max(t) d(u_1, u_2) = k \cdot a \cdot d(u_1, u_2) \end{aligned}$$
> So as long as $a < \frac{1}{k}$ we are done.

k appears at the third line to help us "peel off" f and gain access to the distance between the two functions u_1, u_2. There is a restriction on a (related to k) by the Banach Contraction method, and by Picard Iteration we don't need that constraint. But really the function of k is the same.

Exercise 4.

The system in question is $f^{\pm} = \begin{pmatrix} -1 \pm x_2 \\ x_2 \mp x_1 \end{pmatrix}$.

My first idea after seeing the question is to first solve the ODE and see its analytic behavior, then see how it behaves at the boundary.

Let's focus on the upper half and let the ODE be

$$\dot{x} = A^{+}x + b$$

where

$$A^{+} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

To solve it we move $A^{+}x$ to the left and multiply both sides with integrating factor e^{-tA}, as it is done in exercise 17 chapter 2 in the book. Since $e^{tA} = Pe^{\Lambda}P^{*}$ for the same unitary matrix P if we diagonalize A, they commute and the ODE becomes

$$\frac{d}{dt}(e^{-tA}x) = e^{-tA}(\dot{x} - A^{+}x) = e^{-tA}b \quad \Rightarrow \quad x = e^{tA}\int e^{-tA}b\,dt$$

halfway through this, I regretted even computing this with hand since the eigenvalues of A^{+} are complex and the eigenvectors are very tedious to compute. Anyway, I shall put some of my halfway computation from which we'll know that this route does not work (at least I cannot read anything useful from this expression):

$$x = e^{tA} \cdot \begin{pmatrix} -\lambda_1 \frac{\sqrt{3}}{6} i(1 + \sqrt{3}i)e^{\lambda_1 t} + -\lambda_2 \frac{\sqrt{3}}{6} i(1 - \sqrt{3}i)e^{\lambda_2 t} \\ \frac{\sqrt{3}}{3} i \left(\lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}\right) \end{pmatrix}.$$

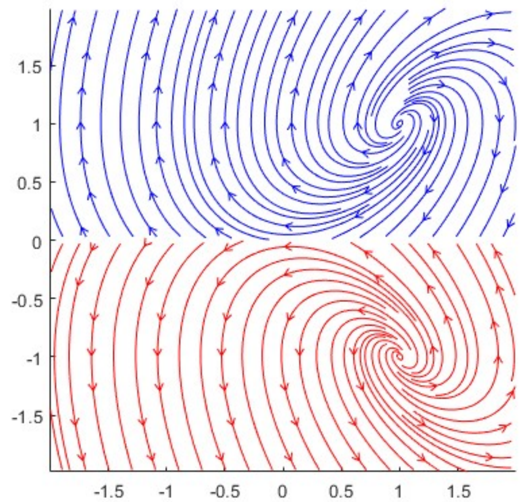
So I decided to read some basic features through the matrices themselves, so back to

$$f^{\pm} = \begin{pmatrix} -1 \pm x_2 \\ x_2 \mp x_1 \end{pmatrix} \quad , \quad \sigma(x) = x_2$$

$$A^{+} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \quad , \quad A^{-} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

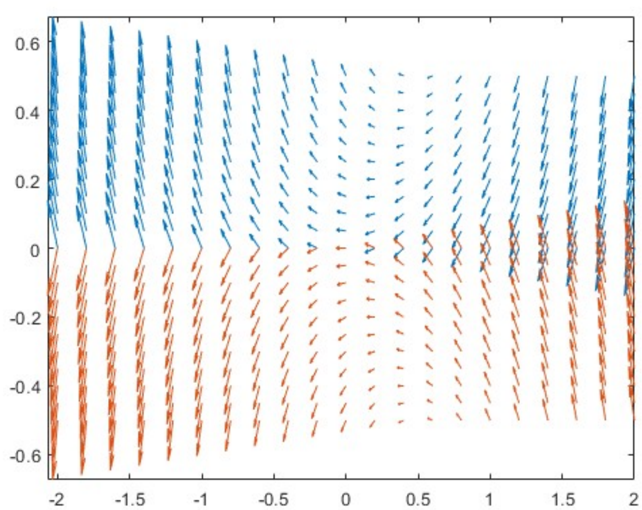
To apply the Uniqueness of the solution theorem we would want to find a Lipschitz condition, which requires at least continuous, but on the boundary (when $x_1 = 0$) f^{+} does not agree with f^{-} except at $x_2 = 0$. So I think that if we defined the boundary it cannot be continuous, so we'd have to find some other way to access the question other than the uniqueness theorem.

Indeed, when I plot the stream slice chart (what Matlab calls it) it yields the following result:



from this, I find that except at the origin, there are some flows "crashing" each other on the right half and going away on the left.

It might occur that it is the case that they are not crashing or leaving in an uncontrolled way if the velocity there is near 0. From the fact that this is a manually grafted and there were things that were cut off, we'd guess that it's not that nice. And indeed when I do the quiver plot it shows there are definitely crashes happening:



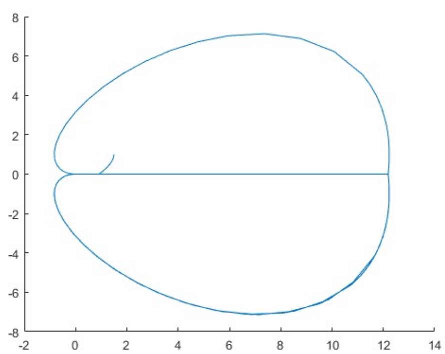
(Note that the topmost orange arrows do starts horizontally.)

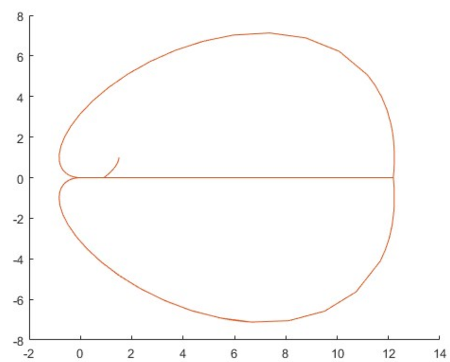
Well at least we can try just combine things together brutally and see how x behaves around the boundary. So I adopted the suggested convex way of combining them with parameter α:

$$\dot{x} = \alpha f^{+}(x) + (1-\alpha) f^{-}(x)$$

on the boundary, and tried to portrait the trajectory of x.

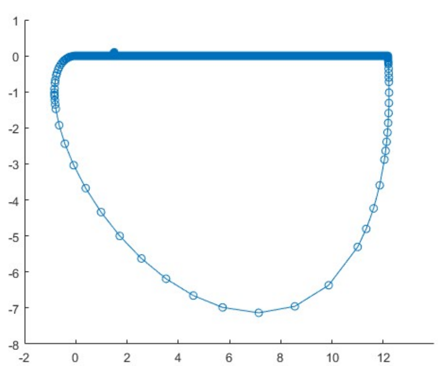
So I did, and I'd have to admit that the solution is not what I think at all. So I tried with staring point (1.5,0.5) and $\alpha = 0.5, 0.3$, and they happen to give the same result, which I would say looks pretty stable. I pick 1.5 such that the curve does not just go off into the sink.

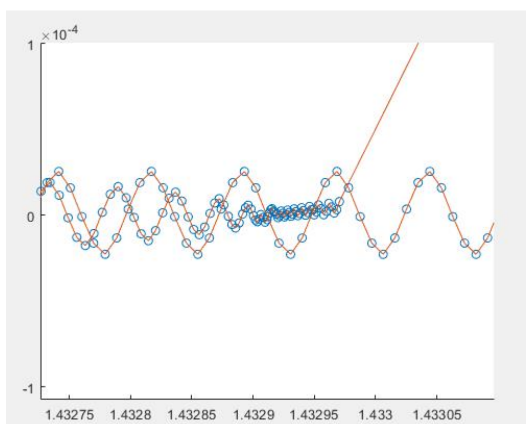




...except that it doesn't make sense.

Just as we defines it we know it shouldn't just go to the left on the boundary. And it took me a long time to realize that I should really zoom in (and each of these graphs take like 2-5 min, so really painful process). The result makes sense because when I've zoomed in of the left graph below (I plotted with circles) and drew another curve.



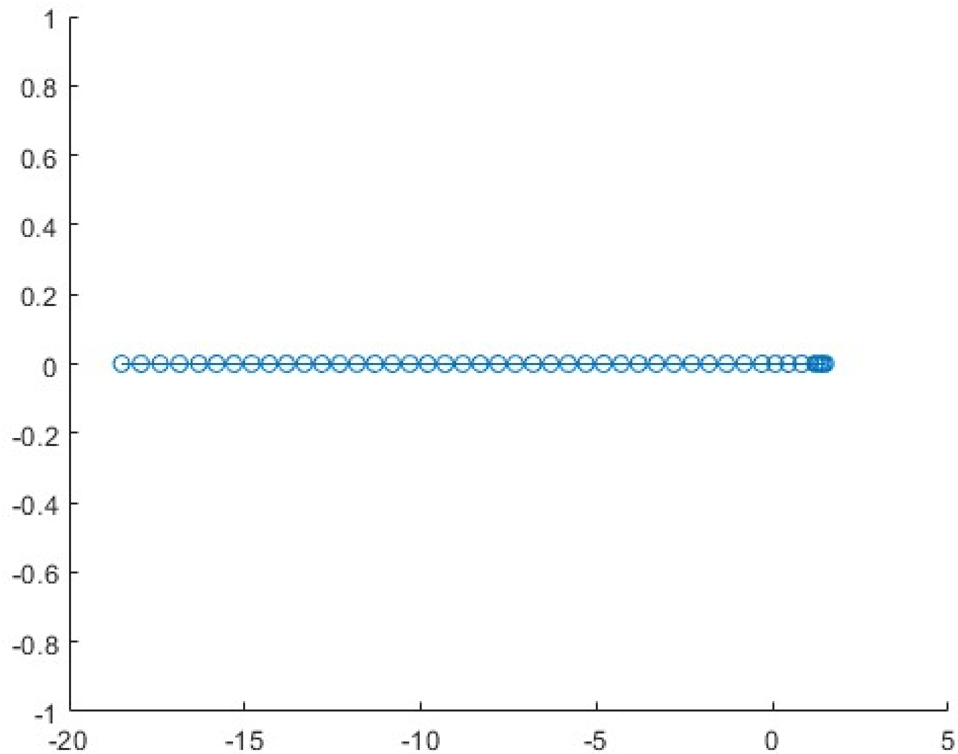


Apologize for the second graph but it was screenshotted when my Matlab crashed after my changing something to nonsense and matlab stop to know what it is doing. The left graph is when $\alpha = 0.3$, and the yellow curve is added with change $\alpha = 0.1$.

The graph shows that the two curves are identical. So what happens is that the step size is not small enough. In fact I think it will never be small enough to really touch 0 numerically.

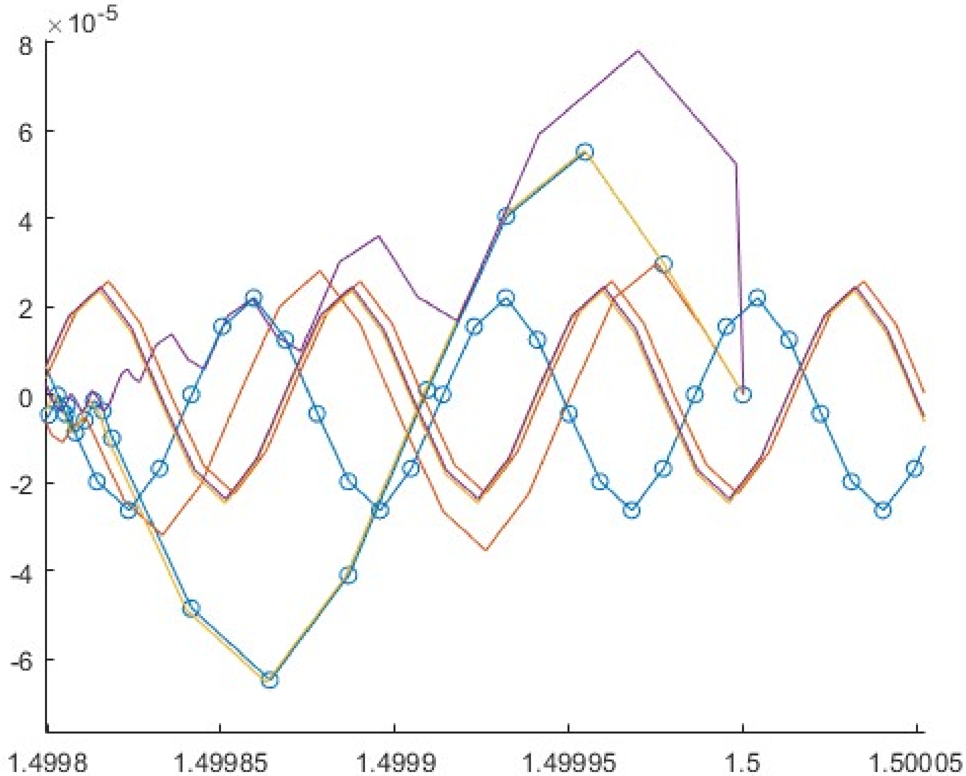
But if it does not touch 0, really the ODE system is like as if the boundary is not defined (since not used).

Now the reasonable thing to do is to start from the boundary (so at least there's some difference). The first thing to do is try it when $\alpha = 0.5$ with starting point $(1.5, 0)$, so really the x_2 velocity should vanish, and indeed:



It just stays on the boundary. (see next page)

After some tryouts (I've decided not to use too many steps, but to use only from time 0 to 20, so matlab will graph each of my curves within 2 to 3 minutes), I plotted the following graph with starting point $(1.5, 0)$.

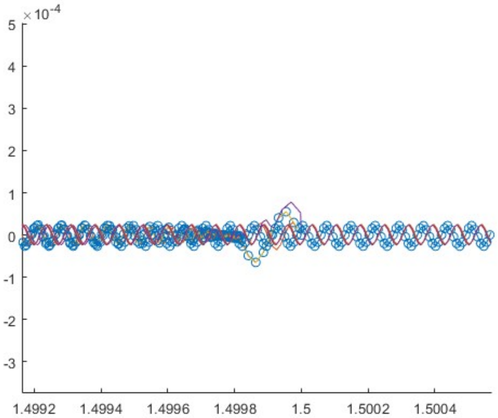


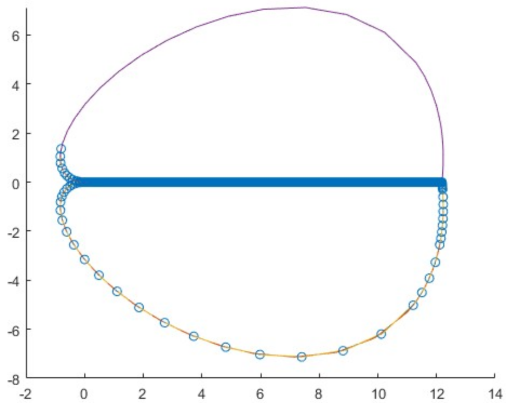
So I'll explain. There are in total 4 curves, and they all starts from the point (1.5,0). They were colored blue, orange, yellow, and purple. (and no green! that's a mixure of yellow and blue.)

- The blue curve is when $\alpha = 0.4$. I drew this first and see what'll happen next.
- The orange curve is when $\alpha = 0.41$. I want to see whether a tiny change will change much to the system. I didn't expect it to change that much, but then I realized we are at 0.00001 level, so it makes sense that there's some change locally.
- The yellow curve is when $\alpha = 0.401$. This agrees with my guess that the change is continuous.
- The purple curve is when I finally realize that the boundary DOES NOT MATTER in this numerical setting. I manually just let the boundary have derivative $(1, 2)$ and see what happens. Indeed after initial fleeing it soon goes back to roughly the same, in a larger scale.

(next page)

Here's the zoomed out version of the graph:

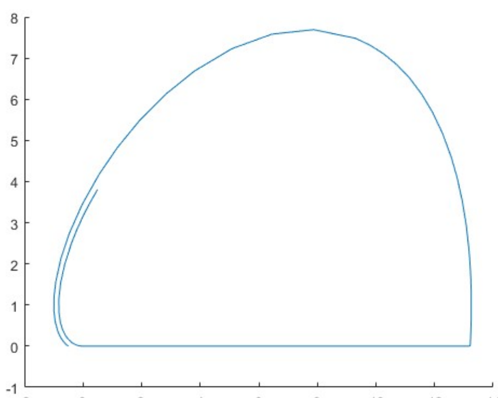


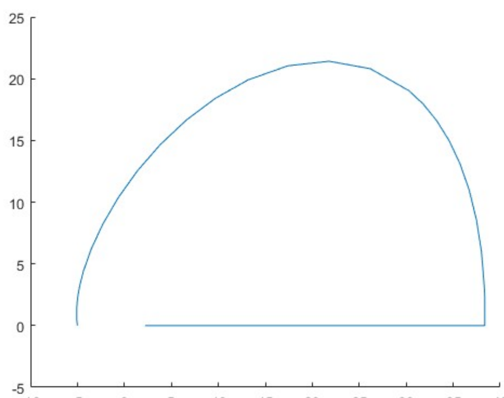


What happens is that when you zoom out a little bit, it will appear periodic around the boundary, but really there's a tiny change to the amplitude each time we go to the left since it's related to x_1. But at this scale no change is detectable.

Zooming out still more, we get the full picture. It might occur that the purple is different after all since it goes through the upper half of the graph. Not so. The seemingly difference is because I used time [0 40] here, and if I do a lot more, they will both go up and down, (see the little tail of blue circles on the left).

I've also tried what happens outside the "hoof" shape, and it is not that interesting. Turns out that it will go through a larger cycle just to get back. It is illustrated below, the left graph is when I start at (-0.5,0), the right side starts at (-5,0):





It really just goes back to the routine one. For the right one it takes too long to plot a finer graph where we can see what happens.

Conclusion of this system:

I think I've finally come to a state where I know this system well, and I'll tell the story below.

Numerically:

The main problem about numerical solving this is that you'd never (probability 0) really touch the boundary again, making the boundary not important at all. What happens is that you start oscillating once you are close to $x_2 = 0$, but since you are always moving left wards you will eventually reach $(-\varepsilon, \delta)$. This negative ε starts to function and, deciding on whether you land above or below the boundary the step you cross the vertical axis (sign of δ), you either go up the hoof or down. Then you follow the original system's flow until you get close to the boundary again.

No matter where you start, due to the two original system's flow chart, you will eventually go near $(\beta, 0)$ for $\beta > 0$ large or small. Then you will march left towards 0 until you've gone in the cycle we see in the largest scale picture.

The most interesting thing is that what starts off different eventually look the same on a larger scale.

Analytically:

The flow does touch the boundary. Deciding on the value of α you will go either up or down. But whichever way you go you are immediately forced to go back. Only the constant velocity leftward is unchanged, so you will arrive the same diverging point a little bit left of $(0, 0)$.

Some difference it makes is that

- if $\alpha = 0.5$ you'd never leave the boundary.
- if $\alpha < 0.5$ then the velocity is towards up and you will always fall in the upper loop.
- if $\alpha > 0.5$ then the velocity is downwards and you will always fall in the lower loop.
- And it might (rarely) happen that you touch the equilibrium $(0, 0)$ and stays there.

Back to the question of whether there's unique solution, I think that there's not since you will, since you have infinite time, it seems like it's possible you'd touch that point again. I cannot tell really since the size of the plane is also infinite. Maybe it will because since we'll touch the boundary, there are necessary intersections of flows, but that's even more confusing to me...

Another thing is that it is definitely a stable system from the graph. I believe it is Lyapnov stable since it will converge to the loop, whereever you start in the plane, except the two sinks and the equilibrium.

Maybe a better system?

Since we're looking for unique solutions, I tried to use the uniqueness theorem and thus need a Lipschitz f on the boundary. One way is to change the boundary, but this won't work since the only point where f^+ and f^- agrees is the origin.

I've also tried to change the ODE system to this: $f^{\pm} = \begin{pmatrix} -1 \pm x_2 \\ x_1 \mp x_2 \end{pmatrix}$, so on the boundary $\sigma(x) = x_2 = 0$, we really have the whole system continuous and Lipschitz. This is guaranteed a unique solution, but it really is another problem. (graph below so we can see that it really looks a good complete flow:)