APPLIED FUNCTIONAL ANALYSIS HOMEWORK 6

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STAT 31210
DUE FRI FEB 17, 2023, 11PM

Discussed with classmates.

Exercise 1. *(7.1) in book*

Proof.

(a):

To get the result we need to get out a form of a fraction to do the principal value arguments. We have

$$\frac{1}{c_n} = \int_{-\pi}^{\pi}(1+\cos x)^n dx \geq \int_{-\delta/2}^{\delta/2}\left(1+\cos\frac{\delta}{2}\right)^n dx = \delta\left(1+\cos\frac{\delta}{2}\right)^n$$

and we plug in to get

$$\int_{\delta<|x|<\pi} c_n(1+\cos x)^n dx \leq \frac{2\pi(1+\cos\delta)^n}{\delta\left(1+\cos\frac{\delta}{2}\right)^n} = \frac{2\pi}{\delta}\cdot c^n \to 0$$

for fixed δ.

(b):

Since $\phi_n \to f$ uniformly, so for $\forall \varepsilon > 0$, $\exists N$ such that for all $n > N$, $g_n := |\phi_n - f| < \varepsilon$ and thus

$$\int_{-\pi}^{\pi} g_n^2 dx \leq 2\pi\varepsilon^2$$

which means $\phi_n \to f$ in L^2 since ε is arbitrary.

(c):

Let

$$f(x) = \begin{cases} 0 & 0 \leq x \leq \delta \\ 1 & 2\pi - \delta \leq x \leq 2\pi \end{cases}$$

and continuous in between, then since ϕ_n is periodic so

$$1 = |f(0) - f(1)| \leq |f(0) - \phi_n(0)| + |\phi(1) - f(1)|$$

and thus at least one of them will be larger than $\frac{1}{2}$ and hence the uniform norm is larger than $\frac{1}{2}$, thus ϕ_n cannot approximate f in the uniform norm, thus $\mathcal{P}$ is not dense in $C[0, 2\pi]$.

$\square$

Exercise 2. *(7.2) in book.*

Proof.

(a): First note that by geometric sum

$$\frac{1}{2\pi}\sum_{n=-N}^{N} e^{in\theta} = \frac{1}{2\pi}e^{-iN\theta}\frac{1-e^{i(2N+1)\theta}}{1-e^{i\theta}} = \frac{1}{2\pi}\frac{e^{-iN\theta}-e^{i(N+1)\theta}}{1-e^{i\theta}}$$
$$= \frac{1}{2\pi}\frac{e^{-i(N+1/2)\theta}-e^{i(N+1/2)\theta}}{e^{-\frac{1}{2}i\theta}-e^{\frac{1}{2}i\theta}} = \frac{1}{2\pi}\frac{\sin((N+1/2)\theta)}{\sin(\theta/2)} = D_N(\theta)$$

so then we can express S_N as

$$\begin{aligned} S_N &= \frac{1}{\sqrt{2\pi}}\sum_{n=-N}^{N}\hat{f}_n e^{inx} \\ &= \frac{1}{2\pi}\sum_{n=-N}^{N}\int_{-\pi}^{\pi} f(t)e^{-int}dte^{inx} \\ &= \int_{-\pi}^{\pi} f(t)\frac{1}{2\pi}\sum_{n=-N}^{N} e^{in(x-t)}dt = f * D_N. \end{aligned}$$

(2):

Again, we first show another form of Fejer kernel:

$$\begin{aligned} \frac{D_0+\cdots+D_N}{N+1} &= \frac{1}{2\pi(N+1)\sin(\theta/2)}\operatorname{Im}\left\{\sum_{k=0}^{N} e^{i(k+1/2)\theta}\right\} \\ (geometric) &= \frac{1}{2\pi(N+1)\sin(\theta/2)}\operatorname{Im}\left\{\frac{1-e^{i(N+1)\theta}}{e^{-\frac{1}{2}i\theta}-e^{\frac{1}{2}i\theta}}\right\} \\ &= \frac{1}{2\pi(N+1)}\frac{1-\cos(N+1)\theta}{\sin(\theta/2)^2} \\ &= \frac{1}{2\pi(N+1)}\frac{\sin((N+1)\theta/2)^2}{\sin(\theta/2)^2} = F_N \end{aligned}$$

and now since convolution is linear we have

$$f * T_N = f * \left(\frac{D_0+\cdots+D_N}{N+1}\right) = \frac{S_0+\cdots+S_N}{N+1} = T_N.$$

(3):

Since D_N is not positive it is not an approximate identity. As for F_n it is positive by above computation, it has integral 1 since D_N has (view it as integral of sum of e^{inx}, which is nonzero if $n=0$ only so the result is true), and it decays fast when far from 0 since D_N is such, which is due to Riemann Lebesgue Lemma.

So F_N is an approximate identity and D_N is not.

By theorem 7.2 $f * F_N \to f$ since $f * F_N - f \to 0$ in the uniform sense.

As for $D_N * f$, first it converges to f if f is Lipschitz: following steps of proof of theorem 7.2 we only have issues when t small, i.e.

$$\int_{|t|\le\delta} (f(x-t) - f(t)) \frac{\sin((N+1/2)t)}{\sin(t/2)} dt$$

cannot be bounded if f is merely continuous. If it's Lipschitz however, we can bound with

$$\int_{|t|\le\delta} \frac{f(x-t) - f(t)}{t} \frac{t}{\sin(t/2)} \sin((N+1/2)t) dt \le 2\delta c \to 0.$$

But we do know that $f * D_N \to f$ in L^2 sense due to Bessel's inequality (since e^{inx} is an orthonormal set of basis).

□

Exercise 3. *(8.1) in book.*

Proof.

(a): Show that X/M is a vector space:

We just prove property by property:

- Associativity:

$$\begin{aligned}((x+M)+(y+M))+(z+M) &= ((x+y)+M)+(z+M)\\ &= (x+y+z)+M\\ &= (x+M)+((y+z)+M)\\ &= (x+M)+((y+M)+(z+M)).\end{aligned}$$

- Commutativity:

$$(x+M)+(y+M)=(x+y)+M=(y+x)+M=(y+M)+(x+M).$$

- Identity: Since $0 \in X$ as X is a vector space we have

$$(x+M)+(0+M)=x+M,\quad \forall x \in X.$$

- Inverse: $\forall x \in X$, we have $x^{-1}=-x$:

$$(x+M)+(-x+M)=0+M.$$

- Field multiplication and vector multiplication: for $\lambda, \mu \in F$ the base field

$$(\lambda \cdot \mu)(x+M)=\lambda\mu x+M=\lambda(\mu(x+M)).$$

- Multiplication identity: For the identity element under multiplication in the base field, denote by 1, we have

$$1\cdot(x+M)=1\cdot x+M=x+M.$$

- Vector distributive:

$$\lambda((x+M)+(y+M))=\lambda((x+y)+M)=(\lambda x+M)+(\lambda y+M)=\lambda(x+M)+\lambda(y+M).$$

- Field distributive:

$$(\lambda+\mu)(x+M)=(\lambda+\mu)x+M=(\lambda x+M)+(\mu x+M)=\lambda(x+M)+\mu(x+M).$$

So indeed it is a linear vector space.

(b): $N \cong X/M$:

Define $\phi(y)=y+M$ for $y \in N$. Then we show it's an isomorphism from N to X/M.

1-1: For $y_1 \neq y_2$, $y_1-y_2 \neq 0$. If $y_1-y_2 \in M$, then

$$y_1=y_1+0=y_2+(y_1-y_2)$$

are two distinct representations as $X = M \oplus N$, contradiction to definition of direct sum. So $y_1 - y_2 \notin M$. Thus, $y_1 + M \neq y_2 + M$ since under vector addition their difference is not the identity element $0 + M$.

Onto: for $x + M \in X/M$, we know that $x \notin M$ and thus we can decompose $x = x_M + x_N$ for $x_M \in M$ and $0 \neq x_N \in N$. But then

$$\phi(x_N) = x_N + M = x + M$$

so $x + M \in \text{Ran}(\phi)$, so the map is onto.

Linear: Using the defined vector addition and multiplication in X/M we have

$$\phi(ax + by) = (ax + by) + M = (ax + M) + (by + M) = a\phi(x) + b\phi(y).$$

So in conclusion ϕ is a linear isomorphism between N and X/M. So the sets are linearly isomorphic.

(c): Is M closed if X/N is finite dimensional:

The result is not true. To find a counter example, we first define an unbounded linear projection P, then let $M := \ker P$, which is a linear subspace because P is linear. Moreover, we will see that the codimension is finite by construction. Then, we show that any unbounded operator has non-closed kernel.

First, by AC we can find an unbounded linear functional f such that there exists a point x_0 for which $f(x_0) = 1$ (always can by scaling).

Then, define

$$P(x) := f(x)x_0$$

we know

$$P(P(x)) = f(f(x)x_0)x_0 = f(x)x_0 = P(x)$$

so P is a projection. By theorem 8.2 in book we have $X = \ker P \oplus \text{Ran } P$ and we define

$$M := \ker P$$

then M is linear subspace since P is a linear operator.

We show that the codimension of M is finite. Since $\ker P \oplus \text{Ran } P$, and we know $\text{Ran } P = \mathbb{C}x_0$ is finite dimensional, so M do has a finite codimension by the isomophism between X/M and $\text{Ran } P$, i.e. M satisfies the condition in problem.

Now we show that for g is linear, if $\ker g$ is closed, then g is bounded.

To show this, we first assume that g is not bounded, then for all n there exists $||x_n|| = 1$ such that $||g(x_n)|| > n$. Moreover, let $e \in X$ be such that $g(e) = 1$. Define

$$y_n := e - \frac{x_n}{g(x_n)}$$

and we know that $||y_n - e|| \to 0$, so $y_n \to e$. But $g(y_n) = 1 - 1 = 0$ for all n, thus $y_n \in \ker g$ with a limit e that is not in the kernel. So $\ker g$ is not closed, contradiction!

So by contrapositive we know that if P is unbounded, $M = \ker P$ is not closed, so the conclusion in problem is false.

$\square$

Exercise 4. *(8.3) in book.*

Proof.

Before the proof, we denote some items that is used. Since $\mathcal{H}$ is Hilbert, so for any projection on it we can decompose $\mathcal{H}$ into the kernel and the range. So for any $x \in \mathcal{H}$, we denote

$$x = y + z = a + b, \quad y \in \ker P, z \in \operatorname{Ran} P, a \in \ker Q, b \in \operatorname{Ran} Q.$$

(a) $\mathcal{M} \subset \mathcal{N} \Rightarrow$ (b) $QP = P$:

Using above notation, for all $x \in \mathcal{H}$ we know

$$P(x) = z$$

and

$$QP(x) = Q(z)$$

where since $z \in \operatorname{Ran} P = \mathcal{M} \subset \mathcal{N} = \operatorname{Ran} Q$ so $Q(z) = z = P(z)$.

Thus, $QP = P$.

(b) $QP = P \Rightarrow$ (c) $PQ = P$:

We want to show $P(b) = P(x) = z$ so we write

$$b = b_y + b_z, \quad b_y \in \ker P, b_z \in \operatorname{Ran} P$$

and assume $z - b_z \neq 0$ we get $P(a) = P(x - b) = P(x) - P(b) = z - b_z \neq 0$ yet $a \in \ker Q$ so $(a, l) = 0$ for all $l \in \operatorname{Ran} Q$. Now, since $Q(k) = k$ for all $k \in \operatorname{Ran} P$ we know that $\operatorname{Ran} P \subset \operatorname{Ran} Q$, which then says for all $l \in \operatorname{Ran} P$, $(a, l) = 0$ so $a \in \ker P$, thus $P(a) = 0$ contradiction. Thus $z - b_z = 0$ and so $PQ = P$.

(c) $PQ = P \Rightarrow$ (d) $||Px|| \leq ||Qx||$:

Note that

$$||Px|| = ||P^2x|| = ||PQx|| \leq ||P|| \cdot ||Qx|| = ||Qx||$$

where the last equality is due to that orthogonal projections has norm 1.

(d) $||Px|| \leq ||Qx|| \Rightarrow$ (e) $\langle x, Px \rangle \leq \langle x, Qx \rangle$ for all $x \in \mathcal{H}$:

Squaring the condition we have

$$\langle Px, Px \rangle \leq \langle Qx, Qx \rangle \Rightarrow \langle x, P^2x \rangle \leq \langle x, Q^2x \rangle$$
$$\Rightarrow \langle x, Px \rangle \leq \langle x, Qx \rangle.$$

(e) $\langle x, Px \rangle \leq \langle x, Qx \rangle$ for all $x \in \mathcal{H} \Rightarrow$ (a) $\mathcal{M} \subset \mathcal{N}$:

For $x \in \operatorname{Ran} P$, let $x = Py$. Then

$$\langle x, x \rangle = \langle x, P^2y \rangle = \langle x, Px \rangle \leq \langle x, Qx \rangle$$

where if $\mathcal{M} = \operatorname{Ran} P = \{0\}$ then the conclusion trivially holds. So we can assume $x \neq 0$ then $Qx \neq 0$ for all x, that is, $x \notin \ker Q$. So $\operatorname{Ran} P \cap \ker Q = \{0\}$.

So we can write $x = a + b$ as is our convention above with $b \neq 0$. But since $(a, b) = 0$ we get

$$||a||^2 + ||b||^2 = ||x||^2 \leq \langle a + b, b \rangle = ||b||^2$$

which means $a = 0$ so $x = b \in \operatorname{Ran} Q$, which means $\mathcal{M} \subset \mathcal{N}$.

$\square$

Exercise 5. *(8.5) in book.*

Proof.

Product rule:

$$\nabla \cdot (\phi v) = \nabla \phi \cdot v + \phi(\nabla \cdot v).$$

First show that $\mathcal{V} \perp \mathcal{W}$: for $w \in \mathcal{W}$ and $v \in \mathcal{V}$ we have

$$\langle w, v \rangle = \int_{\mathbb{T}^3} wv dx = \int_{\mathbb{T}^3} \nabla \phi v dx = \int_{\partial \mathbb{T}^3} \phi v \cdot n dS - \int_{\mathbb{T}^3} \phi(\nabla \cdot v) dx = 0$$

where the second term is 0 because $\nabla \cdot v = 0$, and the first term is 0 because u, ϕ are 2π periodic and hence their dot product is a constant on the surface. Moreover, for any closed surface, the integral of the normal of each direction can be computed by

$$\int_{\partial \Omega} \hat{i} dS = \int_{\Omega} \nabla \cdot \hat{i} dx = 0$$

since $\hat{i}$ is constant, and plus we can decompose the normal vector, so

$$\int_{\partial \mathbb{T}^3} \phi v \cdot n dS = c \int_{\partial \mathbb{T}^3} n dS = 0.$$

Moreover, from multivariate calculus we know that any twice differentiable vector field can be decomposed into a gradient and a curl (Helmholtz decomposition), i.e. for $F \in C^2$

$$F = \nabla \times G + \nabla H$$

where since the divergence of curl is 0, $\nabla \times G \in \overline{\mathcal{V}}$ (closure due to smoothness) and $\nabla H \in \overline{\mathcal{W}}$. We note that since C^∞ is dense in L^2 so the closure is enough.

Now since we can use $F \in C^2$ to approximate L^2 vector fields, so we get our result.

Since $\nabla \cdot v = 0$ so $v \in M = \overline{\mathcal{V}}$ so if we apply P to both sides we get

$$P(v_t) + P[v \cdot \nabla v] + P(\nabla p) = P[v \Delta v]$$

where since we can exchange order of derivative (Clairaut)

$$\nabla \cdot \frac{\partial}{\partial t} = \left(\sum_i \frac{\partial}{\partial x_i} \right) \cdot \frac{\partial}{\partial t} = \frac{\partial}{\partial t} \left(\sum_i \frac{\partial}{\partial x_i} \right) \cdot = \frac{\partial}{\partial t} \nabla \cdot$$

and hence

$$\nabla \cdot v_t = \frac{\partial}{\partial t} \nabla \cdot v = 0$$

and we thus get

$$v_t + P[v \cdot \nabla v] + P(\nabla p) = v \Delta v$$

now note $\nabla p \in N = \ker P$ so we get

$$v_t + P[v \cdot \nabla v] = v \Delta v.$$

$\square$

Exercise 6. *(8.7) in book.*

Proof. $||\phi_y|| \geq ||y||$ because $\phi_y(y) = ||y||^2$; for the other direction use Cauchy to get

$$\phi_y(x) \leq ||x|| \cdot ||y||$$

and dividing by $||x||$ on both sides we get the $||\phi_y|| \leq ||y||$ and hence

$$||\phi_y|| = ||y||.$$

$\square$

Exercise 7. *Prob (8.9) in book.*

Proof.

For all $x \in \mathcal{H}$, we can find a sequence of points y_n in $\mathcal{M}$ that approximates x. Then we decompose

$$y_n = \sum_{\alpha \in I} a_\alpha^n u_\alpha$$

then we can use linearity and continuity of ϕ to pass limit and get (since it's a linear basis, I is a finite set, so we can exchange summation and limit)

$$\lim_{n\to\infty} \phi(y_n) = \lim_{n\to\infty} \sum_{\alpha \in I} \phi(a_\alpha^n u_\alpha) = \sum_{\alpha \in I} \lim_{n\to\infty} \phi(a_\alpha^n u_\alpha) \overset{cts}{=} \sum_{\alpha \in I} \phi\left(\lim_{n\to\infty} a_\alpha^n u_\alpha\right)$$

$$= \phi\left(\sum_{\alpha \in I} \lim_{n\to\infty} a_\alpha^n u_\alpha\right) = \phi(x)$$

thus $\phi(x)$ is determined by $\phi(\mathcal{M})$.

For an orthonormal basis however, c_n converges unconditionally, i.e. square convergence would be equivalent. Necessity is because we need to express the vector whose sum of each dimensional projection = 1; Sufficient due to theorem 6.26.

□