APPLIED LINEAR ALGEBRA HOMEWORK 1

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1. Problem 1

Solution:

(a) Let $\tau = \omega t$, then

$$\frac{d^2\theta}{d\tau^2} = \frac{1}{\omega^2}\frac{d^2\theta}{dt^2} = -\left[\frac{g}{l}\frac{1}{\omega^2} + \frac{g}{l}\frac{A}{\omega^2}\cos(\tau)\right]\sin\theta$$

Letting $\alpha = \frac{g}{l}\frac{1}{\omega^2}, \beta = \frac{g}{l}\frac{A}{\omega^2}$, we get the desired form:

$$\frac{d^2\theta}{d\tau^2} = -(\alpha + \beta\cos(\tau))\sin\theta$$

and the relation between the parameters is $\frac{\beta}{\alpha} = A$.

Let $\dot\theta = \Omega$, the the linearized ODE system is (w.r.t. τ):

$$\begin{cases} \dot\theta = \Omega \\ \dot\Omega = -(\alpha + \beta\cos(\tau))\sin\theta \end{cases} \tag{1.1}$$

(b)

We want to start the pendulum at some where above the pivot that has a very small angle with the vertical line. In terms of θ, we take θ close to π.

Yet since θ is defined in $(-\pi, \pi]$ and I want to eliminate the negative sign in the second line (1.1), I choose $\theta = -\pi + x$, and in practice

$$\sin(-\pi + x) = \sin(\pi + x) = -\sin(x) = -\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right) \approx -x$$

where we used Taylor.

The reason we will continue with only the first degree expansion of sin is because the theorem that says if a system is linearly stable/unstable, it is stable/unstable.

For simplicity reasons, we take $f(\tau) := \alpha + \beta\cos(\tau)$. We use

$$X = \begin{bmatrix} x \\ \Omega \end{bmatrix}$$

as our vector in the ODE, where $x = \theta - \pi$.

Since

$$\begin{cases} \dot{x} = \Omega \\ \dot{\Omega} = f(\tau)x \end{cases}.$$

our ODE system is the following:

$$\frac{dX}{d\tau} = A(\tau)X \quad \text{where} \quad A(\tau) = \begin{bmatrix} 0 & 1 \\ f(\tau) & 0 \end{bmatrix}. \tag{1.2}$$

(c)

The results from ODE theory that is going to explain what happens in later codes is the following:

- Our ODE is $\dot{X} = A(\tau)X$, and we can start this ODE with any initial value $X(0) = X_0$. This give rise to the fundamental matrix solution of X, which we denote by $\Phi(\tau)$ with $\Phi(0) = I$.It satisfies

  $$\frac{d\Phi}{d\tau} = A(\tau)\Phi$$

  What this fundamental matrix solution really means is that we encode all possible initial condition, X_0, as the linear combination of the basis of the space it is in. In our case, we use the standard basis.
  This means that if we want to solve for $X(0) = X_0$, we can simply multiply both side of $\frac{d\Phi}{d\tau} = A(\tau)\Phi$ by X_0 and get the result.
- Since the solution to the ODE is unique, the evolution from $t = 0$ to $t = t'$ is the same as first evolve from $t = 0$ to $t = t_1$, then from $t = t_1$ to $t = t'$.
- The Monodromy matrix $M := \Phi(T)$, where A is periodic T.
- If A is periodic T, then $X(T) = MX_0$. With the second point we can evolve this multiple times and get $X(NT) = M^N X_0$.

  What this tells us is that when we take time large enough, we can always separate time to $t = N_t T + t'$ with $t' < T$. This means the behavior as $t \to \infty$ is largely encoded in the properties of M.

  In particular, if M has an eigenvalue with norm larger than 1 in the complex plane, then after N iterations X will blow up as M^N will.

  On the other hand, if M has two eigenvalues whose norm is less than 1, the result will be stable since the norm of X decays with the scale of M^N.

  When the eigenvalues of M is on the unit circle of the complex plane, then it is likely that the value of the linearlized version of $X(\tau)$ will evolve in a bounded and periodic way as the angle of some step gets closer to a multiple of T. However, this is not likely to be precise in practice due to minimal step size.

Thus, in this question, (as we can see from next result), the determinant of M is 1, which means the two eigenvalues' product is 1, which means that either it blows up or both eigenvalues land on the unit circle, which is Lyapnov stable (since won't decay to 0, and lyapnov due to result curve in phase space).

- (Abel's theorem):

$$\det(\Phi(t)) = \exp\left(\int_{t_0}^{t} \operatorname{tr}(A(s))ds\right)$$

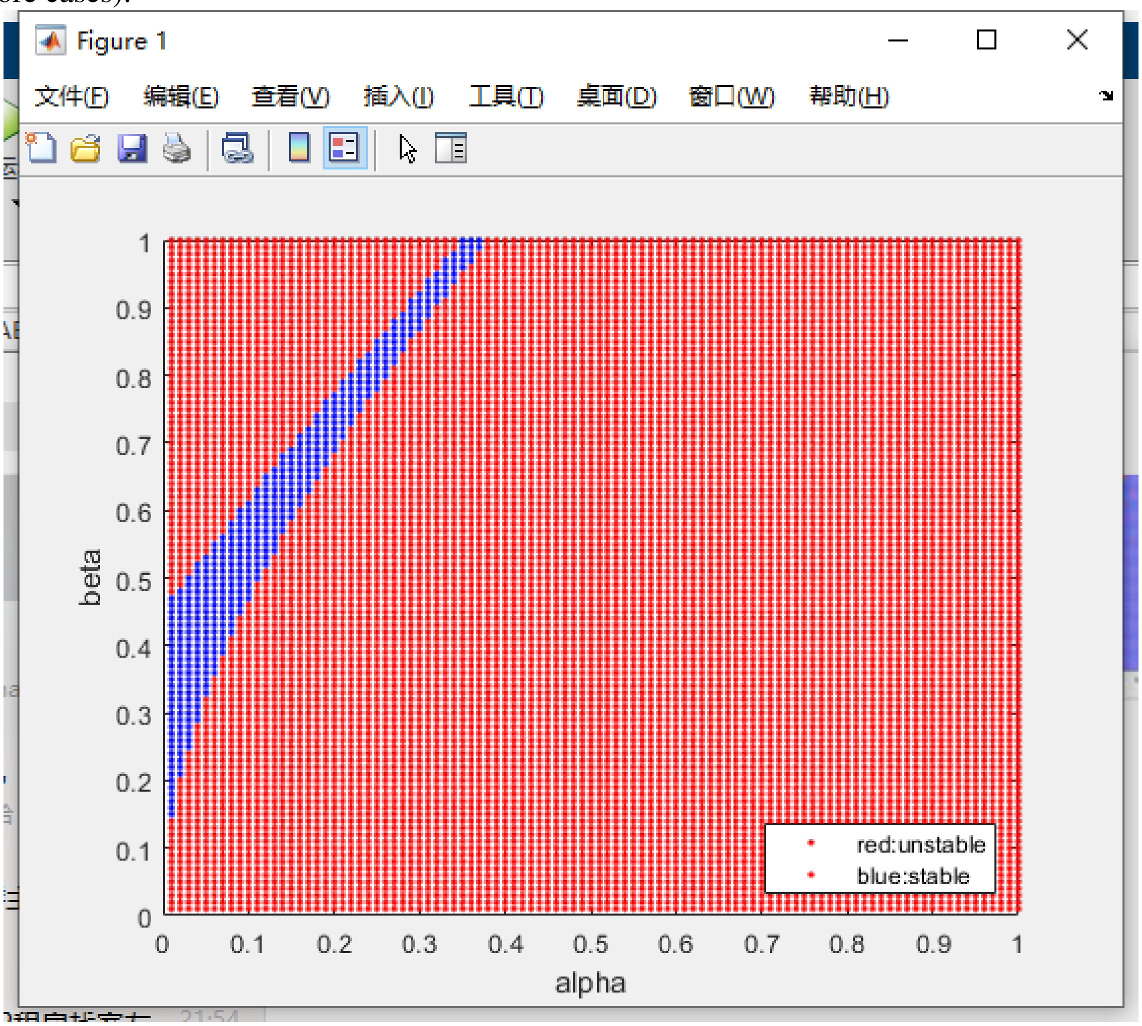
  In our case, since $\operatorname{tr}(A) = 0$ at any time, the determinant of Φ is 1. In particular, $\det(M) = 1$.
- There is a proof of the existence of the solution and what it is in book, theorem 2.36 (and 2.37 for a prettier result). However, it is somewhat unrelated to our practice below, so I just skip it here.

(d)

I will explain with words what I did here, and show some of my codes and the plot of stable and unstable regions.

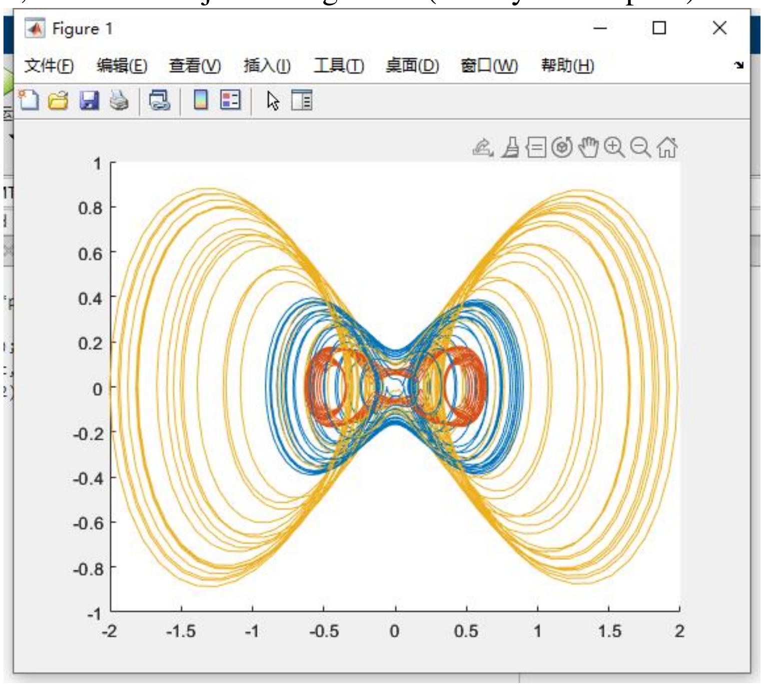
What I did is that I have computed the eigenvalues of M, and since $\det(M) = 1$, it is either unstable(one of the eigenvalue is larger than 1), or linearly Lyapnov stable (complex eigenvalue). What I did then is try to determine the scale of parameters. I first tried $(\alpha, \beta) = (10, 10)$, and the eigenvalue is 10^8, so not really what we need. In the end I went down to $(\alpha, \beta) \in [0, 1] \times [0, 1]$.

The graph is below. I use red to stand for unstable, and blue for linearly Lyapnov stable(I used merely stable in the legend of the graph since back then I have not done more cases):

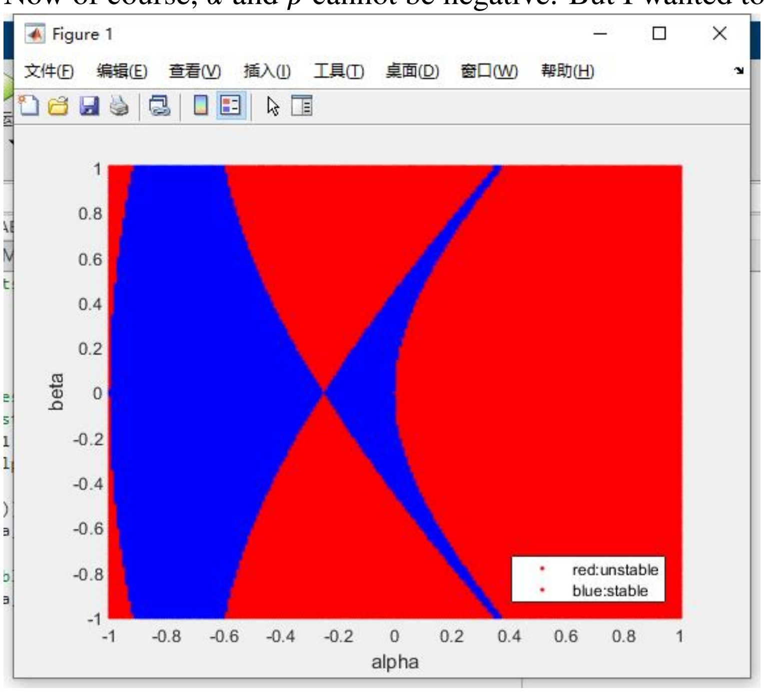


What our theory tells us is that linearly asymptotic stable implies asymptotic stable, and linearly unstable implies unstability. So really we only knew that the red part is unstable. So let's try for a few points in the blue region and see whether it is stable (presumably periodic).

So I chose the point (0.02, 0.30)(orange),(0.24, 0.81)(blue),(0.37, 0.99)(blue) which is in the blue area and get this curve in phase space(xy-plane). Also, I choose the initial condition to be (0.1, 0.1) as we are starting from a small angle with the vertical line, and we want just let it go there(so maybe 0.1 speed):



Very well! They all seem to be at least bounded, or Lyapnov stable numerically. Now of course, α and β cannot be negative. But I wanted to try:



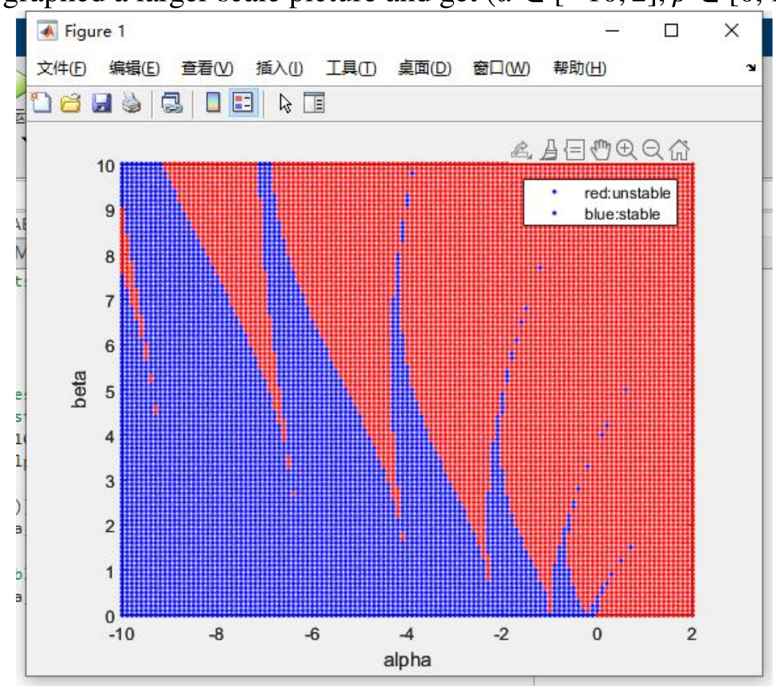
To my surprise, the negative part of α behaves more stably, and it is symmetric with respect to β. This makes sense.

α is really $\frac{g}{l\omega^2}$, and none of them should be negative, whereas $\beta = \frac{gA}{l\omega^2}$, where the amplitude A is symmetric with respect to 0 because starting with a negative amplitude is just starting with a positive amplitude, but starting at some the other time, since periodic.

I do think that $\alpha < 0$ is meaningful, however, which I shall state in the next part.

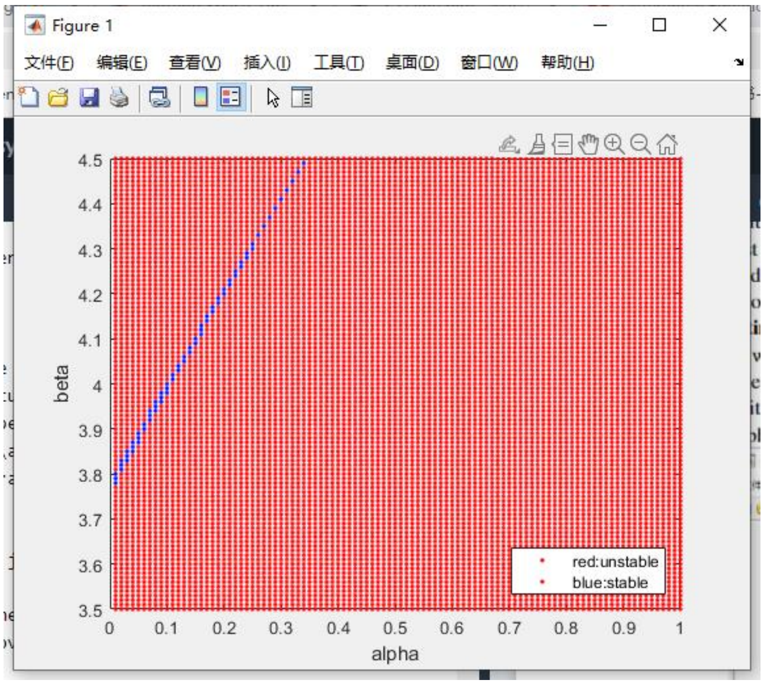
(e)

Starting at the bottom: suppose that we start the inverted pendulum from the bottom, what would happen? Will it go up? It occurs to me that it really is the same problem if we start at the top, but with a flipped gravity. This also need to flip starting amplitude as well, so we were in the $\alpha \leq 0, \beta > 0$ region as $\beta = (-g) \cdot (-A) \cdot \frac{1}{l\omega^2}$. So I graphed a larger scale picture and get ($\alpha \in [-10, 2], \beta \in [0, 10]$):

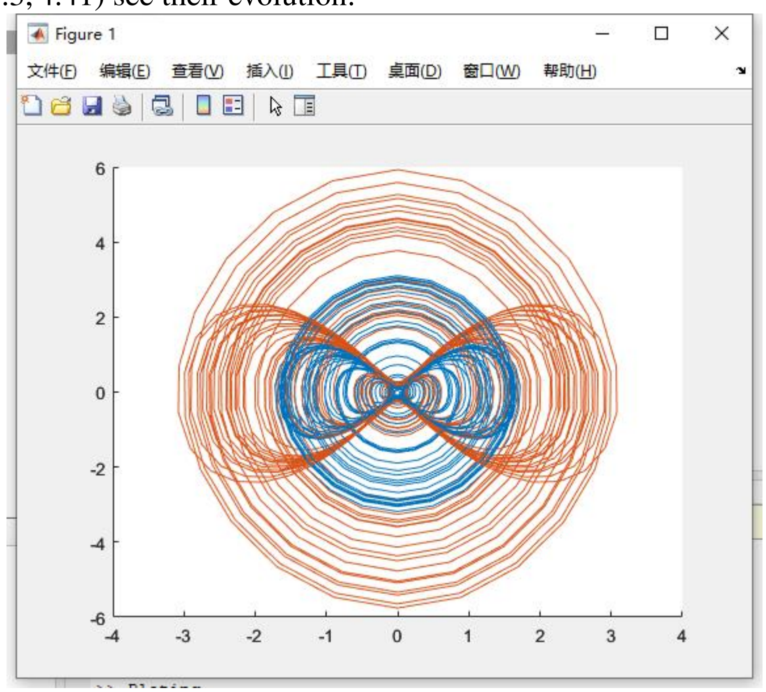


Like wow. This really tells us that at the red part, even if we start from the bottom, the amplitude will cause it to move radically.

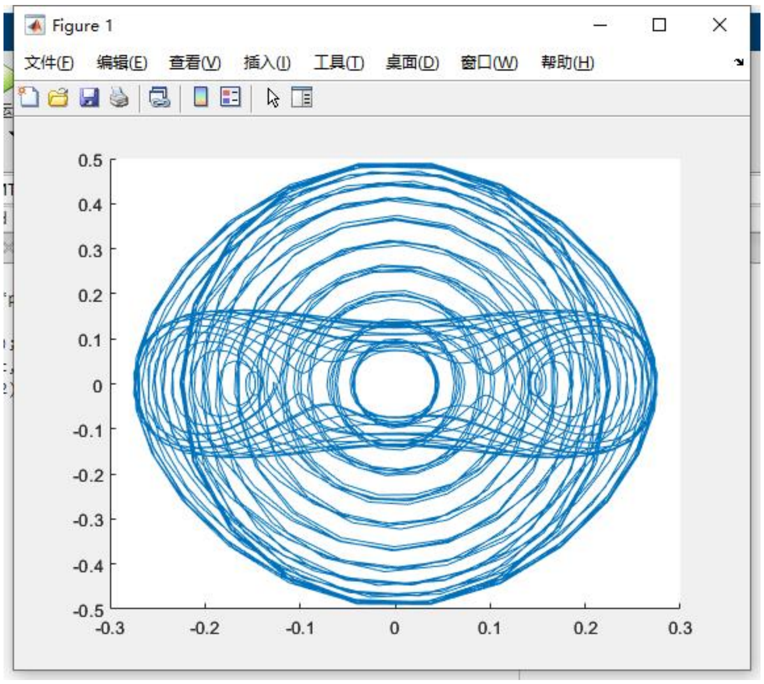
Another region of possible stability: But look at the graph. Interesting things happened: the second "pedal" of the blue region reaches out to the positive part. So I plotted again near that part:



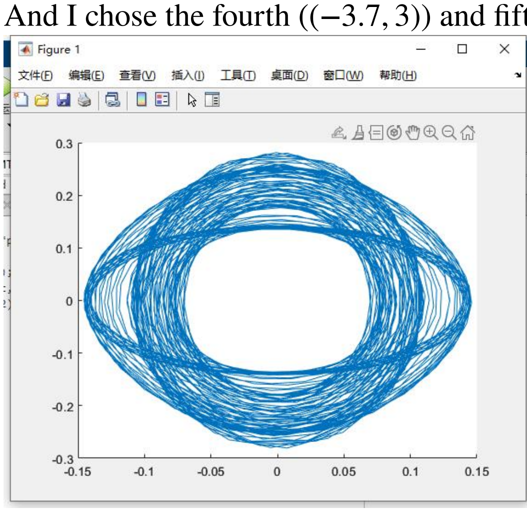
Indeed there's a possible stable region. I pick two points $(0.05, 3.88)$(blue) and $(0.3, 4.41)$ see their evolution:

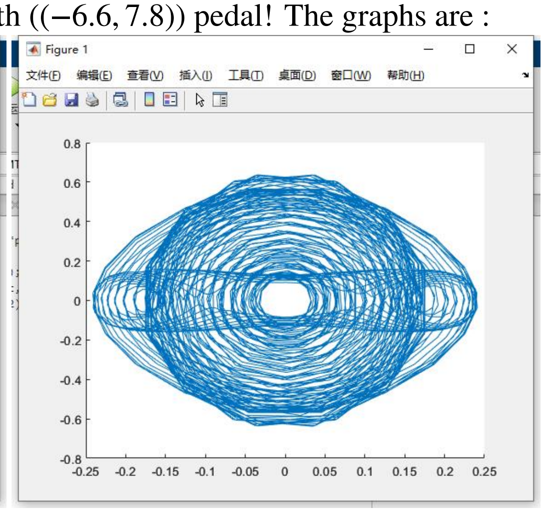


Indeed they are stable! What's more, they contain a outer circle that the stable part below does not have. I'm guessing that it's because it's the second pedal, and I cannot find exact positive values for the third pedal. But I can choose a negative α point there and try plot it (the point is $(-2.1, 2.9)$):



And I chose the fourth $((-3.7, 3))$ and fifth $((-6.6, 7.8))$ pedal! The graphs are :

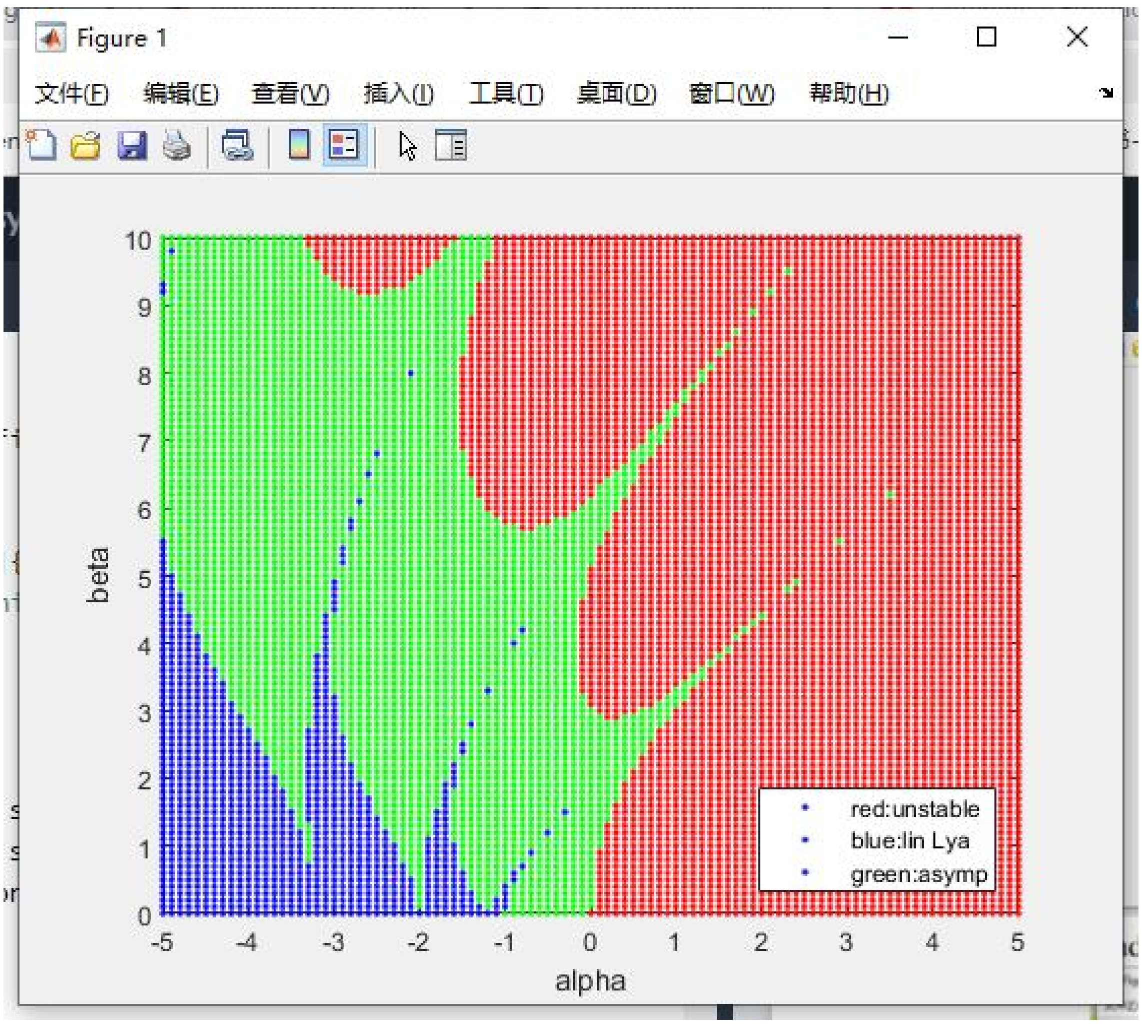




I have no way to explain this.

With damping: I added damping to see whether I can get a promised stable region. This can be done since the trace of A will not be zero (add damping is to add $-b$ at $A(2,2)$), and it's likely we get a linearly asymptotic stable region, which implies asymptotic stable.

So I did, with damping = (-2):



It actually gives us a stable region. This makes sense since with damping, it just should cease to 0. So maybe the interesting part becomes when even with air-friction and so on, the inverted pendulum gains enough energy from the oscillator that it keeps on going-unstably.

Also, since l is always on the denominator, making a longer pendulum will scale down the α and β, thus get closer to a stable region.

(f)

The following refers to all color-dot graphs in the parameter space:

The blue region is linearly Lyapnov stable. We cannot know analytically how it behaves. Numerically it is Lyapnov stable.

The green region is asymptotically stable.

The red region is unstable.

The reasons are all stated above.