Lyapunov Exponents

Jiaheng Chen, Ran Tang, Thomas Wang, Tommenix Yu

STAT

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Outline

Chaotic system

- *It may happen that slight differences in the initial conditions produce very great differences in the final phenomena; a slight error in the former would make an enormous error in the latter. Prediction becomes impossible and we have the fortuitous phenomena.* (Henri Poincaré 1914).

Description of chaotic systems

"Nearby trajectories eventually separate."

Definition

A flow ϕ exhibits sensitive dependence on an invariant set X if there is a fixed r such that for each $x \in X$ and $\forall \epsilon > 0$, there is a nearby $y \in B_\epsilon(x) \cap X$ such that $|\phi_t(x) - \phi_t(y)| > r$ for some $t \geq 0$.

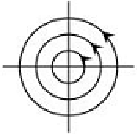
But is it enough?

Example

(Example 7.2 in book) In polar coordinate the system on $\mathbb{R}^2$

$$\begin{cases} \dot{\theta} = r \\ \dot{r} = 0 \end{cases}$$

In this example the system is just concentric orbits rotating with speed proportional to its distance to the origin:



But since the orbits farther away runs much faster, sensitive dependence will occur for any annulus.

General characterization of Chaotic system

- **Sensitive dependence**: "Nearby trajectories eventually separate."
- **Transitive**: "Wanders everywhere."

Definition

A flow ϕ is topologically transitive on an invariant set X if for every pair of nonempty, open sets $U, V \subset X$ there is a $t > 0$ such that $\phi_t(U) \cap V \neq \emptyset$.

Theorem

A flow ϕ is transitive $\iff$ ϕ has a dense orbit in X.

- Always exists close enough point on the same orbit with time far away.
- Helps Wolf's and Kantz's method.

Lyapunov Exponents

Modification of sensitive dependence.

- **Sensitive dependence** : yes/no check;
- **Lyapunov Exponents**: quantitative measure.

The distance grows exponentially locally.

$$\frac{|\phi_{t_0+t}(x) - \phi_{t_0+t}(x+\delta)|}{|\phi_{t_0}(x) - \phi_{t_0}(x+\delta)|} \sim ce^{\lambda t}$$

- Care about λ;
- Look at locally because of attractors (Rosseler).

Caveats

- Infinitesimally close orbits separate exponentially.
- Caveat: simply linearize the system is not good enough.
- We **cannot** simply borrow the tools such as the Jacobian matrix and the Floquet theory for studying a aperiodic orbit.
- We will focus on a particular orbit, $\phi_t(x_0)$, the *fiducial trajectory*, of a flow ϕ on an $n-$dimensional phase space M.

Tangent Bundle

- Given a manifold M, for each point $x \in M$, let T_xM denote the set of tangent vectors at x,i.e., the tangent space at x. Since there is a tangent space attached to every x, we can define make the following definition.

Definition

the *tangent bundle* of M is

$$TM = \{(x, v) : x \in M, v \in T_xM\}.$$

Fundamental Matrix

- For a trajectory $\phi_t(x_0 + \epsilon_0)$ starting near x_0, and get that the initial deviation vector v_0 evolves into

$$v(t) = D_x \phi_t(x_0) v_0. \tag{1}$$

- Putting 1 in the ODE for ϕ gives

$$\dot{v} = Df\left(\varphi_t\left(x_0\right)\right) v \equiv A(t) v. \tag{2}$$

- Therefore, since (1) and (2) holds for any initial vector v_0, the fundamental matrix solution of (2) is

$$\Phi\left(t; x_0\right) = D_x \varphi_t\left(x_0\right)$$

satisfying

$$\dot{\Phi} = A(t)\Phi, \quad \Phi\left(0; x_0\right) = I. \tag{3}$$

Fundamental Matrix Continued

- for any vector v the solution to 2 is $\Phi(t; x_0)v$. The fundamental matrix is a linear operator

$$\Phi\left(t; x_0\right) : T_{x_0}M \to T_{\varphi_t(x_0)}M.$$

Takeaway: if $\phi_t(x_0)$ is *periodic*, then $\phi_T(x_0) = x_0$, so the monodromy matrix maps $T_{x_0}M$ back to itself. Therefore, we could calculate the Floquet multipliers. On the other hand, for an *aperiodic* trajectory, we cannot assume that $T_{x_0}M = T_{\varphi_t(x_0)}M$, so an equation of the form $\lambda v = \Phi v$ does not make sense.

Lyapunov exponents: Definition

Definition

The **Lyapunov spectrum** of the dynamical satisfying 3 is the set of limit points of

$$Sp(x, v) = \left\{ \lambda = \lim_{j\to\infty} \frac{1}{t_j} \log |\Phi(t_j; x)\, v| \text{ for some sequence } t_j \underset{j\to\infty}{\to} \infty \right\}.$$

- This definition makes sense because $\log \frac{|\Phi v|}{t}$ is bounded from both above and below in positive time by the following lemma.

Lemma

Suppose $\Phi(t; x)$ is the fundamental matrix solution of 3 and $\|A(t)\| \leq K$ for all $t \geq 0$. Then for any ν there are positive constants c and c' such that

$$c' e^{-Kt} \leq |\Phi(t; x) v| \leq c e^{Kt}$$

for all $t \geq 0$.

Lyapunov Exponents: Remarks

- Remark: the Lyapunov spectrum is dependent upon both the fiducial trajectory and the initial deviation vector.
- Two special limits : infimum limit and supremum limit:

$$\limsup_{t\to\infty} s(t) \equiv \lim_{T\to\infty}\left(\sup_{t>T} s(t)\right), \quad \liminf_{t\to\infty} s(t) \equiv \lim_{T\to\infty}\left(\inf_{t>T} s(t)\right).$$

- since any limit point of a bounded sequence are bounded by $\liminf$ and $\limsup$ and we are considering a continuous system, the Lyapunov spectrum is a closed interval between $\liminf$ and $\limsup$.
- A Lyapunov spectrum degenerates to a point when they coincide, in this case, we say it is *regular*.

Lyapunov Exponents

- The special case is the largest growth rate, which is exactly the lim sup.

Definition

The **Lyapunov exponent** is the supremum limit

$$\mu(x, v) := \limsup_{t\to\infty} \frac{1}{t} \log |\Phi(t; x)v| = \sup Sp(x, v).$$

We can introduce a more general notation.

Definition

The **characteristic exponent** for a function $f(t)$ is

$$\chi(f) \equiv \limsup_{t\to\infty} \frac{1}{t} \log |f(t)|.$$

In this notation, $\mu(x, v) = \chi(\Phi(t; x)v)$.

A Simple Example

Consider the linear one-dimensional ODE

$$\dot{v} = (\cos(\log|t|) + \sin(\log|t|))v,$$

which has general solution of the form

$$f(t) = \exp(t\sin(\log|t|))v_0.$$

In this case, the one-dimensional fundamental matrix is just the scalar $\exp(t\sin(\log|t|))$, so the Lyapunov spectrum is

$$\left\{\lim_{j\to\infty} \frac{1}{t_j} t_j \sin\left(\log|t_j|\right)\right\} = [-1, 1],$$

and the Lyapunov exponent is

$$\limsup_{t\to\infty} \sin(\log|t|) = 1.$$

Three basic properties

- Property 1:
$$\chi(cf) = \chi(f)$$
- Property 2:
$$\chi(f+g) \le \max(\chi(f), \chi(g))$$
- Property 3:
$$\chi(fg) \le \chi(f) + \chi(g).$$

Lemma 1

(Lemma 7.10 in textbook)
The Lyapunov exponent is independent of the choice of norm on $\mathbb{R}^n$.

Proof Outline:
There exist constants s and $S > 0$ such that for every vector v,

$$s|v|_1 \leq |v|_2 \leq S|v|_1.$$

Therefore,

$$\chi\left(s|\Phi(t;x)v|_1\right) \leq \chi\left(|\Phi(t;x)v|_2\right),$$

Then by Property 1, we have $\mu_1 \leq \mu_2$.

Lemma 2

(Lemma 7.11 in textbook)
If $\varphi_t(x)$ is a bounded trajectory of a C^2 flow φ on an n-dimensional manifold, then it has at most n distinct Lyapunov exponents.

Proof Outline:
Suppose there are two different exponents $\mu_1 > \mu_2$ for linearly independent vectors v_1 and v_2.
Since the equation $\dot{v} = A(t)v$ is linear, the length of any linear combination $v = \alpha v_1 + \beta v_2$ grows asymptotically at the rate μ_1, provided only that $\alpha \neq 0$.

Lemma 3 (Lyapunov basis)

It is conventional to order the exponents so that

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n.$$

Any set of independent vectors $\{v_1, v_2, \ldots, v_n\}$ so that

$$\sum_{i=1}^{n} \mu_i (x, v_i)$$

is as small as possible is called a **Lyapunov basis**.

Lemma

(Lemma 7.12 in textbook)
If $\Phi = [v_1, v_2, \ldots, v_n]$ is any fundamental matrix solution of equation $\dot{v} = A(t)v$ obeying $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, then there is a unit upper triangular matrix U such that ΦU is a Lyapunov basis.

Lemma 4

(Lemma 7.13 in textbook)
If $\varphi_t(x_o)$ is a bounded orbit of the flow φ that is not forward asymptotic to an equilibrium, then it has a zero Lyapunov exponent.

Proof Outline: Consider $v(t) = f(\varphi_t(x_o))$,

$$\frac{d}{dt}v(t) = \frac{d}{dt}f(\varphi_t(x_o)) = Df(\varphi_t(x))\frac{d}{dt}\varphi_t(x_o) = Df(\varphi_t(x_o))\,v(t).$$

Thus $v(t)$ is a solution of $\dot{v} = A(t)v$ with initial condition $v_0 = f(x_0)$.

Theorem (Lyapunov)

(Theorem 7.15 in textbook)
Suppose $\varphi_t(x)$ *is a bounded orbit of a flow* φ *and* $[v_1, v_2, \ldots, v_n]$ *is an independent set of vectors with Lyapunov exponents* $\mu_i = \mu(x, v_i)$*. If the limit*

$$\delta = \limsup_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} Df\,(\varphi_s(x))\, ds$$

exists, then

$$\delta \leq \sum_{i=1}^{n} \mu_i .$$

Proof Outline:

$$\det(\Phi(t;x)) = \exp \int_0^t \operatorname{tr} Df\,(\varphi_s(x))\, ds \quad \text{(Abel's theorem)}$$

$$P(t) = \Phi(t;x)P(0) \implies \delta = \chi(\det\Phi(t;x)) = \chi(\det P(t)).$$

$$\implies \chi(\det P) \leq \sum_{j=1}^{n} \max_{1\leq i\leq n} \chi\,(P_{ij}) = \sum_{i=1}^{n} \chi\,(v_i)$$

Compute the maximal Lyapunov exponent

How to compute $\mu(x_0, v_0) \equiv \limsup_{t\to\infty} \frac{1}{t} \ln |v(t)|$?

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Need to integrate both

- **The original system** : $\dot{\phi}_t(x_0) = f(\phi_t(x_0)),\ \phi_t(x_0)|_{t=0} = x_0;$
- **The linearized system**:

$$\dot{v} = Df\left(\phi_t\left(x_0\right)\right) v,\ \ v|_{t=0} = v_0,\ |v_0| = 1.$$

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- Almost any initial vector v_0 will have some component along the direction of the maximal Lyapunov direction.

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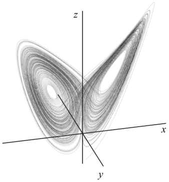
- Expect: this quantity will rapidly converge to the maximal exponent

Example: Lorenz system

(Example 7.18 in book) Lorenz system

- The original system:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$



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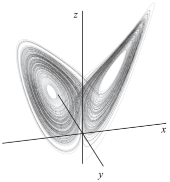

- The linearized equations for a vector $v \in T_x\mathbb{R}^3$:

$$\dot{v} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix} v$$

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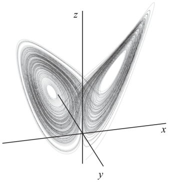

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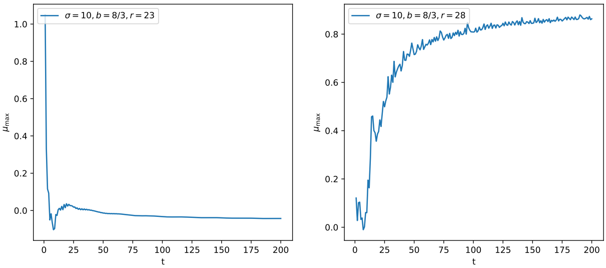
- Integrate Lorenz system and linearized system **simultaneously**

Example: Lorenz system

Numerical results (reproduce Figure 7.5 in book):



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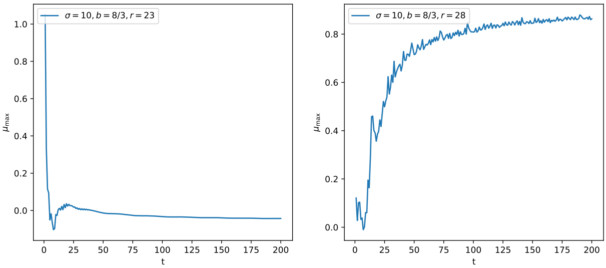

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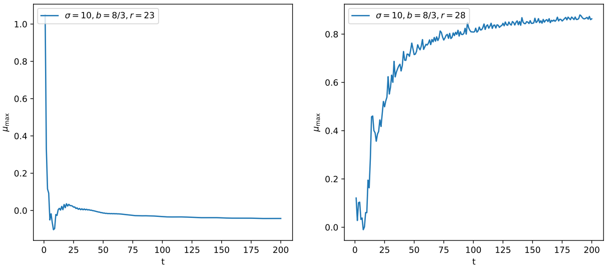

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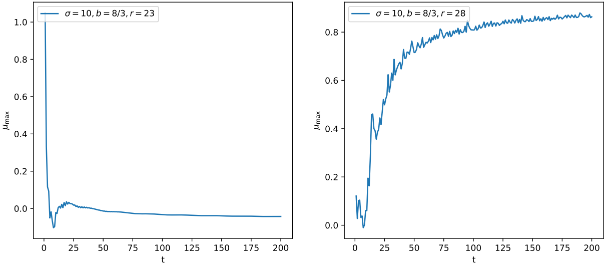

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- Remark: at every step, we normalize the vector v and add the accumulated scaling factor back to obtain $\frac{1}{T} \ln |v(T)|$
- Question: How to compute all of the Lyapunov exponents?

Compute all of the Lyapunov exponents

A sketch:

- The linear system: $\dot{v} = Df\left(\phi_t\left(x_0\right)\right)v =: \mathbf{A(t)}v$

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Lemma (Lyapunov transformation)

$P, P^{-1}, \dot{P}$ are bounded, $P \in C^1 \Longrightarrow$
Lyapunov exponents keep the same under this transformation.

Theorem (Perron triangulation)

There is an orthogonal transformation such that B is **upper triangular**. *Moreover, if $A(t)$ is bounded, then the characteristic exponents for B are the same as those of A.*

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Main idea: $\Phi(t) = Q(t)R(t)$, let $v(t) = Q(t)w$ define a new basis

Compute all of the Lyapunov exponents

- The transformed system: $\dot{w} = B(t)w$

Theorem

If $B(t)$ is a uniformly bounded, upper triangular matrix, and the limits

$$\mu_i = \lim_{t \to \infty} \frac{1}{t} \int_0^t b_{ii}(s) ds$$

exist, then $\dot{x} = B(t)x$ has a regular Lyapunov spectrum with exponents μ_i.

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- Remark: QR procedure can be turned into an effective computational strategy.

Largest Lyapunov exponent

$$\mu_{\max}(T) \approx \frac{1}{T} \ln |v(T)|$$

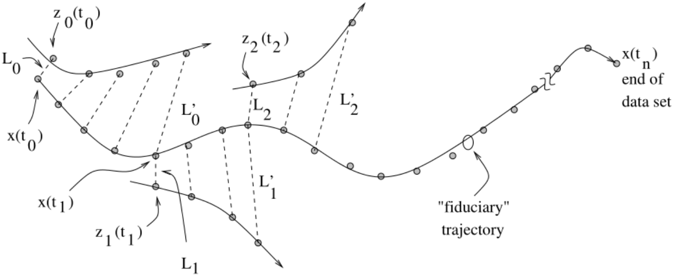
Why do we get the largest Lyapunov exponent in practice?

$$\delta(0) = \sum_{i=1}^{n} \alpha_i \delta_i(0) \quad \Rightarrow \quad \delta(t) = \sum_{i=1}^{n} \alpha_i \delta_i(0) e^{\lambda_i t} \sim c e^{\lambda_1 t}$$

which is why we need to find the Lyapunov basis to find all exponents.

Largest Lyapunov exponent: Wolf's method

Instead of evaluating at just 1 point, we evaluate the average of many starting points.



Can do in practice: use same trajectory since it is dense (remember transitive of flow).

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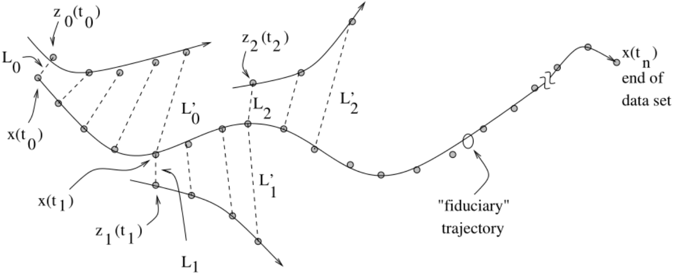
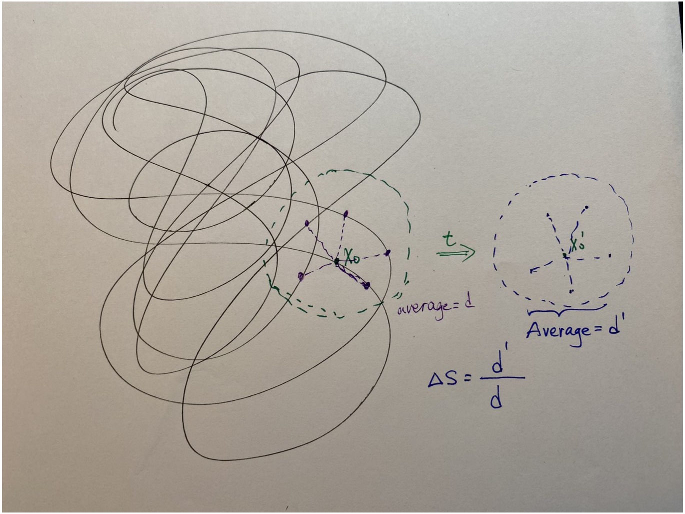


$$\lambda_1 = \frac{1}{N\Delta t} \sum_{i=1}^{M-1} \log_2 \frac{L'_i}{L_i}$$

where M is how many time you do it, and $N\Delta t = t_n - t_0 = T$, the total time.

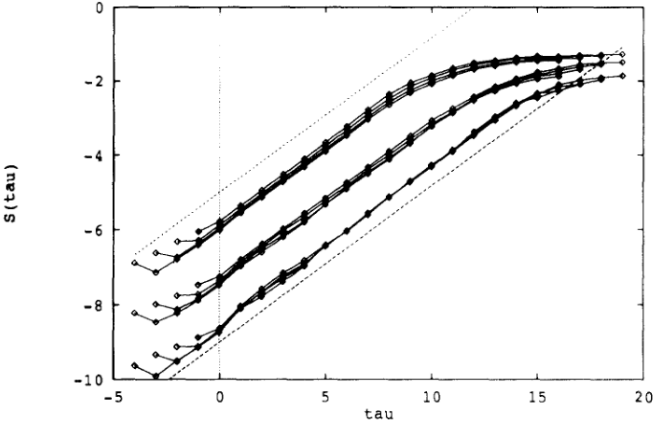
Largest Lyapunov exponent: Kantz's method

Instead of evaluating at just 1 point for each start time, we look at many points near each starting point.



Largest Lyapunov exponent: plottings and results

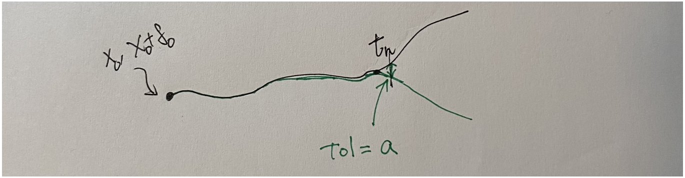
The plot from Kantz's paper is like this (after taking log), where the slope is λ_1:



Back to Chaos: Predicting weather

Having a positive Lyapunov exponent is neither sufficient or necessary for the system being chaotic. But it is a signiture.

Suppose we have two really close initial condition x_0 and $x_0 + \delta_0$, and tolerance a. Then we can predict the behavior up to t_n.



Back to Chaos: Predicting weather

Now, iphone tells me about 10 days' weather in the future.

If we compute the t_n from last slide we will get the Lyapunov time:

$$T = \frac{1}{\lambda} \log \left(\frac{a}{\delta_0} \right)$$

If we want to extend that to 100 days, then we need a multiple of 10 on both sides. But the weather system is fixed so is λ, and we need new initial difference

$$\delta = \frac{1}{e^{10}} \delta_0$$

so not very reliable.

References I

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Differential dynamical systems, revised edition.

Kantz, H. (1994).
A robust method to estimate the maximal lyapunov exponent of a time series.
Physics letters A, 185(1):77–87.

[Kantz, 1994] [D. Meiss, 2017]

Other resources include:

https://www.youtube.com/watch?v=92-ilwuwMTM&t=1s (Wolf):

https://www.youtube.com/watch?v=22VVVn1zPdM (Kantz):

https://www.youtube.com/watch?v=_R-edBK71dc&t=208s And other sources are included in materials.