APPROXIMATION THEORY HOMEWORK 5

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Discussed with classmates.

Exercise 1.

Proof.

(a): On a set with finite measure, L^2 implies L^1 so everything I write below is well-defined. For the change of integrals that's just because both before and after the interchange is well-defined (actually has a value).

Just plug in we have for all $f, g \in L^2[-1,1]$ (since it's on $\mathbb{R}$ the inner product is the real one)

$$\begin{aligned}\langle S_n[f], g\rangle &= \int_{-1}^{1}\left[\sum_{j=0}^{n}\frac{p_j(x)}{||p_j||_2^2}\int_{-1}^{1}p_j(t)f(t)dt\right]g(x)dx\\ &= \int_{-1}^{1}\int_{-1}^{1}f(t)g(x)\sum_{j=0}^{n}\frac{p_j(x)p_j(t)}{||p_j||_2^2}dtdx\end{aligned}$$

note that this step means the kernel of this operator is symmetric with respect to x, t, so it should be self-adjoint (even compact since both way L^2), but continue computation:

$$= \int_{-1}^{1}\int_{-1}^{1}f(t)g(x)\sum_{j=0}^{n}\frac{p_j(x)p_j(t)}{||p_j||_2^2}dxdt = \int_{-1}^{1}\left[\sum_{j=0}^{n}\frac{p_j(t)}{||p_j||_2^2}\int_{-1}^{1}p_j(x)g(x)dx\right]f(t)dt$$
$$= \langle f, S_n[g]\rangle.$$

It is finite rank because if we pack all number term into one coefficient we get

$$S_n[f](x) = \sum_{j=0}^{n} C_{f,j}p_j(x) \in \operatorname{span}\{p_0, \dots, p_n\} = P_n$$

so it's rank is less than $n+1$ so finite rank.

(b): Notice that for each $0 \le i \le n$ we have

$$S_n[p_i](x) = \sum_{j=0}^{n}\frac{p_j(x)}{||p_j||_2^2}\int_{-1}^{1}p_j(t)p_i(t)dt = \frac{\int_{-1}^{1}p_i(t)^2dt}{||p_i||_2^2} = 1.$$

Thus the eigenvectors are p_i and all eigenvalues are 1. And for the other dimensions they're all in the kernel, so all other infinitely many eigenvalues are 0.

(c):

Note first that H is self-adjoint too because it's just a multiplication operator, i.e.

$$\langle Hf, g\rangle = \int_{-1}^{1} H(t)f(t)g(t)dt = \langle f, Hg\rangle$$

and thus we have

$$\langle S_n[H[f]], g\rangle = \langle H[f], S_n[g]\rangle = \langle f, H[S_n[g]]\rangle$$

and similarly

$$\langle H[S_n[f]], g\rangle = \langle f, S_n[H[g]]\rangle$$

so summing up we get

$$\langle (S_nH + HS_n)f, g\rangle = \langle f, (S_nH + HS_n)g\rangle$$

so $T_n^* = T_n$ is self-adjoint.

As for finite dimension, let's defined the $2n+2$ dimensional space such that is

$$S = \operatorname{span}\{p_0, \dots, p_n, Hp_0, \dots, Hp_n\}$$

Then first we know $S_n[H[f]] \in S$ for the same reason as in (a). As for $H[S_n[f]] \in S$, this is first because $S_n[f] \in S$ and applying the multiplier shifts this into $\operatorname{span}\{Hp_0, \dots, Hp_n\} \subset S$. Thus it's at most $2n+2$ dimensional.

(d):

Compute we get

$$\begin{aligned} T_n[f] &= \sum_{j=0}^{n} \frac{p_j(x)}{||p_j||_2^2} \int_{-1}^{1} p_j(t)f(t)H(t)dt + H(x)\sum_{j=0}^{n} \frac{p_j(x)}{||p_j||_2^2} \int_{-1}^{1} p_j(t)f(t)dt \\ &= \int_{-1}^{1} \sum_{j=0}^{n} \frac{p_j(x)p_j(t)}{||p_j||_2^2}(H(t) + H(x))f(t)dt \end{aligned}$$

and thus the trace is

$$\int_{-1}^{1} \sum_{j=0}^{n} \frac{p_j(t)p_j(t)}{||p_j||_2^2}(H(t) + H(t))dt = 2\sum_{j=0}^{n} \int_{0}^{1} \frac{p_j(t)p_j(t)}{||p_j||_2^2}dt = n+1$$

because $\frac{p_j(t)p_j(t)}{||p_j||_2^2}$ is even in t, since each polynomial is either even or odd.

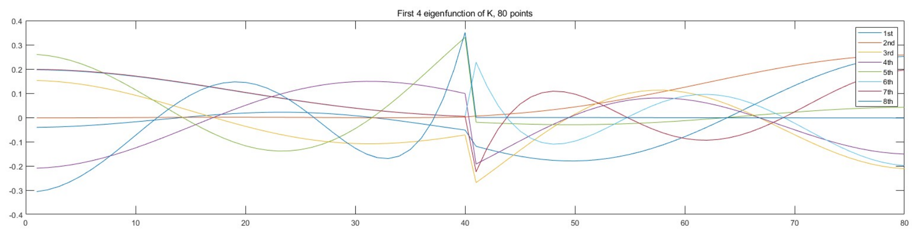
(e): Just use the Legendre quadrature if we denote

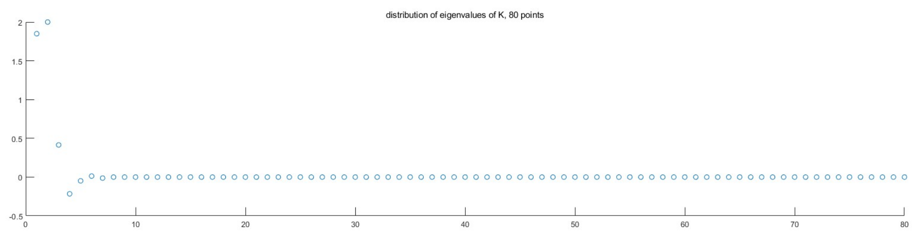
$$k(x, y) = \sum_{j=0}^{n} \frac{p_j(x)p_j(y)}{||p_j||_2^2}(H(x) + H(y))$$

then we know

$$\int_{-1}^{1}\sum_{j=0}^{n}\frac{p_j(x)p_j(t)}{||p_j||_2^2}(H(t)+H(x))f(t)dt=\sum_{i=1}^{80}w_i k(x_i,x_j)f(x_i)$$

thus we construct the matrix and diagonalize it to get eigenfunctions and eigen values (generated by file "TommenixYu_q1e"):





and to check that the trace is $n+1=4$ we just report the trace, which is indeed 4 (generated by file "TommenixYu_q1e"):

```
ans =

   4.00000000000002
```

$\square$

Exercise 2.

Proof.

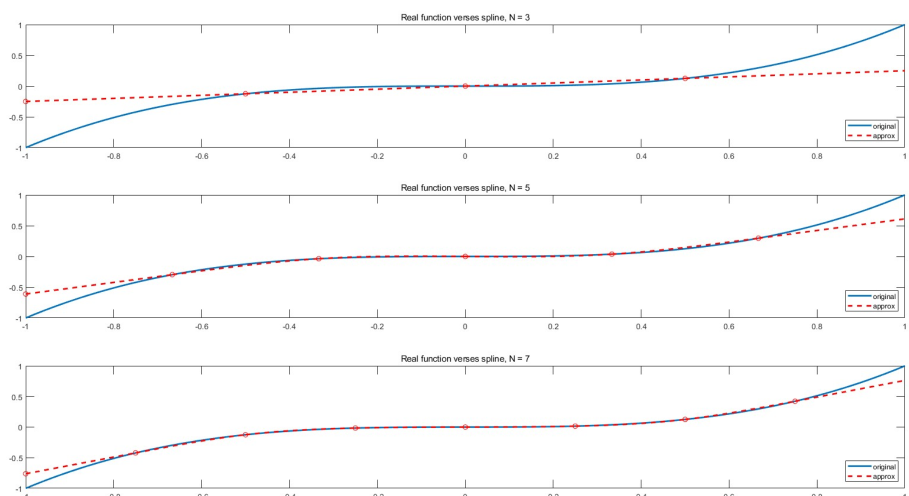
(a):

Notice that on each piece we just pick $y = x/4$, then it is a set of spline interpolation that satisfies the condition we need (both ends are linear, interpolation points match, first and second derivatives match).

Moreover, we know such spline is unique, so the only result is just $y = x/4$ on all parts.

(b):

The results are (both generated by file "TommenixYu_2b"):



and the coefficients for each piece are represented as rows in the following matrices:

```
>> TommenixYu_q2b
Coefficients (rows) for N = 3

C =

         0         0    0.2500         0
         0         0    0.2500         0
         0         0    0.2500         0
         0         0    0.2500         0

Coefficients (rows) for N = 5

C =

         0         0    0.9444    0.3333
   -1.5000   -3.0000   -1.0556   -0.1111
    1.5000         0   -0.0556         0
    1.5000         0   -0.0556         0
   -1.5000    3.0000   -1.0556    0.1111
         0         0    0.9444   -0.3333

Coefficients (rows) for N = 7

C =

         0         0    1.3625    0.6000
   -2.8000   -6.3000   -3.3625   -0.5813
    2.0000    0.9000    0.2375    0.0187
    0.8000         0    0.0125         0
    0.8000         0    0.0125         0
    2.0000   -0.9000    0.2375   -0.0187
   -2.8000    6.3000   -3.3625    0.5813
         0         0    1.3625   -0.6000
```

(c): From the plot above it's easy to see that the least approximated points are $x = \pm 1$. This is because

(1) It is the farthest point from any interpolating points (x_1 through x_n);
(2) All the middle points need to cope with both endpoints not only linearly, but with respect to 2nd derivatives;
(3) We are not asking for informations at the endpoints. So one could imagine a curve that explodes at the endpoints and that would not even affect our result.

Due to the above reasons it is reasonable the maximum is attained at the endpoints.

(d):

For $n > 3$ odd, we use the natural splines with end intervals are $\frac{n-1}{2}$ degree polynomials, and that at each middle point derivatives up to the $n-1$th match.

For n even we just add one more point close enough to x_n and use the natural spline for $n+1$. So we'll just focus on n odd below.

For the other results, yes they can be generated, as introduced below:

Proposition 0.1. *For $k = \frac{n+1}{2}$, given $f \in H^k$, $x_1, \dots, x_n \in [a, b]$ and $f_i = f(x_i)$. Then if s is the natural interpolant, then $\langle f^{(k)} - s^{(k)}, s^{(k)} \rangle = 0$.*

Proof. since $s^{(k)}$ is polynomial of order $k-1$ on each interval, we have

$$\int_{x_i}^{x_{i+1}} (f^{(k)} - s^{(k)}) s^{(k)} dx = -\int_{x_i}^{x_{i+1}} (f^{(k-1)} - s^{(k-1)}) s^{(k+1)} dx + (f^{(k-1)} - s^{(k-1)}) s^{(k)} \Big|_{x_i}^{x_{i+1}}$$

so we focus on the boundary term and note that since $f^{(k-1)}$ and $s^{(k-1)}$ both have $k-1$th smoothness at those points we end up with only $s^(k)$ at the endpoints, which is 0 because $s \in P_{(k-1)}$ for endpoints intervals.

As for the first term we can keep doing the above steps and get vanished boundary terms (last boundary vanish because $s = f$ at points) untill we get $s^{(n+1)}$ in the interval, for which we know vanishes. Thus the whole thing vanishes and we have orthogonality. □

Radial basis function

As for the radial basis function that's almost the same also: Now since we can write polynomials as polynomials plus each other

$$s(x) = \sum_{j=1}^{N} \alpha_j (x - x_j)_+^n + \sum_{j=0}^{n} \beta_j x^j$$

and the choice of natural polynomials on the first interval means that $\beta_{k+1} = \cdots = \beta_n = 0$. On the last interval we have

$$s(x) = \sum_{j=1}^{n} \alpha_j (x - x_j)^5 + \sum_{j=0}^{k} \beta_j x^j$$

and our assumptions that quadratic and cubic terms vanish on this inteval means

$$\sum_{j=1}^{n} \alpha_j = 0; \quad \sum_{j=1}^{n} \alpha_j x_j = 0; \quad \ldots \quad \sum_{j,\ldots,s} (\alpha_j x_j) \ldots (\alpha_s x_s) = 0$$

To be more clear we can write $x_+^n = \dfrac{|x|^n + x^n}{2}$ so on the whole interval we have

$$s(x) = \sum_{j=1}^{n} \frac{\tilde{\alpha}_j}{2} |x - x_j|^n + \sum_{j=0}^{k} \tilde{\beta}_j x^j$$

because the higher order terms vanish by above condition. So interpolating spline are of the form

$$s(x) = \sum \alpha_j \phi(|x - x_j|) + p$$

where $p \in P_k$. And here we've chosen $\phi = r^n$.

□