APPLIED DYNAMICAL SYSTEM HOMEWORK 5

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STAT 31410
DUE TUESDAY, NOV. 15, 11PM

General ideas were discussed with many classmates in casual talks. Especially, I discussed with Muyi Chen on the second problem, which he has really good codes that I learned from. And I worked with Tim Su on the third problem, to split the jobs.

Exercise 1.

The ODE system is

$$\begin{cases} \dot{x} = 2x + y^2 \\ \dot{y} = -2y + x^2 + y^2 \end{cases}$$

At around $x = 0$, there's two tangent manifolds $W^s_{loc}(0)$ and $W^u_{loc}(0)$. Let $x = h_s(y)$ on $W^s_{loc}(0)$ and $y = h_u(x)$ on $W^u_{loc}(0)$ then we know that if $(x_0, y_0) \in W^s_{loc}(0)$, i.e.

$$(x_0, y_0) = (h_s(y_0), y_0)$$

then

$$(x(t), y(t)) = (h_s(y(t)), y(t))$$

therefore we have

$$\dot{x} = h_s'(y(t)) \cdot \dot{y}$$

and by plugging in we have

$$2h_s(y) + y^2 = h_s'(y)(-2y + h_s^2(y) + y^2). \tag{1}$$

Now, by Taylor, for $|y| << 1$,

$$h_s(y) = h_s(0) + h_s'(0)y + \frac{1}{2}h_s''(0)y^2 + \dots$$

since $h_s(0) = h_s'(0) = 0$ by assumption and theorem, we have

$$h_s(y) = \alpha y^2 + \beta y^3 + \gamma y^4 + \dots$$

then by plugging back in (1) and matching power, then

$$\begin{aligned} 2\alpha y^2 + 2\beta y^3 + 2\gamma y^4 + O(y^5) + y^2 &= (2\alpha y + 3\beta y^2 + 4\gamma y^3)(-y + y^2 + O(y^4)) \\ &= -4\alpha y^2 + (2\alpha - 6\beta)y^3 + (3\beta - 8\gamma)y^4 + O(y^5) \end{aligned}$$

which yields

$$\alpha = -\frac{1}{6}, \beta = -\frac{1}{24}, \gamma = -\frac{1}{80}.$$

Using the same method, I compute the unstable manifold here too.

We have with a similar method the following:

$$\dot{y} = h_u(x(t))\dot{x}$$

which is equivalent to

$$-2h_u(x) + h_u^2(x) + x^2 = h_u'(x)(2x + h_u(x)^2)$$

with the same Taylor approximation we assume

$$h_u(x) = \alpha x^2 + \beta x^3 + \gamma x^4 + \dots$$

Writing it out we get:

$$-2\alpha x^2 - 2\beta x^3 - 2\gamma x^4 + O(x^5) + \alpha^2 x^4 + O(x^5) + x^2 = (2\alpha x + 3\beta x^2 + 4\gamma x^3 + O(x^4))(2x + O(x^4))$$

which means

$$(-2\alpha + 1)x^2 - 2\beta x^3 + (-2\gamma + \alpha^2)x^4 + O(x^5) = 4\alpha x^2 + 6\beta x^3 + 8\gamma x^4 + O(x^5)$$

and which yields

$$\alpha = \frac{1}{6}, \beta = 0, \gamma = \frac{1}{360}.$$

Plotting the quiver plot, streamslice plot,the local stable manifold curve

$$x = -\frac{1}{6}y^2 - \frac{1}{24}y^3 - \frac{1}{80}y^4$$

and the local unstable manifold curve

$$y = \frac{1}{6}x^2 + \frac{1}{360}x^4$$

gives the following graph (local stable manifold is the green curve, unstable manifold is the red curve):

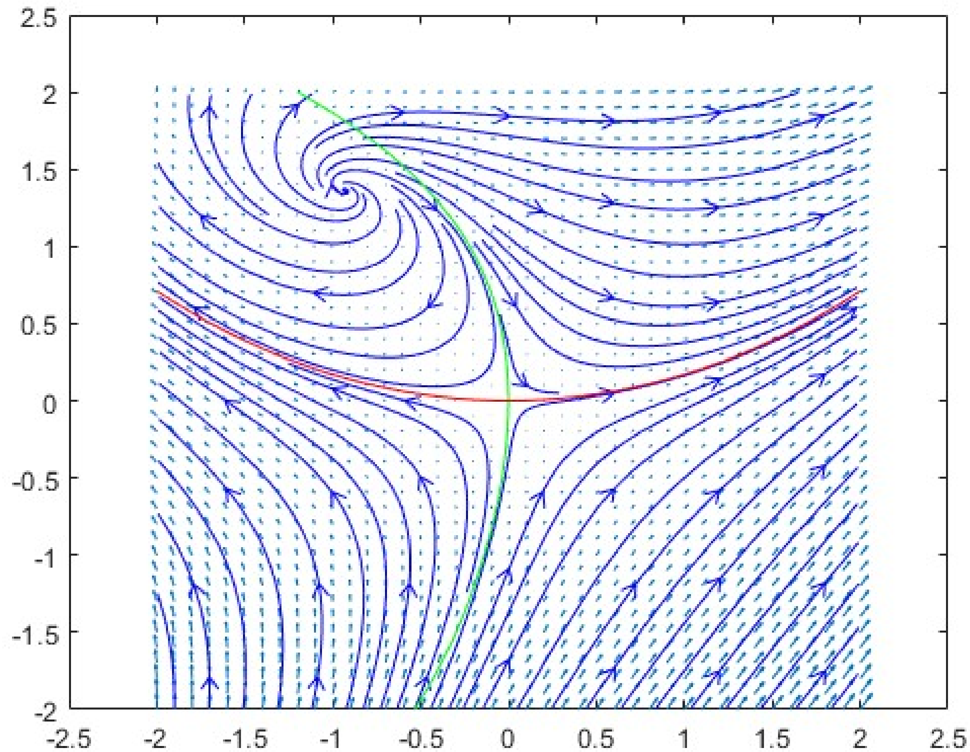
(see next page)

To get the global stable manifold, we only need to compute

$$\{\phi_t(x) | x \in W_{loc}^s \cap U, t \leq 0\}$$

for the expression of W_{loc}^s above and small enough U.

To do this in practice, we just need to invert time and plot the curve for a point on the local stable manifold, say $(h_s(y), 0.001)$ and $(h_s(y), -0.001)$.



Which is the same as the one in the book.

Exercise 2.

The ODE system is

$$\dot{N} = rN\left(1 - \frac{c}{r}N\right)\left(\frac{N-\mu}{\nu+N}\right) - \frac{\alpha N P}{\beta + N}$$
$$\dot{P} = \chi\frac{\alpha N P}{\beta + N} - \delta P$$

(a): N is the prey and P is the predator. This is because if $N = 0$, then $\dot{P} = -\delta P \leq 0$, which exactly represents lack of food.

Contrarily, if $P = 0$, $\dot{N} = rN\left(1 - \frac{c}{r}N\right)\left(\frac{N-\mu}{\nu+N}\right)$ which is a good curve by itself. The existence of P will cause some negative trend to the population.

The strong Allee effect this model represents says that the the growth of a population is fast when the population is small, slow when larger, and negative when reached maximum capacity.

We can see that this fits the equation for N well: the first term of $\dot{N}$ is

$$\frac{-cN^3 + (r + c\mu)N^2 - r\mu N}{N + \nu} \sim -cN^2 + (r + c\mu)N - r\mu =: P(N)$$

which is a downward quadratic equation with zero points $N = \dfrac{r + c\mu \pm (r - c\mu)}{2c} = \mu$ or $\dfrac{r}{c}$, and thus (note that all values are not precise because we neglected the ν term, which is small, but this gives us the behavior):

- Since μ is very small, we won't bother with $N \leq \mu$;
- When N is smaller than roughly $\left(\frac{r}{c} + \mu\right)/2$, $\dot{N}$ is positive and thus N is increasing;
- When N is roughly between $\left(\frac{r}{c} + \mu\right)/2$ and $\frac{r}{c}$, $\dot{N}$ is positive but is decaying, which represents survival competition when the population is larger;
- When N is larger than the capacity $\approx \frac{r}{c}$, then the population is decreasing.

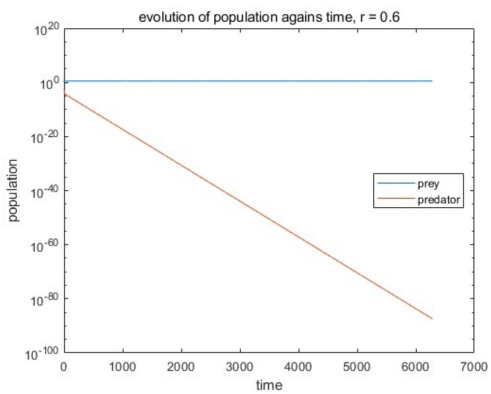
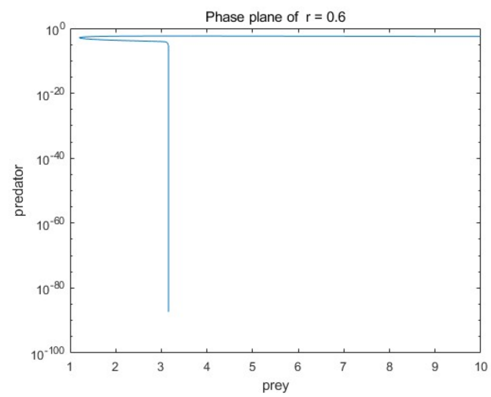
Thus, it satisfies the Strong Allee effect.

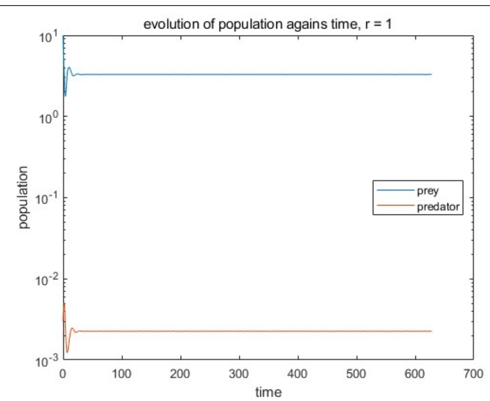
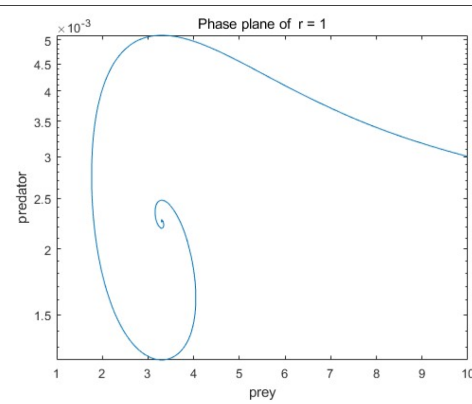
One observation is that the the parameter χ represents the conversion rate of food to population of predator. More specifically, each predator need to consume $\dfrac{1}{\chi}$ number of preys to survive.

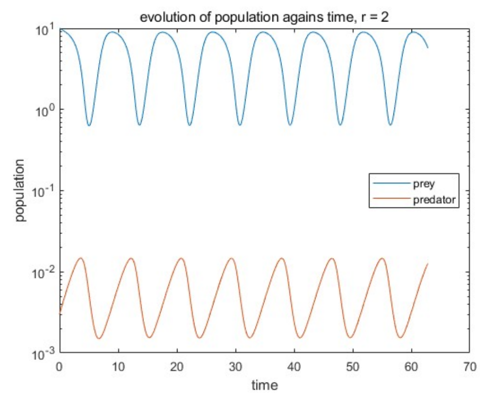
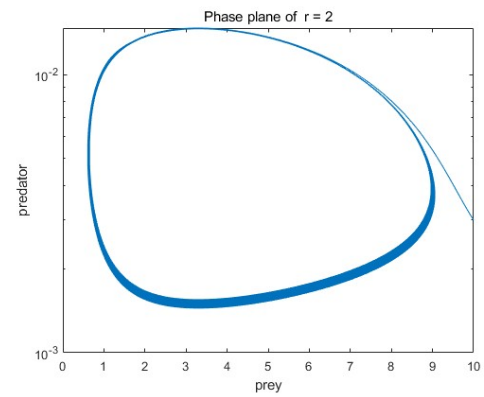
(b):

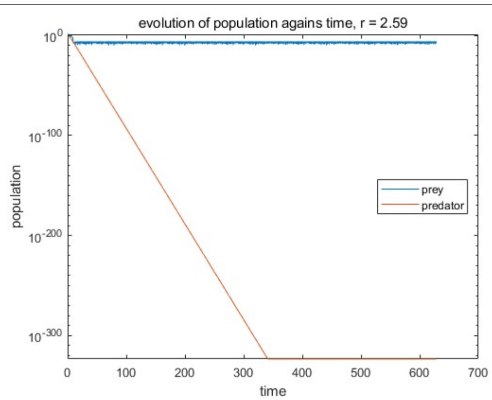
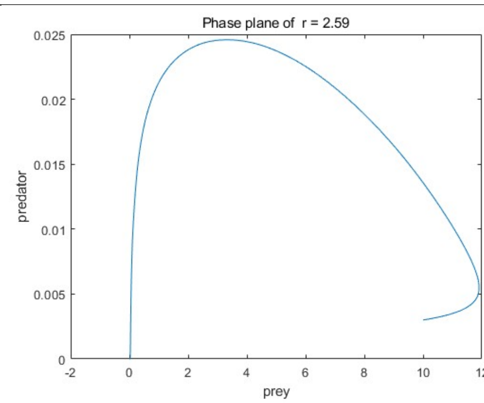
A good thing about the analysis above is that we know the population of prey versus predators is about $\chi = 0.004$, so the difference between the populations is at least around 200 times. This tells us that we should use a semilogy plot to plot the population vs. time plot, which is what I did to find the bifurcation parameter. Also, it helps me to choose a suitable starting point $(10, 0.003)$, which, for instance

I find that there are roughly 4 types of behaviors of the populations with different choice of parameters $0 < r < 3$:

<table>
<tr><td>$r = 0.6$</td><td>$r = 0.6$</td></tr>
<tr><td>Population - time plot</td><td>Phase space plot</td></tr>
<tr><td></td><td></td></tr>
<tr><td>decreasing</td><td>towards the equilibrium at $P = 0, N \approx 3$</td></tr>
</table>

<table>
<tr><td>$r = 1$</td><td>$r = 1$</td></tr>
<tr><td>Population - time plot</td><td>Phase space plot</td></tr>
<tr><td></td><td></td></tr>
<tr><td>stable population</td><td>The creation of a new equilibrium at about $N \approx 3.5, P \approx 0.002$</td></tr>
</table>

$r = 2$	$r = 2$
Population - time plot	Phase space plot
	
Oscillation; Nice delay affect: predator's population reaches peak and valley sometimes after the prey.	The creation of a limit cycle. Or conversely, the elimination of the limit cycle as r decreases.

$r = 2.59$	$r = 2.59$
Population - time plot	Phase space plot
	
starts to decay since the orbit goes into 0.	The destruction of a limit cycle. This is the exact graph of homoclinic bifurcation.

The critical points I found were $r_1 = 0.6495$ and

$$r_H = 1.5 \quad \text{and} \quad r_h = 2.59.$$

I will explain the last 2 first and the first one later.

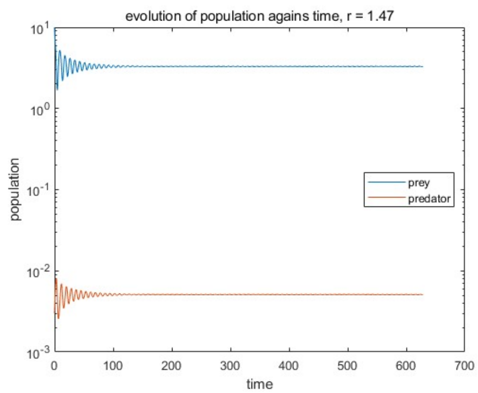
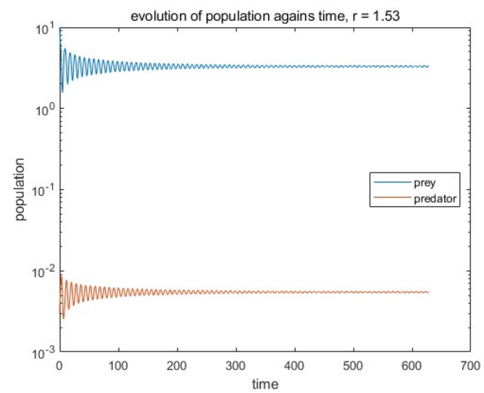
There are in general two ways to find the bifurcation point numerically, one is to see via the population versus time plot, and the other is to see the curve in the phase plane and check what's happening there.

It turns out that population-time plot is good for checking the large scale behavior rather than actually finding the bifurcation point.

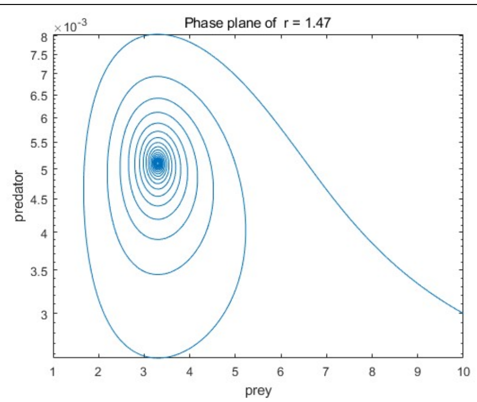
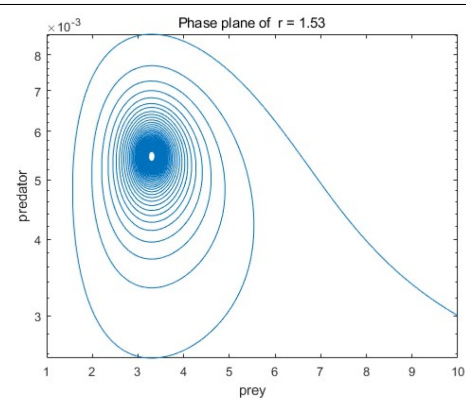
First, I tried to find the Hopf Bifurcation point.

We know from the beginning that this time the behavior should go from a steady line to an oscillating trend, and the phase plane should go from an equilibrium to a limit cycle.

The $r_H = 1.5$ I find is hard to recognize with population time plot, as is seen:

<table>
<tr><td>Population time plot</td><td>Population time plot</td></tr>
<tr><td>$r = 1.47$</td><td>$r = 1.53$</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>oscillation with smaller amplitude</td><td>oscillation with a little larger amplitude</td></tr>
</table>

So we can see the trend of going to a line by smaller and smaller amplitude, but this doesn't indicate anything peculiar at this point. The phase plane plot is much more convincing:

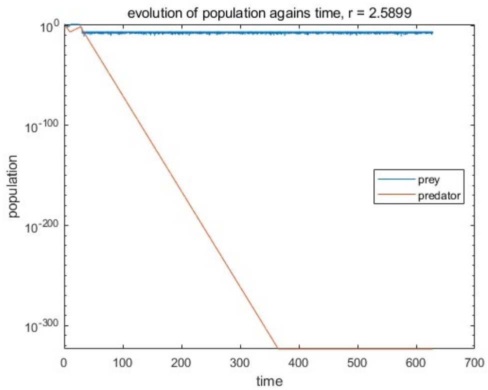
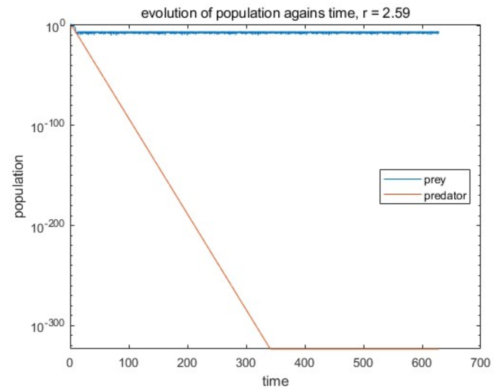
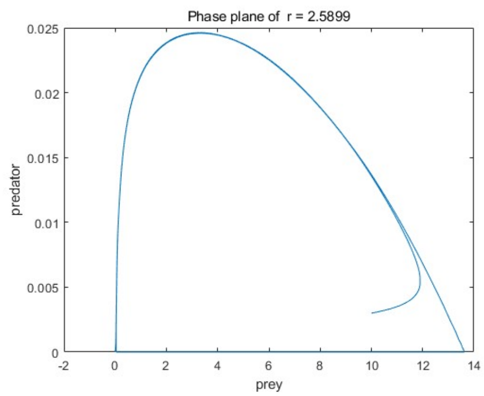
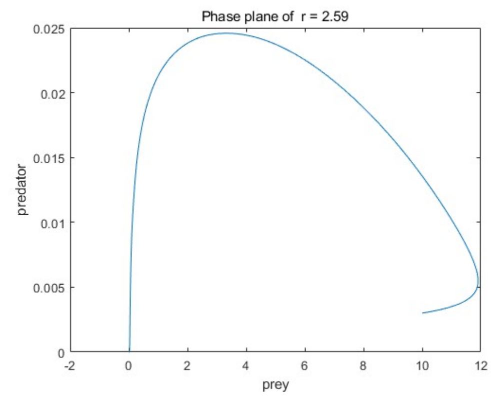
<table>
<tr><td>Phase space plot</td><td>Phase space plot</td></tr>
<tr><td>$r = 1.47$</td><td>$r = 1.53$</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>oscillation with smaller amplitude</td><td>oscillation with a little larger amplitude</td></tr>
</table>

So we can identify the creation/elimination of a limit cycle at around

$$r_H = 1.5.$$

Now let's find the homoclinic bifurcation point.

The $r_h = 2.59$ I find is easy to find through both plots, as is below:

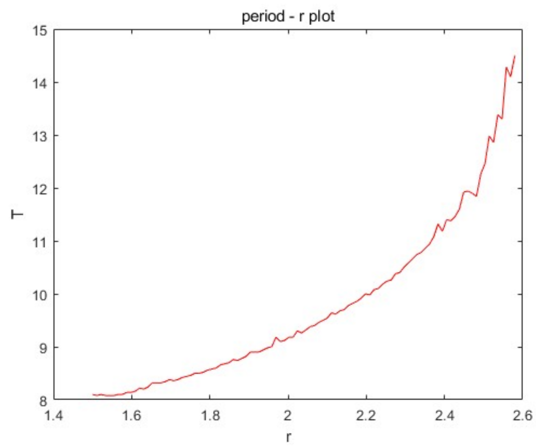
<table>
<tr><td>$r = 2.5899$</td><td>$r = 2.59$</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>There's an initial occurrence of oscillation</td><td>oscillation is gone</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>a limit cycle there is</td><td>no cycles anymore</td></tr>
</table>

So we can identify the creation/elimination of a limit cycle at around

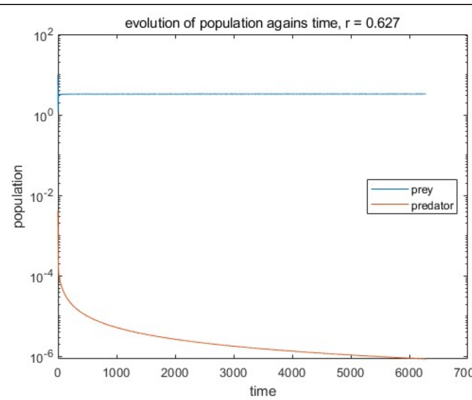
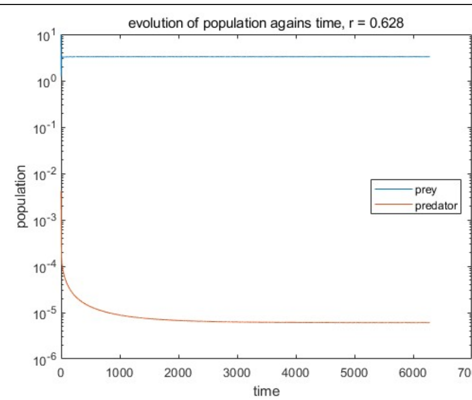
$$r_h = 2.59.$$

It's doable to get r_H analytically by simply using the linearized version of the system, as is taught in class today. Indeed they are both around 8.1 in my output (for which my find period function for the whole system gives exactly 8.1).

Now the question is to find the period against the change of r. So I write up a check of when N goes through the same point the third time (since it's oscillating like sine, and I mean the first time to be at time 0), and the result is (discussed with Muyi Chen):

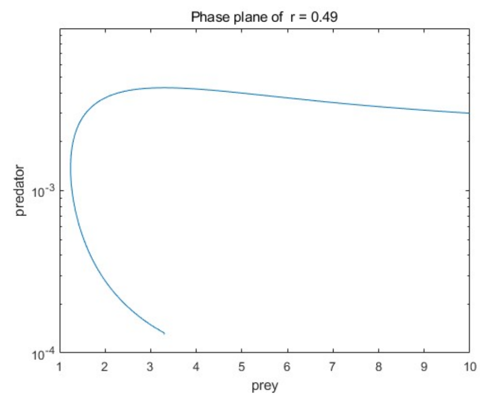
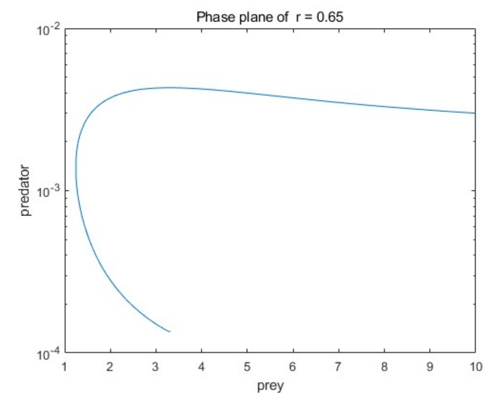
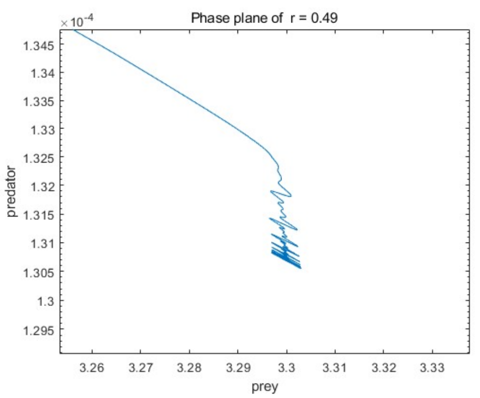
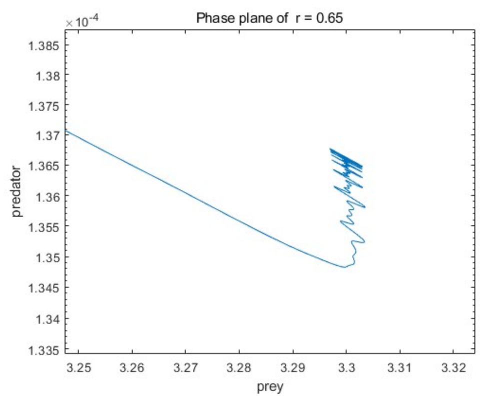


At last, let's find why there's a beginning phase when $r \leq 0.6495$.

Population time plot	Population time plot
$r = 0.627$	$r = 0.628$
	
There's a good amount of decreasing	it seems flat.

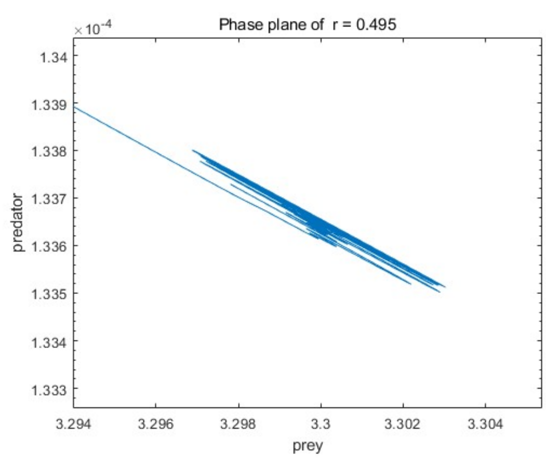
I find that it is relatively hard to find the critical point here with the population-time plot, since it's asking my eyes to detect whether a horizontal line in the semilogy plot is decreasing or genuinely horizontal. Also, it turns out that there's a relatively obvious change at $r = 0.628$ (above graph).

However, when I checked their phase plane plot I realized that's just due to the log-size my time reached, as both time's phase curve looks exactly the same. But by changing r bit by bit I've reached the point $r_1 = 0.6495$:

<table>
<tr><td>Population time plot</td><td>Population time plot</td></tr>
<tr><td>$r = 0.649$</td><td>$r = 0.65$</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>its tail:</td><td>its tail:</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>tail going down</td><td>tail going up</td></tr>
</table>

From above, we know that there's a new equilibrium created somewhere above so that the curve's tail goes up on the right and down on the left.

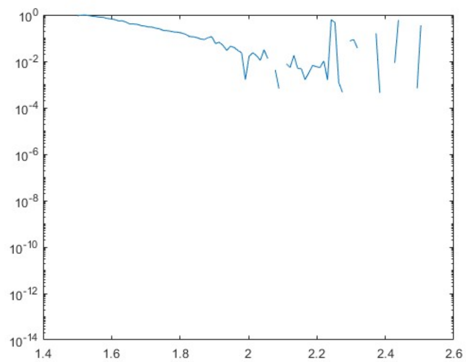
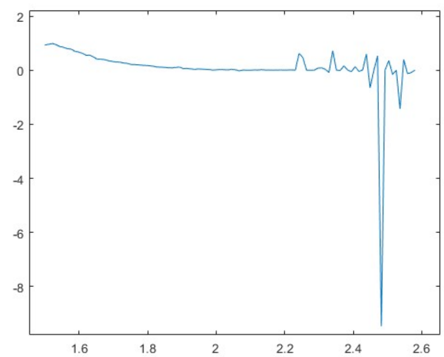
A close look at $r_1 = 0.6495$ gives the tail as this:



which I think represents that the equilibrium below is as strong as the equilibrium above. Therefore there's a critical point here.

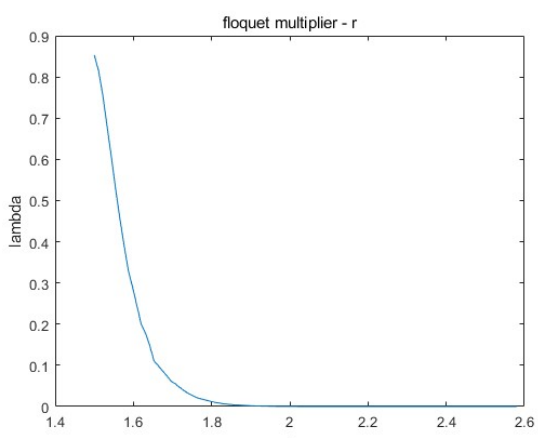
(c):

I did it first following the method from last homework, i.e. using two extra lines to simulate the ODE on the solution. The result I get is not particularly good:

not 1 floquet multiplier	not 1 floquet multiplier
	
Using semilogy plot	Using regular plot.

It turns out that it's gradually unstable when $r \to r_h$. This is because (as matlab directly told me) that eigenvalue to the simulated monodromy matrix is complex, and to plot it can only do the real part of it. Also, there might be some sharp turning points in the plot as r is larger, (see plots around r_h), so the unreliable behavior is with reason.

After discussion with Muyi Chen, I used his method of writing up a little code of symbolic jacobi and use that to compute the floquet multiplier using Abel's theorem. The result this time is fantastic compared to what I got above:



Exercise 3. *(Worked together with Tim Su)*

In this question, we want to prove theorem 5.9. In the setting of the statement of the theorem, it is already assumed that the equilibrium is shifted to the origin by a map $x \rightsquigarrow x+x^*$ where x^* is the unshifted equilibrium.

Theorem 1. *(Local Stable Manifold) Let A be hyperbolic, $g \in \mathbb{C}^k(U)$, $k \geq 1$, for some neighborhood U of O, and $g(x) = o(x)$ as $x \to 0$. Denote the linear stable and unstable subspaces of A by E^s and E^u. Then there is a $\tilde{U} \subset U$ such that local stable manifold of*

$$\dot{x} = Ax + g(x) \tag{2}$$

is

$$W^s_{loc}(0) \equiv \{x \in W^s(0) : \phi_t(x) \in \tilde{U}, t \geq 0\}$$

is a Lipschitz graph over E^s that is tangent to E^s at 0. Moreover, $W^s_{loc}(0)$ is a C^k manifold.

Proof. We divide the proof into three parts:

(1) We prove that there is a unique, forward bounded solution for each point $\sigma \in E^s$ close enough to the origin;
(2) We then show that these solutions actually are on the stable manifold, since they are asymptotic to 0;
(3) We show that these solutions lie on a smooth, Lipschitz graph.

Part 1: We prove that there is a unique, forward bounded solution for each point $\sigma \in E^s$ close enough to the origin, i.e. there exists a unique point $x_\sigma \in \mathbb{R}^n$ that corresponds to σ such that $\phi_t(x_\sigma)$ is forward bounded.

If $y : t \rightsquigarrow y(t)$ is a function $y : \mathbb{R}^+ \to \mathbb{R}^n$, define

$$T : y(t) \rightsquigarrow T(y)(t)$$

an operator $T : C^0(\mathbb{R}^+, \mathbb{R}^n) \to C^0(\mathbb{R}^+, \mathbb{R}^n)$ for a given point $\sigma \in E^s$ of A by

$$T(y)(t) = e^{tA}\sigma + \int_0^t e^{(t-s)A}\pi_s g(y(s))ds - \int_t^\infty e^{(t-s)A}\pi_u g(y(s))ds \tag{3}$$

where π_s and π_u are projections such that

$$\begin{cases} \pi_s : \mathbb{R}^n \to E^s \\ \pi_u : \mathbb{R}^n \to E^u. \end{cases}$$

First, note that T is a self map, which is part of the condition needed for the contraction proof later. Then, note that since y is C^0, since g, π_s, π_u are all C^k, $T(y)$ is C^0.

We will show the following 3 things to conclude part 1:

(a) show that the solution x to (2) is a fixed point of T, provided that the initial condition $\pi_s x(0) = \sigma$;
(b) the map T is a contraction. Since it's a self-map on a complete space $C^0(\mathbb{R}^+, \mathbb{R}^n)$, we can conclude that the solution is unique;
(c) for x defined in (a), $T(x)$ is forward bounded.

(a):

Let $x : t \rightsquigarrow x(t)$ be a solution to the ODE:

$$\begin{cases} \dot{x} = Ax + g(x) \\ \pi_s(x(0)) = \sigma \end{cases}$$

For the purpose of alignment, this is x_σ in the description of part 1. We just use x to denote it for simplicity.

Now we show that $T(x) = x$. We show it by showing

- For x fixed, x satisfies the ODE that $T(x)$ satisfies.
- $T(x)(0) = x(0)$.

Since the solution to the ODE is uniquely defined by the evolution and initial condition, we know that they are the same function, which means x is a fixed point of T.

To show $T(x)$ and x satisfy the same ODE in t, we just take the derivative to $T(x)$:

$$\begin{aligned}
\frac{dT(x)}{dt}(t) =& \frac{d}{dt}\left(e^{tA}\sigma + e^{tA}\int_0^t e^{-sA}\pi_s g(y(s))ds + e^{tA}\int_\infty^t e^{-sA}\pi_u g(y(s))ds\right) \\
=& Ae^{tA}\sigma + Ae^{tA}\int_0^t e^{-sA}\pi_s g(y(s))ds + e^{tA}e^{-tA}\pi_s g(x(t)) \\
&+ Ae^{tA}\int_\infty^t e^{-sA}\pi_u g(y(s))ds + e^{tA}e^{-tA}\pi_u g(x(t)) \\
=& A\left(e^{tA}\sigma + e^{tA}\int_0^t e^{-sA}\pi_s g(y(s))ds + e^{tA}\int_\infty^t e^{-sA}\pi_u g(y(s))ds\right) \\
&+ (\pi_s + \pi_u)g(x(t)) \\
=& AT(x)(t) + g(x(t))
\end{aligned}$$

for which, since $\dot{x}(t) = Ax(t) + g(x(t))$, if $T(x)(0) = x(0)$ we can conclude that they are the same function as the evolution in time is the same.

Now, we show $T(x)(0) = x(0)$:

$$T(x)(0) = \sigma + 0 - \int_0^\infty e^{-sA}\pi_u g(y(s))ds$$

where

$$\begin{aligned}\int_0^\infty e^{-sA}\pi_u g(x(s))ds &= \int_0^\infty e^{-sA}\pi_u(\dot{x}(s) - Ax(s))ds \\ &= \int_0^\infty e^{-sA}\pi_u\dot{x}(s)ds - \int_0^\infty e^{-sA}\pi_u Ax(s)ds \\ \left(\text{using } ds \mapsto de^{-sA}\right) \qquad &= \int_0^\infty e^{-sA}\pi_u\dot{x}(s)ds + (A^{-1}A)\int_0^\infty \pi_u(x(s))de^{-sA} \\ &= \int_0^\infty e^{-sA}\pi_u\dot{x}(s)ds + \pi_u(x(s))e^{-sA}\Big|_0^\infty - \int_0^\infty e^{-sA}\pi_u\dot{x}(s)ds \\ &= \pi_u(x(s))e^{-sA}\Big|_0^\infty = O(e^{-\infty}) - \pi_u(x(0)) = \pi_u(x(0))\end{aligned}$$

where we've used $A\pi_u = \pi_u A$ since E^u is invariant under A.

Plugging back we get

$$T(x)(0) = \sigma + 0 - \int_0^\infty e^{-sA}\pi_u g(y(s))ds = \sigma + \pi_u(x(0)) = x(0)$$

which concludes (a).

(b): In order to prove the map T is a contraction, we need to first review an inequality, namely:

$$\begin{cases} |e^{tA}\pi_s x| \le Ke^{-\alpha t}|\pi_s x| \\ |e^{-tA}\pi_u x| \le Ke^{-\alpha t}|\pi_u x| \end{cases} \tag{4}$$

where π_s and π_u are the same projections defined in part 1.

(**Note**: This in-equality can be found on our Oct 17th lecture notes)

Now, since $g \in C^k(U)$ for $k \ge 1$, which means g is C^1. Then, by definition, let $\epsilon > 0$, there exists some $\delta > 0$ such that $|x| \le \delta$ implies $||Dg(x)|| \le \epsilon$, where $||\cdot||$ is the sup-norm (i.e. $||Dg(x)|| = sup_{x\in U}|Dg(x)|$).

By the mean value theorem in higher dimensions, there exists some $c \in (x, y)$, where $x, y \in B_\delta(0)$ (the open ball centered at 0 with radius δ, the same δ as above) such that:

$$|g(y) - g(x)| \le |Dg(c)||y - x| \le \epsilon|y - x| \tag{5}$$

(**Note**: The equality case only holds if U is convex)

Thus, by substitution and the basic inequality of integrals, we can bound our map T defined in part 1 as:

$$|T(y) - T(x)| \le \int_0^t |e^{(t-s)A}\pi_s[g(y(s)) - g(x(s))]|ds + \int_t^\infty |e^{(t-s)A}\pi_u[g(x(s)) - g(y(s))]|ds$$

Applying inequalities (4), we obtain the bound below:

$$|T(y)-T(x)| \leq \int_0^t Ke^{-\alpha(t-s)}|\pi_s[g(y(s))-g(x(s))]|ds+\int_t^\infty Ke^{(t-s)\alpha}|\pi_u[g(x(s))-g(y(s))]|ds$$

Now, by inequality (5) derived earlier, we can bound this as:

$$|T(y)-T(x)| \leq \int_0^t Ke^{-\alpha(t-s)}\varepsilon|y-x|ds + \int_t^\infty Ke^{\alpha(t-s)}\varepsilon|y-x|ds \tag{6}$$

Thus, we have the formal bound below:

$$|T(y)-T(x)| \leq K\varepsilon|y-x|[\int_0^t e^{-\alpha(t-s)}ds + \int_t^\infty e^{\alpha(t-s)}ds]$$

(We can pull K, ε, and $|y-x|$ out since they have nothing to do with s)

Now, the first integral is easy to compute, the second one can be computed using the definition of an improper integrals, which gives:

$$\int_t^\infty e^{\alpha(t-s)}ds = \lim_{z\to\infty}\int_t^z e^{\alpha(t-s)}ds \tag{7}$$

$$= e^{\alpha t}\lim_{z\to\infty}(\frac{1}{\alpha}e^{-\alpha t} - \frac{1}{\alpha}e^{-z\alpha}) \tag{8}$$

$$= \frac{1}{\alpha} \tag{9}$$

Thus, by substitution, we have:

$$|T(y)-T(x)| \leq K\varepsilon|y-x|(\frac{2}{\alpha} - \frac{1}{\alpha}e^{-\alpha t}) \tag{10}$$

$$\leq K\varepsilon\frac{2}{\alpha}|y-x| \tag{11}$$

Thus, we can conclude that T is a contraction when $\varepsilon < \dfrac{\alpha}{2K}$, so that by contraction mapping principle, we know that T has a unique fixed point.

(c): We prove that $T(x)$ is forward bounded in this section for x defined in part(a). Let's start by defining a closed subset of the function space $C^0(\mathbb{R}^+)$ by

$$V_\delta = \{x \in C^0(\mathbb{R}^+, \mathbb{R}^n) : ||x|| \leq \delta\}$$

where δ here is the same δ used in part(b) and $||\cdot||$ is still the sup-norm.

(**Note**: We define this set in order to give a bounded x)

Then, since $g(x)$ is $o(x)$, recall the definition of $o(x)$, we know that for all $\varepsilon_1 > 0$, there exists some δ_0 such that when x is in the neighborhood of $N(\delta_0)$, we have $|g(x)| \leq \varepsilon_1|x|$. Now, choose $\delta_0 = \delta$ and $N(\delta_0) = V_\delta$ defined above, we have:

$$|T(x)(t)| \leq |e^{tA}\sigma + \int_0^t e^{(t-s)A}\pi_s\varepsilon_1|x(s)|ds - \int_t^\infty e^{(t-s)A}\pi_u\varepsilon_1|x(s)|ds \tag{12}$$

Imposing inequality (4) in part(b), we can bound $|T(x)(t)|$ as:

$$|T(x)(t)| \leq Ke^{-\alpha t}|\sigma| + K\varepsilon_1 \int_0^t e^{-(t-s)\alpha}|x(s)|ds + K\varepsilon_1 \int_t^\infty e^{(t-s)\alpha}|x(s)|ds \qquad (13)$$

(**Note**: The sign flips when we imposing inequality 3 to the last integral in 11 because we implicitly used the triangle inequality. (i.e. $|-\int_t^\infty e^{(t-s)A}\pi_u\varepsilon_1|x(s)|ds| = |\int_t^\infty e^{(t-s)A}\pi_u\varepsilon_1|x(s)|ds|$))
Recall our definition of V_δ and our computation of integrals in part(b), we can easily get a explicit upper bound of $|T(x)(t)|$ as:

$$|T(x)(t)| \leq Ke^{-\alpha t}|\sigma| + K\varepsilon_1\delta(\frac{1}{\alpha} - \frac{1}{\alpha}e^{-\alpha t}) + K\varepsilon_1\delta\frac{1}{\alpha} \leq K|\sigma| + \frac{2K\varepsilon_1}{\alpha}\delta$$

Then, we can bound our $T(x)$ with δ by simply choosing $|\sigma| \leq \frac{\delta}{2K}$ and $\varepsilon_1 \leq \frac{\alpha}{4K}$. Since $g(x)$ is $o(x)$, the bound on ε_1 also defines the neighborhood of x:

$$\left\{x : |g(x)| \leq \frac{\alpha}{4K}|x|\right\} \cap U$$

which gives a effective definition on our bound on $T(x)$ (i.e. δ).

Thus, we conclude that $T(x)$ is forward bounded by δ.

If we combine our part(b) and part(c), we can observe that our map T is a contraction only when $\varepsilon < \frac{\alpha}{2K}$ and is forward bounded by δ only when $\varepsilon_1 \leq \frac{\alpha}{4K}$. Thus, we can conclude that our T is still a contraction if we let $\varepsilon \leq \frac{\alpha}{4K}$, and it has a unique fixed point in the closed subset V_δ.

Also, let $|\sigma| \leq \frac{\delta}{2K}$, using the result above, we can conclude that there exists a unique fixed point (i.e. solution) for every $\sigma \in E^s$. (as long as $|\sigma| \leq \frac{\delta}{2K}$), which is what we want. □

Remark: The overall strategy in part 1 is to find an operator T that its fixed point solves the equation 1, showing that T is a contraction, and find an explicit forward bound of T. Our tools in the section(b) and section (c) are quite limited, namely the inequality 3 and the fact that $g(x)$ is $o(x)$. We utilize those tools (and the ,mean value theorem) and play around with our $\varepsilon - \delta$ argument to find a valid forward bound of T, and use the contraction-mapping principle to get the result we want.

Part 2: We then show that these solutions actually are on the stable manifold, since they are asymptotic to 0.

This is almost direct with the generalized Gronwall's lemma:

Lemma 2. *(Generalized Gronwall's lemma) Suppose α, M and L are non-negative, $L < \frac{\alpha}{2}$, and there is a nonnegative, bounded, continuous function $u : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying*

$$u(t) \leq e^{-\alpha t}M + L\int_0^t e^{-\alpha(t-s)}u(s)ds + L\int_t^\infty e^{\alpha(t-s)}u(s)ds$$

then $u(t) \leq \frac{M}{\beta}e^{-(\alpha - L/\beta)t}$, where $\beta = 1 - 2\frac{L}{\alpha}$.

We first see how this implies part 2 and then prove the lemma.

Again, as we've shown in (c) above, since $g(x) = o(x)$, $\forall \varepsilon > 0$, there exists a small neighborhood V_δ such that when $x \in V_\delta$

$$|g(x)| \leq \varepsilon |x|.$$

Another bound we'll use is (5.11) in textbook, which is proved in chapter 2.7, hence proof not the main focus here:

$$\begin{aligned} |e^{tA}\pi_s x| &\leq K e^{-\alpha t}|\pi_s x| \\ |e^{-tA}\pi_u x| &\leq K e^{-\alpha t}|\pi_u x| \end{aligned} \qquad \text{for } t \geq 0$$

where the flip of sign is to make E^u stable by inverting time.

Since we've shown x_σ is the unique solution of T, using bounds above yields (details in (c) above):

$$|x_\sigma(t)| \leq K e^{-\alpha t}|\sigma| + K\varepsilon \int_0^t e^{-\alpha(t-s)}|x_\sigma(s)|ds + K\varepsilon \int_t^\infty e^{\alpha(t-s)}|x_\sigma(s)|ds$$

and we can see that Gronwall's lemma directly apply since $x_\sigma(t) \in c^0$.

Matching up terms in the form of the lemma, let

$$u = |x_\sigma(t)|, L = K\varepsilon, M = K|\sigma|.$$

Using the same ε as in part 1 (b), we know $\varepsilon \leq \dfrac{\alpha}{4K}$, which means $L \leq \dfrac{\alpha}{4}$ and $\beta = 1-2\dfrac{L}{\alpha} \geq \dfrac{1}{2}$. This tells us $\dfrac{L}{\beta} \leq \dfrac{\alpha}{2}$ and hence by Gronwall's lemma we get

$$|x_\sigma(t)| \leq 2K e^{-\alpha t/2}|\sigma| \to 0$$

as $t \to \infty$.

So we only need to prove the lemma. Since the proof in the book is pretty clear I just write the same proof with some more explanation.

Proof. (of Gronwall's lemma)

Since u is bounded by assumption,

$$v(t) = \sup_{s>t} u(s)$$

is well defined and is non-increasing, whose limit is the limsup of u. The definition immediately gives us a way to use the condition in the lemma, i.e., $\forall t > 0, \varepsilon > 0$, $\exists T \geq t$ such that

$v(t) \le u(T) + \varepsilon$, plugging into the assumption we have

$$\begin{aligned}
v(t) \le u(T) + \varepsilon &\le e^{-\alpha T} M + L \int_0^T e^{-\alpha(T-s)} u(s) ds + L \int_T^\infty e^{\alpha(T-s)} u(s) ds + \varepsilon \\
&\le e^{-\alpha T} M + L \int_0^T e^{-\alpha(T-s)} u(s) ds + L \int_0^\infty e^{-s\alpha} u(T+s) ds + \varepsilon \\
&\le e^{-\alpha T} M + \left(L \int_0^t e^{-\alpha(T-s)} u(s) ds + L \int_t^T e^{-\alpha(T-s)} u(s) ds \right) \\
&\quad + L \int_0^\infty e^{-s\alpha} u(T+s) ds + \varepsilon \\
&\le e^{-\alpha T} M + L \int_0^t e^{-\alpha(T-s)} u(s) ds + 2\frac{L}{\alpha} v(t) + \varepsilon
\end{aligned}$$

where the last equation is because, since we know $u(s) \le v(t)$ and $u(s+T) \le v(t)$ for $s > t$, we can write out:

$$\begin{aligned}
&L \int_t^T e^{-\alpha(T-s)} u(s) ds + L \int_0^\infty e^{-s\alpha} u(T+s) ds \\
&\le L \cdot v(t) \left(e^{-\alpha T} \int_t^T e^{\alpha s} + \int_0^\infty e^{-\alpha s} ds \right) \\
&\le L \cdot v(t) \left(\frac{1}{\alpha} - \frac{1}{\alpha} e^{-\alpha(T-t)} + \frac{1}{\alpha} \right) \le 2\frac{L}{\alpha} v(t)
\end{aligned}$$

hence the bound on $v(t)$ is justified.

Rearranging the bound we get

$$\left(1 - 2\frac{L}{\alpha}\right) e^{\alpha t} v(t) \le e^{-\alpha(T-t)} M + L \int_0^t e^{-\alpha(T-t)} e^{\alpha s} u(s) ds + \varepsilon e^{\alpha t}$$

where by letting $\beta = 1 - 2\frac{L}{\alpha}$ and $z(t) = \beta e^{\alpha t} v(t)$ denote the left hand side and bound $e^{-\alpha(T-t)} \le 1$ we get

$$z(t) \le M + \varepsilon e^{\alpha t} + L \int_0^t \frac{z(s)}{\beta} ds$$

Now using Gronwall's formula 3.28 we can get it's time dependent version (exercise 3.11): if

$$g(t) \le c(t) + \int_0^t k(s) g(s) ds$$

for continuous $k(t) \ge 0$ and continuously differentiable non-decreasing c on the domain, then

$$g(t) \le c(t) \exp\left(\int_0^t k(s) ds \right).$$

In our case, let $z = g$, $c = M + \varepsilon e^{\alpha t}$, $k = \frac{L}{\beta}$. They satisfy the condition since exponential function is as smooth as you want and k is a positive constant. Applying the result above we get

$$z(t) \leq (M + \varepsilon) \exp\left(\int_0^t L/\beta ds\right) = (M + \varepsilon)e^{tL/\beta}$$

which is equivalent to

$$\beta e^{\alpha t} v(t) \leq (M + \varepsilon)e^{tL/\beta}$$

which, since I can take $\varepsilon \to 0$, we get

$$u(t) \leq v(t) \leq \frac{M}{\beta} e^{-\alpha t} e^{tL/\beta} = \frac{M}{\beta} e^{-(\alpha - L/\beta)t}$$

which is what we want.

□

Part 3:We show that these solutions lie on a smooth, Lipschitz graph.

Let's delay the proof of smoothness because it needs the Uniform Contraction Principle. It's quite easy to show that x_σ lies on a Lipschitz graph.

Proof. Let x_{σ_1} and x_{σ_2} be fixed point of T with σ value σ_1 and σ_2 correspondingly, by the result proved in part(1a), we know that:

$$x_{\sigma_1} - x_{\sigma_2} = e^{tA}(\sigma_1 - \sigma_2) + \int_0^t e^{(t-s)A}\pi_s[g(x_{\sigma_1}) - g(x_{\sigma_2})]ds + \int_t^\infty e^{(t-s)A}\pi_u[g(x_{\sigma_2}) - g(x_{\sigma_1})]ds \tag{14}$$

Then, by the definition of projection π_u, we have

$$\pi_u[x_{\sigma_1} - x_{\sigma_2}] = -\int_t^\infty e^{(t-s)A}\pi_u[g(x_{\sigma_1}) - g(x_{\sigma_2})]ds$$

(because we proved in part 1(c) that T is bounded, so this unstable projection should also be bounded, so all the σ terms have to be destroyed, otherwise it would be unbounded)

Now, by the inequality (6) derived earlier (in part I), we have the bound of:

$$|\pi_u[x_{\sigma_1} - x_{\sigma_2}]| = |-\int_t^\infty e^{(t-s)A}\pi_u[g(x_{\sigma_1}) - g(x_{\sigma_2})]ds| \leq K\varepsilon \int_0^\infty e^{(t-s)\alpha}|x_{\sigma_1} - x_{\sigma_2}|ds \tag{15}$$

Our goal now is to bound $|x_{\sigma_1} - x_{\sigma_2}|$, by expansion 13 and the inequality 12 (in part I), since $g(x)$ is $o(x)$, we have:

$$|x_{\sigma_1} - x_{\sigma_2}| \leq Ke^{-\alpha t}|\sigma_1 - \sigma_2| + K\varepsilon_1 \int_0^t e^{-(t-s)\alpha}|x_{\sigma_1} - x_{\sigma_2}|ds + K\varepsilon_1 \int_t^\infty e^{(t-s)\alpha}|x_{\sigma_1} - x_{\sigma_2}|ds \tag{16}$$

By the Generalized Gronwall's lemma proved in part 2, we have:

$$|x_{\sigma_1} - x_{\sigma_2}| \leq 2Ke^{-\frac{\alpha t}{2}}|\sigma_1 - \sigma_2|$$

(The derivation of the coefficient $2Ke^{-\frac{\alpha t}{2}}$ is shown in part 2 of our proof.)

By substitution, we have that:

$$|\pi_u[x_{\sigma_1} - x_{\sigma_2}]| \leq K\varepsilon \int_0^\infty e^{(t-s)\alpha} 2Ke^{-\frac{\alpha t}{2}} |\sigma_1 - \sigma_2| ds \tag{17}$$

By simple improper integration, we can arrive at:

$$|\pi_u[x_{\sigma_1} - x_{\sigma_2}]| \leq \frac{4K^2\varepsilon}{3\alpha} e^{-\frac{\alpha t}{2}} |\sigma_1 - \sigma_2| \tag{18}$$

Thus, we conclude that the solutions lie on a Lipschitz graph.

Now, for the smoothness of solutions, we mention without proof a result known as the *Uniform Contraction Principle*

Lemma 3. *(Uniform Contraction Principle) Let X and Y be closed subsets of two Banach Spaces (complete normed vector spaces) and let $T \in C^k(X \times Y, X), k \geq 0$ be a uniform contraction map. Then, there is a unique fixed point, $x(y) = T(x(y), y)$, where $x(y) \in X$ is a C^k function of $y \in Y$*

This lemma directly gives the smoothness of the fixed point of T since $g(x)$ is C^k implies that the T constructed in part(1) is also C^k, which means that our solutions lie on a smooth graph.

Another observation is that this lemma implies $D_\sigma(x_0)v \in E^s$ for all v (just take the Jacobian of x with respect to σ at $\sigma = 0$), which means that W^s is tangent to E^s, which completes the proof. □

Remark Part 3 of the proof is actually not that hard. One thing to keep in mind is that the solutions lie on a lipschitz graph basically means that the unstable projections are Lipschitz. Then we use all the bounds derived in part(1) to get the desired result.

I omit the proof of the uniform contraction principle because the higher dimension case (i.e. the fixed point is C^k for $k \geq 2$) is quite hard to prove.