MEASURE THEORETIC PROBABILITY III HW 1

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STAT 38300
DUE THU MAR 30TH, 2023, 11PM

Discussed with classmates.

Exercise 1.

Proof.

(⇐:)

Let $A \in \mathcal{G}$, $B \in \mathcal{H}$, $K \in \mathcal{I}$. Then we have, on the one hand

$$\begin{aligned}P(A \cap B \cap K) &= \int_{B\cap K} \mathbb{1}_A d\mathbb{P} = \int_{B\cap K} \mathbb{E}[\mathbb{1}_A|\mathcal{H} \wedge \mathcal{I}]d\mathbb{P} \overset{condition}{=} \int_{B\cap K} \mathbb{E}[\mathbb{1}_A|\mathcal{I}]d\mathbb{P} \\ &= \int_K \mathbb{1}_B \mathbb{E}[\mathbb{1}_A|\mathcal{I}]d\mathbb{P} = \int_K \mathbb{E}\left[\mathbb{1}_B \mathbb{E}[\mathbb{1}_A|\mathcal{I}]\Big|\mathcal{I}\right]d\mathbb{P} \\ &= \int_K \mathbb{E}[\mathbb{1}_B|\mathcal{I}]\mathbb{E}[\mathbb{1}_A|\mathcal{I}]d\mathbb{P}\end{aligned}$$

where the last step is since $\mathbb{E}[\mathbb{1}_B|\mathcal{I}]$ is $\mathcal{I}$ measurable, and taking out what is known.

But on the other hand

$$P(A \cap B \cap K) = \int_K \mathbb{1}_{A\cap B} d\mathbb{P} = \int_K \mathbb{E}[\mathbb{1}_{A\cap B}|\mathcal{I}]d\mathbb{P}$$

and this means that not only is $\mathbb{E}[\mathbb{1}_{A\cap B}|\mathcal{I}]$ $\mathcal{I}$ measurable by definition, it also satisfies that for any $K \in \mathcal{I}$ we have

$$\int_K \mathbb{E}[\mathbb{1}_{A\cap B}|\mathcal{I}]d\mathbb{P} = \int_K \mathbb{E}[\mathbb{1}_B|\mathcal{I}]\mathbb{E}[\mathbb{1}_A|\mathcal{I}]d\mathbb{P}$$

thus $\mathbb{P}(A|\mathcal{I})\mathbb{P}(B|\mathcal{I})$ is a version of $\mathbb{P}(A \cap B|\mathcal{I})$.

(⇒:)

Using again that $A \in \mathcal{G}$, $B \in \mathcal{H}$, $K \in \mathcal{I}$, we know $\mathbb{P}(A|\mathcal{I})$ is $\mathcal{I}$ measurable so it is $\mathcal{I} \vee \mathcal{H}$ measurable, so we only need to show for $B \cap K$ the conditional expectation property holds since our choice of sets are arbitrary.

But notice that the above argument really forms a loop of equalities, so we just write it in the order we want and use the condition to get:

$$\begin{aligned}\int_{B\cap K}\mathbb{E}[\mathbb{1}_A|\mathcal{I}]d\mathbb{P} &= \int_K \mathbb{1}_B\mathbb{E}[\mathbb{1}_A|\mathcal{I}]d\mathbb{P} = \int_K \mathbb{E}\Big[\mathbb{1}_B\mathbb{E}[\mathbb{1}_A|\mathcal{I}]\Big|\mathcal{I}\Big]d\mathbb{P} = \int_K \mathbb{E}[\mathbb{1}_B|\mathcal{I}]\mathbb{E}[\mathbb{1}_A|\mathcal{I}]d\mathbb{P} \\ &\overset{Condition}{=} \int_K \mathbb{E}[\mathbb{1}_{A\cap B}|\mathcal{I}]d\mathbb{P} = \int_K \mathbb{1}_{A\cap B}d\mathbb{P} = P(A\cap B\cap K) = \int_{B\cap K}\mathbb{1}_A d\mathbb{P} \\ &= \int_{B\cap K}\mathbb{E}[\mathbb{1}_A|\mathcal{H}\vee\mathcal{I}]d\mathbb{P}\end{aligned}$$

and we are done since $B\cap K$ is in $\mathcal{I}\cap\mathcal{H}$, which is a π system, and hence it works for $\sigma(\mathcal{I}\cap\mathcal{H}) = \mathcal{I}\vee\mathcal{H}$

□

Exercise 2. *10.1.*

Proof.

M_n is a Martingale:

- $M_n \leq 1$ so $\mathbb{E}[M_n] < \infty$.
- We write out explicitly the σ-algebra that lies under the process. It is

$$\Omega = \{0,1\}^{\infty} \subset l^{\infty}$$

where 0 stands for white ball picked and 1 stood for black ball picked.

As an example, we define

$$[b_1, \dots, b_n]_n : \left\{ (b_1, \dots, b_n, a_{n+1}, a_{n+2}, \dots) \Big| a_i \in \{0,1\}, \forall i \geq n+2 \right\}$$

where the index means how many values are fixed. Thus, using this notation we can write out

$$\mathcal{F}_1 = \sigma\left([1]_1, [0]_1\right) = \{\Omega, \emptyset, [1]_1, [0]_1\}$$

and similarly

$$\mathcal{F}_2 = \sigma\left([1,1]_2, [0,1]_2, [1,0]_2, [0,0]_2\right)$$

and etc. And we check that M_n is $\mathcal{F}_n$ measurable for all point $\omega \in [a_1, \dots, a_n]_n$ we know $X(\omega)$ is a constant thus $X^{-1}(B) \in \mathcal{F}_n$ for all $B \in \mathcal{B}(\mathbb{R})$.
- We check that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$. But we've just check that X_n is $\mathcal{F}_n$ measurable, so we only need to get that for any $A \in \mathcal{F}_n$ the integral is the same. But note that $\mathcal{F}_n$ is a finite sigma algebra, and we've explicitly constructed the generating elements of the σ-algebra, so we only check that the integral equality on $S := [a_1, \dots, a_n]_n$ holds. We have

$$\int_S X_n d\mathbb{P} = \frac{B_n + 1}{n+2}$$

which is how many 1s (black balls picked) inside plus the original black ball. On the other hand we have

$$\int_S \mathbb{E}[X_{n+1}|\mathcal{F}_n] d\mathbb{P} = \int_S X_{n+1} d\mathbb{P} = \frac{B_n+1}{n+2}\frac{B_n+2}{n+3} + \frac{n-B_n+1}{n+2}\frac{B_n+1}{n+3} = \frac{B_n+1}{n+2}$$

so they agree on all $[a_1, \dots, a_n]_n$, which generates the whole $\mathcal{F}_n$, so we are done.

Distribution of B_n:

We use induction to do this. When $n = 1$ we have $\mathbb{P}(B_1 = 0) = \mathbb{P}(B_1 = 1) = \frac{1}{2}$.

If this holds for all $m = n-1$, then for B_n we have

$$\mathbb{P}(B_n = k) = \mathbb{P}(B_{n-1} = k-1)\frac{k-1}{n+1} + \mathbb{P}(B_{n-1} = k)\frac{n-k+1}{n+1} = \frac{1}{n+1}$$

and induction follows.

Distribution of θ:

It's the uniform distribution from $[0, 1]$ since $\mathbb{P}([a, b]) = \mathbb{P}\left(B_n \in [a', b']\right)$ where a', b' is the closest points that makes $\frac{a'}{n+2} \leq a \leq b \leq \frac{b'}{n+2}$. and thus

$$\mathbb{P}([a, b]) = (b' - a')\frac{1}{n+2} \to b - a$$

so it's uniform distribution.

<u>N_n^θ is a Martingale:</u>

- By binomial theorem it is bounded by 1 so $\mathbb{E}[N_n^\varepsilon] \leq 1 < \infty$.
- They are also measurable with respect to $(\Omega, \mathcal{F} := \{\mathcal{F}_n\})$ since they are measurable maps of B_n, who are measurable.
- We check that they satisfy the condition.

  Again, we only need to check for $S := [a_1, \ldots, a_n]_n$, and we for convience assume $B_n = k$

$$\int_S \mathbb{E}\left[N_{n+1}^\theta | \mathcal{F}_n\right] d\mathbb{P} = \int_S N_{n+1}^\theta d\mathbb{P}$$

$$= \mathbb{P}\left(a_{n+1} = 1\right) \cdot \frac{(n+2)!}{(k+1)!(n-k)!}\theta^{k+1}(1-\theta)^{n-k} + \mathbb{P}\left(a_{n+1} = 0\right) \cdot \frac{(n+2)!}{(k)!(n-k+1)!}\theta^k(1-\theta)^{n-k+1}$$

$$= \frac{k+1}{n+2} \cdot \frac{(n+2)!}{(k+1)!(n-k)!}\theta^{k+1}(1-\theta)^{n-k} + \frac{n-k+1}{n+2} \cdot \frac{(n+2)!}{(k)!(n-k+1)!}\theta^k(1-\theta)^{n-k+1}$$

$$= \left[\frac{(n+1)!}{k!(n-k)!}\theta^k(1-\theta)^{n-k}\right] \cdot (\theta + (1-\theta)) = \frac{(n+1)!}{k!(n-k)!}\theta^k(1-\theta)^{n-k} = \int_S N_n^\theta d\mathbb{P}$$

  and so we are done.

$\square$

Exercise 3. *Ex 10.2.*

Proof.

We want to show that $\log Z_n - n\alpha$ is a supermartingale. But since by definition $X_n := \sum \varepsilon_n$ is a submartingale ($p \geq 1/2$), then so is $Z_n = (C \cdot X)_n$. In particular Z_n is integrable and measurable with respect to the filtration $\mathcal{F}_n$. So since $n\alpha$ is constant (thus measurable) under $\mathcal{F}_n$ the only thing we need to show is the inequality in the definition.

Now we have

$$\begin{aligned}
&\mathbb{E}\left[\log(Z_{n+1}) - (n+1)\alpha | \mathcal{F}_n\right] - \mathbb{E}\left[\log(Z_n) - n\alpha \middle| \mathcal{F}_n\right] \\
&= \mathbb{E}\left[\log(Z_{n+1}) - \log(Z_n) \middle| \mathcal{F}_n\right] - \alpha = \mathbb{E}\left[\log\left(1 + \frac{C_{n+1}\varepsilon_{n+1}}{Z_n}\right) \middle| \mathcal{F}_n\right] - \alpha
\end{aligned}$$

So we define $f_n = \dfrac{C_{n+1}}{Z_n}$ and compute the conditional expectation to get

$$\mathbb{E}\left[\log\left(1 + \frac{C_{n+1}\varepsilon_{n+1}}{Z_n}\right) \middle| \mathcal{F}_n\right] - \alpha = p\log(1 + f_n) - q\log(1 - f_n) - \alpha$$

where by taking derivative over f_n we get that the maximal of the above expression occurs at $f_n = p - q$ and the exact value is

$$p\log(1 + f_n) - q\log(1 - f_n) \leq (p + q)\log 2 + p\log p + q\log q = \alpha$$

and we are done.

Notably using supermartingale property we get

$$\mathbb{E}[Z_N / Z_0] \leq \sum^{N} \alpha = N\alpha$$

And of course the best strategy is $C_{n+1} = (p - q)Z_n$.

$\square$

Exercise 4. *10.3*

Proof.

Being a stopping time means $\{T = n\} \in \mathcal{F}_n$ (or $\leq$ but that's equivalent).

So

$$\{S \wedge T = n\} = (\{S \geq n\} \cap \{T = n\}) \cup (\{S = n\} \cap \{T \geq n\}) \in \mathcal{F}_{n-1}$$

note that either one of the above union sets is $\emptyset$ or they are the same, but that does not affect the fact that they are unions and intersections of sets in $\mathcal{F}_n$, since $\{S \geq n\} = \{S \leq n-1\}^c \in \mathcal{F}_{n-1} \subset \mathcal{F}_n$.

Similarly we have

$$\{S \vee T = n\} = (\{S \leq n\} \cap \{T = n\}) \cup (\{S = n\} \cap \{T \leq n\}) \in \mathcal{F}_n.$$

And

$$\{S + T = n\} = \bigcup_{i=0}^{n} (\{S = i\} \cap \{T = n - i\}) \in \mathcal{F}_n.$$

So they are all stopping times.

$\square$

Exercise 5. *10.4*

Proof.

We define

$$\mathbb{1}_S(n,\omega) := \mathbb{1}_{S,n} = \mathbb{1}_{\{S\geq n\}} = \begin{cases} 1 & n \leq S(\omega) \\ 0 & \text{elsewhere} \end{cases}$$

where we know $\mathbb{1}_{S,n}$ is previsible because $\{S \geq n\} = \{S \leq n-1\}^c \in \mathcal{F}_n$.

then we notice that

$$\mathbb{1}_{(S,T]}(n,\omega) = \mathbb{1}_T(n,\omega) - \mathbb{1}_S(n,\omega)$$

but since both terms on the right is previsible, so is their difference.

Now for the next part we have

$$\mathbb{E}[X_{T\wedge n} - \mathbb{E}[X_0] - X_{S\wedge n} + \mathbb{E}[X_0]] = (\mathbb{1}_{(S,T]}(n,\omega) \bullet [X - \mathbb{E}[X_0]])_n$$

because we can just separate cases ($T \leq n$ or $S > n$ or in between) and see that these coincides for each case:

- $T \leq n$: Both sides just use T and S;
- $S > n$: Both sides are 0;
- $S \leq n < T$: T on both sides is changed into n, where the right hand side truncation happens by the truncation of $\bullet$.

Thus, $X - \mathbb{E}[X_0]$ is a supermartingale since X is. Thus

$$\mathbb{E}[X_{T\wedge n} - X_{S\wedge n}] = (\mathbb{1}_{(S,T]}(n,\omega) \bullet (X - \mathbb{E}[X_0]))_n \leq \mathbb{E}[X_0] - \mathbb{E}[X_0] = 0.$$

□

Exercise 6. *10.5*

Proof.

Note that

$$\mathbb{P}(T > kN) = \mathbb{P}(T > kN; T > (k-1)N)$$

because under the condition $T \leq (k-1)N$ it's impossible that the first happen. Then we use induction.

For $k = 1$,

$$\mathbb{P}(T > kN) \leq 1 - \varepsilon$$

by taking $n = 0$ in the given form.

Now assume that $k \leq m$ holds, for $k = m + 1$, we have

$$\begin{aligned}
\mathbb{P}(T > (m+1)N) &= \mathbb{P}(T > (m+1)N; T > mN) \\
&= \int_{\{T>mN\}} \mathbb{1}_{\{T>(m+1)N\}} d\mathbb{P} = \int_{\{T>mN\}} \mathbb{E}\left[\mathbb{1}_{\{T>(m+1)N\}} \middle| \mathcal{F}_{mN}\right] d\mathbb{P} \\
&= \mathbb{E}\left[\mathbb{1}_{\{T>(m+1)N\}} \middle| \mathcal{F}_{mN}\right] \int_{\{T>mN\}} 1 d\mathbb{P} \\
&= \mathbb{P}(T > mN + N | \mathcal{F}_{mN}) \cdot \mathbb{P}(T > mN) \\
&\leq (1-\varepsilon)(1-\varepsilon)^m = (1-\varepsilon)^{m+1}
\end{aligned}$$

and thus by induction we are done.

□

Exercise 7. *10.6*

Proof.

Martingale theory makes it intuitive because what we're finding is that a consecutive of 11 letters come in the form "ABRACADABRA", for which we note that the last 4 is the first 4 letters of the same word, and the last 1 letter is another starter of the word. Thus it's expectation should be

$$\mathbb{E}[T] = 26^{11} + 26^4 + 26.$$

In other words, one people gain 26^{11} dollars, then another people coming and seeing "ABRA" will win 26^4, then the last people win 26 dollars. And left side is because after all wins and losses essentially there's 1 dollar bet at each $t \le T$.

To prove this, we first try to fit into a model for which we can use theorem 10.10c.

Here, the index j indicates that we're only focusing on the person that comes at j.

Let

$$\varepsilon_n^j : \Omega \to \{f_j(A), \dots, f_j(Z)\}$$

be iid random variables with uniform probability where $f_j : \{A, \dots, Z\} \to \mathbb{R}$ is a map that both makes the sums a Martingale, and makes the question easy. We will specify that later. Moreover, let $\varepsilon_n^j \in \mathcal{F}_n$ then if we define

$$Y_n^j := \sum_{i=1}^{n} \varepsilon_i^j$$

then Y_n^j is a Martingale if our choice of f_j Guarantees that.

We now specify T. Just by what it is we define Here the not j-indexed terms are not yet defined, but roughly they are just sums of the indexed ones. This serves as a intuition here.

$$T = \inf\left\{ n > 10 \middle| \left[f^{-1}(\varepsilon_{n-10}), f^{-1}(\varepsilon_{n-9}), \dots, f^{-1}(\varepsilon_n) \right] = \left[ABRACADABRA \right] \right\}$$

where we note that even though f is in general not invertible, but for the exact spelling of $ABRACADABRA$ we really can do it because only that changes the game. After we define f below we'll see why.

Let's fix one person and see the total state into the system for the person that came at time j person at time n. Then we have

$$C_n^j = \begin{cases} 0 & n < j \\ 1 & n = j \\ 26^{n-j+1} & \text{Preceding letters are exactly the first n-j of ABRACADABRA} \\ 0 & \text{else} \end{cases}$$

and with this definition we can already define just any martingale X and apply. But this will lead in a disaster of computation, which we do not like: So we try to find a martingale that makes our stake exactly T, the stopping time.

Thus, we define the Y_n^j to be the total gain/loss the j-th person get from this game. Thus, we have

$$Y_n^j = \begin{cases} 0 & n < j \\ 26^{n-j+1} - 1 & j \le n \le n+10 \ and \ \ [\varepsilon_{n-j}^j, \varepsilon_{n-j+1}^j, \dots, \varepsilon_n^j] = [ABRAC \dots] \text{ first j+1 term} \\ -1 & \text{else} \end{cases}$$

and we can specify the probability in the middle case as ε_n are uniform:

$$\varepsilon_n^j = Y_n^j - Y_{n-1}^j = \begin{cases} 26^{n-j+1} - 26^{n-j} & \mathbb{P} = 1/26 \\ -26^{n-j} & \mathbb{P} = 25/26 \end{cases}$$

and we show that Y_n^j is a Martingale. Since we have ε_n^j is $\mathcal{F}_n$ measurable, the first 2 conditions of Martingale is trivial (since expectation is 0). Now for the third case, since $\mathcal{F}_n$ is made up of minimal elements of the first n outcome:

- if $Y_{n-1}^j = 26^{n-j-1} - 1$, call the corresponding set B, then
$$\begin{aligned} \int_B Y_n^j d\mathbb{P} &= \frac{1}{26}\left(26^{n-j} - 1 + 26^{n-j+1} - 26^{n-j}\right) + \frac{25}{26}(26^{n-j} - 1 - 26^{n-j}) \\ &= 26^{n-j} - 1 = Y_{n-1}^j \end{aligned}$$
- and if $Y_{n-1}^j = -1$, call the corresponding set A, then
$$\int_A Y_n^j d\mathbb{P} = -1 = Y_{n-1}^j$$
- and for 0 of if it just so happens that $n = j+1$ then
$$\int_{n=j+1} Y_n^j d\mathbb{P} = \frac{25}{26}(-1) + \frac{1}{26} \cdot 26 = 0 = Y_{n-1}^j$$
- If $n+1 < j$ then obviously both sides are 0.

Thus we conclude for all sets in $\mathcal{F}_n$ we have the equality, so

$$\mathbb{E}[Y_n^j | \mathcal{F}_n] = Y_{n-1}^j$$

and hence Y_n^j is a Martingale.

And finally we can define X_n, we define it as the total gain/loss of all people in the game:

$$X_n = \sum_{i=1}^{T} Y_n^j$$

and we know it's a Martingale because all the summands are.

So now can check the criterion of theorem 10.10c:

- $\mathbb{E}[T] < \infty$: For $N \geq 11$ we know that the probability

$$\mathbb{P}(T \leq n + N | \mathcal{F}_n) \leq \mathbb{P}(f^{-1}(\varepsilon_{n+1}) = A; f^{-1}(\varepsilon_{n+2}) = B; \dots ; f^{-1}(\varepsilon_{n+11}) = A) = c > 0$$

  because everything's discrete and we can at least compute the probability c.
  Thus by last problem we know $\mathbb{E}[T] < \infty$.
- $|X_n - X_{n-1}| = \left|\sum_{i=1}^{11} 26^i\right| \leq= K_1$.
- $T_n \leq 1 = K_2$.

and thus theorem 10.10c tells us that

$$\mathbb{E}[(T \bullet X)_T] = \mathbb{E}[T \bullet X] = \mathbb{E}[X_T] - \mathbb{E}[X_0].$$

Now the idea is that the total money bet on the game and total money won is equal. For explanations, the "real money" bet on the game is T since only 1 dollar is put into the game at each time, and the total money won is $26^{11} + 26^4 + 26$ by above discussion.

Now we make this computation rigorous and compute: We know $\mathbb{E}[X_0] = 0$ and since

- $Y_n^n = 26 - 1$
- $Y_n^{n-3} = 26^4 - 1$
- $Y_n^{n-10} = 26^{11} - 1$
- for rest j, we have $Y_n^j = -1$

we have

$$\mathbb{E}[X_T] = 26^{11} + 26^4 + 26 + \mathbb{E}[T] \cdot (-1)$$

since the first are the total gains, and the last term comes from the fact that each Y_n^j comes with a minus 1, at all time, and we only add T of them. Thus we get the result

$$\mathbb{E}[T] = 26^{11} + 26^4 + 26.$$

$\square$