MEASURE THEORETIC PROBABILITY III HW 4

TOMMENIX YU
ID: 12370130
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Discussed with classmates.

Exercise 1.

Proof.

conv in prob $\Rightarrow \forall$ subsequence conv a.s.:

If we can show that for the whole sequence, there is a subsequence that converges a.s., then since all subsequence of X_n converges in probability, so we will have the result. So we only show the result for the whole sequence.

To prove convergence almost surely for some subsequence is to prove

$$\mathbb{P}\left(\limsup_{n\to\infty} |X_{\phi(n)} - X| > 0\right) = 0$$

where we can rewrite

$$\left\{\limsup_{n\to\infty} |X_{\phi(n)} - X| > 0\right\} \subset \left\{\omega \,\middle|\, |X_{\phi(n)}(\omega) - X(\omega)| \geq \frac{1}{n} i.o.\right\}$$

since for any ω in the left side it has $|X_{\phi(n)}(\omega) - X(\omega)| = c > 0$ and so for any $n > \dfrac{1}{c}$ the inequality in right side holds, so it holds i.o..

Thus, if we define

$$A_n := \left\{|X_{\phi(n)} - X| \geq \frac{1}{n}\right\}$$

then

$$\left\{\omega \,\middle|\, |X_{\phi(n)}(\omega) - X(\omega)| \geq \frac{1}{n} i.o.\right\} = \limsup_{n\to\infty} A_n$$

by definition. Now we just find suitable subsequence $\phi(n)$ to satisfy the Borel-Cantelli condition.

So we use convergence in probability to find $\phi(n) > N = N(n)$ such that

$$\mathbb{P}\left(|X_{\phi(n)} - X| \geq \frac{1}{n}\right) \leq 2^{-n}$$

and so

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = 2 < \infty$$

which implies that $\mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = 0$. Now by monotonicity of measure we get

$$0 \le \mathbb{P}\left(\limsup_{n\to\infty} |X_{\phi(n)} - X| > 0\right) \le \mathbb{P}\left(\limsup_{n\to\infty} A_n\right) = 0$$

which means

$$\mathbb{P}\left(\limsup_{n\to\infty} |X_{\phi(n)} - X| > 0\right) = 0$$

and we are done.

conv in prob $\Leftarrow$ $\forall$ subsequence conv a.s.:

If all subsequence has a subsequence that converges a.s., then suppose X_n does not converge to X a.s. then there's an infinite subsequence that is at least ε away from X in infinite norm a.s.. But then there's no subsequence that converge to X of the above spotted subsequence, so we know $X_n \stackrel{as}{\to} X$.

Now we show $X_n \stackrel{as}{\to} X \Rightarrow X_n \stackrel{p}{\to} X$:

conv a.s. $\Rightarrow$ conv in prob:

$X_n \stackrel{as}{\to} X$ means that

$$\mathbb{P}\left(\forall \varepsilon > 0, \exists N s.t. \forall n > N, |X_n - X| < \varepsilon\right) = 1$$

and we can move the universal quantifier outside to get

$$\forall \varepsilon > 0, \mathbb{P}\left(\exists N s.t. \forall n > N, |X_n - X| < \varepsilon\right) = 1$$

which then implies

$$\forall \varepsilon > 0, \lim_{n\to\infty} \mathbb{P}\left(|X_n - X| < \varepsilon\right) = 1$$

which is what we want.

□

Exercise 2. *Ex 13.1*

Proof.

($\Leftarrow$:)

By (i) we know that there is a uniform bound on the L^1 norm of $X \in C$. Thus we know that for any $X \in C$ we have

$$\mathbb{P}(|X| > k) \leq K^{-1}A$$

since otherwise the norm is larger. But then for any $\varepsilon > 0$ we know $\exists\ \delta$ and we pick K large such that $\mathbb{P}(|X| > K) < \delta$, then we have by (ii)

$$\mathbb{E}[|X|; X > K] \leq \varepsilon$$

which satisfies the condition for UI.

($\Rightarrow$:)

Given C is a UI family and then fix ε we can find the corresponding K.

Then (i) holds because every $X \in C$ is integrable because

$$\mathbb{E}[|X|] \leq K + 1.$$

For (ii), we use the same ε and check that

$$\mathbb{E}[|X|; F] \leq \mathbb{E}[|X|; |X| > K] + K \cdot \mathbb{P}(F) < \varepsilon + K\mathbb{P}(F)$$

and thus if $\mathbb{P}(F)$ is small enough, say less than δ, then we have the result.

$\square$

Exercise 3. *Ex 13.2.*

Proof.

For all $X + Y \in C + D$, we verify the two conditions in the last question.

(i) $\mathbb{E}[|X+Y|] \le \mathbb{E}[|X| + |Y|] \le \mathbb{E}[X] + \mathbb{E}[Y] = A + B$.

(ii) We pick $\varepsilon' = \varepsilon/2$ and find the corresponding δ_x and δ_y for X and Y (Since C, D are UI families), then take $\delta = \min\{\delta_x, \delta_y\}$, then we have for any $\mathbb{P}(F) < \delta$ and $F \in \mathcal{F}$, we get

$$\mathbb{E}[|X+Y|; F] \le \mathbb{E}[|X|; F] + \mathbb{E}[|Y|; F] \le \varepsilon.$$

Then by last problem we are done.

□

Exercise 4. *13.3*

Proof.

Fix ε.

Again, by 13.1 we denote
$$A := \sup_{X \in C} \mathbb{E}[|X|] < \infty$$
then by (ii) we have that we can uniformly in X choose $\delta > 0$ based on ε such that
$$\mathbb{P}(F) < \delta \Rightarrow \mathbb{E}[|X|; F] < \varepsilon.$$

So we can choose K large such that $KA < \delta$. Note that here we've already throw away all dependence on any particular $X \in C$, so the proof should be identical to that based on one X.

Since Y is a version of $\mathbb{E}[X|\mathcal{G}]$, we have by Jensen that
$$|Y| \leq \mathbb{E}[|X||\mathcal{G}]$$
and taking expectation on both sides we gain
$$K\mathbb{P}(|Y| > K) \overset{Markov}{\leq} \mathbb{E}[|Y|] \leq \mathbb{E}[\mathbb{E}[|X||G]] = \mathbb{E}[X]$$
and thus
$$\mathbb{P}(|Y| > K) < \delta$$
but $\{|Y| > K\} \in \mathcal{G}$, so that from the definition of conditional expectation
$$\mathbb{E}[|Y|; |Y| \geq K] \leq \mathbb{E}[|X|; |Y| \geq K] < \varepsilon.$$

So we are done as $(Y, \mathcal{G})$ pair is arbitrary. □

Exercise 5. *14.1*

Proof.

Follow the hint we get

$$\left|\mathbb{E}[X_n|\mathcal{F}_n] - \mathbb{E}[X|\mathcal{F}_\infty]\right| \leq \left|\mathbb{E}[X_n|\mathcal{F}_n] - \mathbb{E}[X|\mathcal{F}_n]\right| + \left|\mathbb{E}[X_n|\mathcal{F}_n] - \mathbb{E}[X|\mathcal{F}_n]\right|$$
$$\stackrel{Jensen}{\leq} \mathbb{E}[|X_n - X||\mathcal{F}_n] + \left|\mathbb{E}[X_n|\mathcal{F}_n] - \mathbb{E}[X|\mathcal{F}_n]\right| \to 0$$

where the first term goes to 0 because $X_n \to X$ a.s.; The second term goes to 0 because of 14.2 Levy's upward theorem. So the result holds.

□