PDE HOMEWORK 8

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Discussed with classmates.

Exercise 1.

Proof.

1.

Plug in we have

$$c\sqrt{a}u = cv = \Delta v = \Delta\sqrt{a}u = \nabla\cdot\left[\nabla\sqrt{a}u + \sqrt{a}\nabla u\right] = \left(\Delta\sqrt{a}\right)u + 2\left(\nabla\sqrt{a}\right)\nabla u + \sqrt{a}\Delta u$$
$$= \left(\Delta\sqrt{a}\right)u + 2\left(\frac{\nabla a}{2\sqrt{a}}\right)\nabla u + \frac{a\cdot\Delta u}{\sqrt{a}} = \left(\Delta\sqrt{a}\right)u + \frac{1}{\sqrt{a}}(\nabla\cdot a\nabla u) = \left(\Delta\sqrt{a}\right)u$$

thus $c = \dfrac{\Delta\sqrt{a}}{\sqrt{a}}$ satisfies the condition. Check that it is true.

2.

Plug in we have

$$\begin{aligned}\nabla\cdot a\nabla\frac{v}{w} &= \nabla a\cdot\nabla\frac{v}{w} + a\cdot\nabla\left[\frac{w\nabla v - v\nabla w}{w^2}\right]\\ &= \nabla a\cdot\nabla\frac{v}{w} + a\frac{w\Delta v + \nabla v\nabla w - v\Delta w - \nabla v\nabla w}{w^2} + a\frac{2w\nabla w[w\nabla v - v\nabla w]}{w^4}\\ &= \nabla a\cdot\nabla\frac{v}{w} + a\frac{cvw - cwv}{w^2} + a\frac{2\nabla w}{w}\nabla\frac{v}{w} = \left(\nabla a - a\frac{2\nabla w}{w}\right)\nabla\frac{v}{w}\end{aligned}$$

thus to make above 0 we only need to make

$$\nabla a - a\frac{2\nabla w}{w} = 0 \Rightarrow a = w^2.$$

Thus $a = w^2$.

3.

Note that for $w = \sqrt{a}$, then we plug in $c = \dfrac{\Delta\sqrt{a}}{\sqrt{a}}$ from part 1 to get

$$\frac{\Delta\sqrt{a}}{\sqrt{a}}\sqrt{a} = cw = \Delta w = \Delta\sqrt{a}$$

so w is indeed a solution of (2). Thus, assume that v is a solution to (2), and we've shown $\sqrt{a}$ is a solution of (2), by part 2 we know $\dfrac{v}{w} = \dfrac{v}{\sqrt{a}}$ is a solution of (1), since here $a = w^2$ exactly.

This, plus what we've done in part 1, shows that for $v = \sqrt{a}u$, we have:

u is a solution of (1) $\iff$ v is a solution of (2).

This, plus the boundary condition (which interchanges alike, since both 0), means that for $v = \sqrt{a}u$, $c = \dfrac{\Delta\sqrt{a}}{\sqrt{a}}$, we have:

$$\begin{cases} \nabla \cdot a\nabla u = 0 & U \\ u = 0 & \partial U \end{cases} \iff \begin{cases} -\Delta v + cv = 0 & U \\ v = 0 & \partial U \end{cases}$$

and thus the 2 problems has "same" solutions up to $\sqrt{a}$. But if we can show continuity and coercivity, then by Lax-Milgram we know that the system on left attains unique solution, which is 0. So we define

$$B[u,v] := \int a\nabla u\nabla v dx \quad \left(= -\int \nabla\cdot(a\nabla u)\, v dx\right)$$

and get

- Continuity:

$$B[u,v] \le ||a||_\infty ||\nabla u||_2 ||\nabla v||_2 \le C||\nabla u||_{H^1}||\nabla v||_{H^1}$$

  by what the norm is.
- Coercivity: Use Holder and then Poincare to get

$$B[u,u] \ge a_0||\nabla u||^2 \ge a_0||u||^2.$$

Thus we can apply Lax Milgram and see that the solution is unique.

Alternatively, just note we can define $A := \text{diag}(a, a, \dots, a)$ and then the operator becomes $\nabla\cdot a\nabla = \nabla\cdot A\nabla$ is elliptic, then use maximum principle on the set U to get that the unique solution is 0.

Counterexample:

Just take easy 1d Helmholtz equation say $\Delta u + u = 0$ on $U = (0, \pi)$. Then $u = 0$ and $u = \sin x$ are two different solutions that vanishes on the boundary, thus we are done.

$\square$

Exercise 2.

Proof.

So let's just do integration by parts and see what's different.

Formally, if we want $f = -\Delta u + u$ then they'd better be the same with respect to all test functions in H^1, so we compute

$$\int_U -\Delta uv + uvdx = \int_U \nabla u \nabla v + uvdx - \int_{\partial U} \nabla uvdx = \int_U \nabla u \nabla v + uvdx + \int_{\partial U} \frac{1}{\alpha(x)} uvdx$$

where the second integral is justified in the sense that on the boundary u, v are their trace. So we just define the bilinear form

$$B[u,v] := \int_U \nabla u \nabla v + uvdx + \int_{\partial U} \frac{1}{\alpha(x)} uvdx$$

and declare that u is a weak solution to the problem if

$$B[u,v] = \int_U fvdx$$

for all $v \in H^1$.

Now we try to show existence and uniqueness of this by Lax-Milgram, which gives us both at once.

To show continuity, we note that from Holder and bound on traces (5.5, Theorem 1)

$$B[u,v] \leq ||\nabla u||_2 \cdot ||\nabla v||_2 + ||u||_2 \cdot ||v||_2 + \frac{C'}{\alpha_0} ||\nabla u||_{H^1} \cdot ||\nabla v||_{H^1} \leq C ||\nabla u||_{H^1} \cdot ||\nabla v||_{H^1}$$

and thus we're done.

For coercivity, we note that the boundary integral is positive so

$$B[u,u] = \int_U |\nabla u|^2 + u^2 dx + \int_{\partial U} \frac{u^2}{\alpha(x)} dx \geq ||u||_{H^1}$$

and thus we're also done. Now Lax Milgram gives us unique solution directly.

□

Exercise 3.

Proof.

Step 1:

We introduce the cut-off function ζ. Let the support of u be U. The good thing about a compactly supported u is that we do not have to worry about the unboundedness of the derivative of ζ anymore: we can just fix some small enough $\delta > 0$ such that ζ is 1 on V, where

$$V := \left\{ x \in U \middle| \operatorname{dist}(x, \partial U) \geq \delta \right\}.$$

Because we can pick fixed δ, we just use $\nabla\zeta$ is bounded below.

Step 2:

Define

$$\alpha + \beta := \int \nabla u \nabla v + c(u) v dx = \int f v =: \gamma$$

for $f \in L^2$.

Note that from here, if we are justified to let $v = \Delta u$ then we do integral by part and everything will just follow. The only issue is that that's not legal since what we're proving is just that $u \in H^2$, so we go through the toil below.

Step 3:

Since we cannot define Δu, how about let's just do finite difference? We define

$$v := -D_k^{-h}\left(\zeta^2 D_k^h u\right)$$

for k fixed such that what we really want is $v = \partial_k^2 u$, heuristically. Here the notation means

$$D_k^h u(x) = \frac{u(x + he^k) - u(x)}{h}$$

so we need h small enough, but not as strict as we need for the general case since u is defined outside it's support, and we'll left to choose h small for later in bounding β. So we have $v \in H_0^1$ just because everything in it's definition is H^1.

Step 4:

We get our result here. To prove H^2 we try to find it's second weak derivative and show that it is H_0^2 bounded. Note that we've half constructed it already with finite sums, now we really do the calculations.

By Evans 5.8.2 (or just Cauchy Schwartz) to get

$$\int v^2 = \int \left| D_k^{-h} \zeta^2 D_k^h u \right|^2 \leq C \int_U \left| \nabla \left(\zeta^2 D_k^h u\right) \right|^2.$$

We have

$$\alpha = -\int_U \nabla u \cdot \nabla \left(D_k^{-h} \zeta^2 D_k^h u \right) dx \overset{discrete\ ibp}{=} \int_U D_k^h (\nabla u) \cdot \nabla (\zeta^2 D_k^h u) dx$$
$$= \int_U (D_k^h \nabla u) \cdot \nabla (\zeta^2 D_k^h u) dx$$

which is the bad term with two derivatives, not caring about whether discrete or ∇. So we can only use ellipticity.

Now we use result proven in class that D_k^h and ∇ commutes to get

$$\int_U (D_k^h \nabla u) \cdot \nabla (\zeta^2 D_k^h u) dx = \int_U \zeta D_k^h \nabla u \cdot \zeta D_k^h \nabla u dx + \int_U 2\nabla\zeta \cdot \zeta D_k^h \nabla u \cdot D_k^h u dx$$

where we use elliptic to deal with the first, and good bad ε argument to deal with the second, i.e. since $\nabla\zeta$ and ζ are bounded

$$\int_U 2\nabla\zeta \cdot \zeta D_k^h \nabla u \cdot D_k^h u dx = C_1 \int_U (\varepsilon D_k^h \nabla u) \frac{D_k^h u}{\varepsilon} dx \le C_2 \left(\varepsilon^2 ||D_k^h \nabla u||_2^2 + \frac{||D_k^h u||_2^2}{\varepsilon^2} \right)$$

and we pick $C_2 \varepsilon^2 \le \frac{1}{2}$ to put that term in the elliptic term, and $\frac{C_2}{\varepsilon^2}$ is just a constant so we get:

$$\frac{1}{2} \int_U \left| \zeta D_k^h \nabla u \right|^2 \le C \left(|\nabla u|^2 + |D_k^h u|^2 \right) + |\beta|.$$

Now we bound β. Writing things out we have

$$\beta = \int c(u) v dx = -\int c(u) D_k^{-h} \left(\zeta^2 D_k^h u \right) dx \overset{discrete\ ibp}{=} \int c_k^h(u) D_k^h u D_k^h u dx$$

where $c_k^h(u) \to c'(u)$ as we take limit later since it is smooth, so for small h it is close to $c'(u)$, in particular positive after taking the limit. So we have

$$\beta \le C_3 ||D_k^h u||_2^2.$$

But we can use the same good bad ε trick again to bound γ:

$$\gamma = \int_U f v dx \le C_4 \left(\frac{1}{\varepsilon^2} ||f||_2^2 + \varepsilon^2 ||v||_2^2 \right)$$

where we only need to estimate $||v||_2^2$, for which we use Theorem 3 in 5.8.2 (or painfully write out quotient and then bound both terms)

$$\int_U \left| D_k^{-h} \left(\zeta^2 D_k^h u \right) \right|^2 dx \le C_5 \int_V |D_k^h u|^2 + \zeta^2 |D_k^h \nabla u|^2 dx \le C_6 \left(|D_k^h u|^2 + |D_k^h \nabla u|^2 \right)$$

so ok we pick $C_6 \varepsilon^2 = \frac{1}{4}$ to control the terms and combine estimates on α, β, γ to get the final estimate:

$$\frac{1}{4} \int_U \zeta^2 |D_k^h \nabla u|^2 \le C \left(||f||_2^2 + ||u||_2^2 + ||D_k^h u||_2^2 \right)$$

where C is independent of h, which is the only important thing that matters here. Also, what does it matter if we use $|D_k^h \nabla u|^2$ or $|\nabla u|^2$ so we just replace with that (or 5.8.2 to be rigorous).

Now take $h \to 0$ by Evans 5.8.2 Theorem 3(ii) we can pass the limit and get

$$\int_U \zeta^2 |\partial_k \nabla u|^2 \leq C \left(||f||_2^2 + ||u||_{H_0^1(U)}^2 \right)$$

so ok that's inconveniently a square there but we can bound the square root by Cauchy-type estimates and get:

$$||u||_{H^2(U)} \lessapprox C \left(||f||_2 + ||u||_{H_0^1(U)} \right)$$

and thus we are done.

$\square$