APPLIED LINEAR ALGEBRA HOMEWORK 3

TOMMENIX YU
STAT 31430
DUE FRIDAY, OCT. 28, 3PM

1. Written Assignment

Exercise 1.1. *Let $A \in \mathcal{M}_n(\mathbb{R})$ be a given symmetric matrix, and let*

$$R_A(x) = \frac{< Ax, x >}{< x, x >}, x \in \mathbb{R}^n$$

denote the Rayleigh quotient.

(a) *Show that Show that the gradient ∇R_A is given by*

$$\nabla R_A = \frac{2}{\langle x, x \rangle} Ax - \frac{2\langle x, Ax \rangle}{\langle x, x \rangle^2} x$$

for $x \in \mathbb{R}^n$.

(b) *Conclude that if $v \in \mathbb{R}^n$ s an eigenvector of A corresponding to an eigenvalue $\lambda \in \mathbb{R}$, then*

$$(\nabla R_A)(v) = 0.$$

Proof.

(a): We simply calculate the gradient. Since we are eventually taking the partial derivative on a quotient, we will use the quotient rule somewhere. For that reason we first compute some handy results:

$$u := \langle x, Ax \rangle = \left\langle x, \begin{pmatrix} \sum_{i=1}^n x_i a_{1i} \\ \vdots \\ \sum_{i=1}^n x_i a_{ni} \end{pmatrix} \right\rangle = \sum_{j=1}^n \sum_{i=1}^n a_{ji} x_i x_j,$$

$$l := \langle x, x \rangle = \sum_{i=1} x_i^2.$$

And for the partial derivative with respect to x_k for the upper and lower parts are:

$$u_k := \frac{\partial}{\partial x_k} \langle x, Ax \rangle = 2a_{kk}x_k + \sum_{i \neq k} a_{ki} x_i + \sum_{j \neq k} a_{jk} x_j = 2 \sum_{i=1}^n a_{ki} x_i$$

where the last equality is due to the fact that A is symmetric. Similarly we have:

$$l_k := \frac{\partial}{\partial x_k} \langle x, x \rangle = 2x_k$$

and so

$$\begin{aligned}
\frac{\partial}{\partial x_k} R_A(x) &= \frac{u_k l - l_k u}{l^2} \\
&= \frac{\left(2\sum_{i=1}^n a_{ki} x_i\right)\langle x, x\rangle - 2x_k \sum_{j=1}^n \sum_{i=1}^n a_{ji} x_i x_j}{\langle x, x\rangle^2} \\
&= \frac{2}{\langle x, x\rangle}(Ax)_k - \frac{2\langle x, Ax\rangle}{\langle x, x\rangle^2} x_k
\end{aligned}$$

Now we compute:

$$\nabla R_A(x) = \begin{pmatrix} \frac{\partial}{\partial x_1} R_A(x) \\ \vdots \\ \frac{\partial}{\partial x_n} R_A(x) \end{pmatrix} = \frac{2}{\langle x, x\rangle}(Ax) - \frac{2\langle x, Ax\rangle}{\langle x, x\rangle^2} x$$

simply by plugging in.

(b): Let's just plug in and check:

$$\begin{aligned}
(\nabla R_A)(v) &= \frac{2}{\langle v, v\rangle}(Av) - \frac{2\langle v, Av\rangle}{\langle v, v\rangle^2} v \\
&= \frac{2}{\langle v, v\rangle}(\lambda v) - \frac{2\langle v, \lambda v\rangle}{\langle v, v\rangle^2} v \\
&= \frac{2\lambda}{\langle v, v\rangle} v - \frac{2\lambda\langle v, v\rangle}{\langle v, v\rangle^2} v = 0
\end{aligned}$$

□

Exercise 1.2.

(a) Fix $n \geq 2$. $B \in \mathcal{M}_n(\mathbb{R})$ be a given matrix of rank $r \geq 1$, and suppose that $B = V\tilde{\Sigma}U^T$ is a SVD factorization for B with $U, V \in \mathcal{M}_n(\mathbb{R})$ being two orthogonal matrices. Let $\mu_1, \ldots, \mu_r$ denote the non-zero singular values of B. Show that

$$B = \sum_{i=1}^{r} \mu_i v_i u_i^T.$$

(b) Download the image http://sipi.usc.edu/database/download.php?vol=misc&img=5.1.12 from the USCI SIPI database [1]. Use the command

```
image=im2double(imread('5.1.12.tiff')}
```

in Matlab or Octave to load this image file. The image data is contained in pixel values in the 256×256 matrix image. Display the image using the command

```
imshow(image)
```

Run the command

```
[v,sigma,u]=svd(image)
```

to compute the SVD factorization of the matrix image. Compute a "low rank approximation" by taking the first 50 terms in the expansion you derived in (a) as follows:

```
simple=zeros(256,256)
for i=1:50
    simple=simple+sigma(i,i)*v(:,i)*u(:,i)'
end
```

Display the image "simple" by running the command

```
imshow(simple)
```

What is the rank of the matrix simple? How many numerical values must you store to re- construct simple, compared to the $65536 = 256^2$ pixel values in the data image? Try varying the rank of the approximation by changing the number "50" in the for loop above. Write a paragraph summarizing your computations and observations.

Optional: Compute the matrix norm kimage-simplek 2 for varying values n, and draw a plot indicating how the error changes as n increases.

Proof.

(a):

On the one hand, we have

$$B = V\tilde{\Sigma}U^T = \begin{pmatrix} | & & | \\ v_1 & \ldots & v_n \\ | & & | \end{pmatrix} \cdot \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix} \cdot \begin{pmatrix} | & & | \\ \tilde{u}_1 & \ldots & \tilde{u}_n \\ | & & | \end{pmatrix}$$

where $\mu_k = 0$ if $k > r$ and $\tilde{u}_i = (u_{i1}, \dots, u_{in})$ is the rows of U. Writing things out explicitly we get:

$$B = \begin{pmatrix} v_{11} & \dots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{n1} & \dots & v_{nn} \end{pmatrix} \cdot \begin{pmatrix} \mu_1 u_{11} & \mu_1 u_{21} & \dots & \mu_1 u_{n1} \\ \mu_2 u_{12} & \mu_2 u_{22} & & \vdots \\ \vdots & & \ddots & \\ \mu_n u_{1n} & \dots & & \mu_n u_{nn} \end{pmatrix} = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$$

where

$$b_{ij} = \sum_{k=1}^{n} \mu_k u_{ik} v_{jk} = \sum_{k=1}^{r} \mu_k u_{ik} v_{jk} =: \sum_{k=1}^{r} c_{ijk}.$$

On the other hand,

$$\mu_k v_k u_k^T = \mu_k \begin{pmatrix} v_{1k} \\ \vdots \\ v_{nk} \end{pmatrix} \cdot \begin{pmatrix} u_{k1} & \dots & u_{kn} \end{pmatrix} = \begin{pmatrix} c_{11k} & \dots & c_{1nk} \\ \vdots & \ddots & \vdots \\ c_{n1k} & \dots & c_{nnk} \end{pmatrix}$$

which means

$$\sum_{k=1}^{r} \mu_k v_k u_k^T = \begin{pmatrix} \sum_{k=1}^r c_{11k} & \dots & \sum_{k=1}^r c_{1nk} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^r c_{n1k} & \dots & \sum_{k=1}^r c_{nnk} \end{pmatrix} = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nn} \end{pmatrix} = B$$

(b): The original picture and the compiled one are below:

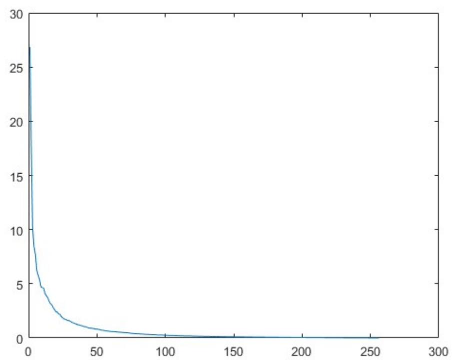
Original Picture

Compiled Picture for $n = 50$

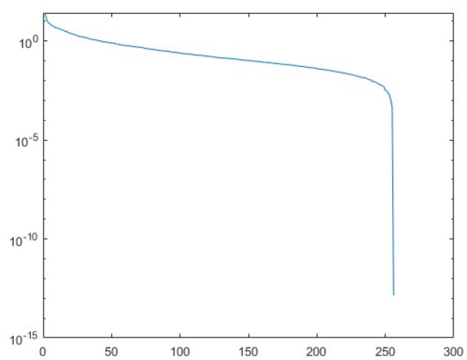
Since we only need to store the first 50 columns of both V and U, and the first 50 singular values. This yields $2 * 256 * 50 + 50 = 25650$ numerical values to store.

What I did with the little experiment is that I tried with the powers of 2 rank. At around rank 32 the picture is still barely visible, and at around rank 8 the shape of things (clock, book, picture) are distinguishable, at lower lever the picture starts to lose cognitive value.

The plot of norm verses rank is below:



Original plot



Semilogy plot

Where the semilogy plot behaves linearly (except at the end) says it decays exponentially.

□

Exercise 1.3.

(a) (2.25) Plot the image of the unit circle of $\mathbb{R}^2$ by the matrix

$$A = \begin{pmatrix} -1.25 & 0.75 \\ 0.75 & -1.25 \end{pmatrix}$$

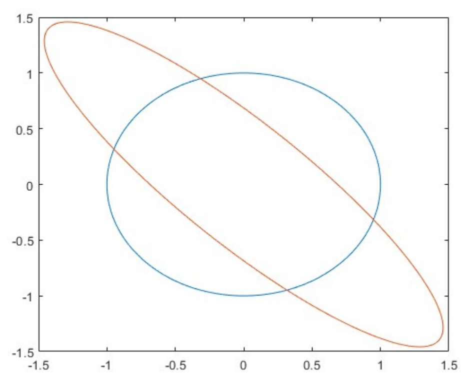
to reproduce Figure 2.2. Use the Matlab function svd.

(b) (2.26) For different choices of m and n, compare the singular values of a matrix A=rand(m,n) and the eigenvalues of the block matrix $B = \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}$. Justify.

Proof.

(a):I used the svd function to compute svd, and plotted the modified ellipsoid. It is indeed the same as Figure 2.2. Codes and figure are attached.

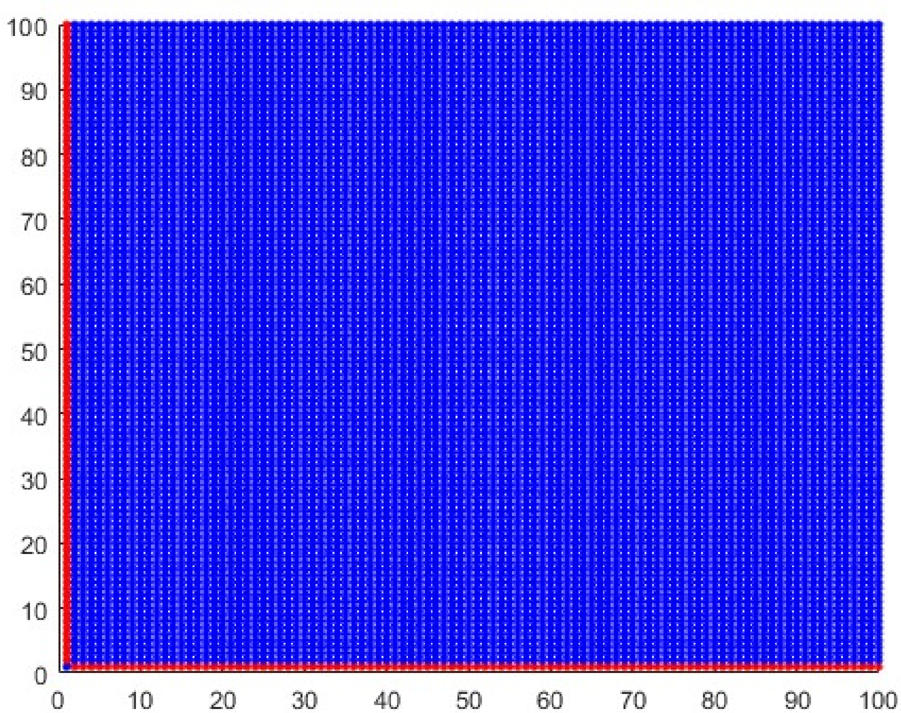
```
clear all;
r=1;
theta = 0:pi/100:2*pi;
xunit = r * cos(theta);
yunit = r * sin(theta);
vc = [xunit,yunit];
plot(xunit, yunit);
hold all
A = [-1.25,0.75;0.75,-1.25];
[v,sigma,u]=svd(A);
xell = sigma(1,1)*r*cos(theta);
yell = sigma(2,2)*r*sin(theta);
ell = [xell;yell];
ell2 = v*ell;
plot(ell2(1,:), ell2(2,:))
```



(b): What I did is that I first tried with $m = n = 10$ and checked that the eigenvalues are a list of all the singular values and their negation, with extra 0 to fill the necessary terms.

So what I did was to order the eigenvalues in a descending order and cut it off at the length of the list of singular values of A. Ideally, the cut eigenvalue list of B will be the same of the singular value of A list. So I computed the norms of the difference between two lists, and if it is larger than the tolerance 10^{-7}, it is marked red, otherwise blue. Below is the code and the result.

```
tol = 0.0000001;
for m = 1:100
    for n = 1:100
        A = rand(m,n);
        [v,sigma,u]=svd(A);
        S = diag(sigma);
        B = [zeros(m,m),A;A.',zeros(n,n)];
        E = sort(eig(B),'descend');
        N = norm(S-E(1:length(S)));
        if N < tol
            plot(m,n,'.', 'Color', 'b', 'MarkerSize', 10);
            hold all
        else
            plot(m,n,'.', 'Color', 'r', 'MarkerSize', 10);
            hold all
        end
    end
end
shg
```



Error occurs when only one of m or n is one, but that really is because the way I get the diagonal list (with diag) returns (in this case) not a list but the whole matrix with only the first term as non-zero. And that term is really the same with the first of the eigenvalue list.

So our conclusion from coding is clear: The eigenvalues of B are the singular values of A and their negatives. Let's prove it:

We denote u_i and v_i as the ith column of U and V in the SVD of A. Then, Au_i is the ith column of $AU = V\tilde{\Sigma}U^*U = V\tilde{\Sigma}$, which is $\lambda_i v_i$. Similarly, $A^T v_i = \lambda_i u_i$.

Now, let $s_i^{\pm} = \begin{pmatrix} v_i \\ \pm u_i \end{pmatrix}$, then

$$Bs_i^{\pm} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} \cdot \begin{pmatrix} v_i \\ \pm u_i \end{pmatrix} = \begin{pmatrix} \pm Au_i \\ A^T v_i \end{pmatrix} = \begin{pmatrix} \pm\lambda_i v_i \\ \lambda_i u_i \end{pmatrix} = \pm\lambda_i s_i^{\pm}.$$

So we have already find all eigenvalues of B. And that they are length 2 eigenvectors that are orthogonal to each other is just due to U and V are orthogonal matrices.

So our former numerical observation is indeed correct.

$\square$

Exercise 1.4. *Fix $n \geq 1$ and suppose that $A \in \mathcal{M}_n(\mathbb{R})$ satisfies $||Ax||_2 = ||x||_2$ for all $x \in \mathbb{R}^n$. Show that A is a unitary matrix.*

Proof.

Since A is a real valued matrix, it is unitary if it is orthogonal, which is what we'll prove.

To prove orthogonal, we only need to prove that each column of A has norm 1 and each two columns are orthogonal (sometimes called orthonormal).

Let a_i denote the i-th column of A, then

$$||a_i||_2 = ||Ae_i||_2 = ||e_i||_2 = 1$$

so every column has norm 1.

Now we prove that each two column vectors of A are orthogonal. But since we are doing things with the Euclidean norm, two vectors are orthogonal corresponds exactly to the fact that the angle they form is 90-degrees (which is invariant under shifting). This is shown in undergraduate linear algebra class and to see why we can just use formula $\cos\theta = \dfrac{u \cdot v}{|u| \cdot |v|}$.

To show this, we note that for each $1 \leq i, j \leq n$, $i \neq j$, we have $||a_i||_2 = ||a_j||_2 = 1$ due to reasons above and that

$$||a_i + a_j||_2 = ||A(e_i + e_j)||_2 = ||e_i + e_j||_2 = \sqrt{2}.$$

Now, we denote the origin by point O, denote the endpoint for vector a_i by point B and the endpoint for vector $a_i + a_j$ by C. By the above discussion, $|OB| = 1$, $|OC| = 1$, and $|BC| = |a_i + a_j - a_i| = |a_j| = 1$, where $|\cdot|$ denotes the length of a line section if the input is a line section.

But we know that there is a unique plane P in which OBC as points lies in. Further, the angle between OB and BC is the same as the angle of the two line segments in P. But then the angle $\theta \in [0, \pi)$ between OB and BC is $\dfrac{\pi}{2}$ since

$$\cos\theta = \frac{|OB|^2 + |BC|^2 - |OC|^2}{2|OB||BC|} = 0$$

by the law of cosines in a triangle.

Therefore a_i and a_j are orthogonal (in the Euclidean sense, but that is the same as in the $||\cdot||_2$ sense), so we are done. But to be explicit the proof is:

$$\langle a_j, a_i \rangle = \frac{\langle a_j, a_i \rangle}{|a_j| \cdot |a_i|} = \cos\theta = 0.$$

And hence A is orthogonal. $\square$

Exercise 1.5. *(Comparing vector p-norms) Fix $n \geq 1$ and let $x \in \mathbb{R}^n$ be given.*

(1) Show that $||x||_2^2 \leq ||x||_1 ||x||_\infty$.
(2) Show that for all $p \geq 1$ and $s \geq 0$, we have

$$||x||_{p+s} \leq ||x||_p.$$

(3) Fix $n \geq 1$ and let $x \in \mathbb{R}^n$ be given. Show that $\lim_{p\to\infty} ||x||_p = ||x||_\infty$.

Proof.

(a):

$$||x||_2^2 = \sum_{i=1}^n x_i^2 \leq \sum_{i=1}^n \left(|x_i| \cdot \max_{1\leq j\leq n} |x_j| \right) = \sum_{i=1}^n |x_i| \cdot \max_{1\leq j\leq n} |x_j| = ||x||_1 ||x||_\infty.$$

(b): Let $\tilde{x}_i = \dfrac{|x_i|}{||x||_p}$, then we have

$$\sum_{i=1}^n \tilde{x}_i^p = \sum_{i=1}^p \frac{|x_i|_p}{||x||_p^p} = \frac{\sum_{i=1}^n |x_i|^p}{\sum_{i=1}^n |x_i|^p} = 1$$

and since each $\tilde{x}_i \geq 0$, $\tilde{x}_i^p \leq 1$ and $\tilde{x}_i \leq 1$, which means $0 \leq \tilde{x}_i^{p+s} \leq \tilde{x}_i^p \leq 1$, and thus

$$\sum_{i=1}^n \tilde{x}_i^{p+s} \leq \sum_{i=1}^n \tilde{x}_i^p = 1.$$

On the other hand,

$$\frac{||x||_{p+s}}{||x||_p} = \left(\frac{\sum_{i=1}^n |x_i|^{p+s}}{||x||_p^{p+s}} \right)^{\frac{1}{p+s}} = \left(\sum_{i=1}^n \tilde{x}_i^{p+s} \right)^{\frac{1}{p+s}} \leq 1$$

so

$$||x||_{p+s} \leq ||x||_p.$$

(c): Let $|x_m| = \max_{1\leq i\leq n} |x_i|$, then we can prove for both directions:

$$\lim_{p\to\infty} ||x||_p = \lim_{p\to\infty} \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \geq \lim_{p\to\infty} \left(|x_m|^p \right)^{\frac{1}{p}} = |x_m| = ||x||_\infty.$$

$$\lim_{p\to\infty} ||x||_p = \lim_{p\to\infty} \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \leq \lim_{p\to\infty} \left(n|x_m|^p \right)^{\frac{1}{p}} = \lim_{p\to\infty} n^{1/p} |x_m| = ||x||_\infty.$$

And so we are done.

$\square$

Exercise 1.6. *Fix $p \geq 1$, and we choose p' such that $\frac{1}{p} + \frac{1}{p'} = 1$ (and $p' = \infty$ if $p = 1$). The exponent p' is often called the* <u>*conjugate exponent*</u> *to p. For $x, y \in \mathbb{R}^n$, let $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ denote the usual Euclidean inner product. Hölder's inequality on $\mathbb{R}_n$ states that for all $x, y \in \mathbb{R}^n$, $\langle x, y\rangle \leq ||x||_p||y||_{p'}$ (you don't have to show this – we'll post a quick proof to Canvas in the coming days). In this exercise, we'll show that*

$$||x||_p = \sup_{y\in\mathbb{R}^n, y\neq 0} \frac{\langle x, y\rangle}{||y||_{p'}}$$

(1) Explain why Hölder's inequality implies that for all $x \in \mathbb{R}^n$,

$$||x||_p \geq \sup_{y\in\mathbb{R}^n, y\neq 0} \frac{\langle x, y\rangle}{||y||_{p'}}.$$

(2) Show that the opposite inequality holds, i.e. that for all $x \in \mathbb{R}^n$,

$$||x||_p \leq \sup_{y\in\mathbb{R}^n, y\neq 0} \frac{\langle x, y\rangle}{||y||_{p'}}.$$

Proof.

(a): $\forall y \in \mathbb{R}^n$, Hölder's inequality says $\langle x, y\rangle \leq ||x||_p||y||_{p'}$, which implies

$$||x||_p \geq \frac{\langle x, y\rangle}{||y||_{p'}}.$$

Since the above holds for all y we have

$$||x||_p \geq \sup_{y\in\mathbb{R}^n, y\neq 0} \frac{\langle x, y\rangle}{||y||_{p'}}.$$

(b): If we just plug in the equality condition for Hölder's inequality, we should be done. And indeed:

Let y be chosen such that $|y_i| = c|x_i|^{p-1}$. Here we only take $c = 1$ and choose individual y_i such that it has the same sign as x_i, which means

$$y_i \cdot x_i = \text{sign}(x_i)|x_i|^{p-1} \cdot \text{sign}(x_i)|x_i| = |x_i|^p$$

where this can be done since $x_i \in \mathbb{R}$.

Now we just plug in and check that the result holds. In this case,

$$||x||_p||y||_{p'} = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{p-1}{p}}$$

$$= \sum_{i=1}^{n} |x_i|^p = \sum_{i=1}^{n} |y_i| \cdot |x_i| = \langle x, y\rangle$$

which means that for some y,

$$||x||_p = \frac{\langle x, y\rangle}{||y||_{p'}}$$

which implies

$$||x||_p \leq \sup_{y\in\mathbb{R}^n, y\neq 0} \frac{\langle x, y\rangle}{||y||_{p'}}.$$

□