PDE

ABSTRACT. There's four main topics of this course.

(i) Classical linear PDEs with "explicit solutions."
(ii) Method of characteristics to solve non-linear scalar first order equations.
(iii) Theory of second order linear PDEs.
(iv) Non-linear PDEs: Variational principles.

The range covers Chapter 1,2,3,4,6,8 of Evans' book, where as Chapter 5 is a reading exercise.

CONTENTS

1. 3/21: PDE; Transport equation; Elliptic equation

Before we start we first introduce some notations. We consider points $x \in U$, the space time, and functions $u : U \to \mathbb{R}$ as the object of interest. Here usually $U \subset \mathbb{R}^n$.

Def 1.1.

- *The total derivative* $D(u) := (\partial_{x_1} u, \dots, \partial_{x_n} u)$.

- *D^2u = matrix of second order derivatives.*
- *D^ju = j-th order derivatives.*
- *The gradient is*

$$\nabla u = Du^T = \begin{pmatrix} \partial_{x_1} u \\ \vdots \\ \partial_{x_n} u \end{pmatrix}.$$

Def 1.2. *A PDE is such that $\forall x \in U$ we have*

$$F\left(D^k u(x), \dots, Du(x), u(x), x\right) = 0$$

where just by definitions above we know that

$$F : \mathbb{R}^{n^k} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}.$$

Of course we also have PDE systems, which we define as

Def 1.3. *A PDE system is such that $\forall x \in U$ we have*

$$F\left(D^k u(x), \dots, Du(x), u(x), x\right) = 0; \quad u = (u^1, \dots, u^m)$$

where just by definitions above we know that

$$F : \mathbb{R}^{mn^k} \times \cdots \times \mathbb{R}^{mn} \times \mathbb{R}^m \times U \to \mathbb{R}^m.$$

If U is bounded, we need some form of boundary conditions. Now before we start investigating particular questions we keep in mind 3 questions that we'd ask ourselves:

(1) What u solves the set of constraints?
(2) Is u uniquely defined?
(3) Is u stable with respect to small changes in F?

These questions have no general answer!

Now we go to chapter 2 and consider solutions to Four kind of PDEs:

- Transport equation:

$$\frac{\partial u}{\partial t} + b \cdot \nabla u = 0; \quad u(x,t) \in \mathbb{R}$$

  where $t > 0$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^n$.
- Elliptic equations:

$$-\Delta u = f$$

  where $x \in \mathbb{R}^n$, $n \geq 2$ and $\Delta = \sum_{j=1}^{n} \partial_{x_j}^2$.
- Parabolic equation:

$$\partial_t u - \Delta u = 0$$

  where $t > 0$ and $x \in \mathbb{R}^n$.

- <u>Wave equation</u>:
$$\partial_t^2 u - \Delta u = 0$$
where $t > 0$ and $x \in \mathbb{R}^n$.

Remark 1.4. *Note that there's really just minimal difference between the last three equations, but their solution, as we'll see later, has much difference. The reason we choose to use* $-\Delta$ *with negative sign is because that makes it a positive operator. In this perspective, since the only difference between the wave equation and the Elliptic equation is the sign on* ∂_t^2*, we note that the complexity comes from the tension of a positive operator competing with a negative one.*

1.1. **Transport equation.**

What we are interested is the following question:

$$\begin{cases} \partial_t u + b \cdot \nabla u = f(x,t) \\ u(x,0) = g(x) \end{cases}$$

for $t > 0, x, b \in \mathbb{R}^n$ and $f \in C^0, g \in C^1$.

We use a simplified version of method of characteristics to solve it (not complete answering all the questions).

Step 1: when $f = 0$.

We just define $z(s) := u(x + sb, t + s)$, where the choice really is to note that our PDE has form $(1, b) \cdot (\partial_t, \nabla)u = 0$. Now taking derivative we have

$$\dot{z}(s) = \left(b_1\partial_1 + \cdots + b_n\partial_n + \partial_t\right) u(x + sb, t + s) = f(x + sb, t + s) = 0$$

and hence $u(x + sb, t + s)$ is a constant in s. So we can take $s = -t$ and get

$$u(x,t) = u(x - bt) = g(x - bt)$$

is one of the ansatzs. And we can check that indeed this u satisfies the PDE. Just by looking at the results we know that there's a propagation of singularities, that is, the solution is as smooth as g, the boundary.

We can be as sloppy as we want till here, but we still don't know anything about it's uniqueness.

Step 2: $f \neq 0$.

Really it's the same, we get

$$\dot{z}(s) = f(x + sb, t + s)$$

so

$$z(s) = z(0) + \int_0^s f(x + \tau b, t + \tau)d\tau$$

and taking $s = -t$ we get

$$g(x - bt) = u(x, t) + \int_0^{-t} f(x + \tau b, \tau + t) d\tau$$

and using $z = \tau + t$ we get

$$u(x, t) = g(x - bt) + \int_{-t}^{0} f(x + \tau b, \tau + t) d\tau = g(x - bt) + \int_0^t f(x + (z - t)b, z) dz.$$

1.2. **Elliptic equation.**

We call this either Laplace equation (RHS = 0) or Possion equation (RHS = f).

Note that we can write the Laplacian as

$$\Delta = \nabla \cdot \nabla = \mathrm{tr}(\nabla \otimes \nabla) = \sum_{i=1}^{n} \partial_i^2.$$

Now recall from calculus that for $\phi(x)$, the energy flux, or just some vector field in $\mathbb{R}^n$, if there's no source inside some area A then the conservation law says that:

Proposition 1.5. *(Conservation law)* $\forall A$ *we have*

$$\int_{\partial A} \phi \cdot \vec{n} ds = 0.$$

Moreover, integral by parts, or divergence theorem says:

Proposition 1.6. *(Divergence theorem)*

$$\int_A \nabla \cdot \phi dx = \int_{\partial A} \phi \cdot \vec{n} ds.$$

We note that in 1D this is the fundamental theorem of calculus as $\vec{n}$ points at the directions corresponding to the signs.

To prove this we can first do it for boxes, which is easy, then for multiple connected boxes, then use Stokes and etc..

Combining the above two propositions we get that

$$\int_A \nabla \cdot \phi dx = 0$$

and if we shrink A to one point we get $\nabla \cdot \phi = 0$ for all $x \in U$.

Phenomenalogically we know that $\phi = -\kappa \nabla u$ where $\kappa \neq 0$. That is, the transportation of heat is porportional to the distance. Thus

$$\nabla \cdot \phi = -\nabla \cdot \kappa \nabla u = -\kappa \Delta u = 0.$$

This is essentially how this equation comes from.

Now, for the Poisson equation, we have

$$\nabla \cdot \phi = f$$

and then the conservation law says

$$\int_{\partial A} \phi \cdot \vec{n} ds = \int_A f ds$$

which just says that how many energy is created inside then how many will go out.

Thus using the same method we have

$$\nabla \cdot \phi = f \Rightarrow -\Delta u = \frac{f}{\kappa}.$$

Now, notice that

$$\Delta(\alpha f + \beta g) = \alpha \Delta f + \beta \Delta g$$

so Δ is linear, this give us the superposition property of elliptic equations.

Proposition 1.7. *(Superposition)*

$$\begin{cases} -\Delta u_1 = f_1 \\ -\Delta u_2 = f_2 \end{cases} \Rightarrow -\Delta(\alpha u_1 + \beta u_2) = \alpha f_1 + \beta f_2.$$

From this property we can simplify the question since we can always write f as the superposition of $f(y)$ at each $y \in U$. To be specific note

$$f(x) = \delta * f = \int_{\mathbb{R}} \delta(x-y) f(y) dy$$

where $f(y)$ stands in place of α and δ stands in place of f.

Now, let's say that $-\Delta G(x, y) = \delta(x - y)$, then we have

$$u = \int G(x, y) f(y) dy$$

if we can justify the change of limits.

2. 3/23: Solution to Poisson Equation

Poisson equation is:

$$-\Delta u = f.$$

2.1. **Guessing a solution.**

Now to solve it remember that

$$\Delta \sum_j \alpha_j u_j = \sum_j \alpha_j \Delta u_j$$

which means if we have

$$-\Delta u = f, -\Delta u_j = f_j \quad \Rightarrow f = \sum_j \alpha_j f_j.$$

Since

$$f(x) = \int \delta(x-y) f(y) dy$$

if we have $-\Delta_x G(x,y) = \delta(x-y)$ then formally if we can change the integral with the Laplacian then one ansatz is

$$u(x) = \int_{\mathbb{R}^n} G(x,y) f(y) dy.$$

Note that everything here is formal so we don't really need to justify any of our guesses, but we need to be careful in the later part where we check that the solution is indeed one.

Since the Laplacian is δ, which means it is a source at x, there's no reason to think that a source at one place is different from another, nor is there reason for us to consider angle matters, i.e. we can get

$$R_\theta \Delta R_{-\theta} = \Delta$$

since δ is rotational invariant. Since G is translation invariant we have

$$G(x,y) = G(x-y, 0) =: G(x-y)$$

where the last is a definition (and a slight abuse of notation; Also, it's named G because it's Green's function). Since angle doesn't matter we further define

$$v(r) = \phi(x) := G(x)$$

where $r = |x|$.

Now we think about this r. In the case of $r > 0$, we know that $\Delta \phi(x) = 0$. We'll deal with this situation first then later with $r = 0$.

In the case of $r > 0$:

Since $r^2 = |x|^2 = \sum_i x_i^2$, we have

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}.$$

Thus, using that $\phi(x) = v(r)$ and chain rule

$$\frac{\partial \phi(x)}{\partial x_i} = \frac{\partial v(r)}{\partial x_i} = \frac{\partial v(r)}{\partial r}\frac{\partial r}{\partial x_i} = v'\frac{x_i}{r}$$

and the second derivative is

$$\frac{\partial^2 \phi(x)}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(v'\frac{x_i}{r}\right) = v''(r)\frac{x_i^2}{r^2} + v'(r)\frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right) = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

and summing up we have

$$\Delta\phi(x) = v''(r) + v'\left(\frac{n-1}{r}\right) = 0$$

where the last equality is just what we've guessed, that this is a solution.

So we just solve the equation and get (for $n = 1$ it's piecewise linear and we don't care)

$$v(r) = \begin{cases} b \cdot \log(r) + c & n = 2 \\ b \cdot r^{2-n} + c & n \geq 3 \end{cases} \tag{2.1}$$

which is just solving the ODE.

To get the constant c above we really need initial/boundary conditions, which we do not have. But we have good reason to set it to 0 because if we think about the $n \geq 3$ case, we would have a decay to 0 at infinity only when $c = 0$. So even though for the log there's not so convincing, we'd like it to vanish for higher order situations.

So of the two parameters b and c we know what c is. Now we solve b.

Divergence tells us that

$$\int_{\partial A} \Phi \cdot n d\sigma = \int_A \nabla\Phi dx$$

where $d\sigma$ is the surface integral step. And morally we know (here we choose the positive Green's function, whereas Evans defined δ with a sign difference)

$$-\Delta\phi = \delta \Rightarrow -\nabla \cdot \nabla\phi = \delta$$

and here's our divergence form $\nabla\phi := \Phi$. We deal with this on the unit ball since we can always scale the ball, as the only problematic point is the origin. So on the one hand we have

$$-\int_{B(0,1)} \nabla \cdot \nabla\phi dx = \int_{B(0,1)} \delta dx = 1$$

by definition of what the δ function is. On the other hand we have

$$-\int_{B(0,1)} \nabla \cdot \nabla \phi dx = -\int_{\partial B} \vec{n} \cdot \nabla \phi ds = -\int_{S^1} \frac{\partial \phi}{\partial \vec{n}} ds$$

where the last equality is just because the gradient dot product with the normal is the length of the projection of the gradient along the normal direction, i.e. the directional derivative of ϕ in the direction of n. Since we are on the unit ball each normal is pointing out at the direction of the radius, so we further get

$$-\int_{S^1} \frac{\partial \phi}{\partial \vec{n}} ds = -\int_{S^1} v'(1) ds = \begin{cases} -2\pi b & n = 2 \\ -b(2-n)\gamma(n) & n \geq 3 \end{cases}$$

where $\gamma(n)$ is the coefficient of the surface area of the n-dimensional unit ball, i.e.

$$S_n = \gamma(n) r^{n-1}$$

and the r^{n-1} cancels with the r^{1-n} that we get by taking the derivative of v, see 2.1.

So with the above few computations we get that

$$b = \begin{cases} -\frac{1}{2\pi} & n = 2 \\ \frac{1}{(n-2)\gamma(n)} & n \geq 3 \end{cases}$$

and again plugging into 2.1 we get

$$\phi(x) = v(|x|) = \begin{cases} -\frac{1}{2\pi} \log(|x|) & n = 2 \\ \frac{1}{(n-2)\gamma(n)} |x|^{2-n} & n \geq 3. \end{cases}$$

And we can use superposition to guess our ansatz

$$u(x) = \int_{\mathbb{R}^n} \phi(x-y) f(y) dy.$$

Now we deal with the case when $r = 0$.

This might be tricky at $r = 0$, since we'd love to have $v'' + \frac{1}{r} v' = \delta$ but that's not even well defined.

One way to illustrate this difficulty is that we can (after some computation) get that the spectrum $\sigma(-\Delta) = [0, \infty)$ (which is why this is positive operator), so in particular at $\lambda = 0$ the solution to the equation

$$(-\Delta - \lambda) u = f$$

is not easy to deal with since the range is not closed (all continuous spectrum).

To conquer this difficulty the strategy is to add an α^2 to the equation to make it strictly positive, i.e. we solve

$$(-\Delta + \alpha^2) u = f.$$

The solution to this (adding one ODE to solve) contains the following term along the way:

$$\frac{e^{-\alpha|x|}}{|x|}\frac{1}{\alpha}e^{-\alpha|x|}$$

from which we note that $\alpha = 0$ is indeed problematic. But so long with that.

2.2. **Verifying our solution.**

What we've done above is mostly sloppy, but we must be strict in verifying them.

That is, we want to show that the ansatz we found has $-\Delta u = f$ pointwise (in other wors, it's a strong solution). That is, we want to show

$$\Delta u = \Delta_x \int_{\mathbb{R}^n} \phi(x-y)f(y)dy \overset{?}{=} \int_{\mathbb{R}^n} \Delta_x\left(\phi(x-y)\right)f(y)dy \overset{?}{=} -f$$

where the last 2 inequalities needed verify. Well, the second is that, if we can get the first then it follows by our definition of ϕ, yet we'll see that the way we show the first question mark forces us to put more effort in the proof of the second.

For the following, we will assume that $f \in C_c^2(\mathbb{R}^n)$ and one can extend that to $C_0^2(\mathbb{R}^n)$ without much difficulties. That said, f is our nice guy here.

The first equality

The common path to this kind of exchange of limit is to use Dominated Lebesgue (DCT). But there's no hope of that here since ϕ is problematic (not integrable) at 0 by our explicit expression (and at ∞ when $n = 2$), in other words it's not a nice guy there. But f should be a nice guy and so we try to work on f.

So the trick here is just to use the associativity of convolution, i.e.

$$(f * g)(x) = \int f(x-y)g(y)dy \overset{y:=x-y}{=} \int f(y)g(x-y)dy = (g * f)(x)$$

from which we know

$$\Delta u = \Delta_x \int_{\mathbb{R}^n} \phi(x-y)f(y)dy = \Delta_x \int_{\mathbb{R}^n} \phi(y)f(x-y)dy$$

and thus we try to dominate f.

Proposition 2.1. *(Dominated Lebesgue) Let $f_n : U \to \mathbb{R}$ be such that*

(i) $f_n \to f$ pointwise (a.s. is enough);
(ii) $|f_n| \le g(x)$ for $g \in L^1(U)$.

Then $f \in L^1$ and

$$\lim_{n\to\infty}\int_U f_n(x)dx = \int_U \lim_{n\to\infty} f_n(x)dx = \int_U f\,dx.$$

Now note that for $n \geq 3$ we have

$$\phi(x) = \frac{c}{|x|^{n-2}}; \quad |\nabla \phi| \leq \frac{c}{|x|^{n-1}}; \quad |\nabla^2 \phi| \leq \frac{c}{|x|^n}$$

this gives another view of why we cannot just exchange limits for ϕ, as $\Delta\phi$ behaves bad around 0. But we'll see that ϕ itself is actually good and integrable locally:

Proposition 2.2. *$\phi(x) \in L^1(U)$ for U bounded (and not containing 0).*

Proof. The only problem here really comes from around the origin since ϕ is a nice guy everywhere else. At the ε ball around the origin we have for $n = 2$

$$\int_{B_\varepsilon} |\phi(y)| dy = \int_0^{2\pi} \int_0^{\varepsilon} -\frac{1}{2\pi} r \log r dr d\theta = \int_0^{2\pi} \left(-\frac{r^2}{8\pi} (2\log(r) - 1) \right) \Big|_{r=0}^{\varepsilon} d\theta$$
$$= 2\pi \cdot \left(-\frac{\varepsilon^2}{8\pi} (2\log(\varepsilon) - 1) \right) \leq C\varepsilon^2 |\log(\varepsilon)| < \infty.$$

and for $n \geq 3$, remember that the polar change of coordinate formula has r^{n-1} inside

$$\int_{B_\varepsilon} |\phi(y)| dy = \int_S^{n-1} \int_0^{\varepsilon} C_n r^{n-1} \frac{1}{r^{n-2}} dr d\theta^{n-1} = \int_S^{n-1} \int_0^{\varepsilon} C_n r dr d\theta^{n-1} = C\varepsilon^2 < \infty.$$

□

Now we will first show the equality first for one derivative then do this again for the second derivative.

Proposition 2.3. *$\nabla u(x)$ exists and*

$$\nabla u(x) = \int_{\mathbb{R}^n} \phi(y) \nabla f(x-y) dy.$$

Proof. We don't know existence yet so we use the quotient

$$\frac{u(x+he_i) - u(x)}{h} = \int_{V_x} \phi(y) \frac{f(x+he_i) - f(x)}{h} dy$$

where V_x is the support of f. Since it's compactly supported we know ϕ is integrable. Thus, since f' is continuous on that compact support we can as well bound that by

$$\left| \phi(y) \frac{f(x+he_i) - f(x)}{h} \right| \leq M\phi(y) ||f'|| \in L^1(V_x)$$

this, plus the fact that $\dfrac{f(x+he_i) - f(x)}{h} \to \dfrac{\partial f}{\partial x}(x-y)$, means by Dominated Lebesgue we can pass the limit on h and get our result. □

Using the strategy above another time we get that

$$\Delta u = \int \phi(y) \Delta f(x-y) dy.$$

This deals with the first equality, but the price we pay is that we'll have to check the integral by parts later.

The second equality:

We want to show now

$$\int_{\mathbb{R}^n} \phi(y) \Delta f(x-y) dy = -f$$

and we must realize that 0 is the only problematic point in the integral (singularity at infinity vanishes as f vanishes). So we let $B_\varepsilon = B(0, \varepsilon)$ and separate our domain into

$$\mathbb{R}^n = B_\varepsilon \cap (\mathbb{R}^n \backslash B_\varepsilon)$$

and on B_ε we have $|\Delta f| \leq C$ which gives us

$$\left| \int_{B_\varepsilon} \phi(y) \Delta f(x-y) dy \right| \leq C \int_{B_\varepsilon} |\phi(y)| dy \leq C\varepsilon \to 0$$

by results in proposition 2.2 above ($\varepsilon |\log \varepsilon = O(\varepsilon)$).

For the other part of the integral we let $U = \mathbb{R}^n \backslash B_\varepsilon$ and again the trick is to find some kind of divergence form: if you find them you win.

We notice that

$$\int_U (u\Delta v - v\Delta u) dx = \int_U \nabla (u\nabla v - v\nabla u) = \int_{\partial U} \left(u\frac{\partial v}{\partial \tilde{n}} - v\frac{\partial u}{\partial \tilde{n}} \right) d\sigma$$

where the first inequality is because we've canceled $\nabla u \nabla v$ terms.

Now we consider the normal vector $\tilde{n}$ of ∂U. They are just $-n$ where n are normal vectors of $\partial B := \partial B_\varepsilon$ since they have the same boundary but occupy different sides, i.e. $\tilde{n} = -n$.

Thus we have

$$\begin{aligned}
&\int_U \phi(y) \Delta f(x-y) dy - \int_U \Delta \left(\phi(y) \right) f(x-y) dy \\
&= \int_{\partial U} \phi(y) \left(\frac{\partial f}{\partial \tilde{n}}(x-y) \right) - \left(\frac{\partial \phi}{\partial \tilde{n}}(y) \right) f(x-y) ds \\
&\overset{(a)}{\to} 0 - \int_{\partial B_\varepsilon} \left(\frac{\partial \phi}{-\partial n}(y) \right) f(x-y) ds = \int_{\partial B_\varepsilon} \left(\frac{\partial \phi}{\partial n}(y) \right) f(x-y) ds \\
&= \int_{\partial B_\varepsilon} \left(\frac{\partial \phi}{\partial n}(y) \right) f(x) ds + \int_{\partial B_\varepsilon} \left(\frac{\partial \phi}{\partial n}(y) \right) \left[f(x-y) - f(x) \right] ds \\
&\overset{(b)}{\to} f(x) \int_{\partial B_\varepsilon} \left(\frac{\partial \phi}{\partial n}(y) \right) ds + 0 \overset{(c)}{=} -f(x)
\end{aligned}$$

where we justify (a) via a similar polar coordinate integration as in the proof of proposition 2.2, the only difference being there's no $\int_0^\varepsilon dr$ so we have one less order of ε, that is

$$\int_{\partial U} \phi(y) \left(\frac{\partial f}{\partial \tilde{n}}(x-y) \right) \leq C \cdot ||f'|| \cdot \varepsilon |\log \varepsilon| \to 0$$

and (b) is due to the fact that $y \in B_\varepsilon$ is small and f continuous, (c) is due to similar reason we've shown above:

$$1 = \int_{B(0,1)} \delta dx = -\int_{B(0,1)} \nabla \cdot \nabla \phi dx = -\int_{\partial B} \vec{n} \cdot \nabla \phi ds = -\int_{S^1} \frac{\partial \phi}{\partial \vec{n}} ds.$$

Thus we have

$$\int_U \phi(y) \Delta f(x-y) dy - \int_U \Delta(\phi(y)) f(x-y) dy = -f(x)$$

but $\Delta\phi = 0$ on U by definition so we have what we want:

$$\int_U \phi(y) \Delta f(x-y) dy = -f(x).$$

What we haven't yet done is to show that the solution is unique. Also, note that

$$u = \phi * f$$

is at least as smooth as the smoother one of the two convolved functions, thus since we have $f \in C^2$ we know u is very smooth.

3. 3/28: Uniqueness of standard Poisson; Regularity properties

One thing to note from last time is that we have

$$\Delta_x[\phi(x-y)] = \Delta_y[\phi(x-y)$$

since the sign flips back, then the integral by parts on y can be done.

3.1. **Uniqueness of solutions.**

Last time we've found a solution of the Poisson equation, today we show that this is unique. Note that to show this we really want to show if there's 2 solution, then their difference is 0. But their difference is harmonic, so we start by studying properties of harmonic functions.

Theorem 3.1. *(Mean Value Theorem) For $U \subset \mathbb{R}^n$ open and $u \in C^2(\overline{U})$ harmonic, we have*

$$u(x) = \fint_{\partial B(x,r)} u ds = \frac{1}{|\partial B|}\int_{\partial B} u ds$$

and

$$u(x) = \fint_{B(x,r)} u ds.$$

Proof. We define the auxiliary function

$$\phi(r) := \fint_{\partial B(x,r)} u ds = \fint_{\partial B(0,r)} u(x+zr) ds(z)$$

and we see that if $\phi'(r) = 0$, that it's constant in r we know that the integral is constant on all r, but the integral is $u(x)$ as $r \to 0$, so that's all we need to show. The way we take derivatives is solve the derivative by directly putting things inside since $u \in C^2$ and hence the derivative is a continuous function on a compact set, hence bounded, then integrable, so DCT can be passed. Now we have

$$\begin{aligned}
\phi'(r) &\overset{DCT}{=} \fint_{\partial B(0,1)} z \cdot \nabla u(x+rz) dS(z) = \fint_{\partial B(0,1)} \frac{\partial u}{\partial n}(x+rz) dS(z) \\
&= \fint_{\partial B(x,r)} \frac{\partial u}{\partial n}(y) dS(y) = \fint_{\partial B(x,r)} (n \cdot \nabla u)(y) dS(y) = \frac{1}{|\partial B(x,r)|}\int_{B(x,r)} (n \cdot \nabla u)(y) dS(y) \\
&= \frac{1}{|\partial B(x,r)|}\int_{B(x,r)} \Delta u(y) dS(y) = 0
\end{aligned}$$

Thus we conclude for the surface integral. But for the whole ball we really just integrate over all radius, that is

$$\begin{aligned}
\fint_{B(x,r)} u ds = \frac{1}{|B|}\int_{B(x,r)} u ds &= \frac{1}{|B|}\int_0^r \left[\int_{\partial B(x,r)} u(r,\theta) d\theta\right] r^{n-1} dr \\
&= \frac{1}{|B|}\int_0^r |S^{n-1}| u(x) dr = u(x)
\end{aligned}$$

where $|S^{n-1}|$ is the volumn of the sphere. □

The powerful thing is that we even have a converse to it. It's not strictly converse for now, but we will see later that it is (no extra regularity required).

Theorem 3.2. *(Converse Theorem) For $u \in C^2(\overline{U})$, if $u(x) = \fint_{\partial B(x,r)} uds$ for all x, r, then $\Delta u = 0$ point wise.*

Proof. Define ϕ the same as above, then we know $\phi'(r) = \frac{|B|}{|\partial B|} \fint_B \Delta u ds = 0$ where the last is by assumption. But this really means that $\Delta u = 0$ on any ball, since if $\exists x$ such that $\Delta u > 0$ then exists neighborhood N such that the integral of Δu is larger than 0 on the ball, contradiction. □

Remark 3.3. *We do not have an iff statement here because we need to assume $u \in C^2(\overline{U})$. We'll take that off later.*

Theorem 3.4. *(Strong Maximum principle) Let $U \subset \mathbb{R}^n$ be open, bounded, connected, and $u \in C^2(U) \cap C^0(\overline{U})$, $\Delta u = 0$ on U. Then if exists $x_0 \in U$ such that $\forall x \in U$ we have $u(x_0) \geq u(x)$, that x_0 is a maximum, then u is constant on U.*

Note that we can easily have the weak version of the max principle.

Corollary 3.5. $\max_{x \in \overline{U}} u(x) = \max_{x \in \partial U} u(x)$.

Proof. (of Strong max principle)

By MVT we have $u(x_0) = \fint_{B(x_0,r)} udy \leq u(x_0)$ which means the inequality is strict, but this means u is constant every where in the r ball. But the space is open and connected so for every y there is a path connecting x_0 and y such that the path can be covered by balls. And thus $u(x_0) = u(y)$. □

Note that since u is continuous on the boundary the boundary is also constant.

This result is not true for wave equation though.

Similarly we get

Corollary 3.6. *The minimum is also attained on the boundary.*

The proof is just by taking $u = -u$. This, weirdly, means harmonic functions are a lot like complex functions.

Application: Uniqueness of Poisson

Proposition 3.7. *If $u \in C^2(U) \cap C^0(\overline{U})$ is a solution of*

$$\begin{cases} \Delta u = f & x \in U \\ u = g & x \in \partial U \end{cases}$$

for $f, g \in C^0$, then it is unique.

Proof. If $u \neq v$ are both solutions, then $w = u - v$ is harmonic that is 0 on ∂U, this means $w = 0$ everywhere, so $u = v$, contradiction. □

3.2. **Regularity properties.**

What we want to show is that harmonic and MVT are really the same, so we need to show the regularity is not needed. The way we do this, surprisingly, is to show that the regularity is implied. This is also analogous to holomorphic functions.

Proposition 3.8. *(Miracle) Assume $u \in C(U)$ such that $u(x) = \fint_{B(x,r)} u ds$, then $u \in C^\infty$.*

Remark 3.9. *Recall that if there is a source f in the ball, then the solution is not c^∞ but relies on the regularity of the source. But the result is local, and very good locally.*

Proof. (of miracle)

We use the usual convolution trick. Let η be a smooth bumping that is positive, has integral 1, supported on $[-1, 1]$, and depends only on $|x|$, so we write $\eta(x) = \tilde{\eta}(|x|)$. That is, the family $\eta_\varepsilon := \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right)$ is a family of good kernels. Thus we know $u * \eta_\varepsilon \to u$.

But the surprising result is that actually for any ε we do have $u * \eta_\varepsilon = u$, thanks to harmonicity. So since

$$u = u * \eta_\varepsilon = \int_{\mathbb{R}^n} \eta_\varepsilon(x - y)u(y)dy \in C^\infty$$

we are done. What's left is to prove the miracle:

$$\begin{aligned} u * \eta_\varepsilon &= \int_{B(x,\theta)} \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right) u(x - y)dy = \frac{1}{\varepsilon^n}\int_0^\varepsilon \tilde{\eta}\left(\frac{r}{\varepsilon}\right)\left[\int_{\partial B(x,r)} u(x - y)ds\right] r^{n-1}dr \\ &= \frac{1}{\varepsilon^n}\int_0^\varepsilon \tilde{\eta}\left(\frac{r}{\varepsilon}\right)\left[u(x)\cdot|\partial B|\right] r^{n-1}dr = u(x)\cdot\int_0^\varepsilon \tilde{\eta}_\varepsilon(r)|\partial B|r^{n-1}dr = u(x) \end{aligned}$$

since

$$|\partial B| = \int_{\partial B} 1 dS.$$

□

Theorem 3.10. *If u is harmonic in U, then u is real analytic, i.e.*

$$u(x) = \sum_{|\alpha|=1}^{\infty} \frac{1}{\alpha!} D^\alpha u(x_0)(x - x_0)^\alpha$$

for terms like usual.

We know the derivatives are well defined, and thus we only need to show convergence. (last time).

Theorem 3.11. *If $\Delta u = 0$ and $B_{x_0}(r) \subset U$, we have*

$$|D^\alpha u(x_0)| \le \frac{C_k}{r^{n+k}} ||u||_{L^1(B_{x_0}(r))}, k = |\alpha|$$

where

$$C_0 = \frac{1}{\alpha(n)}; \qquad C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)}$$

where $\alpha(n) = n\gamma(n) = |B(0,1)|$.

Proof. When $k = 0$, we have

$$|u(x)| = \left|\frac{1}{|B|}\int_B u(y)dy\right| \le \frac{1}{\alpha(n)r^n}||u||_{L^1(B_{x_0}(r))}.$$

When $k = 1$ note that

$$\Delta\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i}\Delta u = 0$$

and we can use harmonic properties on the partial derivative. So we have

$$\left|\frac{\partial u}{\partial x_i}(x_0)\right| = \left|\frac{1}{|B(x_0,r/2)|}\int_{B(x_0,r/2)}\frac{\partial u_i}{\partial x_i}ds\right| = \frac{2^n}{\alpha(n)r^n}\left|\int_{\partial B(x_0,r/2)} u\cdot n_i ds\right|$$
$$\le \frac{2^n}{\alpha(n)r^n}|\partial B|\cdot ||u||_{L^\infty(B(x_0,r/2))} \le \frac{2n}{r}\cdot ||u||_{L^\infty(B(x_0,r/2))}$$

now this is the step where we use the IH, since we've reduced $|u_i|$ to $|u|$. Using the fact that $B(x,r/2) \subset B(x_0,r) \subset U$ we get

$$|u(x)| \le ⨍_{B(x,r/2)}|u(x)|dx = \frac{2^n}{\alpha(n)r^n}\cdot ||u||_{L^1(B(x,r/2))} \le \frac{2^n}{\alpha(n)r^n}\cdot ||u||_{L^1(B(x_0,r))}$$

and thus we bound the above last step with

$$\left|\frac{\partial u}{\partial x_i}(x_0)\right| \le \frac{2n}{r}\cdot ||u||_{L^\infty(B(x_0,r/2))} \le \frac{2n}{r}\frac{2^n}{\alpha(n)r^n}\cdot ||u||_{L^1(B(x_0,r))}.$$

The rest is just induction likely. The bound of k^k is roughly like the $(k-\nu)^\nu$ bound method in Chebyshev approximations. ☐

Application: Uniqueness

Theorem 3.12. *(Liouville) If $u : \mathbb{R}^n \to \mathbb{R}$ is bounded, then it's a constant.*

Proof. Fix $x_0 \in \mathbb{R}^n$, we have

$$|Du(x_0)| \leq \frac{c_1}{r^{n+1}} ||u||_{L^1(B_{x_0}(r))} \leq \frac{c_1 \alpha(n)}{r} ||u||_{L^\infty(B_{x_0}(r))} \to 0$$

as $r \to \infty$. Note that r can be arbitrarily large since u is defined everywhere. $\square$

This immediately gives:

Proposition 3.13. *Any bounded solution u of $-\Delta u = f$ is of the form*

$$u = \int \phi(x-y) f(y) + C.$$

4. 3/30: Boundary Value Problem

One note for last time is that when $n \geq 3$, the expression

$$u = \int_{\mathbb{R}^n} \phi(x - y) f(y) dy + C$$

is indeed bounded for $f \in C_c^2$.

Now we show the analycity of harmonic functions.

Proof. (Theorem 3.10)

Let $x_0 \in U$ and $r = \frac{1}{4}\operatorname{dist}(x_0, \partial U)$ and define constant

$$M := \frac{1}{\alpha(n) r^n} ||u||_{L^1(B_{x_0}(r))}$$

Let $B(x, r) \subset B(x_0, 2r) \subset U$ and thus by theorem 3.11 we have the bound

$$||D^\alpha u||_{L^\infty(B(x,r))} \leq M \cdot \left(\frac{2^{n+1} \cdot n}{r} \right)^{|\alpha|} |\alpha|^{|\alpha|}$$

where we can use Sterling's formula to get

$$|\alpha|^{|\alpha|} \leq C \cdot e^{|\alpha|} |\alpha|! \leq C \cdot (en)^{|\alpha|} \alpha!$$

and thus we get

$$||D^\alpha u||_{L^\infty(B(x,r))} \leq CM \cdot \left(\frac{2^{n+1} \cdot en^2}{r} \right)^{|\alpha|} \alpha!.$$

This, together with Taylor's formula gives us

$$R_N(x) = u(x) - \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{D^\alpha u(x_0)(x - x_0)^\alpha}{\alpha!} = \sum_{|\alpha|=N} \frac{D^\alpha u(x_0 + t(x - x_0))(x - x_0)^\alpha}{\alpha!}$$

for some $t \in [0, 1]$. We want to show this is small, so we can choose $|x - x_0| \leq r_0 := \frac{r}{2^{n+2} n^3 e}$ such that the value goes to 0.

But really we can just compute and get

$$|R_N(x)| \leq CM \sum_{|\alpha|=N} \left(\frac{2^{n+1} \cdot en^2}{r} \right)^N \left(\frac{r}{2^{n+2} n^3 e} \right)^N = O\left(\frac{CM}{2^N} \right) \to 0$$

which means that the convergence of Taylor series is verified. □

So we want to investigate the boundary value problems for Poisson equation

$$\begin{cases} -\Delta u = f & U \\ u = g & \partial U \end{cases}$$

and we want to focus only on the half space. But let's see that later.

Our main tool is Green's formula:

$$\int_U \Delta u \cdot v - \Delta v \cdot u = \int_U \nabla(v \cdot \nabla u - u \cdot \nabla v) = \int_{\partial U} v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n} d\sigma$$

and let's just try plugging in $\phi = v$ and we see roughly we're doing fine, but let's now do the details of this, which is a review. Let $V_\varepsilon = U - B(x, \varepsilon)$ and $v = \phi(x - y) = \phi(y - x)$. We have

$$\int_{V_\varepsilon} u(y)\Delta\phi(y - x) - \phi(x - y)\Delta u = \int_{\partial V_\varepsilon} u(y)\Delta\frac{\partial \phi}{\partial n}(y - x) - \phi(x - y)\frac{\partial u}{\partial n}(y)$$

and since the second term has by integration that

$$\int_{B_\varepsilon} \phi(x - y)\frac{\partial u}{\partial n}(y) = o(1)$$

we are only left with the first, which we know is

$$\int_{B_\varepsilon} u(y)\frac{\partial \phi}{\partial n}(y - x) = u(x) + o(1).$$

Thus, integrating on the whole domain, and get off what is 0 (note $\Delta\phi = 0$ on V_ε) we get

$$\begin{aligned} u(x) &= \int_{\partial U} \phi(y - x)\frac{\partial u}{\partial n}(y) - u(y)\frac{\partial \phi}{\partial n}(y - x)ds(y) - \int_U \phi(y - x)\Delta u(y)dy \\ &= \int_{\partial U} \phi(y - x)\frac{\partial u}{\partial n}(y) - g(y)\frac{\partial \phi}{\partial n}(y - x)ds(y) - \int_U \phi(y - x)f(y)dy \end{aligned}$$

which holds for all $u \in C^2(\overline{U})$ and $x \in U$. So we only need to know what is $\dfrac{\partial u}{\partial n}$ on ∂U.

Remark 4.1.

- *This is the integral method and is useful in many places;*
- *What we've done is we've removed any degree of freedom inside the domain, left with only the boundary;*
- *This is basically Schur complement, since that's just putting the boundary matrix on the left top part;*
- *We call this <u>Dirichlet condition</u>, and if we are given $\dfrac{\partial u}{\partial n}$ we call that <u>Neumann</u> condition. Both are sufficient.*

But really, how do we cure the term $\dfrac{\partial u}{\partial n}$? The best part is that we don't cure it, we kill it.

Note that $\Delta\phi = -\delta$ on $\mathbb{R}^n$. But here we have a boundary and thus we replace this ϕ by something that's adapted to our U. How do we cheat here? We let

$$\begin{cases} -\Delta G(x, y) = \delta_y(x) & U \\ G(x, y) = 0 & \partial U \end{cases}$$

and if this thing is well defined, then using the same argument as we've done above, we know that

$$u(x) = \int_{\partial U} -g(y)\frac{\partial G}{\partial n}(x,y)ds(y) - \int_U G(x,y)f(y)dy$$

(with possible sign error).

Only solving G is hard, alas.

To solve G, we introduce $\phi_x(y)$, the solution of the problem

$$\begin{cases} -\Delta\phi_x = 0 & U \\ \phi_x(y) = \phi(x-y) & \partial U \end{cases}$$

and expect that the solution to be smooth. If we can do so (and that is what we'll assume, details in book), then we can just define

$$G(x,y) = \phi(x-y) - \phi_x(y)$$

which means it satisfies the above condition, tricky heh. Note that in this way we also get around the δ-function completely.

So we compile the above argument into one representation theorem:

Theorem 4.2. *The $u \in C^2$ solution of*

$$\begin{cases} -\Delta u = f & U \\ u = g & \partial U \end{cases}$$

is

$$u(x) = \int_{\partial U} -g(y)\frac{\partial G}{\partial n}(x,y)ds(y) - \int_U G(x,y)f(y)dy.$$

Ok, but can we say more about this G?

Theorem 4.3. *When G is defined we have*

$$G(y,x) = G(x,y).$$

Proof. We define

$$\begin{cases} v(z) = G(x,z) & U \\ w(z) = G(y,z) & U \end{cases}$$

and do the usual trick to let $V_\varepsilon := U - [B(x,\varepsilon) \cup B(y,\varepsilon)]$ then on V_ε we have $\Delta v = \Delta w = 0$ and we have by the usual argument that

$$\int_{\partial B(x,\varepsilon)} \partial_n v \cdot w - \partial_n w \cdot v = -\int_{\partial B(y,\varepsilon)} \partial_n v \cdot w - \partial_n w \cdot v$$

and one of the terms vanishes because IBP and the other yields $w(x)$ and $v(x)$, we get

$$w(x) = v(y)$$

if we add back to the integral on everywhere on U. $\square$

Let us now see an explicit construction of G when $U = \mathbb{R}^n_+ : \{x_n > 0\}$. We want to construct $G(x, y) = \phi(x - y) - \phi_x(y)$ but from physics we know that if there is a source at (y, x) where $y \in \mathbb{R}^{n-1}$ and $x \geq 0$, then a negative source at $(y, -x)$ can make the boundary 0, which means we can just define

$$G(x, y) = \phi(x - y) - \phi(\tilde{x} - y)$$

where

$$x = (x', x_n); \tilde{x} = (x', -x_n).$$

If we let $f = 0$ then we can get

$$u = \int_{\partial U} \frac{\partial G}{\partial n} g$$

and to show this is the solution we first compute

$$\partial_{y_n} G(x, y) = \partial_{y_n} \phi(y - x) - \partial_{y_n} \phi(y - \tilde{x}) = -\frac{1}{n\alpha(n)} \left(\frac{y_n - x_n}{|y - x|^n} - \frac{y_n + x_n}{|y - \tilde{x}|^n} \right)$$

and so

$$\frac{\partial G}{\partial n}(x, y) = -\frac{2x_n}{n\alpha(n)} \frac{1}{|x - y|^n}.$$

So we have the Poisson Formula:

$$u = \int_{\partial \mathbb{R}^n_+} K(x, y) g(y) dy$$

where

$$K(x, y) = \frac{2x_n}{n\alpha(n)} \frac{1}{|x - y|^n}$$

and our task next time is to prove that this u solves the equation.

5. 4/4: Representation theorem; Energy method; Variational method; Heat Equation

5.1. **Poisson on halfspace.**

Last time we've shown a uniqueness result (Theorem 4.2), so now we think about existence for the halfspace $U = \mathbb{R}^n_+$.

Theorem 5.1. *The Poisson formula*

$$u = \int_{\partial \mathbb{R}^n_+} K(x,y)g(y)dy$$

where

$$K(x,y) = \frac{2x_n}{n\alpha(n)}\frac{1}{|x-y|^n}$$

is indeed a solution for g bounded.

Note that it's not a uniqueness result at all.

Proof.

The idea really is to note that K is an approximate identity.

So we can check that

$$\int_{\partial \mathbb{R}^n_+} K(x,y)dy = 1$$

for $x \in U$. Now, for the ball of radius R, we denote the upper semicircle to be S^R_+ and the lower semicircle to be S^R_-. Moreover, we denote the line $[-R,R]$ with certain direction to be S^R_0. Then we have for the aforementioned ϕ has

$$-\int_{S^R_+ \cup S^R_-} \frac{\partial \phi}{\partial n} = -\int \Delta\phi = 1$$

and since the only source is in U we have

$$-\int_{S^R_+ \cup S^R_0} \frac{\partial \phi}{\partial n} = 1.$$

But if we take R large then $x \approx 0$, which means that the effect of the source on the upper and lower semicircle is symmetric, hence the integral is the same, so

$$-\int_{S^R_0} \frac{\partial \phi}{\partial n} = -\int_{S^R_-} \frac{\partial \phi}{\partial n} = \frac{1}{2}$$

and since $K = \partial_{y_n} G$ where $G = \phi - \tilde{\phi}$ so

$$\int_{\partial U} K ds = \frac{1}{2} + \frac{1}{2} = 1.$$

For $x \notin \partial U$, we know that K is a kernel, hence it's smoothing things, by DCT we can pass the limit and get

$$\partial_x^\alpha u = \int_{\partial U} \partial_x^\alpha K(x, y) g(y) dy$$

and taking twice we have

$$\Delta u = \int (\delta K) g = 0$$

in U since $\Delta K(x, y) = \delta_{x=y\in\partial U}$. This finishes the Δu check.

Now we do the usual trick and let $I_\delta = B(x_0, \delta) \subset \partial U$ and $J_\delta = \partial U \backslash I_\delta$, then

$$u(x) = \int_{J_\delta} K(x, y) g(y) dy + \int_{I_\delta} K(x, y) \left(g(x_0) + [g(y) - g(x_0)] \right) dy$$

where

$$\left| \int_{J_\delta} K(x, y) g(y) dy \right| \leq \frac{2||g||_\infty x_n}{\gamma(n)} \int_{J_\delta} \frac{dy}{|x - y|^n} \to 0$$

even though the integral diverges, we can let $x_n \to 0$ faster than the integral, which really just barely diverges.

For the other terms we know $[g(y) - g(x_0)] \to 0$ by continuity, and

$$\int_{J_\delta} K(x, y) g(x_0) dy \to 0$$

since

$$\int_{J_\delta} K(x, y) dy \to 0$$

is an approximate identity. So the only term we still care about is

$$\int_{I_\delta} K(x, y) g(x_0) dy = \int_{\partial U} K(x, y) g(x_0) dy = g(x_0) \int_{\partial U} K ds = g(x_0) \to g(x).$$

This finishes the boundary check. □

5.2. **Energy method.**

Here we use another method to show what we've shown already by Liouville.

Theorem 5.2. *There's at most one result to the Poisson equation with f and g.*

Proof. Let $w = u - v$, both solutions to the system, then if we can use harmonic analysis we are done. But let's see the other way.

We write everything in divergence form and note that

$$\nabla(w \nabla w) = (\nabla w)^2 + w \Delta w = (\nabla w)^2 \geq 0$$

using which we can get

$$0 \le \int_U ((\nabla w)^2) = \int_U \nabla(w\nabla w) = \int_{\partial U} w\frac{\partial w}{\partial n} = 0$$

since $w = 0$ on the boundary (if it's Neumann condition the derivative is 0). Thus $\delta w = 0$ and hence $w = c$ on the whole U. But now the Dirichlet condition tells us $u = 0$ where as Neumann condition does not help. □

5.3. **Variational method/Dirichlet Principle.**

Define the functional

Def 5.3.

$$I[w] = \int_U \left(\frac{1}{2}|\nabla w|^2 - f \cdot w\right) dx.$$

Now we have the following theorem about a equivalent condition for u being a solution of the system.

Theorem 5.4. *u solves the system $\iff$ u is a minimizer of I in $\mathcal{A}$, where the admissible set*

$$\mathcal{A} := \left\{ w \in C^2(\overline{U}); w = g \text{ on } \partial U \right\}.$$

Proof.

($\Rightarrow$):

Let $w \in \mathcal{A}$, and $u \in \mathcal{A}$ be a solution, then we just multiply the system by $u - w$ in the following way:

$$0 = \int_U (-\Delta u)(u-w) - f \cdot (u-w) \overset{IBP}{=} \int_U \nabla u \cdot \nabla(u-w) - f \cdot (u-w)$$

where the IBP holds exactly because $u - w$ vanishes on the boundary, so the boundary terms vanish. Then we have

$$\int_U |\nabla u|^2 - f \cdot u = \int_U \nabla u \cdot \nabla w - f \cdot w \overset{Cauchy}{\le} \int_U \frac{1}{2}\left(|\nabla u|^2 + |\nabla w|^2\right) - fw$$

which means

$$I[u] \le I[w]$$

so u is indeed a minimizer.

($\Leftarrow$):

We know that u is a critical point, and arbitrary $v \in C^2(\overline{U})$ that vanishes on ∂U. Then $u + \tau v \in \mathcal{A}$ is a small perturbation added onto u, and so we've reduced the I into a function on τ, namely

$$i(\tau) := I[u + \tau v] = \int_U \frac{1}{2}|\nabla u|^2 - f \cdot u + \tau(\nabla u \cdot \nabla v - v \cdot f) + \frac{\tau^2}{2}|\nabla v|^2$$

and the only thing we shall need is that u is a critical point of I, so $i'(0) = 0$, which we can compute to be

$$\int_U \nabla u \cdot \nabla v - v \cdot f = 0$$

which by an IBP we have

$$\int_U (\Delta u - f)v = 0.$$

But v is arbitrary so $\Delta u - f$ inside U, we are done. □

The key here is that $\frac{1}{2}|\Delta u|^2$ is a convex function.

5.4. **Heat Equation.**

The heat equation is

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = 0 & \mathbb{R}^n \times \mathbb{R}_+ \\ u(x,0) = u_0(x) & \mathbb{R}_+. \end{cases}$$

We see that it's not elliptic, but rather parabolic, or hyperelliptic, which means that it's smooth.

Now we define Fourier transform that we'll use later:

Def 5.5. *For $u \in L^1(\mathbb{R}^n)$ we define*

$$\left(\underset{x\to\xi}{\mathcal{F}} u\right)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} u(x) dx \in L^\infty$$

where the last result is because $e^{-ix\xi}$ is bounded by 1.

Def 5.6. *In the same manner we define*

$$\left(\underset{\xi\to x}{\mathcal{F}^{-1}} u\right)(x) = \check{u}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\xi} u(\xi) d\xi.$$

And a basic fact is that

Proposition 5.7.

$$\int \frac{e^{ix\xi}}{(2\pi)^n} = \delta(x).$$

Moreover, the reverse transform of δ is 1.

We've shown the tension between $L^1 \to L^\infty$ by $\mathcal{F}$, and we'll try to extend this to L^2 next time.

6. 4/6: Fourier Analysis; Application to PDEs

6.1. **Fourier Analysis.**

For $u \in L^1$ it's obvious from the definition that $\hat{u}, \check{u} \in L^\infty$. What we want to today is to extend the definition to functions that are even merely L^2. But first let's see some properties.

Theorem 6.1. *(Plancherel) For $u \in L^2 \cap L^2$, we have $\hat{u}, \check{u} \in L^2$ and the isometry:*

$$\frac{1}{(2\pi)^n}||\hat{u}||_{L^2}^2 = ||u||_{L^2}^2 = (2\pi)^n||\check{u}||_{L^2}^2.$$

In a sense this means that Fourier transforms does not lose mass within the L^2 norm.

Proof.

We first state three useful properties of Fourier transforms.

(1) We have

$$\int v(y)\hat{w}(y)dx = \int v(x)\left(\int e^{-ixy}w(y)dy\right)dx \overset{Fubini}{=} \int\int v(x)e^{-ixy}dxw(y)dy$$
$$= \int \hat{v}(y)w(y)dy.$$

(2) A plain fact of plain waves:

$$\mathcal{F}\{e^{-\varepsilon|x|^2}\}(\xi) = \left(\frac{\pi}{\varepsilon}\right)^{n/2} e^{-\frac{|\xi^2|}{4\varepsilon}} \to \delta.$$

This is related to the uncertainity principle because as $\varepsilon \to 0$ the original function goes to 1, wider and wider, and the transform goes to δ, thinner and thinner.

(3) If $w = u * v$, then $\hat{w}(\xi) = \hat{u}(\xi) \cdot \hat{v}(\xi)$. Just plug in it's not hard to check.

Now, we apply the above to $w = u * v$ with $v(x) = \bar{u}(-x)$, then the first properties tells us

$$\int \hat{w}(\xi)e^{-\varepsilon|\xi|^2}d\xi = \left(\frac{\pi}{\varepsilon}\right)^{n/2}\int e^{-\frac{|x|^2}{4\varepsilon}}w(x)dx$$

and we want to to have that if we take $\varepsilon \to 0$ we get

$$\int \hat{w}(\xi)d\xi = (2\pi)^n w(0).$$

So let's now justify. For the Right hand side we've shown in midterm last semester that the thing timed with w is an approximate identity, but for that even to work we need $w(0)$ defined.

We use Young's inequality and get $w \in L^1 \cap C^0$ where L^1 is because both u and v are L^1, and for C^0 it's because, first we know L^∞ as both L^2, and some more toils show C^0, so yeah we skip something. Thus the RHS convergence is shown, and we need to show for the left side.

For the left hand side, since the limit of the right hand side is defined we can use MCT to get

$$LHS \to \int \hat{w}(\xi)e^{-\varepsilon|\xi|^2}d\xi.$$

We know that $\hat{w}(\xi) = |\hat{u}(\xi)|^2$ just by plugging in, and this tells us

$$\int |\hat{u}|^2(\xi)d\xi = w(0) = (2\pi)^n \int u(x)v(0-x)dx = (2\pi)^n \int |u|^2(x)dx.$$

And the other equality is similar. $\square$

Def 6.2. *(Extension to L^2.) For $u \in L^2$, we can find $u_k \to u$ such that $u_k \in L^1 \cap L^1$. Since it converge u_k is Cauchy. But since we have*

$$||\hat{u}_k - \hat{u}_l||_{L^2} = C||u_k - u_l||_{L^2}$$

which means that $\hat{u}_k$ is Cauchy too. So we can define

$$\hat{u} := \lim_{k\to\infty} \hat{u}_k \in L^2.$$

Now we can really denote this as

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{ix\xi}u(x)d\xi$$

even though this clearly does not make any sense. We can also justify the formula using integral by parts though.

Also, we can check that if $v_k \to u$ then the limit of $\hat{v}_k$ is the same limit, so it's really well defined.

The good thing of this definition is that we finally have that the inverse of transform is the function itself.

Theorem 6.3. *For $u \in L^2(\mathbb{R}^n)$, we have*

$$\hat{\check{u}} = \check{\hat{u}} = u.$$

Proof.

Using our favorite density argument we know that if we can show the result for $u \in L^2 \cap C$, then we have the result.

But for continuous functions everything is fine since we have

$$\begin{aligned}\int \hat{u}(\xi)e^{iz\xi}d\xi &= \lim_{\varepsilon\to 0}\int \hat{u}(\xi)e^{iz\xi}e^{-\varepsilon|\xi|^2}d\xi \\ &= \lim_{\varepsilon\to 0}\left(\frac{\pi}{\varepsilon}\right)^{n/2}\int u(x)e^{-\frac{|x-z|^2}{4\varepsilon}}dx \to u(z)\end{aligned}$$

So we are done (up to almost sure convergence if we use density to apply to L^2, but that's obvious). $\square$

We know Schwarz class is mapped to Schwarz class by Fourier transform, and the Dual of Schwarz classes are Temperate distributions.

Something good to know is

- $$\mathcal{F}\delta = 1.$$
- $$\mathcal{F}^{-1}1 = \delta.$$
- $$\delta = \int \frac{e^{ix\xi}}{(2\pi)^n} d\xi; \quad \delta' = \int \frac{i\xi e^{ix\xi}}{(2\pi)^n} d\xi; \quad \delta^{(n)}(x) = \int \frac{(i\xi)^n e^{ix\xi}}{(2\pi)^n} d\xi.$$

We end our discussion with some main properties that we'll use for Fourier transform:

Proposition 6.4.

- $$\widehat{u(\cdot + h)}(\xi) = e^{ih\xi}\hat{u}(\xi).$$
- $$\widehat{D^\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi).$$

For some intuition about the last property we note that

$$u' \sim \frac{u(x+h) - u(x)}{h} \overset{\mathcal{F}}{\rightsquigarrow} \frac{e^{ih\xi} - 1}{h}\hat{u}(\xi) \to i\xi$$

6.2. **Applications to PDEs.**

Application 1: Elliptic equations.

For the equation $-\Delta u + \alpha^2 u = f$ where α is the absorption, we can solve it directly using the technique of Fourier transform applied to both sides:

$$\mathcal{F}(-\Delta u + \alpha^2 u) = \mathcal{F} f$$

and we can separately compute

$$\mathcal{F}(-\Delta u) = -\sum_{j=1}^{n} (i\xi_j)^2 \hat{u} = |\xi|^2 \hat{u}$$

since $\mathcal{F}$ is linear

$$\mathcal{F}(\alpha^2 u) = \alpha^2 \mathcal{F}(u)$$

where we have

$$(|\xi|^2 + \alpha^2)\hat{u} = \hat{f}$$

and hence

$$\hat{u} = \frac{1}{|\xi|^2 + \alpha^2}\hat{f}$$

where we use the convolution property to get

$$u(x) = \int \left(\mathcal{F}^{-1} \frac{1}{|\xi|^2 + \alpha^2} \right)(x-y) f(y) dy.$$

So the only difficulty boils down to finding the inverse of the Fourier multiplier. In 1D case we can get

$$B(x) = \frac{1}{2\alpha} e^{-\alpha|x|}$$

and for higher dimensions it's B_α the Bessel potentials, which Jeremy is interested.

Application 2: Heat Equation.

Again, we use the almost same method to guess the solution for the Heat equation system:

$$\begin{cases} \partial_t u - \Delta u = 0 & x \in \mathbb{R}^n, t > 0 \\ u = g & t = 0 \end{cases}$$

and we define the Fourier transform only in x. Note that using a DCT of some form we can easily just get

$$\widehat{\partial_t u} = \partial_t \hat{u}$$

thus the problem becomes

$$\begin{cases} (\partial_t - |\xi|^2)\hat{u} = 0 & \xi \in \mathbb{R}^n, t > 0 \\ \hat{u} = \hat{g} & t = 0 \end{cases}$$

and there's not much to say but to solve the ODE and get

$$\hat{u}(\xi, t) = \hat{g}(\xi) e^{-t|\xi|^2}$$

and the reverse Fourier get us to

$$u(x,t) = \int \Phi(x-y,t) g(y) dy$$

where

$$\Phi(x,t) = \mathcal{F}^{-1}\left(e^{-t|\xi|^2}\right)(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4\pi}}.$$

In particular, for t small Φ is an approximate identity, and for t large it's spread out. This should make intuitive sense for heat diffusion. Conservation law for this is $\displaystyle\int \Phi(x,t)dx = 1$ for $t > 0$.

Theorem 6.5. *If $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then*

- $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$;
- $(\partial_t - \Delta)u = 0$;
- $\displaystyle\lim_{(x,t^+)\to(x_0,0)} u(t,x) = g(x_0)$ *for all* $x_0 \in \mathbb{R}^n$.

Proof.

- The fact that Φ is smooth gives C^∞, as before.
- Just discussed above.
- we have
  $$u(x,t) - g(x_0) = \int \Phi(x-y,t)(g(y)-g(x_0))dy = \int_{|y-x_0|<\delta} + \int_{|y-x_0|\geq\delta} \to 0$$
  where the first integral uses continuity of g and the second uses that Φ is an approximate identity (goes to 0 at other points).

$\square$

Application 3: Duhamel Principle.

We now move onto the problem

$$\begin{cases} \partial_t u - \Delta u = f & x \in \mathbb{R}^n, t > 0 \\ u = g & t = 0. \end{cases}$$

The key is to note that by adding the boundary condition, it is as if we put the δg source on the boundary, and zero outside, in other words:

$$\begin{cases} (\partial_t + |\xi|^2)\hat{u} = 0 & \xi \in \mathbb{R}^n, t > 0 \\ \hat{u} = \hat{g} & t = 0 \end{cases} \iff \begin{cases} (\partial_t - |\xi|^2)\hat{u} = \delta_{t=0}\hat{g} & \xi \in \mathbb{R}^n, t \geq 0 \\ \hat{u} = 0 & t < 0 \end{cases}$$

and we note the fact that

$$\hat{f}(\xi,t) = \int_0^\infty \hat{f}(\xi,s)\delta(t-s)ds$$

where we have via the same ODE trick that

$$\hat{u}(\xi,t) = e^{-t|\xi|}\hat{g}(\xi) + \int_0^t e^{-(t-s)|\xi|^2}\hat{f}(\xi,s)ds$$

which in turn means

$$u(x,t) = \int \Phi(t,x-y)g(y)dy + \int_0^t \int_{\mathbb{R}^n} \Phi(t-s,x-y)f(s,y)dyds$$

and it's an exercise to chefck that this u indeed solves the problem.

We briefly go through the same method for Heat equation as we've done for Elliptic equation.

Here, we have the mean value formula in the form

$$(\partial_t - \Delta)u = 0 \iff u(x,t) = \frac{1}{4r^n}\int_{E(x,t;r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}dyds$$

where the "Heat ball" is defined to be

$$E(x,t;r)\{(y,s)|s \leq t, \Phi(x-y,s-t) \geq \frac{1}{r^n}\}.$$

And indeed we can check

Theorem 6.6. *For $u \in C_1^2(U_T) \cap C(\overline{U_T})$ where U_T is the cylinder, and Γ_T being it's boundary, and $(\partial_t - \Delta)u = 0$, we have $u \in C^\infty(U_T)$ and the maximum principle*

$$\max_{\overline{U}_T} u = \max_{\Gamma_T} u.$$

7. 4/11: Last bit on Heat Equation; Wave equation

For heat equation

$$\begin{cases} \partial_t u - \Delta u = f & x \in \mathbb{R}^n, t > 0 \\ u = g & t = 0 \\ u = h & t > 0, x \in \partial U \end{cases}$$

we use the same energy method to show that the solution is unique.

Proposition 7.1. *If there's a solution, then it's unique.*

Proof. Define energy

$$e(t) = \int_U w^2(x,t)dt$$

then via DCT we get

$$\dot{e}(t) = 2\int w\partial_t w = 2\int w\Delta w \overset{ibp}{=} -2\int |\nabla w|^2 + 0$$

so we know $\dot{e}(t) \leq 0$ and thus $e(t) \leq e(0) = 0$ hence $w = 0$ and the solution is unique. □

What if we want to do the backward heat equation? Just by thinking of it this should not be done because it's very ill conditioned to ask what has happened for the heat diffusion before some time t. Our problem now is

$$\begin{cases} \partial_t u - \Delta u = 0 & x \in \mathbb{R}^n, t > 0 \\ u = g & t = T \\ u = 0 & T > t > 0, x \in \partial U \end{cases}$$

and thus if we define $v(t) = u(T - t)$ then we can check that

$$\begin{cases} \partial_t u + \Delta u = 0 & x \in \mathbb{R}^n, t > 0 \\ u = g & t = 0 \\ u = 0 & T > t > 0, x \in \partial U \end{cases}$$

and the idea is that adding the Laplacian does not give us a good result because the eigenvalue of the operator is then negative. That is

$$(\partial_t + \lambda)u = 0$$

only for $\lambda > 0$. This might be well illustrated in Fourier domain: We have the equation

$$(\partial_t - |\xi|^2)\hat{v} = 0$$

and solving the ODE gives

$$\hat{v} = \hat{g}(\xi)e^{t|\xi|^2}, t > 0$$

which blows up when $t \to \infty$. Thus it's bad and is dominated by noise $\hat{g}$.

But there's still hope! We might still have uniqueness.

Theorem 7.2. *If $u, v \in C^2(\hat{U}_T)$ solutions of the problem*

$$\begin{cases} \partial_t u - \Delta u = f & x \in \mathbb{R}^n, t > 0 \\ u = g & t = T \\ u = h & t > 0, x \in \partial U \end{cases}$$

then $u = v$ in U_T.

Proof.

Let $w = u - v$ then we have

$$\begin{cases} \partial_t w - \Delta w = 0 & x \in \mathbb{R}^n, t > 0 \\ w = 0 & t = T \\ w = 0 & t > 0, x \in \partial U \end{cases}$$

and we use the same energy term

$$e(t) = \int_U w^2(x,t)dt$$

to get

$$\dot{e}(t) = -2 \int |\nabla w|^2 < 0$$

and the problem is that we do not know whether it's the case that before T there's a true decay. But let's do it twice to get

$$e''(t) = \partial_t 2 \int_U w \Delta w = 2 \int_U (\Delta w)^2 + w \Delta^2 w \overset{ibp}{=} 4 \int (\Delta w)^2$$

and if we compute

$$(e'(t))^2 = \left(2 \int w \Delta w < 0\right)^2 \overset{C.S.}{\leq} e(t) \cdot e''(t)$$

whence if we can find t in some interval (t_1, t_2) such that $e(t) > 0$ then

$$f(t) = \log e(t)$$

is well defined inside. Taking derivative of f we have

$$f'' = \frac{e \cdot e'' - (e')^2}{e^2} \geq 0$$

which means that f is convex. Now we just use continuity of w and extend the interval such that $e(t_2) = 0$ and thus by convexity

$$f\left((1-\tau)t_1 + \tau t_2\right) \leq (1-\tau) f(t_1) + \tau f(t_2)$$

and taking exponential back yields (note that we're really formally writing t_2 as a close enough point, then by continuity everything holds)

$$e\left((1-\tau)t_1 + \tau t_2\right) \leq \left[e(t_1)\right]^{1-\tau} \cdot \left[e(t_2)\right]^{\tau}$$

so since $e(t_2) = 0$ and the other term is bounded, we know that $e \equiv 0$ on $[t_1, t_2]$ where at t_1 it's continuity again. □

Note that this give us a bound even for v, if we know the error at $t = 0$ is ε, and the error at T is bounded by M, then the error at time $\frac{T}{2}$ should be $\sqrt{\varepsilon}$ by the same formula

$$e(\tau T) \leq e^{1-\tau}(0)e^{\tau}(T).$$

7.1. **Wave Equation.**

The problem is

$$\square\, u := (\partial_t^2 - \Delta)u = 0$$

plus condition.

Our way to tackle this is first show for $n = 1$, then for n odd, then for all n by dimension reduction.

Case 1: $n = 1$:

Now we face the Cauchy Problem:

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & x \in \mathbb{R}, t > 0 \\ u = g & t = 0 \\ \partial_t u = h & t = 0 \end{cases}$$

where we can decompose

$$\partial_t^2 u - \Delta u == (\partial_t + \partial_x)(\partial_t - \partial_x)u$$

and it turns into 2 transport equation such that we have

$$(\partial_t + \partial_x)v = 0 \Rightarrow v(x,t) = v(x-t,0)$$

and using the same trick computing give us

$$(\partial_t - \partial_x)u = v \Rightarrow u(x,t) = u(x+t,0) + \frac{1}{2}\int_{x-t}^{x+t} v(y,0)dy$$

and plugging in initial condition yields the D'Alembert formula:

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y,0)dy.$$

It's a trivial exercise to check that

Theorem 7.3. *For $g \in C^2$, $h \in C^1$, then $u \in C^2$ solves the PDE.*

Now the problem we face is that for wave transportation there's no maximum principle due to superposition of waves. We use rather the principle of interference.

For the half-line, we extend the D'Alembert formula by imagining a negative wave symmetric to the one we have, so that our wave bounces back from 0.

That is, if our problem is defined in this way:

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & x \geq 0, t > 0 \\ u = g & t = 0 \\ \partial_t u = h & t = 0 \end{cases}$$

then we define

$$\tilde{u} = \begin{cases} u x \geq 0 \\ -u(-x,t) x < 0 \end{cases}$$

and we can check that $\square\, u = 0$ on $\mathbb{R} \times (0, \infty)$. Using D'Alembert on $\tilde{u}$ yields:

$$u(x,t) = \begin{cases} \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y,0)dy & x \geq t > 0 \\ \frac{1}{2}\left(g(x+t) - g(-x+t)\right) + \frac{1}{2}\int_{-x+t}^{x+t} h(y,0)dy & t \geq x \geq 0. \end{cases}$$

n is odd:

In this case we do Fourier transform and get

$$(\partial_t^2 + |\xi|^2)\hat{u} = 0; \quad \hat{u}_{t=0} = \hat{g}; \quad \partial_t \hat{u}_{t=0} = \hat{h}$$

and just solve the equation yields

$$\hat{u}(\xi, t) = \hat{g}(\xi)\cos|\xi|t + \frac{\hat{h}(\xi)}{|\xi|}\sin|\xi|t$$

whence rewriting gives

$$\hat{u}(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi + |\xi|t)}\left(\frac{1}{2}\hat{g} - i\frac{\hat{h}(\xi)}{|\xi|}\right)d\xi + \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(x\cdot\xi - |\xi|t)}\left(\frac{1}{2}\hat{g} + i\frac{\hat{h}(\xi)}{|\xi|}\right)d\xi$$

and the only hard part is to find

$$\mathcal{F}^{-1} e^{i(x\cdot\xi + |\xi|t)}$$

which now is not a Gaussian and thus not easy to find. One way to do it is just to do it and is an exercise on distributions. But that's not what we'll do for now. We just use the method fo spherical mean to solve it.

Define

$$U(x; r, t) := \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u(y,t)dS(y) = \fint_{\partial B(x,r)} u(y,t)dS(y).$$

Then by continuity we know

$$u(x,t) = \lim_{r\to 0} U(x; r, t).$$

Now we define some auxiliary terms G, H from g, h.

Lemma 7.4. *(Euler- Poisson- Darboux) Let $u \in C^m$ for $m \geq 2$ be a solution of $\square\, u = 0$. Then $U \in C^m(\overline{\mathbb{R}_+} \times (0, \infty))$ for fixed x and*

$$\left(\partial_t^2 - \partial_r^2 - \frac{n-1}{r}\partial_r\right)U = 0, \mathbb{R}_+ \times (0, \infty).$$

Proof. Just compute.

$$\partial_r U = \fint \frac{\partial u}{\partial n}(x+ry,z)dS = \frac{1}{|\partial B|}\int_{\partial B(x,r)} \frac{\partial u}{\partial n}(x,t)dS = \frac{|B|}{|\partial B|}\fint_B \Delta u(y,t)dy$$

and we see the inside shows up what we want. Keeping it up we have

$$r^{n-1}\partial_r U = \frac{1}{\gamma(n)}\int_{B(x,r)} \partial_t^2 u(y,t)dy$$

and just taking derivative on the integral term we go from ball to sphere and have

$$\partial_r\left(r^{n-1}\partial_r U\right) = \frac{1}{\gamma(n)}\int_{\partial B(x,r)} \partial_t^2 u(y,t)dy = r^{n-1}\partial_t^2 \fint_{\partial B(x,r)} u(y,t)dS(y) = r^{n-1}\partial_t^2 U.$$

Rearrange then result follows. □

Now we actually start doing for odd n, where really everything is the same so we use $n=3$ to illustrate. We let

$$\tilde{U} = r\cdot U(x;r,t)$$

then

$$\Box\tilde{U} = 0$$

and since we are considering of sending $r\to 0$ we have by the half space D'Alembert when $r\le t$ that

$$\tilde{U}(x;r,t) = \frac{1}{2}\left(\tilde{G}(r+t)+\tilde{G}(t-r)\right) + \frac{1}{2}\int_{-r+t}^{r+t}\tilde{H}(y)dy$$

and plugging back to u by sending $r\to 0$ yields

$$u = \partial_t\left(\frac{t}{\gamma(3)t^2}\int_{\partial B(x,t)} g ds\right) + \frac{t}{\gamma(3)t^2}\int_{\partial B} h dS.$$

Moreover, to compute the middle term we have

$$\partial_t \fint_{\partial B(x,t)} g ds = \partial_t\left(\frac{1}{|\partial B|}\int_{\partial B(0,1)} g(x+tz)ds(z)\right) = \frac{1}{|\partial B|}\int_{\partial B(0,1)} z\cdot\nabla g(x+tz)ds(z).$$

So we finally have the <u>Kirkholl's formula</u>:

$$u = \frac{1}{4\pi t^2}\int_{\partial B(x,t)} \left[th(y)+g(y)+(y-x)\cdot\nabla g(y)\right]dS(y).$$

Note that the intuition that there should be the sphere $\partial B(x,t)$ is because the amplitude at point x and time t is of course the superposition of all waves on the sphere at time 0.

8. 4/13: Dimensional reduction; Energy method

Today we finish most of Wave equation.

8.1. **Dimensional reduction.**

We have proven that for odd n the D'Alembert formula holds, but oddness is needed because when we do $\tilde{U} = rU$ (only for $n = 3$), similar change of variable only apply to odd n.

Now for even dimensions, the trick is to view the problem as if it is a problem in higher dimension. We will illustrate this by $\mathbb{R}^2$ embedded in $\mathbb{R}^3$ by letting

$$\bar{u}(x_1, x_2, x_3, t) := u(x_1, x_2, t)$$

such that $\bar{u}$ is actually independent of x_3.

This strategy often does not work because this immediately create an unintegrable function that is destructive in integration related problems. But here it's just fine. So we have

$$\Box_{2D} u = 0 \Rightarrow \Box_{3D} \bar{u} = 0$$

and if we define $\bar{x} = (x_1, x_2, 0)$ and $\bar{g}$, $\bar{h}$ in the same manner, and let $\bar{B} = B(\bar{x}, t)$, then we have by the same middle step in D'Alembert's deduction that

$$u(x,t) = \bar{u}(\bar{x}, t) = \partial_t \left(\frac{t}{|\partial \bar{B}|} \int_{\partial \bar{B}} \bar{g} ds \right) + \frac{t}{|\partial \bar{B}|} \int_{\partial \bar{B}} \bar{h} ds$$

and if we think about it we are extending the value on the plane to the whole space vertically. So at time t a point is only affected by the ball $B(x,t)$ at time 0, thus effects from the horizontal line cancels with each other. As for the horizontal effects from skewed upward directions, they are mitigated by the angle. More precisely, if we map a small part on the sphere onto the disc, there will be a change of measure $ds = \sqrt{1 + |\nabla\phi|^2}$. Thus we have the Poisson formula by projection onto the disc: (note the integral is in 2D disc)

$$u(x,t) = \frac{1}{2\pi t} \int_{B(x,t)} \frac{g(y) + th(y) + [\nabla g(y)] \cdot (y - x)}{(t^2 - |x-y|^2)^{\frac{1}{2}}} dy.$$

Remark 8.1. *Note that*

$$\int_{\partial B} = \int_B \delta_{\partial B}$$

and hence in our case the integral

$$\int_B \frac{1}{(t^2 - |x-y|^2)^{\frac{1}{2}}}$$

acts like a delta function.

Here are 2 theorems that we need to check to just convince ourselves.

Theorem 8.2. *For $n = 3$, $g \in C^3$ and $h \in C^2$, then the u described in the Kirchhoff formula is $u \in C^2(\mathbb{R}^3 \times [0, \infty)$ and $\Box\, u = 0$.*

Theorem 8.3. *For $g \in C^{m+1}$ and $h \in C^m$, we have that $u \in C^2$ described above by Poisson formula and Kirchhoff formula, where*

$$\begin{cases} m = \frac{n+1}{2} & n \geq 3 \text{ is odd} \\ m = \frac{n+2}{2} & n \geq 2 \text{ is even.} \end{cases}$$

Now we go to the Non-homogenous problem and use again the Duhamel Principle. Remember that for the problem

$$\begin{cases} \Box\, u = 0 & t > 0 \\ \partial_t u = g & t = 0 \\ u = 0 & t = 0 \end{cases} \iff \begin{cases} \Box\, u = g \cdot \delta_0(t) & t \geq 0 \\ u = 0 & t < 0 \end{cases}$$

where we take in the ∂_t term because the wave equation has ∂_t^2 term here.

Now we consider the function $u(x, t, s)$ where t is the point of evaluation and s is the point of of the emitting source. We have by Duhamel that

$$\begin{cases} \Box\, u(x, t, s) = 0 & \\ \partial_t u(x, s, s) = g & t = s \\ u(x, s, s) = 0 & t = s \end{cases} \iff \begin{cases} \Box\, u = f(t) = \int \delta(t - s) f(s) ds & \\ \partial_t u = 0 & t = s \\ u = 0 & t = s \end{cases}$$

Then using the same method we get the following:

Theorem 8.4. *For the problem (note how we shift back to our usual notation of u)*

$$\begin{cases} \Box\, u = f(t) & \\ \partial_t u = 0 & t = 0 \\ u = 0 & t = 0 \end{cases}$$

we have

$$u(x, t) = \int_0^t u(x, t; s) ds.$$

The solution for 3D and 1D is listed:

- 3D:
$$u(x, t) = \frac{1}{4\pi} \int_{B(x,t)} \frac{f(y, t - |x - y|)}{|x - y|} dy.$$
- 1D:
$$u(x, t) = \frac{1}{2} \int_0^t \int_{x-s}^{x+s} f(y, t - s) dy ds.$$

Now a good exercise is for the equation

$$\square u = Vu = f$$

and if we view the problem as

$$u = \int Vu ... = Ku$$

then we can just do it by solving the eigenvalue of K!

8.2. **Energy method.**

We will cover uniqueness of solution and finite speed propagation.

Proposition 8.5. *If u is a solution to the wave equation, then it is unique.*

Proof. Let $u = w - v$ be the difference of 2 solutions, then even though we have absolutely no maximum principle, we still can find a suitable energy.

Note that integral by parts twice on the term

$$\partial_t^2 \int u^2$$

yields a different sign and if we use the same method as we use for Heat equation we will get

$$\partial_t^2 \int u^2 - \int |\nabla u|^2 = 0$$

which does not help us.

So we be clever and just note that $\partial_t^2 = \partial_t(\partial_t)$ and we try to create $(\partial_t u)^2$. So we have

$$\int_{\mathbb{R}^n} (\partial_t u)\partial_t^2 u = \frac{1}{2}\partial_t \int_{\mathbb{R}^n} (\partial_t u)^2$$

and

$$\int \partial_t u(-\Delta u) = \int \partial_t \nabla u \cdot \nabla u = \frac{1}{2}\partial_t \int |\nabla u|^2$$

and combining we have

$$0 = \int_{\mathbb{R}^n} (\partial_t u) \cdot (\partial_t^2 u - \Delta u) = \frac{d}{dt} \int_{\mathbb{R}^n} \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2$$

where we recognize that the inside is nothing but kinetic energy plus potential energy.

So we know $0 = e(0) = e(T)$ and hence $\partial_t u = \nabla u = 0$ which means $u = 0$ so the solution is unique. $\square$

Note that this can be more general, i.e. we can work the same on

$$\partial_t a(x)\partial_t - \nabla \cdot b(x)\nabla.$$

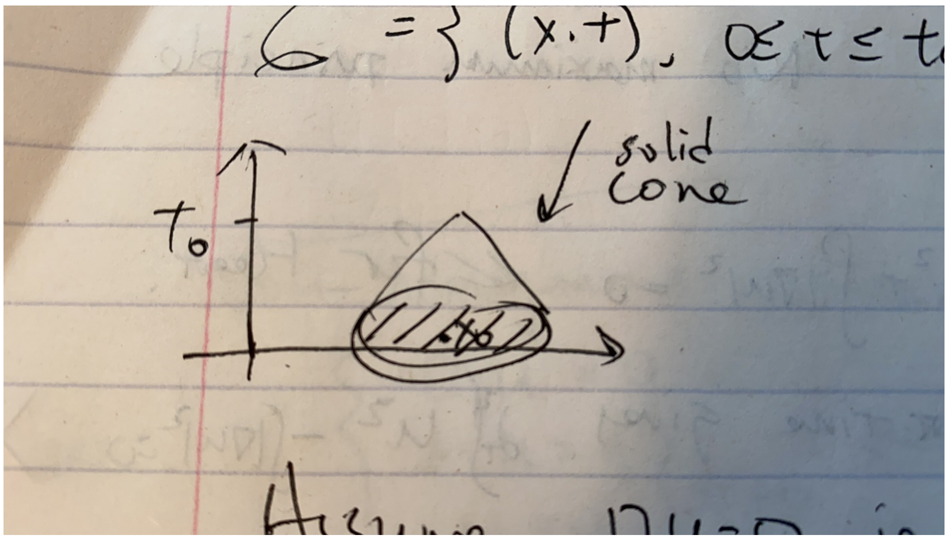
But of course adding those changes something, and that something turns out to be the speed of propagation. This explains why what ever h and g we choose, the speed of propagation is always 1.

Finite speed propagation

Define the domain of dependence

$$C := \{(x,t), 0 \leq t \leq t_0, |x - x_0| \leq t_0 - t\}$$

then this is nothing but the solid cone



Theorem 8.6. *If $\Box\, u = 0$ in the vicinity of C (because we will need derivative on ∂C to vanish), then if u and $\partial_t u$ vanishes at $t = 0$ on $B(x_0, t_0)$, then $u = 0$ in C.*

Note that this means exactly that what is outside of the cone cannot affect what is inside, in other words, the speed cannot be faster than how the cone shrinks, so speed is bounded.

Proof. Let

$$e(t) = \frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2 + |\nabla u|^2 dx \geq 0$$

then by assumption we have $e(0) = 0$. So if we can show $\dot{e} \leq 0$ we win. But nothing is really special about it so we use DCT to take derivative and get

$$\dot{e}(t) = \int_B u_t \cdot u_{tt} + \nabla u_t \cdot \nabla u dx - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} u_t^2 + |\nabla u|^2 dx$$

where the first term is taking derivative inside, and the second is taking on the integral boundary, that's where the negative sign comes from. Continuing with computation we have by IBP that

$$\dot{e}(t) = \int u_t \cdot \Box\, u + \int_{\partial B} \partial_t u \partial_n u - \frac{1}{2}(u_t^2 + |\nabla u|^2) dx$$

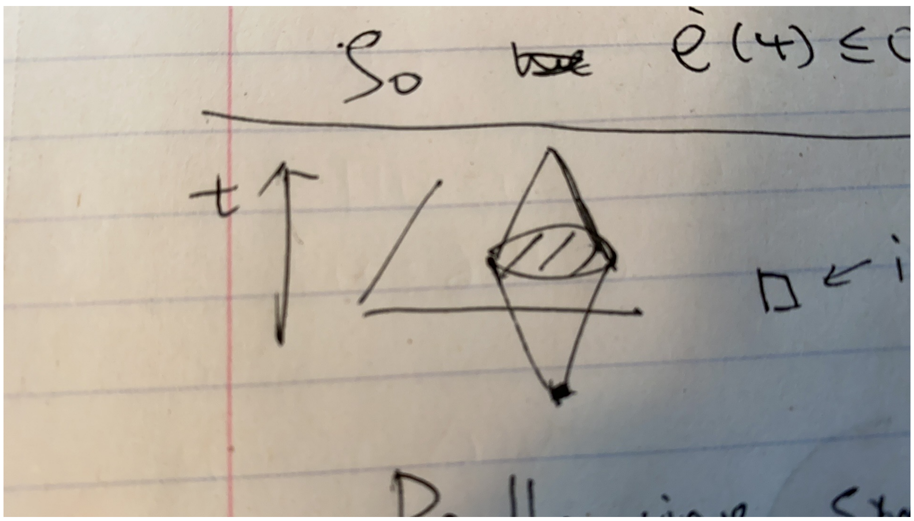
the first term is 0 by condition, and for the second we use Cauchy to get

$$|u_t \partial_n u| \overset{C.S.}{\leq} \frac{1}{2}\left(u_t^2 + |\partial_n u|^2\right) \leq \frac{1}{2}\left(u_t^2 + |\nabla u|^2\right)$$

where the second inequality is just because $\partial_n u$ is a projection onto $\vec{n}$ of ∇u.

But then we are done since $e' \leq 0$. $\square$

Note that $\square$ is really invariant with respect to the map $t \mapsto -t$, so we have that this also work for negative progression of time. It's quite obvious because that is how wave propagate. So on the double cone below we have the exact same theorem:



Note that singularity also propagates with time in the case of wave equation, and that is why it's a lot harder to deal with.

8.3. **Method of Characteristics.**

Now we try to convert PDE to ODEs using this method. It worked pretty well when it works but it only works when we have a Non-linear first order PDE on scalar quantities, i.e.

$$F(Du, u, x) = 0$$

for $x \in U$ the spacetime. So really

$$F : \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}, F \in C^1(\mathbb{R}^n \times \mathbb{R} \times U).$$

But what is our needed boundary condition? Well, consider the easiest case where

$$F(p, z, x) = p$$

then $\frac{du}{dx} = 0$ so u is a constant: we only need one boundary condition at one endpoint.

An important thing to ponder on is what, after all, is a PDE: It really is to investigate how a function looks like in all places where only a Cauchy surface Σ is known to us. If we can find how a vector field β emerges out of the function then we are done by solving local ODEs, say using Euler method.

In this case, we only need the following to solve everything:

$$\begin{cases} x(t) \\ u(x(t)) = z(t) \\ \nabla u(x(t)) = p(t). \end{cases}$$

9. 4/18: METHOD OF CHARACTERISTICS

In this case we need to assume that $u(x)$ is scalar, $x \in U \subset \mathbb{R}^n$, and we need the first order constraints only:

$$F(\nabla u(x), u(x), x) = 0$$

then the method of Characteristics is just to find a fake time such that the question becomes an ODE problem.

The heuristics is the following: define the new variables

$$\begin{cases} x(s) \\ z(s) = u(x(s)) \\ p(s) = \nabla u(x(s)) \end{cases}$$

and thus we note that

$$\dot{z} = \partial_s u(x(x)) = \nabla u \cdot \dot{x} = p \cdot \dot{x}$$

and for our equation it becomes

$$F(p, z, x) = 0$$

where by chain rule we have

$$\frac{dF}{ds} = \partial_p F \cdot \dot{p} + \partial_z F \cdot \dot{z} + \partial_x F \cdot \dot{x} = 0$$

where we note that the first and third is a genuine dot product where the second is just multiplication. Plugging in $\dot{z}$ we have

$$\partial_p F \cdot \dot{p} + (\partial_z F \cdot p + \partial_x F)\dot{x} = 0$$

where we always have $\dot{x}\dot{p} - \dot{p}\dot{x} = 0$ so if we can let

$$\partial_p F = \dot{x}; \quad \text{and} \quad (\partial_z F \cdot p + \partial_x F) = -\dot{p}$$

we are done. We can of course define them so there's a constant factor in front, but that would be unnecessary as we can reparametrize s so that the constant get kicked off (constant is the speed of flow). So we have a new system:

$$\begin{cases} \dot{x} = \partial_p F \\ \dot{p} = -(\partial_z F \cdot p + \partial_x F) \\ \dot{z} = p \cdot \partial_p F \end{cases}$$

and we need to assume G is Lipschitz to really do the problem. But that's all formal.

What we need to do is that we need some initial condition for this to work, and so we need the initial condition on a non-trajectory surface Σ such that for all points we need we can find the trajectory it is in and fetch data from where the trajectory intersects Σ, thus there cannot be more than 1 source of initial data, nor can trajectories meet.

Example 9.1.

For Quasilinear function

$$F(\nabla u(x), u(x), x) = b(x, u) \cdot \nabla u + c(x, u) = 0$$

we have

$$F(p, z, x) = b \cdot p + c = 0$$

and so the system only requires that

$$\dot{x} = \partial_p F = b; \dot{z} = p \cdot \partial_p F = b \cdot p = -c.$$

Example 9.2.

For a particular case, let $b = (1, 1)^T$, $c = -u^2$, so the 2D question is

$$\begin{cases} (1,1) \cdot \nabla u = u^2 & U = \{x_2 > 0\} \\ u = g & \{x_2 = 0\} \end{cases}$$

so our boundary $\Sigma = \{x_2 = 0\}$. Note that $z = u$ so $c = -z^2$ and since our initial condition is on the x_1 axis the initial condition is

$$x_1(s) = x_0 + s; x_2(s) = s$$

for which we compute $\dot{z} = -c = z^2$ and the ODE solution is

$$z(s) = \frac{z_0}{1 - z_0 s}; z(0) = z_0$$

let $x = (x_1, x_2)$ be fixed then $x_2 = s$ and $x_0 = x_1 - x_2$, so $z(s) = u(x)$ imples

$$z(0) = u(x_0, 0) = g(x_0) = g(x_1 - x_2)$$

and so

$$u(x) = z(s) = \frac{z_0}{1 - z_0 s} = \frac{g(x_1 - x_2)}{1 - sg(x_1 - x_2)}.$$

As usual the exercise is to check that this works. Note that this only is defined when x_2 small, because there is finite time blowup when the denominator goes to 0.

One point here is that since we already know p along dimensions in Σ, and so there's only one initial condition for which we don't know: the one pointing outside of Σ. But that is exactly what $F = 0$ applied to the boundary tells us. So we do see that the conditions are enough.

Example 9.3.

For the fully non-linear system, say

$$\begin{cases} \partial_1 u \partial_2 u = u & U = \{x_1 > 0\} \\ u(0, x_2) = x_2 = g(x_2) \end{cases}$$

we will see that we cannot just omit p. We know $F = p_1 p_2 - z$ and so $\dot{p}_j = p_j$; $\dot{z} = 2p_1 p_2$; $\dot{x}_1 = p_2$; $\dot{x}_2 = p_1$. So the solution to ODE is

$$p_1 = p_1^0 \cdot e^s; p_2 = p_2^0 \cdot e^s$$

and

$$z = z_0 + p_1^0 p_2^0 (e^{2s} - 1)$$

for x they are

$$x_1 = p_2^0 (e^s - 1); x_2 = x_0 + p_1^0 (e^s - 1)$$

where as $(x_1, x_2)(0) = (0, x_0)$ tells us $g(x_0) = x_0^2$ and

$$p_2^0 = \partial_2 u = \partial_2 g = 2x_0$$

and we use $F = 0$ to get

$$p_1^0 = \frac{z_0}{p_2^0} = \frac{x_0}{2}$$

so

$$\begin{cases} x_1(s) = 2x_0(e^s - 1) \\ x_2(s) = \frac{x_0}{2}(e^s - 1) \\ z(s) = x_0^2 e^{2s} \end{cases}$$

the goal is to find x_0, s such that $(x_1, x_2)(s) = x$which means

$$x_0 = x_2 - \frac{x_1}{4}; e^s = \frac{x_1 + 4x_2}{4x_2 - x_1}$$

and we see that $x_2 \neq 0$ to satisfy the above condition. But this is reasonable because all the trajectories behaves like ce^{x_1} for $c \neq 0$. So the x_1 axis is never touched. In this case the solution is

$$u(x) = z(s) = \frac{(x_1 + 4x_2)^2}{16}.$$

9.1. **Local theory.**

What we do here is we straighten Σ onto a flattened surface so that all the algebras are as easy as above. First note that Σ is already a graph with respect to some dimension x_n (if not we can always cut it into pieces).

It is a graph so

$$x_n = \gamma(x_1, \dots, x_{n-1})$$

and we just use an invertible map

$$y_j = \Phi(x_j) = \begin{cases} x_j & j \leq n-1 \\ x_n - \gamma(x_1, \dots, x_{n-1}) & j = n \end{cases}$$

then $\Phi(\Sigma)$ is flattened. We also have

$$D\Phi = \begin{pmatrix} I_{n-1} & 0 \\ -D\gamma & 1 \end{pmatrix}$$

such that $\det(D\Phi) = 1$. So this is a diffeomorphic map and we define it's inverse map

$$U = \Psi(V); V = \Phi(U).$$

The only heavy algebra here is that we need to take $D\Phi$ into account every time. But the later computations are easier.

So if we let $v(y) = u(x)$ then $\nabla u(x) = (D\Phi)^T \cdot \nabla v$ or just written as $Du = Dv \cdot D\Phi$. So

$$0 = F(\nabla u, ux) = F((D\Phi)^T \circ \Psi \cdot \nabla v, v(y), y) =: G$$

and the equation is

$$\begin{cases} G(\nabla v, v, y) = 0 \\ v = g \circ \Phi = h. \end{cases}$$

Our last point is just repeating what we've said: that the derivative of one of the directions pointing out of the surface is unknown, and we've labeled it $p_n(0)$. We can in principle solve it, but there's no guarantee that there is just a unique solution. In general, if there's multiple solution then each solution corresponds to a different theory. For example, consider the following system:

$$|\nabla u|^2 = 1$$

and thus the solution is

$$p_n(0) = \pm\sqrt{1 - \sum_{j=1}^{n-1} p_j^2(0)}$$

and so there's 2 theories here.

10. 4/20: LOCAL PROBLEM; JACOBI HAMILTON PROBLEM

Lemma 10.1. *(Non-characteristic boundary) Assume $\partial_{p_n} F(p^0, z_0, x_0) \neq 0$, then $\exists q(y)$, a parametrization of Σ, such that*

- *$F(q(y), g(y), y) = 0$.*
- *For y close to x_0, $y_n = 0$.*
- *$q_j(y) = \partial_j g(y)$ for $1 \leq j \leq n-1$.*

Note that the above is n constraints in total, and our assumption are simply saying that there are some potential for x to get out of our surface Σ.

Proof. As for the proof we just use the implicit function theorem. Define $G(p, y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ (here y is the extended y) where

$$G_i(p, y) = p_i - \partial_i g(y)$$

and

$$G_n(p, y) = F(p, g(y), y).$$

Thus $G(p^0, x_0) = 0$ and

$$D_p G(p^0, x_0) = \begin{pmatrix} I_{n-1} & 0 \\ D_{n-1} F & \partial_{p_n} F \end{pmatrix}$$

which is invertible because the right bottom term is non-zero. Then we can just apply implicity function theorem and say that there exists such $y \mapsto q(y)$ locally such that $G(q(y), y) = 0$. □

Lemma 10.2. *(local invertibility) Assume that $(\partial_{p_n} F)(p^0, z_0, x_0) \neq 0$. Then $\exists 0 \in I \subset \mathbb{R}$, a neighborhood W of x_0 in $\mathbb{R}^{n-1}$ and a neighborhood V in $\mathbb{R}^n$ such that $\forall x \in V$, $\exists s \in I$ and $y \in W$ such that $x = X(y, s)$, the flow, where X is the trajectory of x.*

Intuitively this is obvious, since we want to find a trajectory in which x is on, it's corresponding point in Σ and the fake time s associated with it.

Proof. The map X is such that $X : (y, s) \mapsto X(y, s) = x$ and thus

$$D_{y,s} X(x_0, 0) = \begin{pmatrix} I_{n-1} & \nabla_p F \\ 0 & \partial_{p_n} F \end{pmatrix}$$

and what we want is just by inverse function theorem that there is such points y, s. □

Theorem 10.3. *(Local existence) For F smooth, $\partial_{p_n} F \neq 0$, then P_n^0 exists, where $u(x)$ is defined as:*

$$x \rightsquigarrow (y, s) \rightsquigarrow z(s) = u(x) \rightsquigarrow p(s) = p(x)$$

and $u \in C^2(V)$, $u = g$ on W, and such that $F(\nabla u(x), u, x) = 0$.

(just by above this is obvious. What is obvious is that the PDE holds by ODE solution, what is harder is that $p = \nabla u$ as we hope. Look at book.)

10.1. **Applications.**

Example 10.4.

let's see why it is indeed local. We have the scalar conservation laws:

$$G(\nabla_x u, \partial_t u, u, x, t) = u_t + \mathrm{div}(F(u)) = 0$$

where F is some flow. In other words

$$u_t + F'(u)\nabla_x u = 0$$

so its linear in u. And $u = g$ at $t = 0$ (in Σ). We have $\dfrac{\partial G}{\partial p_{n+1}} = 1 \neq 0$. We can rewrite our condition into

$$G(p, p_t z, x, t) = p_t + F'(z) \cdot p = 0$$

and we get

$$p_{n+1} = -F'(z) \cdot p$$

(the admissible point), then the equation by plugging in is

$$\dot{x}_i = F_i'(z); \dot{z}(s) = \partial_{\tilde{p}} G \cdot \tilde{p} = p = 0$$

which is the conservation law. We get that $\dot{t} = 1 \Rightarrow s = t$ and $z(s) = g(x_0)$, and ODE is

$$x(s) = F'(g(x_0)) \cdot s + x_0$$

and thus $u(x,t) = g(x - t \cdot F'(u(x,t)))$ where taking derivatives yields

$$\begin{cases} \partial_i u = \partial_j g(x - t \cdot F'(u(x,t))) \left(\delta_{ij} - t \cdot F_j''(u) d_i u\right) \\ \partial_t u = \partial_j g(x - t \cdot F'(u(x,t))) \left(F_j - t \cdot F_j''(u) d_t u\right) \end{cases}$$

and solving yields

$$u_t + F'(u) \cdot \nabla u = -t\nabla g(x - tF'(u)) \cdot F''(u) \cdot (u_t + F'(u)\nabla u)$$

and thus $u_t + F'(u) \cdot \nabla u = 0$ when

$$1 + t\nabla g(x - tF'(u)) \cdot F''(u) \neq 0$$

and for t small this of course can hold.

But how about for longer times?

Example 10.5.

For $n = 1$, it's 1D, so if $F'' > 0$, $g' < 0$ then the above will fail for t large just by sign.

For $n \geq 2$, we have the burger's equation:

$$\begin{cases} \partial_t u + u\partial_x u = 0 \\ \text{or (with smoothness difference) } \partial_t u + \frac{1}{2}\partial_x u^2 = 0 \end{cases}$$

where

$$g = \begin{cases} 1 & x \le 0 \\ 1-x & x \in (0,1) \\ 0 & x \ge 1 \end{cases}$$

and this is just the wave flapping at the beach. If we list the x, p, z there's $2n+1$ equations in the false space, and our original is n dimensional system, so the projection onto the original space leads to possible crossing of trajectories, which can be bad.

Example 10.6. *Hamilton- Jacobi equations*

Let

$$G(\nabla u, \partial_t u, u, x, t) = u_t + H(\nabla u, x) = 0$$

and we similarly convert to new system to get

$$G(p, p_{n+1}, z, x, t) = p_{n+1} + H(p, x)$$

where since $\dfrac{\partial G}{\partial p_{n+1}} = 1$ we know that $t = 0$ is non-characteristic, thus the solution is fine locally. There's some issues with global theory.

For

$$(\partial_t^2 - c^2(x)\Delta)u = 0$$

it's usually dealt with

$$u(x,t) = a(x,t) \cdot e^{1\frac{s(x,t)}{\varepsilon}}$$

where the exponential can be $e^{i\frac{wt-kx}{\varepsilon}}$. Now we formally plot the ansatz into the PDE to leading order and get

$$(\partial_t S)^2 - c^2(x)|\nabla S|^2 = 0$$

where the high frequency by ε is killed. We have

$$\partial_t S = \pm\sqrt{c^2(x)|\nabla S|^2}.$$

Example 10.7.

For Schordinger equation we extend to

$$\left(i\hbar\partial_t + \frac{1}{2}\hbar^2\Delta - V(x)\right)u = 0$$

and we plug in

$$u = ae^{i\frac{s(t,x)}{\hbar}}$$

to see that it is Hamiltonian.

11. 4/25: Legendre transform; Starting on Elliptic equations

11.1. **Legendre transform.**

We start with the Lagrangian $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$L = \frac{1}{2}mv^2 - V(x)$$

where $\frac{1}{2}mv^2$ is the kenetic energy, and $V(x)$ is the potential energy. But what is this even? Turns out that this is related to the total energy (of course)

$$H = \frac{1}{2}mv^2 + V(x) = \frac{1}{2m}p^2 + V(x)$$

where $p = mv$. This is related to the standard Hamilton-Jacobi equation

$$\partial_t u + H(\nabla u, x) = 0$$

then the above H will do for this.

But theory first. Let's suppose we have L, not necessarily of the above form, to be convex and such taht $\frac{L(q)}{|q|} \to \infty$ as $|q| \to \infty$. The idea is just that the slope must cover a whole range of numbers. We can think of it just as the quadratic function x^2. An important insight is that at any point there is a supporting hyperplane tangent to the graph.

Def 11.1. *The Legendre transform of L is*

$$L^*(p) = \sup_{q \in \mathbb{R}^n}(pq - L(q)), p \in \mathbb{R}^n.$$

Note that if L is smooth enough, and since at infinity $L \gg p$, a linear map, so the sup must be attained in a ball of large radius R, so it's sup is just attained as a maximum.

Moreover, if $p = DL(q^*)$ at some position $q^* = q^*(p)$, i.e. p is the slope of L at q^*, then $L^*(p) = pq^* - L(q^*)$.

Def 11.2. *The Hamiltonian is*

$$H(p) = L^*(p).$$

Theorem 11.3. *Assume that L is convex, $\frac{L(q)}{|q|} \to \infty$, and define $H = L^*$, which we've shown exists above. Then H is convex, $\frac{H(p)}{|p|} \to \infty$, and $H^* = L$. That is, Lagendre transform is an involution.*

Remark 11.4. *Note that for the first 2 results the convexity is not needed. However, if we want $H^* = L$ then by the first result H must be convex, so we'd better have L be convex.*

Moreover, for a non-convex graph of L one way to find the convex hull of it is to take the Legendre transform twice.

Proof. Just apply and show convexity of H: for $0 \le \tau \le 1$, we have

$$\begin{aligned} H(\tau p + (1-\tau)p') &= \sup_q \big((\tau p + (1-\tau)p')\cdot q - (\tau + 1 - \tau)L(q)\big) \\ &\le \sup_q (\tau p \cdot q - \tau L(q)) + \sup_q \big((1-\tau)p' \cdot q - (1-\tau)L(q)\big) \\ &= \tau H(p) + (1-\tau)H(p'). \end{aligned}$$

For the asymptotic result, note that for all $\lambda \in \mathbb{R}$, $q = \dfrac{\lambda p}{|p|}$, we get

$$\frac{H(p)}{|p|} \ge \frac{\lambda |p| - L\left(\frac{\lambda p}{|p|}\right)}{|p|} \overset{\lambda\to\infty}{\gtrsim} \lambda$$

where the last part is due to the asymptotic behavior of L. But then we see $\dfrac{H(p)}{|p|} \to \infty$.

Now the last part is to show duality.

$L(q) \ge H^*(q)$:

The slick idea to show this is to note that

$$H(p) = \sup_{\tilde{q}} (p \cdot \tilde{q} - L(\tilde{q})) \Rightarrow H(p) + L(q) \ge p \cdot q$$

for all p, q. So the trick is just to shift term and get $\forall p$

$$L(q) \ge (p \cdot q - H(p)) \Rightarrow L(q) \ge \sup_p (p \cdot q - H(p)).$$

$L(q) \le H^*(q)$:

Here is the only place we use convexity. We write out

$$H^*(q) = \sup_p (p \cdot q - \sup_r (p \cdot r - L(r))) = \sup_p \inf_r (p \cdot (q - r) + L(r))$$

where the inf comes from negative sign. Note that L convex implies that $\exists s \in \mathbb{R}^n$, the slope of the hyperplane at point q, such that

$$L(r) \ge L(q) + s(r - q)$$

which means $\forall r$

$$L(r) + s \cdot (q - r) \ge L(q)$$

and so

$$\sup_p \inf_r (p \cdot (q - r) + L(r)) \overset{p=s}{\ge} \inf_r (s \cdot (q - r) + L(r)) \ge L(q).$$

$\square$

Remark 11.5. *Note that no regularity is needed except convexity. Thus, even the convex extension (∞ at both ends) is fine for the proof. This can be related to solving the dual problem in optimization.*

Theorem 11.6. *(Hopf- Lax formula)*

Let u be given by

$$u(x,t) = \min_{y\in\mathbb{R}^n}\left(tL\left(\frac{x-y}{t}\right) + g(y)\right).$$

Then, at (x,t) *where u is differentiable, we have*

$$\frac{\partial u}{\partial t} + H(\nabla u(x,t))) = 0.$$

Note that in the formula, as $t \to 0$ we have $y \to x$ due to asymptotic property, and thus $u \to g$ at the boundary.

As a hint on how to prove the above:

Lemma 11.7. *For*

$$u(x,t) = \min_{y}\left((t-s)L\left(\frac{x-y}{t-s}\right) + u(y,s)\right)$$

Note that this is a local estimation from $t = s$ to $t = s + \varepsilon$. This, plus a similar argument as that of Banach contraction, yields the result.

Prove that $\frac{\partial u}{\partial t} + H \leq 0$ and $\frac{\partial v}{\partial t} + H \geq 0$, and I have no idea what that is.

Note that u, from it's expression, is not smooth in general, but we can always change our point of view and say ok, it can be our weak solution if not.

11.2. **Second order elliptic equations.**

For $u(x) \in \mathbb{R}$, real and scalar valued, we have

$$\begin{cases} -\nabla\cdot A\cdot\nabla u + b\cdot\nabla u + cu = f & x \in U \subset \mathbb{R}^n \\ u = 0 & x \in \partial U \end{cases}$$

where

$$A(x) = (a_{ij}(x)); \qquad b(x) = (b_j(x)); \qquad c = c(x)$$

moreover, we let U be nice and smooth, but no other conditions. Note that we have covered some special cases of this solution:

- When $A = 0$ then we can use the method of characteristics.
- When $b = c = 0$, $A = I$, then note

  $$\Delta = \nabla\cdot I\cdot\nabla$$

  so it's the Laplace equation!
- When $b = c = 0$, $A = \begin{pmatrix} 1 & 0 \\ 0 & -I \end{pmatrix}$ then we have

  $$\Box = \partial_t^2 - \Delta = -\nabla\cdot A\cdot\nabla$$

  so this time it's wave equation!

So we say a problem is elliptic if all the eigenvalues of A are strictly positive. If some of them are 0 this means in some direction we are back to method of characteristics.

Let's have a further look at the problem:

$$\begin{aligned} Lu &= -\nabla \cdot A \cdot \nabla u + b \cdot \nabla u + cu \\ &= -\sum_{i,j} a_{ij}\partial_{ij}^2 u + (b + \nabla \cdot A) \cdot \nabla u + cu \\ &= -\sum_{i \le j} \tilde{a}_{ij}\partial_{ij}^2 u + \tilde{b}\nabla u + cu \end{aligned}$$

where the last line we have

$$\tilde{a}_{ij} = \begin{cases} a_{ij} + a_{ji} & i \neq j \\ a_{ii} & i = j \end{cases}$$

and $\tilde{b} = b + \nabla \cdot A$. The lesson from this is that we can WLOG assume A to be symmetric. Now u can be of any reasonable space, so let's say the space of distributions. Then Fourier transform still passes and

$$\Delta u \to |\xi|^2 \hat{u}; \qquad \nabla \cdot A \cdot \nabla u \to [\xi^T A \xi]\hat{u}.$$

Just to verify this we consider the wave equation, for whom A is defined above. We call the transform of $t \to \omega$, the frequency, and $x \to \xi$ as usual to get

$$\Box u \to (-\omega^2 + |\xi|^2)\hat{u} = \left[\begin{pmatrix} \omega \\ \xi \end{pmatrix}^T A \begin{pmatrix} \omega \\ \xi \end{pmatrix} \right] = 0$$

but then we see that what is in front is not always positive, and will be 0 (in which case our solution fails to hold) where $|\omega|^2 = |\xi|^2$, the light cone.

This is illustrated in the Fourier transform even:

- For $\Delta u = f$ we have $\hat{u} = \dfrac{\hat{f}}{|\xi|^2}$.
- For $\Box u = f$ we have $\hat{u} = \dfrac{-1}{\omega^2 - |\xi|^2}\hat{f}$.

And we do see that the light cone is problematic for the second expression.

Def 11.8. *A PDE is elliptic if (below are equivalent conditions):*

- *For some $\theta > 0$ we have*
$$\xi^T A \xi \ge \theta |\xi|^2, \forall x \in U, \xi \in \mathbb{R}^n.$$
- *Equivalently, we can let $\tilde{\xi} := \dfrac{\xi}{|\xi|}$ then*
$$\tilde{\xi}^T A \tilde{\xi} \ge \theta.$$
- *Or we just say all the eigenvalues of A is positive.*

Note that this term on involves the highest order derivatives. Of course this might be problematic since say we have

$$(-\Delta + c)u = f \Rightarrow (|\xi|^2 + c)\hat{u} = \hat{f} \Rightarrow \hat{u} = \frac{\hat{f}}{|\xi|^2 + c}$$

then since e^{ic} roughly is the frequency, we are very well for high frequencies, but not so good for low frequencies, say $c = -5$.

Remark 11.9. *We want to have a theory that works for minimal regularity, say $a_{ij}, b_i, c \in L^\infty$, given that we know how to multiply distributions, i.e. $\delta_x \cdot \delta_y$ is not defined.*

But we do not do this because we want generality. We do this because we have our theory first.

And the theory is about weak solutions. The idea is test functions. One way to say $a = b$ is just to say $a - b = 0$. This is equivalent to saying

$$\int (a-b)\delta dt = 0$$

or just evaluating point wise, which is what we've been doing all the time. But we can test it with any other test functions, as long as they make sense. So we might consider $a = b$ in the sense that

$$\int (a-b)v dt = 0$$

for v in test class. In our scenario it becomes

$$\langle Lu, v\rangle = \langle f, v\rangle.$$

Formally speaking, we want

$$\int_U Luv dx = \int_U f v dx$$

and if we recall our definition of Lu, which is just full of divergence forms, we need to do IBP. Well since we're doing IBP then we'd better let our test function v vanishes on the boundary. So we have

$$\int_U -\nabla \cdot A \cdot \nabla u = \int_U A\nabla u \cdot \nabla v - \int_{\partial U} A \cdot \nabla u \cdot \vec{n} \cdot v ds$$

and only the first term is left. But for the first order term IBP just move the ∇ from u to v and does not change the order, we left it as it was. So what we want to show is:

$$\int_U A\nabla u \cdot \nabla v + b \cdot \nabla uv + cuv dx = \int_U f v dx.$$

As one might have expected, if we're in Hilbert space, then Riesz will solve the problem after some trials. So we generalize that idea to the <u>bilinear form</u> :

$$B[u,v] = \int_U A\nabla u \cdot \nabla v + b \cdot \nabla uv + cuv.$$

That is, we just put what we need into it and hope for the best.

We deal with the three part separately and see whether $B[u,u] \geq 0$, positive definite:

- Second order term: if the problem is elliptic then $B[u,v] \geq \theta|\nabla u|^2$ and for $c > 0$ its fine.
- Last term: if $c > 0$ then we're good.
- Middle term: we extended Riesz for this.

So we need the bilinear form to exist, and we want v to be the test function, so we use $u, v \in H_0^1(U)$ where

$$H_0^1(U) := \left\{ u \in L^2(U); \nabla u \in L^2(U;\mathbb{R}^n); u = 0 \text{ on } \partial U \right\}.$$

The hard problem, as often is in functional analysis, is that for a mere L^2 function the boundary is not defined: $u|_{\partial U}$ is not defined. We cannot extend, as we will do for continuous classes, use approximation, but we can use the trace operator to conquer this! It's all well recorded in book.

For a 1D taste of things, we have

$$u(x) = u(0) + \int_0^x u'(y)dy$$

and hence

$$u(0) \leq c \int_0^1 u^2(x) + (u')^2(x)dx$$

then by a density argument we can approximate the boundary with C functions. Note that this only works for H_0^1 functions as for L^2 this can be very wrong.

12. 4/27: EXISTENCE OF WEAK SOLUTION

Def 12.1. *We say that $u \in H_0^1(U)$ is a weak solution of the problem*

$$\begin{cases} Lu = f & U \\ u = 0 & \partial U \end{cases}$$

if

$$B[u, v] = \langle f, v \rangle_{Dual(H_0^1, H^{-1})}, \quad \forall v \in H_0^1(U)$$

where as before

$$B[u, v] = \int_U A\nabla u \cdot \nabla v + b \cdot \nabla uv + cuv.$$

Note $\langle f, v \rangle_{Dual}$ is defined when $f \in (H_0^1)^* = H^{-1}$, and that is defined as

$$H^{-1}(U) := \left\{ f \middle| f = f_0 - \nabla_x \cdot F, f_0 \in L^2(0), F \in L^2(U, \mathbb{R}^n) \right\}$$

where since F is just in L^2 we note that the divergence above is weak derivative. Now we start really doing things.

Def 12.2. *(Usual inner product on H_0^1) We use the norm*

$$\langle u, v \rangle_{H_0^1(U)} = \int_U (\nabla u \cdot \nabla v + uv) dx.$$

Theorem 12.3. *(Riesz) For H Hilbert space and it's dual H^*, then $\forall u^* \in H^*$ we know $\exists! u \in H$ such that $u^*(v) = \langle u, v \rangle, \forall v \in H$.*

Theorem 12.4. *Say $B[u, v] = B[v, u]$ is symmetric (this means $b = 0$) and assume that $c \geq c_0 > 0$ is a nice guy. Then, $B[u, v]$ is an inner product on $H_0^1(U)$ equivalent to the usual one.*

This means that for all $f \in H^*$ by Riesz $\exists! u \in H$ such that

$$\langle f, v \rangle = B[u, v]$$

which means exactly that there exists a unique weak solution to the PDE.

Proof.

Need equivalence of norm, which makes H_0^1 Hilbert under this inner product too. So we do estimations: (continuity)

$$B[u, v] \leq c||u||_{H^1} \cdot ||v||_{H^1}$$

and thus

$$B[u, u] \leq c||u||_{H^1}^2.$$

Moreover, the other direction is by

$$B[u,u] = \int_U A\nabla u \cdot \nabla u + cu^2 \geq \theta \int_U |\nabla u|^2 + c_0 \int_U u^2 \geq \min\{\theta, c_0\}||u||_{H^1}^2$$

which means that the norms generated by the inner product is equivalent to the H^1 norm. This implies the inner product is equivalent by parallelogram law. $\square$

So we're done with the case when $b = 0$, so now when $b = 0$, we extend Riesz to Lax-Milgram theorem.

Theorem 12.5. *For $B : H \times H \to \mathbb{R}$ a bilinear form such that for $\alpha, \beta > 0$ for which we know*

(1) $|B[u,v]| \leq \alpha||u|| \cdot ||v||$ (continuity).
(2) $\beta||u||^2 \leq B[u,v]$ (coercivity).

Let $f \in H^*$, then $\exists! u \in H$ such that $B[u,v] = \langle f, v\rangle, \forall v \in H$.

Proof. The trick here is that we apply Riesz to both sides, then do as we see fit.

First, say that $u \in H$ is fixed, then the bilinear form $B[u,v] : H \to \mathbb{R}$ can be seen as a functional that takes v to a real number, so by Riesz there exists some w such that $B[u,v] = \langle w, v\rangle$.

Since this works for all u, we define the operator A such that $A(u) = w$. We see that it is linear since B is. Moreover, we know $B[u,v] = \langle Au, v\rangle$. We now investigate A to see that it is in fact a nice guy. We know

$$||Au||^2 = ||Au|| \cdot ||Au|| = B[u, Au] \leq \alpha||u|| \cdot ||Au||$$

which means $|A|$ is bounded above. Moreover, we have

$$\beta||u||^2 \leq B[u,u] = \langle Au, u\rangle \Rightarrow \beta||u|| \leq ||Au||$$

and so $|A|$ is bounded from below. Thus by a priori estimate $\ker(A) = \{0\}$ and $\mathrm{Ran}(A)$ is closed. So we still need to show that the range of A is everything to get that A can be inverted. But since it's closed, if $u \perp \mathrm{Ran}\, A$ we know

$$0 = \langle Au, u\rangle = B[u,u] \geq \beta||u||^2$$

and thus $u = 0$. So the only vector orthogonal to $\mathrm{Ran}\, A$ is the null vector, hence it is the whole space H.

Since A is invertible, then by open mapping A^{-1} is bounded. Now we deal with the other functional.

Fix f then we use Riesz again to write

$$\langle f, v\rangle := \langle w, v\rangle$$

for some $w = w(f)$. Then we define $u = A^{-1}w$ to get

$$B[u, v] = \langle Au, v\rangle = \langle w, v\rangle = \langle f, v\rangle.$$

That means we've constructed a weak solution u.

Now we show uniqueness. We have

$$B[u_1 - u_2, v] = \langle f, v\rangle - \langle f, v\rangle = 0$$

for any test function v. So we just take $v = u_1 - u_2$ to get

$$||u_1 - u_2||^2 = 0$$

so the solution is unique. □

All's well and good except we still need to find some B that satisfies the above 2 conditions. This is not hard either.

Theorem 12.6. *(Energy Estimate)* $\exists \alpha, \beta, \gamma$ *such that*

- $|B[u, v]| \leq \alpha ||u||_{H_0^1} \cdot ||v||_{H_0^1}$ *(continuity).*
- $\beta ||u||^2_{H_0^1} \leq B[u, u] + \gamma ||u||^2_{L^2}$ *and we see here that the shift of spectrum does deal with a low frequency.*

Proof. Continuity:

$$\begin{aligned}|B[u, v]| &\leq ||A||_\infty \int |\nabla u| \cdot |\nabla v| + ||b||_\infty \int |\nabla u| \cdot |v| + ||C||_\infty \int |uv| \\ &\overset{C.S.}{\leq} \max\{||A||_\infty, ||b||_\infty, ||C||_\infty\} \cdot ||u||_{H_0^1} \cdot ||v||_{H_0^1}\end{aligned}$$

just because each term is less than that.

Coercivity:

We have

$$\begin{aligned}B[u, u] &= \int A\nabla u \cdot \nabla u + b\nabla u \cdot u + cu^2 \\ &\geq \theta ||\nabla u||^2 - ||b||_\infty \int |\nabla u| \cdot |v| - ||C||_\infty \int |uv|\end{aligned}$$

but we note in the middle term $|\nabla u|$ is the actual bad part, while $|u|$ can be covered with the γ term. So if we try to do the normal thing

$$\int |\nabla u| \cdot |v| \leq \frac{1}{2}(||\nabla u||^2 + ||u||^2)$$

it will not be good enough. So instead we write it like this:

$$\int |\nabla u| \cdot |v| = \int \varepsilon |\nabla u| \cdot \frac{1}{\varepsilon} |v| \leq \frac{1}{2}(\varepsilon^2 ||\nabla u||^2 + \frac{1}{\varepsilon^2} ||u||^2)$$

and thus plugging into the above we have

$$B[u,u] \geq \theta||\nabla u||^2 - \frac{||b||_\infty \varepsilon^2}{2}||\nabla u||^2 - \left(||c||_\infty + \frac{1}{2\varepsilon^2}\right)||u||^2$$

thus we can choose

$$\frac{||b||_\infty \varepsilon^2}{2} = \frac{\theta}{2} \Rightarrow \varepsilon^2 = \frac{\theta}{||b||_\infty}$$

to get

$$B[u,u] \geq \frac{\theta}{2}||u||_{L^2}^2 + \frac{\theta}{2}||u||_{L^2}^2 - \gamma||u||_{L^2}^2 = \frac{\theta}{2}||u||_{H_0^1}^2 - \gamma||u||_{L^2}^2$$

where let's just say $\gamma = \frac{\theta}{2} + ||c||_\infty + \frac{1}{2\varepsilon^2}$.

Of course there's a trade off between θ and γ, but we can always choose ε smaller so the trade off is minimal. This usually does not matter. □

Just combining the above 2 theorems we have:

Theorem 12.7. $\exists\gamma \geq 0$ *such that* $\forall\mu \geq \gamma$, $f \in H_0^1(U)$ *and* $\exists!$ *weak solution of*

$$\begin{cases} Lu + \mu u = f & U \\ u = 0 & \partial U. \end{cases}$$

One thing that is overlooked by the book maybe is the following inequality.

Theorem 12.8. *(Poincare inequality) For* $b = 0$ *and* $c \geq 0$ *the above is OK and* $\gamma = 0$. *We know that we have*

$$\int_U u^2 dx \leq C_U \int_U |\nabla u|^2 dx$$

where $u \in H_0^1(U)$. *This basically means that if the gradient is controlled, then u cannot grow too large, which is obvious.*

But then this gives us

$$B[u,u] \geq \theta \int |\nabla u|^2 = \frac{\theta}{2}\int |\nabla u|^2 + \frac{\theta}{2}|\nabla u|^2 \geq \frac{\theta}{2}\int |\nabla u|^2 + \frac{\theta}{2C_u}\int_U |u|^2 \geq \beta||u||_{H_0^1}^2.$$

This yields the coercivity.

Note that this means when there is a leak of information, the total information cannot be too many.

What we have done is that we've shifted L to $L+\mu$. If we have $L+\lambda$, L compact, then we might wonder for which λ can we invert the operator, and the result is first by compactness there's only discrete spectrum, and thus the only problematic point is when $\lambda \in p(L)$. Next time we deal with this by compactness + Fredholm Alternative.

Example 12.9. *(example of the last* -1 *minute).*

For $U = (0, \pi)$,$u(x) = \sin(k\pi)$ satisfies the equation

$$\begin{cases} -u'' = k^2 u \\ u(0) = u(\pi) = 0 \end{cases}$$

and thus $L = -u'' - k^2$ and it's not invertible when $k \in \mathbb{Z}$.

The eigenvalue of this goes to ∞ and thus that of the inverse goes to 0.

13. 5/4: EXISTENCE; REGULARITY THEORY

We continue on last time. Suppose we have the bilinear relation

$$B[u,v] = \int_U A\nabla u \nabla v + b\cdot \nabla uv + cuvdx$$

then from last time we know that

$$|B[u,v]| \le C||u||_{H_0^1}||v||_{H_0^1}$$

and we can introduce

$$B_\gamma[u,v] = B[u,v] + \gamma\int_U uvdx$$

such that

$$|B_\gamma[u,u]| \ge \beta||u||^2_{H_0^1}.$$

The idea is really that by adding the term with γ we are shifting the spectrum to the right. Moreover, this addition can be viewed as applying a compact operator, so the spectrum becomes discrete.

Remember, from Lax-Milgram we know that $\exists! u \in H_0^1$ such that $B_\gamma[u,b] = \langle f, v\rangle$ for all $v \in H_0^1$.

Now we recall our definition that $L = -\nabla A\nabla + b\nabla + c$, and we have shown last time that for some γ, the function

$$\begin{cases} (L+\gamma)u = f + \gamma u =: g & U \\ u = 0 & \partial U \end{cases}$$

has a unique weak solution u for the above, and thus we denote

$$L_\gamma := L + \gamma; \quad u := L_\gamma^{-1} g$$

where the inverse is in the weak sense. We can further write out

$$u = L_\gamma^{-1}(f+\gamma u) = L_\gamma^{-1} f + L_\gamma^{-1} u =: h + Ku$$

where as one might expect the notation k is because it is a compact perturbation.

Now one very important thing is that the only thing we have at control is θ (which also gives β), the control of the highest derivative.

So since $u = h + Ku$ if K is small enough then we can directly invert using $(I-K)^{-1}$ like method, but the whole goal of γ is to cancel the lower order perturbation, so we really do not expect γ to be small.

But if K is compact then we know it has only discrete spectrum, and thus the only problem is whether 1 is an element of its spectrum. If it is then we need to add additional properties on h, if not then we are done. This is heuristics, and we'll see how they are represented.

So we want to show K compact, but in what sense? Since $g \in L^2$ so it's L^2 compact. By our condition we know $L_\gamma^{-1} : L^2 \to H_0^1(U)$ is bounded since

$$B_\gamma[u, v] = \langle f, v \rangle$$

so the H_0^1 norm of u is bounded by the norm of f, after we divide both side taken absolute value.

Remark 13.1. *Do not use open mapping (here you can't use because that's a weak inverse) when you can control the bound in any form.*

Theorem 13.2. *(Rellich Compactness Theorem) Assume U is a bounded open subset of $\mathbb{R}^n$, and $\partial U \in C^1$. For $1 \le p < n$ we have*

$$W^{1,p}(U) \subset\subset L^q(U)$$

for each $1 \le q < p^*$.

Anyways, this theorem gives that the identity operator

$$Id : H_0^1(U) \to L^2(U)$$

is compact for U bounded.

Thus, if we view $L_\gamma^{-1} := id \circ L_\gamma^{-1} : L^2 \to L^2$ is compact, and hence K is compact if we view it as acting on only L^2.

Def 13.3. *For the function $u = Ku + h$, it is Fredholm Alternative if either $\exists!$ solution or for $N := \ker(I - K)$ is finite dimensional and $\dim N = \dim N^*$, where note $N^* = I - K^*$, (here K^* is compact with respect to L^2) and $u = Ku + h$ is a solution iff $hu^* = 0$ forall $u^* \in N^*$.*

So we want to solve $(L - \lambda)u = f$ for every λ, in order to find eigenvalues and check Fredholm, thus we first investigate $\lambda = 0$, in which case we have

$$Lu = f \iff Lu + \gamma u = f + \gamma u \iff u = ku + h$$

and we assume

$$u^* = k^* u^*$$

then

$$0 = (h, u^*) = (L_\gamma^{-1} f, u^*) = \frac{1}{\gamma}(Kf, u^*) = \frac{1}{\gamma}(f, K^* u^*) = \frac{1}{\gamma}(f, u^*)$$

and note that our condition means

$$u^* = K^* u^* = \gamma \left(L_\gamma^*\right)^{-1} u^* \iff L_\gamma^* u^* = \gamma u^* \iff L^* u^* = 0.$$

So $Lu = f$ has weak solution when $(f, u^*) = 0$ for all u^* such that $L^* u^* = 0$.

Generalize this to any λ:

Theorem 13.4. *There's at most a countable set $\Sigma \subset \mathbb{C}$ such that*

$$\begin{cases} Lu = \lambda u + f & U \\ u = 0 & \partial U \end{cases}$$

that has unique weak solution for $f \in L^2(U)$ and $\lambda \in \Sigma$.

Proof. We again write everything to get

$$(L+\gamma)u = (\lambda+\gamma)u + f \iff u = \frac{\gamma+\lambda}{\gamma}Ku + L_\gamma^{-1}f$$

as for why this thing solves the problem we use the fact that it is a Fredholm Alternative. And this is not solvable if $1 \in \sigma\left(\frac{\gamma+\lambda}{\gamma}K\right)$, but for other cases since we know $\left(\frac{\gamma+\lambda}{\gamma}K\right)$ is compact we denote

$$K\phi_i = k_i\phi_i$$

then $\lambda_i = \gamma\left(\frac{1}{k_i} - 1\right)$ and thus $\{\lambda_i\} =: \Sigma$. Since K compact Σ is discrete so at most countable. Moreover, the only accumulation point of k_i is 0 so $\lambda_i \to \infty$. □

Energy estimates

For $\lambda \notin \Sigma$ we have

$$||u||_{L^2} \leq c||L_\gamma^{-1}f||_{L^2} \leq C||L_\gamma f||_{L^2}$$

and to assume $1 \notin \sigma\left(\frac{\gamma+\lambda}{\gamma}K\right)$ we have

$$u = \left(I - \frac{\gamma+\lambda}{\gamma}K\right)^{-1} L_\gamma^{-1}f$$

where we define the inverse like how we'd decompose any compact operator, eigenspace by eigenspace. Thus we have

$$L_\gamma u = (\gamma+\lambda)u + f \leq (\gamma+\lambda)||u|| + ||f||_{L^2} \leq C||f||_{L^2}$$

and so

$$||u||_{H_0^1} \leq C||f||_{H_0^1}.$$

Theorem 13.5. *The above defined $C = C(\lambda, U, L)$ is such that $C \to \infty$ as $\lambda \to \Sigma(U, L)$.*

13.1. **Regularity Theory.** :

We know $u \in H_0^1$, but can it be smoother? Still, our only tool is $\xi A\xi \geq \theta|\xi|^2$.

Motivation is the following formal computation:

$$\int_U f^2 = \int_U (\Delta u)^2 = \int_U \sum_{i,j} u_{ii}u_{jj} = -\int_U \sum_{i,j} u_i u_{ijj} = \int_U \sum_{i,j} u_{ij}^2$$

which gives us directly from only the diagonal to all derivatives!

Since we're summing up positive numbers, we have

$$\int_U u_{ij}^2 \leq ||\Delta u||^2$$

and hence for $-\Delta u = f$, we have $-\Delta u_i = f_i \in L^2$ if $f \in H^1$, thus $u_i \in H^2 \Rightarrow u \in H^3$. So for $f \in H^m$ we have $u \in H^{m+2}$. Yet all these are just hand waving and we need to justify the existence of many of the above.

Even we do not know Δu exists for $u \in H^2$, we know

$$\frac{u(x+h)-u(x)}{h} \in H_0^1.$$

Now for interior regularity we note

$$Lu = f \Rightarrow -\nabla A \nabla u = f - b\nabla u - cu \in L^2$$

since everything on the right is in L^2. We want from RHS L^2, which means u on the left hand side is H^2. But really if we think about it, regularity is a local property, and we only need to show H^2 locally. So there's only two cases: when the point is in the interior and is in the boundary. We only deal with interior here.

Since we're dealing with only the interior we can kill the boundary with a smooth cutoff function that vanishes in the vicinity of ∂U.

Theorem 13.6. *Assume $A \in C^1$, the boundary condition is bounded, and $f \in L^2$ (cannot be H^{-1} since then we at best have $u \in H^1$, but we want H^2) . Then $u \in H^1$, the weak solution of $Lu = f$ is suchc that $u \in H^2_{loc}(U)$ for each $V \ll U$ we have*

$$||u||_{H^2} \leq C(V)\left(||f||_2 + ||u||_2^2\right)$$

where $V \subset \bar{V} \subset U$

So as we mentioned, we introduce the cutoff function $\zeta \in C_c^\infty(U)$ such that $\zeta(V) \equiv 1$.

Theorem 14.1. *For $A \in C^1(U)$, b, c bounded, and $f \in L^2$, and for $u \in H_0^1$ be a weak solution of the equation, then we have $u \in H^2_{loc}(U)$. More over, for $V \subset\subset U$ (compactly embedded) we have*

$$||u||_{H^2(V)} \leq C\left(||f||_{L^2(U)} + ||u||_{L^2(U)}\right).$$

Remark 14.2. *First, we really need $f \in L^2$, that is, to be better than H^{-1} so that we can borrow some kind of smoothness from it.*

Moreover, we need the $||u||_{L^2(U)}$ term because that deals with the issue when u is not unique, i.e. the eigenvalue is not bad, intuitively.

Proof.

Step 0: Bound L^2 norm of ∇u:

We use the cut off function that we've defined above, moreover, just to get good terms we use ζ^2. From the fact that u is a weak solution, we know that

$$\int_U A\nabla u \cdot \nabla v + b\nabla uv + cuv - f v dx = 0$$

holds for all v, so we just pick $v = \zeta^2 u$ and get

$$\int_U A\nabla u \cdot \nabla(\zeta^2 u) + \zeta^2 b\nabla u \cdot u + c(\zeta u)^2 - f\zeta^2 u dx = 0$$

$$\Rightarrow \int_U A\zeta\nabla u \cdot \zeta\nabla u + A\zeta u 2(\nabla\zeta) \cdot u + b\zeta\nabla u \cdot \zeta u + c(\zeta u)^2 - f\zeta^2 u dx = 0$$

and we can use ellipticity to control the first term, i.e. the sum of all other terms. That is

$$A(\zeta\nabla u) \cdot \zeta\nabla u \geq \theta|\zeta\nabla u|^2$$

so the first term is dealt with. We check the other terms and ask are they nice? Since $f \in L^2$, $u \in L^2$, and thus the tail terms we can easily bound with Cauchy Schwartz:

$$\int_U c(\zeta u)^2 - f\zeta^2 u dx \leq C\left(||f||_2 + ||u||_2\right)$$

since $|\zeta| \leq 1$. For the other terms we combine and get

$$|A\zeta u 2(\nabla\zeta) \cdot u + b\zeta\nabla u \cdot \zeta u| \leq C' \, |A\nabla u \cdot u + b\nabla u \cdot u|$$

which is because we know ζ is just a smooth decay function, and thus for any fixed V the decay rate is bounded. Note that this is not true when we are close to the boundary, which causes no problem because we're not dealing with the boundary at all. So we then bound A and b uniformly with another constant to get

$$C' \left| A\zeta \nabla u \cdot u + b\nabla u \cdot u \right| \leq C \left| \zeta \nabla u \cdot u \right| = C \left| \varepsilon \zeta \nabla u \cdot \frac{1}{\varepsilon} u \right| \leq C\varepsilon |\zeta \nabla u|^2 + \frac{C}{\varepsilon}|u|^2$$

And so putting things together we have

$$\begin{aligned}
\theta \int_U |\zeta \nabla u|^2 dx &\leq \int_U A\zeta \nabla u \cdot \zeta \nabla u dx \\
&= -\left(\int_U A\zeta u 2(\nabla \zeta) \cdot u + b\zeta \nabla u \cdot \zeta u + c(\zeta u)^2 - f\zeta^2 u dx \right) \\
&\leq \int_U C_1 \varepsilon |\nabla u|^2 + \frac{C_1}{\varepsilon}|u|^2 dx + C_2 \left(||f||_2 + ||u||_2 \right)
\end{aligned}$$

so we just let $C\varepsilon := \theta/2$ to get the trade off

$$\frac{\theta}{2} \int_U |\zeta \nabla u|^2 dx \leq C \left(||f||_2 + ||u||_2 \right)$$

so we get the intermediate result:

$$||\nabla u||_{L^2(U)} \leq C \left(||f||_{L^2(U)} + ||u||_{L^2(U)} \right) \tag{14.1}$$

Step 1:

We introduce the cut-off function in more detail. In particular we let W be a set such that it is in between U and V, i.e. $V \subset \overline{V} \subset W \subset \overline{W} \subset U$. More over, we require that W is closer to V than it is to U, then we define ζ as a smoothened step function that is supported only on W. This is relevant later, but really a technical issue.

Step 2:

Define

$$\alpha := \int A\nabla u \nabla v = \int \tilde{f} v =: \beta$$

for $\tilde{f} \in L^2$ where $\tilde{f} = f - b\nabla u - cu$.

Note that from here, if we are justified to let $v = \Delta u$ then we do integral by part and everything will just follow. The only issue is that that's not legal since what we're proving is just that $u \in H^2$, and that's just reverse engineering.

Step 3:

Since we cannot define Δu, how about let's just do finite difference? We define

$$v := -D_k^{-h} \left(\zeta^2 D_k^h u \right)$$

for k fixed such that what we really want is $v = \partial_k^2 u$, heuristically. Here the notation means

$$D_k^h u(x) = \frac{u(x + he^k) - u(x)}{h}$$

so we need $0 < h < \frac{1}{2}(V, \partial W)$, which is probably the only place that W is needed. So we have $v \in H_0^1$ just because everything in it's definition is H^1.

Step 4:

We get our result here. To prove H^2 we try to find it's second weak derivative and show that it is H_0^2 bounded. Note that we've half constructed it already with finite sums, now we really do the calculations.

So we cheat a little bit for now (proven later) and use Evans 5.8.2 to get

$$\int v^2 = \int \left| D_k^{-h} \zeta^2 D_k^h u \right|^2 \leq C \int_U \left| \delta \left(\zeta^2 D_k^h u \right) \right|^2$$

and we'll come back to it later.

We have

$$\begin{aligned} \alpha = -\int_U A \nabla u \cdot \nabla \left(D_k^{-h} \zeta^2 D_k^h u \right) dx &\overset{discrete\ ibp}{=} \int_U D_k^h (A \nabla u) \cdot \nabla (\zeta^2 D_k^h u) dx \\ = \int_U A_k^h (D_k^h \nabla u) \cdot \nabla (\zeta^2 D_k^h u) dx &+ \int_U D_k^h (A) \nabla u \cdot \nabla (\zeta^2 D_k^h u) dx \end{aligned}$$

where $A_k^h = A(x + he_k)$, which we know is also elliptic with the same θ (just a shift so nothing changes). So now we think about our equation. The bad terms are the ones with two derivatives, not caring about whether discrete or ∇. For the first integral it's a multiplication of 2 bad terms, so we can only use ellipticity; for the second it is a good term ∇u times a bad term, so usual trick does the job.

Now we assert that D_k^h and ∇ commutes to get

$$\int_U A_k^h (D_k^h \nabla u) \cdot \nabla (\zeta^2 D_k^h u) dx = \int_U A_k^h \zeta D_k^h \nabla u \cdot \zeta D_k^h \nabla u dx + \int_U A_k^h 2 \delta \zeta \cdot \zeta D_k^h \nabla u \cdot D_k^h u$$

where we use elliptic to deal with the first, and good bad ε argument to deal with the second, then we combine with the $D_k^h(A)$ term to have the general bound:

$$\frac{\theta}{2} \int_U \left| \zeta D_k^h \nabla u \right|^2 \leq C \left(|\nabla u|^2 + |D_k^h u|^2 \right) + |\beta|$$

but we can use the same trick again to bound β:

$$\beta \leq C_2 \int_U (|f| + |u| + |\nabla u|) \, |v| dx$$

which is also a 1 derivative times 2 derivative case (v by definition is of second derivative) so ok we use $\frac{\theta}{4}$ to contro them and get the final estimate:

$$\int_U \zeta^2 |D_k^h \nabla u|^2 \leq C \left(||f||_2^2 + ||u||_2^2 + ||\nabla u||_2^2 \right)$$

where C is independent of h, which is the only important thing that matters here.

Now take $h \to 0$ by Evans 5.8.2 Theorem 3(ii) we can pass the limit and get

$$\int_U \zeta^2 |\partial_k \nabla u|^2 \leq C \left(||f||_2^2 + ||u||^2_{H_0^1(U)} \right)$$

so ok that's inconveniently a square there but we can bound the square root by Cauchy-type estimates and get (we make $U \to V$ since that's strictly smaller, and the integrand is positive):

$$||u||_{H^2(V)} \lessapprox C \left(||f||_2^2 + ||u||^2_{H_0^1(U)} \right)$$

where remember from step 0 (14.1) we get a bound for $||\nabla u||_2$ by $||u||_2$ so

$$||u||_{H^2(V)} \lessapprox C \left(||f||_2^{L^2(U)} + ||u||^2_{L^2(U)} \right).$$

Till here the proof is done but we fill in details.

Step 5:

We show first that $||D^h u|| \leq C||\nabla u||$, but that is easy because we have

$$\frac{u(x+he_i)-u(x)}{h} = \int_0^2 \partial_i u(x+the_i)dt$$

by a Taylor-like estimate. Then we take L^2 norm on both sides and use Cauchy Schwarz to get the result.

For the last bit we encode in a lemma:

Lemma 14.3. *If* $\int_V |D_k^h \phi|^2 dx \leq C$ *for* $\phi \in L^2(U)$*, then* $\phi \in H^1(V)$ *and* $||\nabla \phi||_{L^2(V)} \leq C$.

in other words we put in $\phi = u'$ to get the result.

Proof. (lemma)

For any $\psi \in C_c^\infty(V)$, we have by finite ibp that

$$\int_V \phi D_k^{-h} \psi dx = -\int_V D_k^h \phi \cdot \psi dx$$

where we know $D_k^{-h}\psi \to \partial_k \psi$ because that's nice guy. And the only thing to do is to identify $D_k^h \phi$. So we have

$$||D_k^h \phi||_{L^2} \leq C$$

by assumption and thus Sobolev embedding means that there's some subsequence such that

$$D_k^{h_j} \phi \overset{L^2,w*}{\rightharpoonup} v_k$$

for some $v_k \in L^2$.

But then we have

$$-\int_U \phi \partial \psi = -\lim_{h\to 0} \int_V \phi (D_k^{-h}\psi) = \lim_{h \to 0} \int_V D_k^{-h} \phi \cdot \psi \rightharpoonup \int_V v_k \psi = \int_U v_k \psi$$

and that's just the definition of weak derivative so we know $\partial_k \phi = v_k$. □

Now passing the other limits are really justified. □

To wrap up:

For the interior, we have if $A, b, c \in C^{m+1}(U)$, $f \in H^m(U)$, $Lu = f$ and suppose u is a weak solution, then

$$||u||_{H^{m+2}(V)} \leq C \left(||f||_{H^m(U)} + ||u||_{L^2(U)} \right)$$

and thus $u \in H^{m+2}_{loc}$.

For the boundary, we first straighten the boundary, then use the below theorem in book to conclude that $u \in H^{m+2}$ uniformly everywhere.

Theorem 14.4. *For $A, b, c \in C^{m+1}(\overline{U})$, $f \in H^m(U), \partial U \in C^{m+2}$, if $u \in H^1_0$ is a weak solution, then*

$$||u||_{H^{m+2}(U)} \leq C \left(||f||_{H^m(U)} + ||u||_{L^2(U)} \right).$$

15. 5/11: MAXIMUM PRINCIPLE; CALCULUS OF VARIATION

Before we go into topics today, we add a few things.

Def 15.1. *The non-divergence form of the elliptic operator is*

$$Lu = -a_{ij}\partial_{ij}^2 u + b_i \partial_i u + cu.$$

Moreover, we have some smoothness of general classes:

Theorem 15.2. *(Sobolev embedding) For* $\varepsilon > 0$ *we have*

$$H^{m+\frac{n}{2}+\varepsilon} \subset C^m$$

where n is dimension.

The above explains why we just assert $u \in C^2$ in the next part today.

15.1. Maximum Principle.

Theorem 15.3. *(Weak maximum principle): Assume* $u \in C^2(U) \cap C^0(\overline{U})$ *and* $c = 0$.

- *If* $Lu \leq 0$ *in* U*, then*
$$\max_{\overline{U}} u = \max_{\partial U} u$$
which we call u a sub solution.
- *If* $Lu \geq 0$ *in* U*, then*
$$\min_{\overline{U}} u = \min_{\partial U} u$$
which we call u a super solution.

The picture is that in 1D, a upward quadratic function is a subsolution, a downward is a supersolution, where as a harmonic, or linear function is both. See that they attains maximum or minimum correspondingly.

Also, we need the absorption c to be zero, because adding $c > 0$ causes a upward shift, which might conflict if we're finding the minimum.

Proof.

Step 1: Assume that $Lu < 0$ strictly:

Assume $u(x_0) \geq u(x)$ where $x_0 \in U$ and $\forall x \in \overline{U}$. Then since it is a maximal we have $\nabla u(x_0) = 0$ and $\Delta^2 u \preceq 0$.

- If $A = aI$, then $Lu = -a\Delta u + b\nabla u$ which gives
$$Lu(x_0) = -a\Delta u(x_0) = -a \operatorname{tr} \nabla^2 u(x_0) \geq 0$$
a contradiction.

- For $A = P^T DP$. Now we do the change of variable $y = x_0 + P(x - x_0)$ which means $x - x_0 = P^T(y - x_0)$. And the whole idea is to change $\nabla \cdot A\nabla u$ to $\nabla_y^2 u$ with that diagonalization, then we get $\nabla_y \cdot D\nabla_y^2 u \geq 0$ by the same trace argument, note that the matrix D is also elliptic thus we are done by the same trace argument.

so we are done for $Lu < 0$.

When $Lu \leq 0$:

We define $u_\varepsilon(x) = u(x) + \varepsilon e^{\lambda x_1}$ then (recall $Lu \leq 0$)

$$Lu_\varepsilon = Lu + \varepsilon L^e \lambda x_1 \leq \varepsilon e^{\lambda x_1}\left(-\lambda^2 a_{11} + b_1\lambda\right) \leq \varepsilon e^{\lambda x_1}\left(-\lambda^2\theta + ||b||_\infty \lambda\right) \overset{\lambda\to\infty}{\leq} 0$$

since the leading term dominates. Now ε is arbitrary so u also attains maximum at the boundary. $\square$

Corollary 15.4. *For $Lu = f$ on U and $u = 0$ on the boundary, we can thus show directly uniqueness and existence by above.*

Proof. Uniqueness is obvious.

For existence, we note that uniqueness means that the kernel is $\{0\}$, which by Fredholm alternative (we can use because after a shift it is compact, so it is itself compact) that $\{0\}$ is the kernel of L^*, but then $(f, v) = 0$ for all $v \in \ker L^*$, so we are done. $\square$

Theorem 15.5. *(Strong maximum principle)*

For $u \in C^2(U) \cap C^0(\overline{U})$ and $c = 0$, U connected, open, bounded, then if $Lu \leq 0$ and u attains maximum over $\overline{U}$ at some interior point $x_0 \in U$, then u is a constant.

This uses Hopf lemma but we don't have time for it. Another useful bound is the Harnack's inequality:

Theorem 15.6. *(Harnack's inequality) Assume $u \geq 0$ is a C^2 solution of $Lu = 0$, then suppose $V \subset\subset U$ is conneccted, then there exists a constant C such that*

$$\sup_V u \leq C(V) \inf_V u.$$

15.2. **Calculus of Variations.**

As an intuition, this is just elliptic PDEs but with some additional steps.

For $A[u] = 0$, we want to find I such that $A[x] =: I'[u] = 0$ and thus we're finding critical points of functionals. An example that we've seen before is

$$\begin{cases} -\Delta u = f \\ u = 0 \end{cases} \iff I[u] = \int_U \frac{1}{2}|\nabla u|^2 - fu.$$

That's very good and we note that inside the first term is kenetic energy and the second looks like potential energy, maybe, so we note it is a Lagrangian. To formally define it, we have that the Lagrangian is

$$L : \mathbb{R}^n \times \mathbb{R} \times \overline{U} \to \mathbb{R}$$

and we define the action I that integrates the Lagrangian:

$$I[w] := \int_U L(\delta w, w, x) dx$$

where for convenience we also label it $L(p, z, x)$ in line with our notations. This leads us eventually to Euler-Lagrangian equations, but let's deduce it first. For $v \in C_c^\infty(\overline{U})$ smooth we define

$$i(\tau) = I[u + \tau v], \tau \in \mathbb{R}$$

and the fact that u is a critical point of I (we want that) means $i'(0) = 0$. So we still have to do some formal computation, but that's not hard:

$$i(\tau) = \int_U L(\delta u + \tau \nabla v, u + \tau v, x) dx$$

and

$$i'(\tau) = \int_U \nabla_x v \cdot \nabla_p L(p_\tau, z_\tau, x) + v \partial_z L(p_\tau, z_\tau, x) = \int_U \left[-\nabla_x \cdot \nabla_p L + \partial_z L \right] v dx$$

and letting $i'(0) = 0$ we have the <u>Euler-Lagrangian equation</u>:

$$-\nabla_x \cdot \nabla_p L(\nabla u, u, x) + \partial_z L(\nabla u, u, x) = 0.$$

Example 15.7.

- $L(p, z, x) = \frac{1}{2}|p|^2$, then the E.L. is $-\nabla_x \nabla_x u = -\Delta u = 0$.
- $L = \frac{1}{2} p^T A p - z f(x) \Rightarrow \nabla_x A \nabla_x u - f(x) = 0$.
- Let $F(w)$ be the potential energy then we can let $L = \frac{1}{2}|\nabla u|^2 - F(u)$ then $\nabla_p L = p; \nabla_z L = -F'(z) =: f(z)$ and hence the Euler Langrange is

  $$-\Delta u = f(u).$$

  In particular we can pick $F(z) = \frac{c}{2} z^2 \Rightarrow f(z) = cz$ and the equation is $-\Delta u + cu = 0$.

Now for u is a minimizer, we just compute the second derivative and use $i''(0) \geq 0$, a smile curve. Then the idea is try to find the alike form of first order E.L. and use that it's a constant.

So

$$i''(\tau) = \int_U \nabla v \nabla_p^2 L(p_\tau, z_\tau, x) \nabla v + 2 v \nabla_p \nabla_z L(p_\tau, z_\tau, x) \cdot \partial v + v^2 \partial_z^2 L(p_\tau, z_\tau, x) dx$$

and plug in $\tau = 0$ we have

$$
\begin{aligned}
i''(0) &= \int_U \nabla v \nabla_p^2 L(\nabla u, u, x) \nabla v + \int_U \partial_z \left(\nabla_p L(\nabla u, u, x) \cdot 2v \nabla v + v^2 \partial_z L(\nabla u, u, x) \right) dx \\
&= \int_U \nabla v \nabla_p^2 L \nabla v + \int_U \partial_z \left(\nabla_p L \cdot \nabla v^2 + v^2 \partial_z L \right) dx \\
&= \int_U \nabla v \nabla_p^2 L \nabla v + \int_U \partial_z \left(\nabla_x \nabla_p L \cdot + \partial_z L \right) v^2 dx
\end{aligned}
$$

and since we know already that the Euler-Lagrange is 0, in particular a constant in z when $\tau = 0$, so the second integral is 0,and thus

$$
i''(0) = \int_U \nabla v \nabla_p^2 L \nabla v
$$

and that is positive if given $A := \nabla_p^2 L$ is elliptic. Now we can see it generalizes elliptic PDEs.

So we have, up to now, replaced our PDE with a minimization problem of a functional I. This does not guarantee that we have a easier problem, but at least that's some method. We need 2 conditions to find the minimum:

(1) Coercivity: that $L \to +\infty$ at ∞. This prevents the function not having minimums, for instance think of e^{-x} that we've excluded. Moreover, this also guarantees that the minimum is in some compact domain.

To be precise, we write it in the following way: for $1 < q < \infty$, $\alpha > 0, \beta \geq 0$, we require

$$
L(p, z, x) \geq \alpha |p|^q - \beta
$$

where $p \in L^q$ controls the highest order term. This implies

$$
I[w] \geq \alpha ||\nabla w||_{L^q}^q - \beta |u|
$$

and we'll see how we can deal with that next time.

(2) Lower semi-continuity: Just because we want to attain minimum. The statement is self-stating:

$$
\liminf_{n \to \infty} f(x_n) \geq f(x).
$$

We'll see how to use the compactness of $W^{1,q}$ then extract subsequence to converge to minimizer.

16. 5/16: EXISTENCE AND UNIQUENESS OF MINIMIZER

Define the admissible set

$$\mathcal{A}\left\{w \in W^{1,q}, w = g \text{ on } \partial U\right\}$$

We want to find the minimizing sequence such that

$$m = \inf_{u \in \mathcal{A}} I[w] = \lim_{k\to\infty} I[u_k] \leq m + 1$$

and by Coercivity we have

$$||\nabla u_k||_{L^q} \leq C$$

and Poincare means $||u_k||_{W^{1,q}} \leq C$

So we have $u_k \rightharpoonup u$ in $W^{1,q}(U)$, which since $W^{1,q}(U) \subset\subset L^q$ and thus identity function is compact, and we know $u_k \to u$ in L^q strongly. But do we know $I[u_k] \to I[u]$? The answer is no since of course we have the counter example of $\cos(nx) \rightharpoonup 0$.

But we don't need that at all. Instead we just need

$$m = I[u] \leq \liminf I[u_k] \to m$$

for which we've almost used everything we have, but lower semicontinuous is direct.

Def 16.1. *I is sequentially* <u>*weakly semi-lower continuous*</u> *on $W^{1,q}(U)$ when*

$$I[u] \leq \liminf_k I[u_k]$$

for $u \rightharpoonup u$ in $W^{1,q}(U)$.

Theorem 16.2. *For L smooth and bounded below that is convex in p (for any fixed u, x), then $I = \int_U L$ is weakly lower semicontinuous in $W^{1,q}(U)$.*

Proof. For $u_k \in W^{1,q}$ such that $u_k \rightharpoonup u$ then by uniform boundedness we have u_k is bounded since we can bound u with liminf of u_k, and then by compact plus weak converge implies strong convergence.

So we can define

$$I[u_k] = \int_U L(\nabla u_k, u_k, x)dx$$

and we note that for most part we can pass the limit of $u_k \to u$ if ∇u_k is reasonably bounded. Thus, we do the small and large argument.

To do this we use Egorov theorem:

Theorem 16.3. *(Egorov Theorem) For $f_k \to f$ a.e. on A, a bounded measure set, then $\forall \varepsilon > 0$ there exists $E_\varepsilon \subset A$ such that $|E_\varepsilon - A| \leq \varepsilon$ where $f_k \to k$ uniformly.*

In other words, this set E_ε is a good set in which we can pass limits. Moreover, let

$$F_\varepsilon := \left\{ x \in U \,|\, |u(x)| + |\nabla u(x)| < \frac{1}{\varepsilon} \right\}$$

and thus this is also a good set. So we define $G_\varepsilon := E_\varepsilon \cap F_\varepsilon$ then since E_ε and F_ε both goes to the whole set, the limit of their intersection also goes to full measure. So we have

$$I[u_k] \geq \int_{G_\varepsilon} L(\nabla u_k, u_k, x) dx + \int_{U \setminus G_\varepsilon} L dx$$

yet from our assumption of coercivity we know the second part is bounded below by $\beta |U \setminus G_\varepsilon| \to 0$ so this part goes away.

Now for the first part we can use convexity to get

$$\int_{G_\varepsilon} L(\nabla u_k, u_k, x) dx \geq \int_{G_\varepsilon} L(\nabla u, u_k, x) + \nabla_p L(\nabla u, u_k, x) \cdot (\nabla u_k - \nabla u) dx$$

where the second part, again, goes to 0 because it is a combination of

$$\nabla u_k - \nabla u \rightharpoonup 0$$

and the gradient goes to $\nabla_p L(\nabla u, u, x)$ strongly by our assumption that we are on G_ε. In particular the first strong convergence is in L^∞, hence in L^q, so their product converges strongly. As for the first terms we can just pass the limit and get

$$\liminf_k I[u_k] \geq \int_U \mathbb{1}_{G_\varepsilon} L(\nabla u, u, x) dx$$

then use DCT to pass the limit to get the result. □

Theorem 16.4. *(Mazur's trace theorem) For $u_k \in W^{1,q}$ that $\rightharpoonup u$ and $u_k = g$ on ∂U, then $u = g$ on ∂U, which means $u \in \mathcal{A}$.*

To wrap up the above arguments we conclude that:

Theorem 16.5. *FOr L coercive, convex in p and $\mathcal{A} \neq \emptyset$, then there's at least one u minimizing I over $\mathcal{A}$.*

Proof. Morally just above. To be a little precise Poincare tells us that $||\nabla u_k|| \leq C$ implies $||u_k||_{L^q} \leq C$ some constant. And so

$$m = \liminf_k I[u_k] \geq I[u] \geq m.$$

□

Now we show uniqueness. Note that we require that L is not dependent on u. For an intuition of this we note that in many cases if u is a solution then cu is also a solution (very casually explanation). So we require $L = L(p, x)$. Moreover, strictly convex is the same as elliptic, that $\exists \theta$ such that $\xi^T \nabla_p^2 L \xi \geq \theta |\xi|^2$.

Theorem 16.6. *For $L(p,x)$ coercive and strictly convex, the minimizer is unique.*

Proof. The proof is directly what we'd expect. Given two different minimizer we use strictly convex to show that the average is smaller.

Let $v = \dfrac{u+\tilde{u}}{2}$ then. Note that just Taylor gives us

$$L(p) = L(q) + \nabla_p L(q)\cdot(p-q) + \frac{1}{2}(p-q)^T \nabla_p^2 L(sp+(1-s)q)(p-q)$$

where the second order term is larger than $\frac{1}{2}\theta|p-q|^2$ and by plugging in $p=\nabla u$ and $q=\nabla v$ we get

$$I[u] = \int L(\nabla u, u, x)dx \geq I[v] + \int_U \nabla_p L(\nabla v)\nabla\frac{u-\tilde{u}}{2} + \frac{\theta}{2}\int \left|\frac{\nabla(u-\tilde{u})^2}{2}\right|$$

and by using $p=\tilde{u}$ and $q=v$ we get the symmetry result:

$$I[\tilde{u}] = \int L(\nabla \tilde{u}, \tilde{u}, x)dx \geq I[v] + \int_U \nabla_p L(\nabla v)\nabla\frac{\tilde{u}-u}{2} + \frac{\theta}{2}\int \left|\frac{\nabla(u-\tilde{u})^2}{2}\right|$$

and adding up and averaging we get

$$\frac{1}{2}\left|I[u]+I[\tilde{u}]\right| \geq I[v] + \frac{\theta}{8}\int |\nabla(u-\tilde{u})|^2$$

and we know $I[v]=m$, the minimized value, as well as that $|\nabla(u-\tilde{u})|=0$. This, plus the fact that both agree on the boundary means $u=\tilde{u}$, contradiction thus the solution is unique. □

How is this related to the Euler-Lagrange equation? We remember that the weak form of Euler-Lagrange is

$$\int_U \nabla_p L(\nabla u, u, x)\cdot\nabla_x v + \partial_z L v dx = 0$$

and everything's done except that the term $\nabla_x v$ might not be integrable. So we need to find what set must v be in in order for the above to make sense.

Def 16.7. *$u \in \mathcal{A}$ is a weak solution of E-L with $u=g$ on the boundary when the above weak form holds for $\forall v \in W_0^{1,q}(U)$.*

To do this we also need some assumptions:

- $|L(p,z,x)| \leq c(|p|^q + |z|^q + 1)$. This intrinsically means that even though for coercivity means we need to blow up, we really cannot be blowing up too fast.
- $\left(|\nabla_p L| + |\nabla_z L|\right)(p,z,x) \leq c\left(|p|^{q-1} + |z|^{q-1} + 1\right)$ where $q-1$ appears natually because $q' = \dfrac{q}{q-1}$.

Theorem 16.8. *Under the above assumptions, u is a weak solution of E-L.*

17. 5/18: Calculus of Variations; With constraints

17.1. Relation of minimizer and solution of WEL.

We need to find the weak solution of Euler-Lagrange equation. That is, we want to satisfy

$$\int_U \nabla_p L \cdot \nabla_x v + \partial_z L \cdot v dx = 0$$

for all $v \in W_0^{1,q}$. And the main reason we want v in this space is that we want $u + v \in \mathcal{A}$.

We restate the last theorem from last class and prove it.

Theorem 17.1. *Under the below assumptions, u is a weak solution of E-L:*

(1) $|L(p, z, x)| \le c(|p|^q + |z|^q + 1)$. *This intrinsically means that even though for coercivity means we need to blow up, we really cannot be blowing up too fast.*
(2) $\left(|\nabla_p L| + |\nabla_z L|\right)(p, z, x) \le c\left(|p|^{q-1} + |z|^{q-1} + 1\right)$ *where* $q - 1$ *appears natually because* $q' = \dfrac{q}{q-1}$.

Remark 17.2. *Note that the hardest part is not calculations, but to prove that the expression* i' *makes sense.*

Proof. Since we want to show existence of derivative, we start from quotient: let

$$L^t(x) = \frac{1}{t}\left[L(\nabla u + t\nabla v, u + tv, x) - L(\nabla u, u, x)\right]$$

and under which notation we get

$$\frac{i(t) - i(0)}{t} = \frac{I[u + tv] - I[u]}{t} = \int_U L^t(x) dx.$$

Since we are trying to find the minimum of i so if $i(s) = \infty$ then we really don't care, and thus we need conditions that make $i(t)$ finite. Condition (1) helps us here since we have

$$L^t \le C(|\nabla u + t\nabla v|^q + |u + tv|^q + 1) + C'(|\nabla u|^q + |u|^q + 1)$$

and thus $L^t \in L^1(U)$ as $u, v \in W^{1,q}$. So good the quotient is defined as the integral on the right hand side. Now we want to show that $L^t(x)$ is a nice guy as $t \to 0$ so we write out and see what it is: since L is differentiable we have

$$\begin{aligned} L^t(x) &= \frac{1}{t}\int_0^t \frac{d}{ds}\left[L(\nabla u + s\nabla v, u + sv, x)\right] ds \\ &= \frac{1}{t}\int_0^t \left[|\nabla_p L \cdot \nabla v| + |\partial_z L v|\right](\nabla u + s\nabla v, u + sv, x) ds \end{aligned}$$

and we really need this to be dominated that we can pass in the limit of t. So for the first term we have

$$|\nabla_p L \cdot \nabla v| \overset{Young's}{\le} \frac{1}{q'}|\nabla_p L|^{q'} + \frac{1}{q}|\nabla v|^q =: g_1(x) \in L^1(U)$$

where the first terms is L^1 by assumption (2).

Moreover, the second is L^1 also by assumption (2). So we just bound it by $g_1 + |\partial_z Lv|$ and then by DCT we can pass the limit and note that by doing so we are pushing the limit

$$\int_{\mathbb{R}} \lim_{t\to 0} \mathbb{1}_{[0,t]} \cdot ds$$

and thus we have that $\lim_{t\to 0} L^t(x)$ exists and is equal to

$$\lim_{t\to 0} L^t(x) = \nabla_p L(\nabla u, u, x)\nabla_x v + \partial_z Lv$$

thus

$$i'(0) = \int_U \nabla_p L(\nabla u, u, x)\nabla_x v + \partial_z Lv dx$$

where in particular it exists. Since 0 is a minimizer and the derivative exists, it is 0 and hence u is indeed a weak solution of Euler Lagrange. □

17.2. **Regularity theory.**

We just state theorem here. Let $L(p, z, x) = L(p) - f(x)z$, i.e. if $L(p) = \frac{1}{2}p^T Ap$, then the above is $-\nabla A\nabla u = f$, then here the E.L. is

$$-\nabla\nabla_p^2 L\nabla u = f$$

where the Hessian play the role of A and provides ellipticity.

Theorem 17.3. *If u is a W.E.L solution, then $u \in H^2_{loc}(U)$.*

The idea of the proof is the same as before, with the cutoff annd finite difference, etc.

Remark 17.4. *Note that there's no drift terms here, and it's not easy to add a drift term $b\cdot\nabla u$ is hard to remove.*

17.3. **Minimization under constraints.**

Suppose $L(p) = \frac{1}{2}|p|^2$ then $I[u] = \frac{1}{2}\int_U |\nabla u|^2 dx$. Note that we are choosing $q = 2$ here since why not. Then we want to minimize under

$$\mathcal{A} = \left\{u \in H^1_0(U); J[u] = 0\right\}$$

where

$$\mathbb{R} \ni J[u] = \int_U G(u(x))dx$$

(for example $G(u) = 1-u^2$, this is a normalization of $|U|$ since $J[u] = 0$ means $|U| = \int_U u^2$). And as one can see the above makes it an eigenvalue problem.

Theorem 17.5. *For $\mathcal{A} \neq \emptyset$ and $\mathcal{A}$, J defined as above. If $g = G'$ and*

$$|g(z)| \leq C(1 + |z|)$$

(which alternatively means $|G| \leq C(1 + |z|^2)$), then $\exists u \in \mathcal{A}$ such that

$$I[u] = \min_{w \in \mathcal{A}} I[w].$$

Proof. Let $u_k \in \mathcal{A}$ be a minimizing sequence such that $I[u_k] \downarrow m$, then

$$\frac{1}{2}||\nabla u_k||_{L^2}^2 \leq m + 1$$

which in tern by Poincare implies $||u_k||_{H_0^1(U)} \leq C$.

Now since for some u $u_k \rightharpoonup u$ in H^1, we know $u_k \to u$ strongly in L^2 (compact embedding).

If $u \in \mathcal{A}$ then we are done, but we does not know that yet. So we investigate and see what really is $J[u] = 0$. If $J[u] = 0$ means $||u||_2 = 1$, then we are done in this case since the norm cannot escape.

Now in general

$$|J[u]| \leq |J[u] - J[u_k]| \leq \int_U |G(u) - G(u_k)| dx \leq \int_u \int_0^1 |g(su + (1-s)u_k)(u - u_k)| ds dx$$
$$\leq \int_U C(1 + |u| + |u_k|)|u - u_k| dx \to 0$$

since $|u - u_k| \to 0$. Thus $J[u] = 0$ by limit continuity. $\square$

Now let's see how to link this with Lagrange multiplier.

Theorem 17.6. *Let u be as above, then $\exists \lambda \in \mathbb{R}$ such that*

$$\int_U \nabla u \cdot \nabla v dx = \lambda \int g(u) v dx$$

and also $J[u] = \int_U G(u) dx = 0$.

Note that the condition in above theorem is a weak form of

$$\begin{cases} -\Delta u = \lambda g(u) & x \in U \\ u = 0 & \partial U \end{cases}$$

and if $g = u$ we are really dealing with a eigenvalue problem.

Proof. Assume first that $g(u) = 0$ a.e.:

Then

$$\nabla_x G(u) = g(u) \nabla u = 0 \Rightarrow G(u) = C$$

and $J = \int_U G dx = 0 \Rightarrow G = 0$. So $G(0) = 0$ and thus if $u \equiv 0$, then $G(u) = 0$ and thus $J[u] = 0$ so $u \equiv 0$ is in the admissible set. Thus $\lambda \in \mathbb{R}$ is arbitrary and the required statement holds.

But this is sort of obvious because 0 is the most obvious eigenvalue of problem.

<u>Now, assume $g(u) \neq 0$ on set of positive measure:</u>

After skipping arguments on boundary measure and etc, we can choose $w \in H_0^1(U)$ such that

$$\int_U g(u) w dx \neq 0.$$

So we let $v \in H_0^1(U)$ and $u \in \mathcal{A}$. Then if we consider the set that make $J[u] = 0$, and if we add a little perturbation in the direction of v with length t, then we do not have that in general $u + tv \in \mathcal{A}$. So instead of wanting that we want something different, i.e. with 2 perturbations except we can use implicit function theorem to modify the second and make us stay inside $\mathcal{A}$, the right manifold. Making precise we have

$$j(t, s) := J[u + tv + sw] = \int_U G(u + tv + sw) dx.$$

and thus we want to find $s = s(t)$ such that $J[\cdot] = 0$. Since $j(0, 0) = 0$ we note that

$$\partial_s j(0, 0) = \int wG'(u) dx \neq 0$$

and this corresponds to that the determinant is not 0 in IFT.

So there exists $\phi(t)$ such that $j(t, \phi(t)) = 0$ for all small t. And computation yields

$$\frac{d}{dt} j(t, \phi(t)) = \partial_t j + \phi' \partial_s j = 0$$

and plug in $t = 0$ we have

$$\phi'(0) = \frac{-\int g(u) v ds}{\int g(u) w dx}$$

thus everything is well defined.

If we define $i(t) := j(t, \phi(t))$ then $I[u] = i(0)$ is minimal at u implies $i'(0) = 0$. So we only need to compute

$$\begin{aligned} i'(0) &= \int \nabla u (\nabla v + \phi'(0) \nabla w) dx \\ &= \int \nabla u \left(\nabla v - \frac{\int g(u) v ds}{\int g(u) w dx} \nabla w \right) dx \end{aligned}$$

and thus $\lambda = \dfrac{\int g(u) v ds}{\int g(u) w dx}$. $\square$

There's 3 examples in the notes and we only focus on the last one here. So far, $J[u] = 0$ there's one constraints, and we want to make it infinite numbers of constraints (pointwise constraint).

Example 17.7. *Stokes's problem*

We think of water, which is very hard to compress, so we have the inconpressibility constraint: $\nabla \cdot w = 0$, and the variation is

$$I[w] = \int_U \frac{1}{2}|\nabla w|^2 - f w dx.$$

The admissible set is

$$A := \left\{ w \in H_0^1(U; \mathbb{R}^3), \nabla \cdot w = 0 \right\}$$

the results are:

- Existence: $u_k \rightharpoonup u$.
- Uniqueness: $w = \dfrac{1}{2}(u + \tilde{u})$.

And the E.L. theorem:

Theorem 17.8. *There exists scalar function $p \in L^2_{loc}(U)$ such that W.E.L. holds:*

$$\begin{cases} -\Delta u = f - \nabla p & U \\ \nabla u = 0 & U \\ u = 0 & \partial U. \end{cases}$$

APPENDIX A. A

APPENDIX B. B

APPENDIX C. C

Acknowledgements.