APPLIED FUNCTIONAL ANALYSIS HOMEWORK 1

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Discussed with classmates.

Exercise 1. *(1.4) in book*

Proof.

Property (a): $d(x, y) \geq 0, \forall x, y \in X$ and $d(x, y) = 0 \iff x = y$.

From definition we know that

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2) \geq 0$$

by the property (a) of the metric on X and Y.

For the later half, if $d((x_1, y_1), (x_2, y_2)) = 0$, then again since both summand are non-negative they are both 0, in which case $x_1 = x_2$ and $y_1 = y_2$, by property (a) of the metric on X and Y.

If $(x_1, y_1) = (x_2, y_2)$, then $d((x_1, y_1), (x_2, y_2)) = 0$ follows by definition of the L^1 metric and property (a) of the metric on X and Y.

Property (b): $d(x, y) = d(y, x)$.

By definition we have

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2) = d_X(x_2, x_1) + d_Y(y_2, y_1) = d((x_2, y_2), (x_1, y_1))$$

since d_X and d_Y are also symmetric.

Property (c): $d(x, z) \leq d(x, y) + d(y, z)$.

Using the triangle inequality of d_X and d_Y we get

$$\begin{aligned} d((x_1, y_1), (x_3, y_3)) &= d_X(x_1, x_3) + d_Y(y_1, y_3) \\ &\leq \left(d_X(x_1, x_2) + d_X(x_2, x_3)\right) + \left(d_Y(y_1, y_2) + d_Y(y_2, y_3)\right) \\ &= d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)) \end{aligned}$$

the triangle inequality of d. $\square$

Exercise 2. *(1.12) in book.*

Proof. To show that h is continuous, we only need to show that for all open set $U \in \mathcal{O}_Z$ in the topology of Z, it's pre-image in X is also open.

Using f^{-1}, g^{-1}, h^{-1} to stand for pre-image but not the inverse (since might not a function), we get that

$$h^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(V)$$

where $V \in \mathcal{O}_Y$ by continuity of g and $f^{-1}(V) \in \mathcal{O}_X$ by continuity of f, hence we are done. $\square$

Exercise 3. *(1.15) in book.*

Proof. **Every compact subset of a metric space is closed and bounded:**

Theorem 1.62 tells us that every subset of a metric space is compact iff it is sequentially compact, and Theorem 1.59 says that every subset of a metric space is sequentially compact iff it is complete and totally bounded.

Combining them, we get that every compact subset of a metric space is complete and totally bounded.

bounded:

If a subset if totally bounded, it automatically says that it is bounded since for any $\varepsilon > 0$, we can find a finite ε-cover of the set. Then, $\forall x, y \in X$, let V_x and V_y be covers in the finite cover that contains x and y. Then

$$d(x, y) \leq d(x, v_x) + d(v_x, v_y) + d(v_y, y)$$

where v_x and v_y are centers of V_x and V_y. Since the cover is finite $\max_{\forall center} d(v_x, v_y) < \infty$, hence

$$\sup_{x,y} d(x, y) \leq \infty$$

and the subset is bounded.

closed:

So we only need to show that it is closed. But by contradiction if we assume that it is not closed, then $X \backslash S$ (S is the subset) is not open, so $\exists z \in X \backslash S$ such that $\forall \delta > 0$, $\exists z_\delta \in S$ such that $z_\delta \in B_\delta(z)$. Taking $\delta = \dfrac{1}{n}$ we can construct a sequence $(z_n) \to z$ such that $\forall n, z_n \in S$.

But then (z_n) does not have any convergent subsequence in S since if it has, it must be the limit $z \notin S$. Hence S is not sequentially compact, hence not compact by theorem 1.62, contradiction! So S is closed.

In conclusion, S is closed and bounded.

A closed subset of a compact space is compact:

Let X be the compact space and S be a closed subset. $\forall$ open cover C of S. But by definition of induced topology (I found it under example 4.4 in textbook), every open set in a sub topology is the intersection of some open set in the orginal topology and the subset. That is, $\forall O \in C, \exists O' \in \mathcal{O}_X$ such that $O = O' \cap S$. Adopt Axiom of Choice we can find the choice function $f : C \to \mathcal{O}_X$ such that $f(O) \cap S = O$. Let $C' := f(C)$.

Now $C' \cup X \backslash S$ is an open cover for X, which can be reduced to a finite cover $\mathcal{D}'$ of X. If $X \backslash S$ is in the new cover, we subtract it; if not we don't change the finite cover. Denote this processed cover $\mathcal{D}''$. Now let $\mathcal{D} = \{U = U'' \cap S | U'' \in \mathcal{D}''\}$ we obtain a finite subcover $\mathcal{D}$ of C of S. Hence, S is compact. $\square$

Exercise 4. *(1.16) in book.*

Proof. Just use the function in the hint and check the three conditions are satisfied. The function is

$$f(x) = \frac{d(x, G^c)}{d(x, G^c) + d(x, F)}.$$

(a) $0 \leq f(x) \leq 1$:

Since $G^c \cap F = \emptyset$, the 2 terms in the denominator cannot both be 0, so the function is well defined for all $\mathbb{R}^n$.

Since $d(x, G^c) \geq 0$ and $d(x, F) + d(x, G^c) \geq 0$, $f(x) \geq 0$;

since $d(x, F) + d(x, G^c) \geq d(x, G^c) \geq 0$, $f(x) \leq 1$.

(b) $f(x) = 1$ for $x \in F$:

When $x \in F$, $d(x, F) = 0$ and $d(x, G^c) \neq 0$, so $f(x) = \dfrac{d(x, G^c)}{d(x, G^c)} = 1$.

(c) $f(x) = 0$ for $x \in G^c$:

When $x \in G^c$, $d(x, G^c) = 0$ so $f(x) = \dfrac{d(x, G^c)}{d(x, G^c) + d(x, F)} = 0$.

□

Exercise 5. *(1.20) in book.*

Proof.

Banach ⇒ every absolute convergent series converge:

For $\sum x_n$ absolute convergent, we use the fact that convergent implies Cauchy to get that $\forall \varepsilon$, $\exists N$ such that

$$\sum_{k=n>N}^{m} ||x_k|| < \varepsilon$$

for all $m \geq n \geq N$.

By triangular inequality we get

$$d\left(\sum_{i=1}^{m} x_i, \sum_{j=1}^{n} x_j\right) = \left\|\sum_{i=1}^{m} x_i - \sum_{j=1}^{n} x_j\right\| = \left\|\sum_{k=n>N}^{m} x_k\right\| \leq \sum_{k=n>N}^{m} ||x_k|| < \varepsilon$$

which means that $\sum x_i$ is Cauchy. But the space is Banach, which means it is complete with respect to the norm metric, which is the one we're using, so it converges.

Banach ⇐ every absolute convergent series converge:

For the purpose of contradiction, assume that the space is not Banach, that is, there exists a sequence (y_n) that is Cauchy but does not have a limit in the space.

Now since y_n behaves pretty bad let's just pick a subsequence of it, call it z_n, such that for $x_n = z_n - z_{n-1}$ with $x_1 = 0$ we have $||x_n|| \leq \frac{1}{2^n}$. This is possible since y_n is Cauchy and we can just let $\varepsilon = \frac{1}{2^n}$ each time and find the corresponding N, then let $z_n = y_{N+1}$.

Now by our construction, $\sum ||x_n||$ converges and thus $\sum x_n$ converges to some point x. Hence $z_n = z_1 + \sum x_n = z_0 + x := y$ by our construction. So a subsequence of y_n has a limit y, and by Cauchy property we know that $\forall \varepsilon, \exists N$ such that $\forall m \geq N$ we have $||y_m - y|| < \varepsilon$, which means that it converges. In other words, y_n has a limit in the space, contradiction! So the space is indeed Banach. $\square$

Exercise 6. *(1.27) in book.*

Proof. For the purpose of contradiction, we assume that the sequence does not converge to x. This means by definition that $\exists \varepsilon > 0$ such that there are infinitely many x_is out side of the ball $B_\varepsilon(x)$. Rename these infinite points (must be countable since it's subset of a sequence) as y_i, with the original order.

Then, since the space is compact metric space, it is sequentially compact and hence y_i has a converging subsequence (z_i). But the limit of (z_i) is x by assumption, yet this cannot be true since none of the element in (z_i) is in the ball $B_\varepsilon(x)$. Contradiction! So $x_n \to x$ as $n \to \infty$. $\square$