SET THEORY

ABSTRACT. We'll cover some infinitary combinatorics, then maybe ultrafilters.

CONTENTS

1. 9/26: ALMOST DISJOINT FAMILY

We sometimes consider questions of consistency, and how we build models. For instance, models with CH or that without.

Def 1.1. *For κ an infinite cardinal, $A, B \leq \kappa$, we say A, B are almost disjoint if $|A \cap B| < \kappa$.*

If $\kappa = \aleph_0$ then $A \cap B$ would be finite.

Def 1.2. *An almost disjoint family is $f \subset \mathscr{P}(\kappa)$ such that for all $A \in f$, $|A| = \kappa$ and for all $A, B \in f$, A, B are almost disjoint.*

The only similarity of almost disjoint families and disjoint families are their names. Let's consider how they can differ each other.

Example 1.3. *Disjoint family of $\mathbb{N}$.*

We know there are countable ones by arranging $\mathbb{N}$ as $\mathbb{N} \times \mathbb{N}$. On the other hand, we know each number is mapped to some set or not, and since sets are disjoint we've found an injection from a subset of $\mathbb{N}$ to the disjoint family, so at most countable.

Another way to prove this is using regularity of $\aleph_1$.

Def 1.4. *A cardinal λ is regular if we can divide it into $< \lambda$ pieces, and at least one of the pieces will have size λ.*

Note that this is the pigeon-hole principle. Now, to show there's no uncountable disjoint family of $\mathbb{N}$ we suppose there is one family of $\aleph_1$, call it $\{f_\alpha : \alpha < \aleph_1\}$. Since every set in the family is infinite, therefore non-empty, and there's only countably many of them, we can choose $n_\alpha \in f_\alpha$ and then partition the f_α according to the choice of n. Then one of the partition must have uncountably many elements, in particular 2. So the family will not be disjoint. Contradiction.

For a review of cardinal arithmetics, we have $\kappa + \lambda$ is the cardinality of $A \cup B$ for A, B disjoint with $|A| = \kappa, |B| = \lambda$. Product is direct product, and

$$\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}.$$

Now one example of a non-regular set is $\aleph_\omega$, which is a countable union of lesser cardinals, by definition. Note that if a cardinal is the successor of some cardinal, then it's regular.

Def 1.5. *For $(w, <)$ and $(I, <)$ with the same ordering, if for each $x \in w$ we can always find $y \in I$ such that $y > x$, then we say they are cofinal if $|w| = |I|$.*

Now we ask about almost disjoint families. Can there be a countable almost disjoint family? Well, of course. A more interesting question is: Can there be a countable almost disjoint **maximal** family?

Remember that a maximal family is not strictly contained in other almost disjoint families.

Proposition 1.6. *There is non countable almost disjoint **maximal** family of $\mathbb{N}$.*

Proof. Suppose $f = \{A_n : n < \omega\}$ is a countable maximal almost disjoint family, then for each n, we can always denote the "new part" of A_n, namely

$$B_n := A_n - \bigcup_{m<n} A_m$$

which is countable since A_n countable, and we're deleting finite elements.

Now B_n is a partition of $\mathbb{N}$ into countably many disjoint countable pieces. We can just pick $C \subset \mathbb{N}$ by picking one element from each B_n (countable, so doable), thus forming a diagonal argument. That is, $|C \cap A_n| \leq n$ by definition, and hence $f \cap \{C\}$ would be a larger almost disjoint family, which contradicts maximal. □

Q: Why can't we just add one to it until we stop?

A: First, we are only disproving maximal **countable** family. But what may be the largest cardinal of this iterative process? Well, say it's m, then $\aleph_1 \leq m \leq 2^{\aleph_0}$, where if we assume CH we know the exact cardinal.

We can replace $\aleph_0$ by κ regular, and note that in this case

$$B_\alpha := A_\alpha - \bigcup_{\beta<\kappa} A_\beta$$

we have the union is size less than κ by regularity.

But how will we prove this for $\aleph_0$? We use a binary tree to show an almost disjoint family of size $2^{\aleph_0}$. Writing $\aleph_0$ as a binary tree we take each branch as one set in f, then

$$2^{\aleph_0} = |\text{countable binary sequence}| = |\mathscr{P}(\mathbb{N})|$$

so we have a continuum many of branches. For each $\eta \in {}^{\mathbb{N}}\{0, 1\}$ let A_η be the nods along this branch, then they are an almost disjoint family.

Here notation is $3^5 = |{}^5 3|$.

What do we need for this proof to be generalized? We need

(1) The number of branches has $2^{\aleph_0}$ size.
(2) The number of internal nodes has size $\aleph_0$.

How do we compute the second number? We use

$$2^{<\omega} = \sum_{n<\omega} 2^n$$

which is a sum of cardinals, and thus if

$$2^{<\kappa} = \kappa$$

then a similar argument will work.

2. 9/28: FILTER, MARTIN'S AXIOM

We start with the Δ system lemma:

Def 2.1. *$\mathcal{G}$ is a Δ system if the elements of $\mathcal{G}$ all have some common intersection or are disjoint.*

Lemma 2.2. *If $\mathcal{F}$ is any uncountable family of finite sets, then there is an uncountable $\mathcal{G} \subset \mathcal{F}$ which forms a Δ system.*

Note that this is analogous to Ramsay's theorem, in a sense.

Proof. Since what we want to do is just finding an uncountable set, we may as well assume $|\mathcal{F}| = \aleph_1$ by discarding the extra ones. Now, we can use the fact that $\aleph_1$ is regular. Since all element in $\mathcal{F}$ is finite, we can partition them by their size, which are finite, and since $\mathcal{F}$ is regular we can use one of the countable partitions which have uncountable elements, and discard all else in $\mathcal{F}$.

So we have reduced our family $\mathcal{F}$ to an uncountable family of sets with n elements. Now we do an induction on n.

- Base case $n = 1$: In this case, either $|\cup \mathcal{F}| = \aleph_1$ or $|\cup \mathcal{F}| = \aleph_0$. If $|\cup \mathcal{F}| = \aleph_1$ then by AC we can find for each element a set containing it, where, since each set is order 1, we are done as non-intersects.

  If $|\cup \mathcal{F}| = \aleph_0$, then there exist an element that occurs in uncountably many sets by regularity. Then we just pick those uncountable sets as the Δ system.
- When $n = m+1$: If $|\cup\mathcal{F}| = \aleph_1$ and $\exists a \in \cup\mathcal{F}$ such that a is contained in uncountably many sets. In this case we discard all that doesn't contain a, then consider all sets $-\{a\}$, and we're back to $n = m$ so we can find a Δ system.

  If $|\cup\mathcal{F}| = \aleph_1$ and does not $\exists a \in \cup\mathcal{F}$ such that a is contained in uncountably many sets. Then we just ignore one element in each set by AC, then by IH we can find an uncountable Δ system on the un-ignored ones. Now for each set we find another point (can always do by condition) and do the same with the un-neglected ones. Then the only possible contradiction lies with in the two neglected points in each set, so we do for it too and hence we get a Δ system.

  If $|\cup \mathcal{F}| = \aleph_0$ then $\exists a \in \cup\mathcal{F}$ such that a is contained in uncountably many sets. In this case we use the same argument above.

□

2.1. **Martin's Axiom.**

So we ask the question of whether CH fails, i.e. $\aleph_1 < 2^{\aleph_0}$. Moreover, if it holds, then there must be some sense in which $\aleph_1$ looks like $\aleph_0$.

Def 2.3. *A* <u>*partial order*</u> *is a pair* $(\mathbb{P}, \leq)$ *where* $\mathbb{P}$ *is non-empty and* $\leq$ *is a relation that's transitive and reflexive.*

Here in our situation, "$q \leq q$*" is pronounced* <u>*"p extends q"*</u> *and elements of* $\mathbb{P}$ *are called* <u>*conditions*</u>*.*

For instance, we can use $\mathbb{P} = \mathscr{P}(\kappa)$ and $\leq$ is the $\subset$ order, and $p \leq q$ means p has more information.

Def 2.4. *In a partial order* $\mathbb{P}$*, say* $D \subset \mathbb{P}$ *is* <u>*dense*</u> *if* $\forall p \in \mathbb{P}$*,* $\exists q \leq p$ *such that* $q \in D$*.*

Def 2.5. $G \subset \mathbb{P}$ *is a* <u>*filter*</u> *if*

- $\forall p, q \in G$*,* $\exists r \in G$ *such that* $r \leq p$ *and* $r \leq q$*.*
- $\forall q \in G$*,* $\forall p \in \mathbb{P}$*, then* $q \leq p \Rightarrow p \in G$*.*

Def 2.6. <u>$MA(\kappa)$*, Martin's Axiom for* κ</u> *is the statement: whenever* $\mathbb{P}$ *is a non-empty ccc partial order, and* D *is a family of no more than* κ *dense sets, there is a filter* G *on* $\mathbb{P}$ *which meets all of them, i.e.* $G \cap D_\alpha \neq \emptyset$ *for each* $D_\alpha \in D$*.*

And <u>*Martin's Axiom*</u> *is:* $MA(\kappa)$ *for all* $\kappa < 2^{\aleph_0}$*.*

For both the definitions above and below, it's good to keep in mind the following example:

Example 2.7. $\mathbb{P}$ *is the set of* <u>*finite partial functions*</u> *from* $\mathbb{N} \to \{0, 1\}$*, i.e. if the domain of* p *is* $\{2, 3, 17\}$ *then* p *can be*

$$p(2) = 0; \quad p(3) = 1; \quad p(17) = 0.$$

In this case, if we want to find a q with $q \leq p$ then we can think of $\mathrm{dom}(q) \geq \mathrm{dom}(p)$ such that they agree on their common area.

Def 2.8. *In a partial order,* p, q *are* <u>*compatible*</u> *if there's* r *such that* $r \leq p$ *and* $r \leq q$*, and are* <u>*incompatible*</u> *otherwise.*

Def 2.9. *An* <u>*antichain*</u> *is a set of pairwise incompatible elements.*

Def 2.10. *A set is* <u>*ccc*</u> *if every antichain is countable.*

The idea is that pairwise incompatible is not that easy. Imagine if p and q differ at point 3, q and r differs also at 3, then p and r cannot differ at 3.

To think about filters, we note that for our partial function case, if $G \subset \mathbb{P}$ is a filter, then $\cup G$ is a function which we call g.

Then we think of dense sets and MA: suppose $n \in \mathbb{N}$, then

$$D_n := \{p \in \mathbb{P} : n \in \mathrm{dom}\, p\}$$

is dense because for any point in $\mathbb{P}$ either n is in its domain or we can add that in.

Thus, if MA is correct, then there is a filter that intersects D_n for all n, hence $\cup G$ has full domain.

Moreover, note that

$$D_{\text{non-constant}} := \{p : p\text{non-constant}\}$$

is also a dense set since we're dealing with finite partial functions. Then we know $\cup G$ is non-constant.

Proposition 2.11. *$MA(\kappa)$ is not true for $\kappa \geq 2^{\aleph_0}$.*

Proof. Suppose it holds, then we know $\cup G$ is a function with full domain. But note that

$$D := \{p | p \text{ contradict } f\}$$

for any f is dense, and there's at most ${}^{\mathbb{N}}\{0, 1\}$ many of such which means we're having only $2^{\aleph}_0$ many dense sets, so we cannot find any $\cup G$ at all, contradiction. $\square$

Proposition 2.12. *MA($\aleph_0$) is true.*

Proof. We use induction. $D = \{D_n : n < \omega\}$ is any countable family of dense set, and let p_0 be arbitrary since $\mathbb{P} \neq \emptyset$. Now, let p_{n+1} be such that $p_{n+1} \leq p_n$ such that $p_{n+1} \in D_n$ so we get a decreasing sequence $p_0 \geq p_1 \geq p_2 \geq \dots$ Now look at the filter generated by upward closure. $\square$

3. 10/3: Finding almost disjoint set

One question is that we seems to not have used ccc in the proof for $MA(\aleph_0)$. So let's see an example of how the absence of ccc might cause trouble. consider P : the set of finite functions $\mathbb{N} \to \omega_1$, for instance, f might map $\{0, 1, 3\} \mapsto \{\omega, \omega + 1, \omega + \omega\}$. In this case we have an uncountable antichain that just maps 0 to any ω_1.

It's not hard to see that it is an antichain, also it's not hard to see a dense set

$$\{f(a) = \alpha, \alpha < \omega_1 \quad \text{for some } a \in \mathbb{N}\}$$

because for all finite functions either it is in it or we can extend. Now assume we have $MA(\aleph_1)$ here then that means we can find the family $D_{\mathbb{N}}$ in the above way then there is a filter G that meets all of them, in particular we have f which maps a subset of $\mathbb{N}$ onto ω_1, which is impossible.

Now we might ask what is really something special for countable sets such that it is MA? The answer could just be that they are MA!

Now we go back to almost disjoint sets. We ask the following question:

Suppose $A \subset \mathscr{P}(\mathbb{N})$ is some family of countable subsets, can we build a new subset of ω which is almost disjoint from each of them.

One observation is that if we have $\mathbb{N} \in A$ then we just cannot. But first let's go through some sketch, then fill in the details.

First, we try to use the setup of MA. The idea is that given a finite set $s \subset \mathbb{N}$ and finitely many $\{X_1, \dots, X_n\} := S \subset A$ then my partial order is on set (s, S), where it must be satisfied that the set I'm choosing will have intersection with $X \in S$ on no other place than s, and it could be the empty set.

Now we pose the partial order on the sets such that $(t, T) \leq (s, S)$ means $s \subset t$, $S \subset T$ and for all set $X \in S$ we still have $X \cap t \subset s$. In other words, we are adding more X to S, but we cannot intersect the new sets at places that are covered by sets in S, except s.

The idea now is that with MA we can find a filter and through which we can find the union of all first coordinates, which will be almost disjoint from all $X \in A$.

Theorem 3.1. *If $MA(\kappa)$ holds and if $A \subset \mathscr{P}(\mathbb{N})$ and $|A| \leq \kappa$, plus that for all finite $S \subset A$ we have*

$$|\mathbb{N} \setminus \cup S| = \aleph_0$$

then there is an infinite $Y \subset \mathbb{N}$ which is almost disjoint from all $X \in A$. In particular, there is (assuming $MA(\kappa)$) no maximal almost disjoint family of subsets of $\mathbb{N}$ of size κ.

Actually the condition in theorem says that we cannot have sets like $\mathbb{N}$ or evens and odds. Without those obviously wrong cases, we can always do it.

Proof. We still use the partial order that we've just defined. What will be our choice of dense sets? It will be, for each $X \in A$ we define

$$D_X := \{(s, S) : X \in S\}$$

that is, the intersection with X is already fixed.

An observation here is that if G is a filter which meets every D_x then $Y_G \cap X$ is finite (hence almost disjoint) since it is contained in some finite set. Here,

$$Y_G := \bigcup \{s : \exists S : (s, S) \in G\}.$$

$\square$

4. 10/5: PROOF OF LAST THEOREM; ULTRAFILTERS

We prove thoerem 3.1 here.

Proof. Define the partial order P with $(t, T) \leq (s, S)$ if $s \subset t, S \subset T$ and $t - s$ does not include any elements in $\cup S$.

ccc: For two elements to contradict it means that there is no common extension. To show it is ccc we suppose we have an uncountable $\{(s_\beta, S_\beta) : \beta < \omega_1\}$ that is pairwise incompatible. But note (s, S) and (s, S') are not contradictory, and there's only countably many finite subsets of A. So it is ccc.

Now we find our dense sets to construct our filter with MA. We want our set to be almost disjoint and we want it to be infinite. So we find

$$D_\alpha := \{(s, S) : A_\alpha \in S\}$$

and

$$D^n := \{(s, S) : |s| \geq n\}$$

then our filter satisfies the objective. Now for the filter G we take $B := Y_G = \cup\{s : (s, S) \in G\}$, and we are done finding an element that is almost disjoint with A. $\square$

Def 4.1. *For X a set, an ultrafilter on X is $D \subset (X)$ with*

- $X \in D$;
- $A \in D, B \in D \Rightarrow A \cap B \in D$;
- $A \in D \iff X - A \notin D$.

Note that we're only defining it on a particular partial order, the inclusion order. Also note that an ultrafilter is a filter.

Def 4.2. *$f \subset \mathscr{P}(X)$ has finite intersection property(FIP) if the intersection of any finitely many elements of f is non-empty.*

For an example, the cofinite sets are FIP.

Theorem 4.3. *If f has FIP then we can extend it to an ultrafilter.*

The key observation is that suppose f has FIP and $Y \subset X$, then either $f \cup \{Y\}$ or $f \cup \{X - Y\}$ has FIP.

Reason: assume both not FIP, then there is a finite collection of sets whose intersection is outside of Y, but also there's a finite collection of sets whose intersection is outside of $X - Y$. Hence putting things from both collections together we contradict FIP of f.

Now we talk on why we might want to build ultrafilters by induction rather than using Zorn's lemma.

Def 4.4. *An ultrafilter on X with $|X| = \lambda$ is* <u>*good*</u> *if for every function $f : [\lambda]^{<\aleph_0} \to D$, f has a multiplicative refinement $g : [\lambda]^{<\aleph_0} \to D$ such that for any finite $u, v \leq \lambda$, $g(u) \cap g(v) = g(u \vee v)$.*

A <u>*multiplicative refinement*</u> *is such that $\forall$ finite $|u| \leq \lambda$, $g(u) \subset f(u)$.*

5. 10/10: ZFC Axioms

Today we revisit the ZFC Axioms. The idea is we form a model such that we build everything from scratch. It's incredible how much we can build from these little confinements.

(1) Set existence: $\exists x(x = x)$.
This is mainly just to say there is.
(2) Extensionality: sets are determined by their elements:

$$\forall x \forall y \forall z(x \in x \iff z \in y) \to x = y.$$

(3) : Pairing: we can pair two elements:

$$\forall x \forall y \exists z((x \in z \wedge y \in z) \wedge \forall w(w \in z \to w = y \vee w = x)).$$

(4) : Axiom of schema of separation: if P is a property, X is a set, then the set "elements in X with property P" is a set.
Note that this is just a schema, and property P is really irrelevant of X, so we cannot have the russell set. In fact, this schema consists of countably many constraints.
More specifically, for every formula $\phi(x, y)$, we have

$$\forall a \forall X \exists Y(Y = \{x \in X : \phi(x, a)\}).$$

An example here is that we get $A - B$ if we let the property be "not in B". If we let it be "in B" we get intersection. One might think we can construct ordinals, but actually we can't even construct a set with more than 3 elements!
A natural thing to define here is an ordered pair, which is a set such that we can view as an abbreviation

$$(a, b) = \{\{a\}, \{a, b\}\}$$

. We can check that this indeed is an ordered pair, but take note of the case $a = b$.
Similarly we can define ordered triple. But we can just define what is a triple while not having the existence.
(5) Union: if X is a set then $\cup X$ is a set:

$$\forall x \exists y(y = \cup X).$$

(6) Power set: $\mathscr{P}(X)$ exists.
Note here we have abbreviated something: $X \subset Y$ means $\forall z(z \in x \to z \in Y)$ and powerset actually means:

$$\forall x \exists y(z \in Y \iff z \subset X)$$

and here is when we can really get that ordered multiples exist.
(7) Infinity: $\exists$ an infinite set:

$$\exists x(0 \in X \wedge (\forall y \in x \to y + 1 \in x)$$

where the successor is defined to be $y + 1 = y \cup \{y\}$ and we have it by pairing and union.

Note that we can extract that something exists from this.

(8) Replacement Schema: the range of a definable function $\phi(x, y, a)$ applied to a set is a set, i.e. let f be a definable function, the $\forall X, \exists Y = F(X)$.

One can argue that this might "replace" the schema of separation. On way we can do this is that if $\phi(x, a)$ then we map x to itself and if the property does not hold we map the element to an element outside, namely b. Then we intersect with A.

(9) Foundation/Regularity: Every non-empty set has an $\in$ minimal set.

(10) Choice: Every family of non-empty sets has a choice function.

Two questions we might ask myself:

- Does there exist a model in which every thing but the infinity axiom holds?

  Note that the answer is not trivial at all since we can have $\{\omega, 1\}$ then take union we are in trouble.
- Is there any good reason ZFC is infinite? Can we maybe use only finitely many of them?

6. 10/12: CONSTRUCTABLE SETS

In the back of our mind we are always thinking of "if there $\exists$ a model then there exists a model of this kind".

Now, Given some model of set theory, we want to produce a minimalist model.

For X a set, X' is defined as the union of X and the sets such that they consist of all definable subsets of X by ZFC.

First it's obvious that any finite set is in X' since for instance $\{x : x \in \mathbb{N} \wedge x = 1 \vee x = 2\} = \{1, 2\}$ is in the set.

Def 6.1. *For any ordinal α, define $M_0 = \emptyset$, $M_\alpha = \left(\bigcup_{\beta<\alpha} M_\beta\right)'$.*

Note that for ω $M_{\omega+1}$ is still countable since only get definable sets.

Def 6.2. *A set X is constructable if $\exists\alpha$ and $x \in M_\alpha$.*

Here we use $x \in L$ to abbreviate the saying x is constructable. Some properties are:

- Transitivity: $y \in x \in M_\alpha$ then $y \in M_\alpha$;
- $\emptyset \in L$.
- If x is constructable then it's elements are constructable so we have extentsionality on constructable sets.
- For pairing, assume $x \in M_\alpha, y \in M_\beta$ then WLOG $\alpha \leq \beta$ and thus $x, y \in M_\beta$ thus $\{x, y\} \in M_\beta$.

Proposition 6.3. *For any α ordinal, $\alpha \in M_\alpha$.*

Proof is in handouts.

Corollary 6.4. *Infinity is satisfied in L.*

Now, to construct power sets in X' ,consider in L, $\forall y \in L$ which is a subset of X, then each one belongs to some M_β and let β be the least element. Now, by replacement $\exists$ ordinal γ which is a supremium of β, and then working in M_β we can form the set of all subsets of X, this is done by just taking intersection with M_γ.

7. 10/17: Replacement in constructable sets

Today we think of replacement. Denote $\phi_M(x, \dots)$ be the quantification over elements in M. Last time we've done things already till power set, and today we do replacement schema. For $A(x, y : z_1, \dots, z_t)$ a formula where $z_1, \dots, z_t$ are parameters, we can define $A_L(x, y : z_1, \dots, z_t)$ to be the formula relative to L. What we want to show is that if $a_1, \dots, a_t$ are parameters from L, then $A_L(x, y : a_1, \dots, a_t)$ defines a function in L, and that if $u \in L$, then $v = \text{range } \phi$ on u is also in L.

Lemma 7.1. *$\exists w \in L$ such that if $v =$ range ϕ on u, then $v \subset w$.*

First, let's realize that $\mathscr{P}_L(X)$ might not be enough, since that's only things in L.

One way we might go is that for each $x \in u$ let $g(x)$ be the least α such that $\phi(x) \in M_\alpha$ and let $\beta = \sup \alpha$ then $V \subset M_\beta \in L$. We just take $M_{\beta+1}$ and call it a day.

This is naively wrong. The problem lies in the fact that ϕ can be really big.

To overcome the difficulty, we use the following theorem, which is homework.

Theorem 7.2. *Let $A(x_1, \dots, x_n)$ be a formula of ZF in which all variables are bounded and free, that are restricted to be in L. For any $S \in L$ be a set, then $\exists S'$ with $S \subset S'$, $S' \in L$ such that for all $a_1, \dots, a_n \in S'$*

$$A(a_1, \dots, a_n) \iff A_{S'}(a_1, \dots, a_n).$$

This reads "given S, we can always pad a little to get S' constructable where we get the correct answer."

Corollary 7.3. *No finite number of axioms of ZF implies all of ZF.*

Proof. (of corollary) If a theorem is stated for some A, it also works for finitely many of those, but infinitely many does not work since we cannot have $A(a_1, \dots, a_n, b_1, \dots, b_m)$.

Now suppose $\exists T'$ finite then T' is enough to show incompleteness, i.e. $T' \nvdash \text{con}(T')$, that is, it cannot show it's own consistency. But reflection tells us that $ZF \vdash \text{con}(T')$ thus if

$$T' \vdash ZF \vdash \text{con}(T')$$

we have contradiction. □

The rest of the proof is:

Proof. (of lemma) Step 1: Given u, we find α such that $V :=$ range of ϕ on $u \subset M_\alpha$ and WLOG assume $u, a_1, \dots, a_t$ also belong to M_α.

Step 2: Take M_α as the S in theorem, then we obtain S' as there, then $S' \in L$ and thus $S' \in M_\beta$ so we are in good shape to conclude that everything is constructable, and thus $M'_\beta$is enough. □

8. 10/24: More on reflection and transitive collapse

Theorem 8.1. *Let $A(x_1, \dots, x_n)$ be a formula for ZF, we can prove in ZF such that $\forall S, \exists S'$, $S' \supset S$ such that $A(a) \iff A_{S'}(a)$.*

Def 8.2. *A list of formulas $\phi_0, \dots, \phi_l$ is subformula closed if*

(1) Every subformula of ϕ_i occurs in the list.
(2) No formula on the list includes $\forall$.

Proof. The opening move is to let $\phi_0, \dots, \phi_l$ be a finite sub-formula closed list containing A, and for each ϕ_i it takes in some number of parameters. Now, for each relavant finite r and each existential $\phi_j(x) := \exists y \phi_j(x, y)$ where $x = (x_1, \dots, x_r)$ is of length r.

We define the function $F_r : V^r \to$ ordinals by:

- If $\phi_j(a)$ then $F(a)$ is the least α such that $\exists b \in V_\alpha$ such that $\phi_i(a, b)$.(here $a \in V^r$).
- If not then define it to be 0.

Now let $S = S_0$, for each F_r and each $a \in S_0$ we get $F_r(a) \in$ ordinals. Then by replacement there's a sup of ordinals, which is itself an ordinal α_r, and we note there's only finitely many F_r so we take the supreme and plus the minimum α such that $S \in V_\alpha$, then we define the supreme of all these to be β_1.

We let $S_1 = V(\beta_1)$. We get repeatedly that $S' = V(\beta_\omega) \in V$. □

We can do this applied to L by changing $V^r \rightsquigarrow L^r$ and put everything in L. The proof passes through.

Now we start to show something about CH.

Def 8.3. *E is well-founded if for all non-empty $X' \subset X$ has an element y such that $\sim zEy$ for all $z \in X'$.*

Def 8.4. *E is extensional if $x, y \in X$ and $x \neq y$, then $\exists z \in X$ such that $zEx, \sim zEy$ or vise versa.*

Theorem 8.5. *If E is well-founded and extensional relation on a class X, then there exists a transitive class M and isomorphism Π such that*

$$(X, E) \overset{\Pi}{\sim} (M, \in).$$

We do a graph so that we understand:

$$\begin{tikzcd}
 & a = \{d,b,248\} & \\
d = \{1,3\} \arrow[ur] & & b = \{1,2,7\} \arrow[ul] \\
e = \{0\} = 1 \arrow[u] \arrow[urr] \arrow[rr] & & c = \{0,1\} = 2 \arrow[u] \\
 & f = 0 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

and the theorem basically says isomorphic to this:

$$\begin{tikzcd}
 & a = \{\{1\},\{1,2\}\} & \\
d = \{1\} \arrow[ur, two heads] & & b = \{1,2\} \arrow[ul, two heads] \\
e = \{0\} = 1 \arrow[u, two heads] \arrow[urr, two heads] \arrow[rr, two heads] & & c = \{0,1\} = 2 \arrow[u, two heads] \\
 & f = 0 \arrow[ul, two heads] \arrow[ur, two heads] &
\end{tikzcd}$$

where now the arrow represents ε relation.

In the end:

Theorem 8.6. *If $x \in L_\alpha$, α is infinite, and $y \subset X$ (so $y \in L$) then $\exists\beta$ such that $|\beta| = |\alpha|$ and $y \in M_\beta$, then every subset of ω in L is "constructed" by a countable ordinal.*

9. 10/26: Consistency of CH

We today show Godel's proof that AC holds in L. The superstructure is that $V = L \Rightarrow$ AC. If X is infinite we know $|X| = |X'|$ and we can show with transfinite induction that $|\alpha| = |M_\alpha|$. Now we assume $V = L$, and we'll work towards the following theorem:

Theorem 9.1. *If $x \in M_\alpha$, α infinite, and $y \subset x$ in L, then $\exists \beta$ with $|\beta| = |\alpha|$ and $y \in M_\beta$.*

Let's first discuss how the whole thing will work.

Discussion: $|\alpha| = |M_\alpha|$ immediately shows CH now since if β is the next cardinal of α then $\alpha \in M_\beta$ is obvious, but for any $x \in \alpha$ we can express it within $\leq \alpha$ constraints (theorem), which by definition is still in M_β. So $|\mathcal{P}(\alpha)| \leq |M_\beta|$. But from Cantor $|\mathcal{P}(\alpha)| \geq |\beta|$ is obvious so we are done.

Now we gather pieces of the proof of theorem:

1. Transitive collapse as is presented last time.
2. Reflection gives us that sets are transitive or sets that's the same order, but not necessarily both (our choice of S').
3. Absoluteness, which we'll discuss.

Now a sketch of how our proof of theorem will begin: We take α to be infinite and $x \in M_\alpha$, $y \subset X, y \in M_\beta$ for some β. But we a priori know nothing about β and we want to show we can take it to be the same cardinal.

Proposition 9.2. *The statements "β constructs M_β" is ZF which says that if β is an ordinal and there there is a function f defined for all $\gamma < \beta$ such that*

$$f(\gamma) = \left(\bigcup_{\delta<\gamma} f(\delta)\right)'$$

and $f(\beta) = M_\beta$.

Def 9.3. *A formula f of set theory is Δ_0 if it has no quantifiers or is a finite boolean bombination of Δ_0 formula or it's $(\exists x \in y)\phi$ or $(\forall x \in y)\phi$ where ϕ is Δ_0.*

Def 9.4. *If M is a transitive class and for all $a_1, \dots, a_w$ in M we have*

$$\phi_M(a_1, \dots, a_w) \iff \phi(a_1, \dots, a_w)$$

then we say ϕ is <u>absolute</u> for M.

Godel wanted to analysis which formulas are absolute for transitive models and proved Δ_0 formulas have this property.

Lemma 9.5. *$\exists$ a set T such that $M_\alpha \subset T$, $y, M_\beta, \beta \in T$ and T is extensional and $|T| = |\alpha|$, and "β constructs M_β" is valid when relativized.*

Proof. We look at the lemma and see that by letting $S = M_\alpha \cup \{y, M_\beta, \beta\}$, then we want some S' to satisfy two formulas: ""β constructs M_β" is valid when relativized" and "T is extensional", and from reflection we can do so with $|T| = |S| = |M_\alpha| = |\alpha|$. □

Proof. Now we do the proof of theorem. Say T is the extensional set we've found, then we can apply transitive collapse to get a transitive isomorphism F of T onto a $\leq |T|$ size set R.

Since M_α is transitive $F|_{M_\alpha} = \mathrm{Id}(M_\alpha)$ and so $F(x) = x$, also is $y \subset x$. However, the map need not be identity for β, but that's no matter!

Notice that "being an ordinal" is absolute so $F(\beta) =: \beta'$ is still an ordinal and we know $\beta' \subset R$ by transitive so $|\beta'| \leq |\alpha|$ and recall $|T| \leq \alpha$ and F is 1-1 so $y \in M_\beta$ holds when restricted to R (this uses the fact that the construction of F already states that it is $\in$-isomorphism), so "$y \in M_\beta$" is absolute and indeed the result is correct. □

10. 10/31: TOWARDS FORCING

Last time we've seen Godel's proof of consistency of CH. Today we start marching on Cohen's proof that the negation of CH is.

The set up is this: for M a ctm (countable transitive model) for ZFC, and for $(\mathbb{P}, \leq, 1)$ a partial order (1 being the largest element) belonging to M. The idea is to use only those to obtain a generic extension N of M which is also a model of ZFC. For other axioms, the depend on combinatorics of $\mathbb{P}$ in M.

Def 10.1. *G is generic over M if G is a filter on $\mathbb{P}$ and for all dense (here, we are standing at a perspective $\overline{\text{outside}}$ of M) D we have*

$$D \subset \mathbb{P}, D \subset M \Rightarrow G \cap D \neq \emptyset.$$

Lemma 10.2. *If M is countable, $\mathbb{P} \in M$, then $\exists G$ that is $\mathbb{P}$-generic over M.*

Proof. There are only countably (again, from V, since people in M does not know they are countable) D so we can enumerate them in V, and inductively we pick $q_0 \geq q_1 \geq \dots$ for $q_1 \in D_0$ and $q_2 \in D_1$, then take the G filter generated by this. (So the same as Martin's axiom) □

Corollary 10.3. *We can choose $p \in M$ and find a generic G containing p.*

Note that V and M agree on which set are dense, but not necessarily how many.

Lemma 10.4. *For M-transitive model of ZFC and $\mathbb{P} \in M$ is a partial order such that it is "not uninteresting" in the sense that*

$$\forall p \in \mathbb{P}, \exists q, r \in \mathbb{P}(q \leq p, r \leq p, q \perp r)$$

or in other words, there are always further decision to make, and if G is $\mathbb{P}$-generic over M, then $G \notin M$.

The idea is that we really have something outside of M, which is incredible given how much we have.

Proof. Suppose $G \in M$, consider $\mathbb{P} \backslash G =: D$, then $D \in M$ by our axioms. As the letter suggests, D is dense. Why? Reason: If $p \in \mathbb{P}$, then we can find q, r in the same way in the statement of lemma, then at least one is not in G, so it is in D, hence dense.

Now by definition $G \cap D = \emptyset$, contradicting to definition of generic. □

The idea now is to show how to construct another ctm $M[G]$ which will satisfy

- $M \subset M[G]$;
- $o(M) = o(M[G])$;
- $G \in M[G]$.

where $o(M) := \mathrm{Ord} \cap M$. The satisfaction of other axioms will depend largely on $\mathbb{P}$, and we'll also want to show that this is in some sense a minimal extension.

The picture is we want to take "closure" of M added G. So we want to add only those really needed, in other words, those with a "name".

Def 10.5. *τ is a $\underline{\mathbb{P}\text{ name}}$ if τ is a relation (set of ordered pairs) and*

$$\forall \langle \sigma, p \rangle \in \tau [\sigma \textit{ is a } \mathbb{P} \textit{ name and } p \in \mathbb{P}]$$

and note that this is a definition by transfinite recursion.

Def 10.6. *$V^{\mathbb{P}}$ is the class of $\mathbb{P}$ names.*

If M is a trasitive model of ZFC and $\mathbb{P} \in M$, we have

$$M^{\mathbb{P}} := V^{\mathbb{P}} \cap M = \{\tau \subset M : (\tau \text{ is a } \mathbb{P} \text{ name})^M\}$$

and the second equality holds since the statement is absolute.

Def 10.7.

$$\mathrm{val}(\tau, G) = \{\mathrm{val}(\sigma, G) : \exists p \in G(\langle \sigma, p \rangle \in \tau)\}$$

and we abbreviate $\tau_G := \mathrm{val}(\tau, G)$.

This val means a rule of evaluation.

Def 10.8. *For M transitive model of ZFC and $\mathbb{P} \in M$, $G \subset \mathbb{P}$, then*

$$M[G] = \{\tau_G : \tau \in M^{\mathbb{P}}\}.$$

Lemma 10.9. *(minimality of $M[G]$) If in addition N is a transitive model of ZFC containing M and G, then $M[G] \subset N$.*

Example 10.10.

We see that the set of "instruction" τ can look like this:

$$A = \{(1, p), (2, q), (3, r)\}$$

and if $p, q \in G$ but $r \notin G$, we get that $\{1, 2\} \subset M[G]$. Moreover, this is recursively defined so

$$\{(A, p), (B, r)\}$$

is a higher level instruction, such that if p is contained, then A is contained. This builds a transfinite hierarchy to build $M[G]$.

Example 10.11.

(1) $0_G = 0$ for any G. Where 0 is the instruction that really there's no instruction so $\emptyset$ is a $\mathbb{P}$ name.

(2) For instruction $\{\langle 0, p\rangle\}$, we have

$$\mathrm{val}(\{\langle 0, p\rangle\}) = \begin{cases} 0 & p \notin \boldsymbol{G} \\ \{0\} & p \in \boldsymbol{G} \end{cases}$$

(3) $\{\langle 0, 1\rangle\}$ is a name which always evaluates since $\{0\} = 1$.

More generally, any element has a "canonical name" which we call $\check{x}$ defined recursively by

$$\check{x} = \{\langle \check{y}, \mathbb{1}_{\mathbb{P}}\rangle : y \in x\}.$$

11. 11/2: MORE ON NAMES, MORE ON FORCING

We give some examples of names of ordinals. We know, for instance, that we can write the ordinal 2 as $\{\langle \emptyset, 1\rangle, \{\langle 0, 1\rangle\}, 1\}\}$, note that $1 \in G$ always since G is a filter and 1 is the maximal element of the relation.

Proposition 11.1. *If $x \in M$, then the canonical name $\check{x}$ for x always evaluates to x, whatever the generic G is.*

Now we need a name for G, but we just consider $\{\langle \check{p}, p : p \in P\}$ which evaluates to the set G, which is just by definition.

For our homework, we check the following properties: $M[G]$ satisfies pairing, extension, and union.

Example 11.2.

We now give an example to show how forcing works. For M a ctm of ZFC, $\mathbb{P}$ a set of finite partial functions from ω to 2, which is partially ordered by reverse inclusion. Then $f \leq g$ means f has more information. Just as before, if G is a filter on $\mathbb{P}$ then $f_G = \cup G$ is a function with domain $\subset \omega$ and if G is $\mathbb{P}$ generic, it meets with all the dense sets. Since $D_n := \{p \in \mathbb{P}; n \in \operatorname{dom} p\}$ is dense and is an element of M, we know f_G has full domain ω.

Let's verify that $f_G \in M[G]$.

Reason: We know $G \in M[G], f_G = \cup G$, so we have this one we show $M[G]$ satisfies union.

But maybe let's find a name of it:

$$F = \{\langle \{\langle n, m\rangle\}, p\rangle : p \in \mathbb{P}, n \in \operatorname{dom} p, p(n) = m\}$$

So indeed $F_G \in M[G]$. But do we know if $F_G \in M$? We know that $G \notin M$ so let's try counter proof it. Suppose it is the case, then $F_G \in M$, and for ease we call f the corresponding function, then let

$$E = \{p \not\subset f : p \in \mathbb{P}\}$$

then $E \subset \mathbb{P}$ is dense since we can extend finite p functions, but then if E were in M, then it contradict to generic.

12. 11/9: Theorems of Forcing

In the forcing language, we've introduced names. We can think of them as elements or we can think that they reason about truth.

We call elements in $\mathbb{P}$ the forcing conditions, and we'll want to define when p "forces" a statement to hold. This notation, $p \Vdash_{\mathbb{P}} \phi$ seems like the implication notation $\vDash$ but is actually crucially different.

To read, and understand the notation of forcing, we read $p \Vdash_{\mathbb{P}} \phi$ as every time we have a generic G such that $p \in G$, then ϕ is true.

Now, the strategy we will adopt is:

(1) Define a notion of $p \Vdash \phi$.
(2) Try to find a complete sequence: that is, we find a sequence of forcing conditions $p_0 \geq p_1 \geq \dots$ such that each ϕ is decided by some p_i.

We would want the notion of forcing to satisfy the following properties:

- Consistent: not $p \Vdash \phi$ and $p \Vdash \sim \phi$;
- Monotonicity: if $p \Vdash \phi$, $q \leq p$, then $q \Vdash \phi$;
- Deciciveness: $\forall \phi$, $\exists q \in \mathbb{P}$ such that $q \Vdash \phi$ or $q \Vdash \sim \phi$.

The key rule is that we shall say "p forces $\sim A$" iff for all $q \leq p$, q does not force A.

Def 12.1. *If $T_1, \dots, T_n$ are names for $\mathbb{P}$ and $\phi(x_1, \dots, x_n)$ is a first order formula of the language of set theory, then we write $p \Vdash_{\mathbb{P}} \phi(T_1, \dots, T_n)$ if for every generic subset $G \subset \mathbb{P}$ which contains p and $M[G] \vDash \phi(x_1, \dots, x_n)$.*

Theorem 12.2. *(Forcing)*

(1) If G is a generic subset of $\mathbb{P}$ over M, then $\exists$ transitive set $M[G]$ which is a model of ZFC and $M \subset M[G], G \subset M[G]$ and $o(M) = o(M[G])$.
(2) If M is a countable transitive model, then a generic G exists.
(3)
$$M[G] = \{\tau[G] : \tau \in M \& \tau \text{ is a name } \}$$
.
(4) If G is a generic subset of $\mathbb{P}$, then $\forall \phi(t_1, \dots, t_n)$ there is a $p \in G$ such that $p \Vdash \phi$ or $p \Vdash \sim \phi$, and $M[G] \vDash$ the same statement accordingly.

What is remaining to show:

(1) $M[G]$ is indeed ZFC;
(2) Done.
(3) This is just definition;
(4) We'll show the following lemma.

Lemma 12.3. *Let G be an upward closed subset of $\mathbb{P}$, then the following are equivalent:*

- *G is generic for $\mathbb{P}$ over M;*
- *For $\forall$ maximal antichain $I \in M$ of $\mathbb{P}$ we have $|G \cap I| = 1$.*

Proof. ($\Rightarrow$:)

Let I be a maximal antichain, G generic, and let

$$J = \{p \in \mathbb{P} : \exists q \in I : q \leq p\}$$

which is the downward closure of elements of I. Now we need to show J is dense:

Consider $r \in \mathbb{P}$, then it must be compatible with some $i \in I$ by maximality of antichain, so we let s be such that $s \leq i$ and $s \leq r$ since they are compatible, and by definition of J we know $s \in J$.

But J dense means G meets J and by upward closure we know G meets I. But G cannot meet two elements of I since that would contradict consistency, so $|G \cap I| = 1$.

($\Leftarrow$:)

Suppose G is upward closed and $|G \cap I| = 1$, There are 3 things to show in order to show G generic:

(a) upward closed.
(b) Directed ($\forall p, q \in G$, $\exists r \in G$ such that $r \leq p$ and $r \leq q$).
(c) Meet all dense sets.

(a) is given, now we show (b) and with (b) we show (c).

Let J be a dense set and let's prove G meets it. Consider the antichain of elements of J and by Zorn's lemma there is a maximal one I^*, we claim that it is really a maximal antichain of $\mathbb{P}$, but this by assumption means G meets it.

To see this, suppose $r \in \mathbb{P}$, since J dense $\exists q \in J$ with $q \leq r$ and necessarily q is compatible with some elements of I^*, so r must also be compatible with that element, this means we cannot add r to the antichain, hence maximality.

So we have (c). Now to show (b), given $p, q \in G$, then define

$$X := \{r : (r \leq q, r \leq p) \vee (r \perp q) \vee (r \perp p)\}$$

and X is dense because for any $p' \in G$ either p' is consistent with p and q, in which case they have a common lower bound (i.e. $r \leq q \vee p \vee p'$) that is in X, or it is not consistent with p or q, which is captured by $(r \perp q) \vee (r \perp p)$ for $r \leq p'$ (or r stronger than p', hence implies p').

Since X is dense we know G meets X and so if we pick $t \in G \cap X$ then either $t \leq p \vee q$, in which case we are done, or $t \perp q$, but then we can extend $\{t, q\}$ to a maximal chain in I that G meets twice, contradiction, so this case cannot happen. So G is indeed directed.

So we conclude that G is generic. $\square$

13. 11/14: Remains of Forcing; Use forcing to prove CH

Proposition 13.1. *In the example of partial functions, $\forall p \in \mathbb{P}$, p forces that f_G has infinite domain. In other words*

$$p \Vdash \forall_\alpha x \exists_\alpha y (y > x \wedge y \in \operatorname{dom} f_G)$$

where we adopted Cohen's notation of α-labelling.

Proof. By definition, for $p \in \mathbb{P}$, $p \Vdash \forall_\alpha x \theta(x)$ if for all $q \leq p$ and for all $c \in S_\beta$, $\beta \leq \alpha$, q does not force $\sim \theta(c)$. In the above and the following, we use

$$\theta(c) := \exists_\alpha y (y > c \wedge y \in \operatorname{dom} f_G)$$

as an abbreviation.

Now we ask a few yes-no questions, and answer them from the last to the first, which will then get us to our results:

(1) Does p force $\forall x \theta(c)$?
(2) For q defined as above, does q force $\sim \theta(c)$?
(3) For all $r \leq q$, does r not force $\theta(c)$?

Note that $r \leq q$ really is saying that $\operatorname{dom}(r) \supset \operatorname{dom}(q)$ and since c fixed, there will eventually be some $n > c$ for some r' with $n \in \operatorname{dom}(r')$ since otherwise $\operatorname{dom}(r) \subset \{0, 1, 2, \ldots, c\}$ for all $r \leq q$. Thus, $r' \Vdash \theta(c)$.

So the answer to (3) is No; Answer to (2) is No; Answer to (1) is Yes by argument in the beginning. Note we are done by this since c is arbitrary. □

Notice that by Cohen's definition of forcing, we know $\sim (p \Vdash \phi \wedge p \Vdash \sim \phi)$ but we do not a priori know $(p \Vdash \phi \wedge p \Vdash \sim\sim \phi)$.

Corollary 13.2. *G is generic if any one of the following equivalent conditions holds:*

(1) G meets every dense set;
(2) G meets every maximal set;
(3) G meets every pre-dense set.

Corollary 13.3. *We can define names by cases. If $I = \{p_i : I < \alpha\}$ is an antichain in $\mathbb{P}$ and $\{T_i : I < \alpha\}$ is a family of $\mathbb{P}$ names, then there $\exists$ name T such that for every $i < \alpha$ and every generic G, if $p_i \in G$, then $T[G] = \tau_i[G]$.*

Proof. For each $\tau_i = \{\langle \tau_{ij}, p_{ij} \rangle : j < j_i\}$ we define

$$\tau := \{\langle \tau_{ij}, r \rangle : j < j_i, i < \alpha, r \leq p_{ij}, r \leq p_i\}.$$

If I is not maximal we get $\emptyset$. □

Now we do an alternative proof of consistency of CH by forcing. The outline is that we start with our ctm M, and in M let $2^{\aleph_0} \equiv \aleph_1$ and we shall extend to $M[G]$ in which G is a function from ω_1 onto the family of subsets of ω.

We overcome the following:

(1) How do we know each $A \subset \omega$ appear in Ran(G);
(2) Is $\omega_1^M = \omega_1^{M[G]}$?
(3) What about $A \subset \omega$ in $M[G]$ which may be new?

The idea is to show that there's no new elements, and show some set is dense.

In M, let $\mathbb{P} = \{f : f :$ countable ordinals $\to \mathcal{P}(\omega)\}$ and $f_1 \leq f_2$ if f_1 extends f_2.

Now, looking ahead to $M[G]$, g be a function with domain the union of the domain of $f's$ in G, then the domain is the union of ordinals $< \omega_1^M$, hence the union $\subset \omega_1^M$. An observation is that for a countable set in M then it remains countable in $M[G]$.

Corollary 13.4. *If ω_1^M is uncountable in $M[G]$ then it's at least uncountable ordinal so it's $\omega_1^{M[G]}$ and if not it is $\leq \omega_1^{M[G]}$ which means either way $\omega_1^M \leq \omega_1^{M[G]}$ and by Cantor $\mathcal{P}(\omega)^{M[G]}$ has to be uncountable.*

We claim that every $H \subset \omega$ in M is in Ran(g). This is because the set of $p \in \mathbb{P}$ that has A in range is dense and in M.

Lemma 13.5. $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^{M[G]}$.

Def 13.6. *A forcing notion is called $\aleph_1$ complete if any time $p_0 \geq p_1 \geq \dots$ then there is a p such that $\mathcal{P}_i \geq p$ for all $i < \omega$.*

Observe that our $\mathbb{P}$ is $\aleph_1$ complete.

Theorem 13.7. *If $\mathbb{P}$ is a $\aleph_1$ complete forcing notion, then for every generic G, $M[G]$ contains no new ω sequences of members of M. That is, if $\langle a_n : n < \omega \rangle \in M[G]$ and each $a_n \in M$ then already $\langle a_n : n < \omega \rangle \in M$.*

14. 11/28: CONSISTENCY OF CH

We start with model M, aiming for $M[G] \vdash$CH. The idea is to enumerate subsets of ω, i.e. $\mathscr{P}(\omega)$ by $\aleph_1$.

So we had $\mathbb{P}$ partial order of functions whose domains were countable ordinals and

$$f : (\text{countable ordinal})^M \to \mathscr{P}(\omega)^M$$

where $\mathbb{P}$ is countably complete.

Recall that $\mathbb{P}$ is λ complete if whenever we have chain $p_0 \geq \cdots \geq p_\alpha \geq \dots$ of length $< \lambda$, then it has a lower bound. So for example $\aleph_1$ complete means countable sequence has lower bounds. (For this reason it is also called countable complete, although that is an abuse of terms, in some sense.)

So $\mathbb{P}$ is countably complete since ctb union of ctb set is ctb (we're talking about the domain). Now let G be a generic subset of $\mathbb{P}$ and f_G be its associated function. We have verified that f_G has domain ω_1^M and f_G has range $(\mathscr{P}(\omega))^M$, where the first is because adding countable ordinals we get countable ordinals, and we extend by addition.

We want to know what happens when vied from $M[G]$, and there's two things to justify: We want to know that M and $M[G]$ agrees on ω, and that any countable ordinal in M is countable in $M[G]$. Given this, if $\aleph_1^M$ is uncountable in $M[G]$, then it is the least uncountable in $M[G]$, i.e. it is $\aleph_1^{M[G]}$.

But now by Cantor $\aleph_1^M$ is actually uncountable in $M[G]$. This is enough to conclude that whatever $\aleph_1^M \leq \aleph_1^{M[G]}$.

We claim: $\mathscr{P}(\omega)^M = \mathscr{P}(\omega)^{M[G]}$.

The claim is sufficient because:

We know $f_G : \omega_1^M \to \mathscr{P}(\omega)^M$ so the size of range is smaller and if $|\mathscr{P}(\omega)^M| = |\mathscr{P}(\omega)^{M(G)}|$ then

$$|\mathscr{P}(\omega)^{M(G)}| = |\mathscr{P}(\omega)^M| \leq |\omega_1^M| \leq |\omega_1^{M[G]}|$$

and by Cantor we are done.

Now we show the claim.

Lemma 14.1. *If $\mathbb{P}$ is countable complete, then there are in $M[G]$ no new countable sequence of elements of M.*

We first recall that a generic set is such that meets every dense set, or meet every maximal antichain. We also have the idea of dense below p. The idea of <u>predense</u> is such that $\forall r \leq p, \exists d \in \tilde{D}$ such that $e < d, e < r$ for some e. In other words, it is "anything unavoidable given what we have decided."

Suppose that we have $\langle a_n : n < \omega \rangle$ in $M[G]$ where it is a sequence of elements in M. We aregue that for any specific $p \in G$, and n, the set of $q \leq p$ wich "know" that the nth element is forced by elements in a dense set below p.

Why? We use definability of forcing for one relevant formula: A sequence is really a function h in $M[G]$ with domain in ω, and because $\omega^{M[G]} = \omega^M$ we know that for each $n < \omega$, $\exists \check{a}_n$ such that we have a predicate for it in M, and since $h(n) = \check{a}_n$, some p should force it. In fact, there is $p \in G, p \leq q$ for given q that does force it.

So we are aiming toward that $\forall q \in G$, the set of τ names $\{r \in \mathbb{P}, r \Vdash (\tau \in M)\}$ is dense or predense below q.

The idea now is to find a countable chain. We first find such p_0, the 0th value, then find p_1' in the same manner. We now find a common lower bound of the two and call it p_1, there by defining a descending chain of p_α, and countable complete means that the whole sequence is forced by some element q. So we are done.

What we will show next time is that:

Theorem 14.2. *If $\mathbb{P}$ has ccc, then the cardinality does not change, i.e. $\aleph_\alpha^M = \aleph_\alpha^{M[G]}$.*

Now we do some reflection on CH, whether we really need it. One example is the proof that $M \equiv N$ iff they have the same ultrapowers. The first proof of this assumes GCH in two places. The proof of Keisler has structure that goes like this:

(1) $\exists$ infinite λ, there exists a regular good ultrafilter D.
(2) If M is a model, then in the language of size $\leq \lambda$, M^λ/D is λ^+ saturated.
(3) If we have two models both λ^+ saturated and of size λ^+, then they are isomorphic.
(4) If D is regular on λ, then $|M^\lambda/D| = |M^\lambda|$, $M \equiv N$, because:
choose $\lambda \geq \max\{|M|, |N|\}$ and D a regular good ultrafilter on λ, then

$$M^\lambda/D \equiv M \equiv N \equiv N^\lambda/D$$

and

$$|M^\lambda/D| = |M^\lambda| = 2^\lambda = |N^\lambda||N^\lambda/D|$$

and by GCH we conclude $M \cong N$.

In this proof we've used GCH two times, once in step 1 and once in step 4. The use in step 1 can be removed by the theorem below, but the use of GCH on step 4 is not trivially removed.

Theorem 14.3. *There $\exists$ regular good ultrafilters on any infinite λ by independent family $f : \lambda \to \lambda$.*

15. 11/30: INDEPENDENCE OF CH

Theorem 15.1. *If $\mathbb{P} \in M$ and $(\mathbb{P}$ is ccc$)^M$ then forcing with $\mathbb{P}$ preserves cardinals and cofinalities.*

Proof. Step 1 (Approximation): Suppose $A, B \in M$, G is generic, $f : A \to B$ and $f \in M[G]$, then $\exists F : A \to \mathscr{P}(B)$ crutially $F \in M$ and for all $a \in A, f(a) \in F(a)$ and $|F(a)| \leq \aleph_0$ in M.

Note that countable in M makes the problem non-trivial.

Now we fix a name $\check{f}$ for f, i.e. "$\check{f} : \check{A} \to \check{B}$", then we can choose a p that forces it. We define $F : A \to \mathscr{P}(B)$ by

$$F(a) = \{b \in B : \exists q \leq p[q \Vdash (\check{f}(\check{a}) = \check{b})]\}$$

By instance of definability $F \in M$, so we need to show that $f(a) \in F(a)$ and countable.

To show countable, suppose $b \in F(a)$ is witnessed by q_b, but if $b \neq b'$ then they'd be witnessed by a different q and $q_b \perp q_{b'}$. By M knows ccc we have $|F(a)| \leq \aleph_0$ in B.

$f(a) \in F(a)$ because we had a lemma that says we can define names by cases: if we had $\{p_i : i < i_b\} \subset \mathbb{P}$ and names $\{\tau_i : i < i_b\}$, then we could cook up a name that evaluates to τ_i if $p_i \in G_i$. Then $\{p_i : i \leq i_b\}$ is an anti-chain, then we get where G meets it.

Step 2: Suppose λ is an uncountable cardinal in M, then it's necessarily an ordinal in $M[G]$, but what if it is not a cardinal?

Since $M, M[G]$ have the same ordinals, there exists $\alpha < \lambda, f \in M[G], f : \alpha \to \lambda$ so $\exists F : \alpha \to \mathscr{P}(\lambda)$ such that $f(\beta) \in F(\beta)$ and $|F(\beta)| \leq \aleph_0$.

Computing in M, let λ be the range $f \subset \bigcup_{\beta<\alpha} F(\alpha)$ in M, then M knows that each $F(\alpha)$ is countable. So

$$\left|\bigcup_{\beta<\alpha} F(\beta)\right| \leq |\alpha|\aleph_0 < \lambda$$

which contradicts to λ being countable in M. $\square$

What is also useful is existential completeness: If $p \Vdash \exists x\phi(x)$ then $\exists \tau$ a name such that $p \Vdash \phi(\tau)$, which is just by the same method as we've cooked up name with chains of names.

By forcing, we try to make the continuum large, so we'd like to make continuum at least λ large. Suppose our $\mathbb{P}$ is finite approximation to a function from $\lambda \to \{0, 1\}$ and $p \in \mathbb{P}$ is a function with domain $\subset \lambda$ and range $\subset \{0, 1\}$.

Lemma 15.2. *$M[G]$ will have at least λ reals. (where real is just a poetic way of saying elements in continuum).*

Proof. For each $i < \lambda$, let g be the function associated to G, and look at $f_i := f_i(n) = g(i+n)$ which are functions with domain ω and range ≤ 2, and if we have a name for g we also have a name for all of them. Let $\check{a}$ be the name for f_i, then use sublemma: g has domain λ and range 2. The proof is that the appropriate set is dense. $\square$

So each a_i is a subset of ω and we claim that if $i \neq j$ then $a_i \neq a_j$. Now in $M[G]$, there will be at least λ^M many reals distinct from each other. By ccc $\lambda^M = \lambda^{M[G]}$, so from $M[G]$ there are $\lambda^{M[G]}$.

The only thing left for now is to show $\mathbb{P}$ has ccc. Let's suppose we have a failure of ccc, then there exists p_i pairwise contradictory, and let's how that two of them are consistent, which is just by Delta system lemma, since for two finite functions, they have to differ at some finite set, and thus there is a common place of contradiction V_* (possibly empty) where ω many elements have common domains on. But this cannot make all of them pairwise contradictory since V^* is finite, so we are done.

We conclude the above into the theorem:

Theorem 15.3. *For any λ, there exists a model in which $2^{\aleph_0} \geq \lambda$.*

One further question is what actuall value can it be.

Theorem 15.4. *(Konig) If $K \geq 2$, λ infinite then cofinality $(k^\lambda) > \lambda$ and in particular $cf(2_0^\lambda) > \omega$.*

Which seems to be the only restriction.

APPENDIX A. A

APPENDIX B. B

APPENDIX C. C

Acknowledgements.