BROWNIAN MOTION AND STOCHASTIC CALCULUS HW 7

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Discussed with classmates.

Exercise 1. *(5.2)*

Proof.

1. By definition from book, for $t < T$ and $t \in \mathbb{Z}$, $W_t = -2^t + 1$ and $W_T = 1$.

Now we show Martingale: Adapted is due to W_n adapted; integrable because W_n is integrable; and we show the Martingale property:

We want to show $\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}[M_s]$ for $s \le t$:

$$\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}[M_t \mathbb{1}_{T<s}|\mathcal{F}_s] + \mathbb{E}[M_t \mathbb{1}_{s\le T\le t}|\mathcal{F}_s] + \mathbb{E}[M_t \mathbb{1}_{T>t}|\mathcal{F}_s$$
$$= 0 + 0 + 2^t \cdot \frac{1}{2^{t-s}} = 2^s = \mathbb{E}[M_s|\mathcal{F}_s]$$

Thus M_n is a Martingale.

(2):

$$\mathbb{E}[M_n \mathbb{1}_V] = \mathbb{E}[\mathbb{1}_V \mathbb{E}[M_n|\mathcal{F}_m]] = \mathbb{E}[M_n \mathbb{1}_V]$$

by Martingale property.

(3):

$$Q(M_{n+1} = 2^{n+1} \cap M_n = 2^n) = Q(M_{n+1} = 2^{n+1}) = \mathbb{E}[M_{n+1}\mathbb{1}_{M_{n+1}=2^{n+1}}] = 2^{n+1}\frac{1}{2^{n+1}} = 1$$

and

$$Q(M_n = 2^n) = \mathbb{E}[M_n \mathbb{1}_{M_n=2^n}] = 2^n \frac{1}{2^n} = 1$$

thus

$$Q(M_{n+1} = 2^{n+1}|M_n = 2^n) = \frac{Q(M_{n+1} = 2^{n+1} \cap M_n = 2^n)}{Q(M_n = 2^n)} = 1.$$

(4): Note that

$$Q(T = \infty) = \mathbb{P}(M_i \ne 0, \forall i \in \mathbb{Z}^+) = Q(M_1 = 2)\prod_{i=1}^{\infty} Q(M_{i+1} = 2^{i+1}|M_i = 2^i) = 1$$

hence

$$Q(T < \infty) = 1 - Q(T = \infty) = 0.$$

(5): No,

$$\mathbb{E}_Q[M_1] = 2 \cdot Q(M_1 = 2) + 0 = 2 \neq 1 = \mathbb{E}_Q[M_0].$$

$\square$

Exercise 2. *(5.3)*

Proof.

(1): $dX_t = 2dt + dB_t$.

Define $M_t = e^{Y_t}$ where

$$Y_t = \int_0^t A_s dB_s - \frac{1}{2}\int_0^t A_s^2 ds$$

and $A_t = -2$. By discussion in class we know M_t is the solution of

$$dM_t = A_t M_t dB_t$$

and we define the needed measure via Radon-Nikodym by

$$Q_t(V) = \mathbb{E}[M_t \mathbb{1}_V]$$

for $V \in \mathcal{F}_t$, and similarly we define Q on $\mathcal{F}_\infty$.

Now, by Girsanov theorem we know indeed X_t is the Standard Brownian motion under Q. We compute (with initial condition plugged in)

$$\left.\frac{dQ}{d\mathbb{P}}\right|_1 = M_1 = e^{Y_1} = \exp\left\{\int_0^1 -2dB_s - 2\int_0^1 ds\right\} = e^{-2B_1-2}.$$

(2): $dX_t = 2dt + 6dB_t$.

From discussion in class, if we want to shift measure such that X_t is a Standard Brownian Motion, then we'd need to use a measure that, viewed as a Wiener measure, has $\sigma = \sqrt{6} \neq 1$ and hence we'd necessarily have shift to a orthogonal measure, hence not equivalent.

(3): $dX_t = 2B_t dt + dB_t$.

This is the same with the first case. Define M_t, Y_t similarly and $A_t = -2B_t$. By the same argument of Girsanov we get X_t is a Standard Brownian motion under Q (defined in the same way). We compute

$$\left.\frac{dQ}{d\mathbb{P}}\right|_1 = M_1 = e^{Y_1} = \exp\left\{\int_0^1 -2B_s dB_s - 2\int_0^1 B_s^2 ds\right\} = e^{-B_1^2+1-2\int_0^1 B_s^2 ds}.$$

□

Exercise 3. *(5.4)*

Proof.

(1): Notice that

$$M_t = X_t \exp\left\{\int_0^t g(X_s)ds\right\} = \exp\left\{-2m\int_0^t B_s dB_s + \int_0^t g(X_s) - mds\right\}$$

where we define, as in last question, that

$$-2m\int_0^t B_s dB_s + \int_0^t g(X_s) - mds =: Y_t$$

and hence

$$dY_t := -2mB_s dB_s + (g(X_s) - m)ds$$

by Ito's formula we compute

$$dM_t = de^{Y_t} = e^{Y_t}dY_t + \frac{1}{2}e^{Y_t}d\langle Y\rangle_t = -2mB_t e^{Y_t}dB_t + e^{Y_t}\left(g(X_t) - m + 2m^2B_t^2\right)dt$$

and since we want M_t to be a local Martingale, we can make it to be a stochastic integral, i.e. drift term = 0. This requires

$$g(X_t) - m + 2m^2B_t^2 = 0$$

which yields $g(X_s) = m - 2m^2B_t^2 = m + 2m\log(X_s)$ and hence

$$g(x) = m + 2m\log x$$

is a choice of g.

(2): As is computed in part 1, the SDE is

$$dM_t = -2mB_t e^{Y_t}dB_t = -2mB_t M_t dB_t.$$

(3):

$$\langle Y\rangle_t = \int_0^t 4m^2B_t^2 ds =: \int_0^t A_s^2 ds$$

from above computation. Let

$$T_n := \inf\{t : M_t + \langle Y\rangle_t = n\}$$

and let $A_t^{(n)} = \begin{cases} A_t & t < T_n \\ 0 & t \geq T_n \end{cases}$ where we have

$$dM_{t\wedge T_n} = A_t^{(n)}M_{t\wedge T_n}dB_t$$

is a square integrable Martingale as

$$\mathbb{E}[(M_{t\wedge T_n} - 1)^2] = \mathbb{E}\left[\left(\int_0^t A_t^{(n)} M_{s\wedge T_n} dB_s\right)^2\right] = \int_0^t \mathbb{E}\left[(A_t^{(n)} M_{s\wedge T_n})^2\right] ds$$
$$\leq n^2 \mathbb{E}\left[\int_0^t (A_t^{(n)})^2 ds\right] \leq n^3$$

by orthogonality of increments, Variance rule, and Fubini respectively for each step.

By Girsanov's theorem, this allows us to write $\mathbb{P}_n^*$ for each n for $t \leq T_n$ where $A_t = A_t^{(n)}$. Since for $n \neq m$ and $s \leq \min\{m, n\}$ we know $\mathbb{P}_n^* \equiv \mathbb{P}_m^*$ restricted to $\mathcal{F}_s$, we know $\mathbb{P}^*$ for measure on $t \leq T := \lim_{n\to\infty} T_n$ is well-defined.

Now we can define the standard brownian motion with respect to $Q|_{t\leq T} := P^*$ by

$$dB_t = dW_t - A_t dt$$

where W_t is a Q-Brownian motion for $t < T$. Explicit computation tells us

$$M_t = \exp\left\{-mB_t^2 - 2m^2t^2 + mt\right\} \leq \exp\left\{-2m^2t^2 + mt\right\} \leq C(m)$$

is uniformly bounded by $C(m)$ in t. And in particular, $\mathbb{E}[M_t] = \mathbb{E}[M_{t\wedge T_{\lceil C(m)\rceil}}] = 1$ for any t. This guarantees M_t is in fact a Martingale, hence Girsanov is well defined for all t, and $Q := \mathbb{P}^*$ indeed. Thus

$$dB_t = dW_t - A_t dt$$

always holds.

(4): This is shown in last part.

□

Exercise 4.

Proof.

(1):

Let $f(x) = |x|$ for $x \in \mathbb{R}^d$ and compute:

$$\partial_i f(x) = \frac{x_i}{|x|}; \quad \partial_i^2 f(x) = \frac{1}{|x|} - \frac{x_i^2}{|x|^3}$$

thus Ito gives

$$dX_t = \sum_{i=1}^d \frac{B_t^i}{|B_t|} dB_t^i + \frac{1}{2} \sum_{i=1}^d \left(\frac{1}{|B_t|} - \frac{x_i^2}{|B_t|^3} \right) dt = \frac{d-1}{2} \frac{1}{|B_t|} dt + \sum_{i=1}^d \frac{B_t^i}{|B_t|} dB_t^i$$

and hence if we denote

$$dM_t = \sum_{i=1}^d \frac{B_t^i}{|B_t|} dB_t^i = A_t \cdot dB_t$$

where $A_t = \dfrac{1}{|B_t|}(B_t^1, \dots, B_t^d)$. Note that M_t is a Martingale since each component of A_t has length ≤ 1.

(2):

$$\langle M \rangle_t = \int_0^t A_s^2 ds = \int_0^t \frac{(B_s^1)^2 + \cdots + (B_s^d)^2}{|B_s|^2} ds = t$$

and since M_t is continuous Martingale with $\langle M \rangle_t = t$, it is a standard Brownian motion.

□

Exercise 5.

Proof.

Lemma 0.1. *(from Lecture on Oct. 12th) Let B_t be a standard d dimensional Brownian motion starting at x with $r < |x| < R$. Then, define*

$$T_{r,R} := \min\{|B_t| = r \ \text{ or } R\} = \min\{t : B_t \in \partial D\}$$

where we denote D to be the open annulus with inner radius r and outer radius R. Further denote

$$\phi(x) := \mathbb{P}^x\{|B_T| = R\}$$

, then up to linear term we have uniquely that

$$\phi_{r,R}(x) = \begin{cases} \frac{|x|^{2-d} - r^{2-d}}{R^{2-d} - r^{2-d}} & d > 2 \\ \frac{\log|x| - \log r}{\log R - \log r} & d = 2 \end{cases}$$

Proof. (This is done in class, with solving PDE part ignored. So I'd do the same here.) By rotational invariance, $\phi(x) = \phi(|x|)$ and if $\phi(x) = 0$ we know $|x| = r$, while if $\phi(x) = 1$ we know $|x| = R$, and the boundary is continuous. B_t is a Brownian motion so ϕ satisfies MVP by strong Markov property.

Define the conditional property up to when the path hits an ε ball, and

$$\phi(B_{T,\varepsilon}) = \mathbb{P}^x\{|B_T| = R|\mathcal{F}_{T,\varepsilon}\}$$

Then we have

$$\mathbb{P}^x\{|B_T| = R\} = \mathbb{E}^x[\mathbb{P}^x\{|B_T| = R|\mathcal{F}_{T,\varepsilon}\}] = \mathbb{E}^x[\phi(B_{T,\varepsilon})] = MV(\phi, x, \varepsilon)$$

where

$$MV(f; x, \varepsilon) := \int_{|x-y|=\varepsilon} f(y) ds(y)$$

Thus $\phi(x)$ is harmonic on $D_{r,R}$ with boundary value 0 on $\{|y| = r\}$ and 1 on $\{|y| = R\}$ and is rotational invariant.

Thus we have $\Delta\phi(|x|) = 0$ and chain rule gives

$$\phi''(|x|) + \frac{d-1}{|x|}\phi'(|x|) = 0$$

which admits unique solution up to a multiplicative and addition constant:

$$\phi_{r,R}(x) = \begin{cases} \frac{|x|^{2-d} - r^{2-d}}{R^{2-d} - r^{2-d}} & d > 2 \\ \frac{\log|x| - \log r}{\log R - \log r} & d = 2 \end{cases}$$

and we note that they are all harmonic in the space without the origin. $\square$

(1): We find R such that $d(B_R, x) \geq d(B_s, x)$ and $D \subset B_R$ where d is the usual norm, B_R is the ball of radius R. Denote $\tau_R := \min\{t : |B_t| = R\}$ and we know

$$s\mathbb{P}^x\{T_s < \tau\} \geq s\mathbb{P}^x\{T_s < \tau_R\}$$

since $\tau_R > \tau$.

For the other direction, denote $r := \frac{1}{2}d(\partial D, 0)$, and denote S to be the circle with radius r. We know that the probability of B_t reaching S before τ is positive by infinite return in 2d (or via the rectangular paths proven in past homework). Denote this possibility as q_x, and denote the stopping time of above event as T'. Then

$$s\mathbb{P}^x\{T_s < \tau\} \leq q_x s\mathbb{P}^{B_{T'}}\{T_s < \tau_r\}$$

now we use Lemma 0.1 to bound both sides:

$$s\mathbb{P}^x\{T_s < \tau_R\} = s\left(1 - \frac{\log|x| - \log e^{-s}}{\log R - \log e^{-s}}\right) = \frac{s(\log R - \log|x|)}{\log R + s} \to (\log R - \log|x|)$$

as $s \to \infty$.

For the other side,

$$q_x s\mathbb{P}^{B_{T'}}\{T_s < \tau_r\} = \frac{q_x s(\log 2r - \log|B_{T'}|)}{\log 2r + s} \to q_x(\log 2r - \log r) = q_x \log 2.$$

In particular we have

$$(\log R - \log|x|) = \lim_{s\to\infty} s\mathbb{P}^x\{T_s < \tau_R\} \leq \lim_{s\to\infty} s\mathbb{P}^x\{T_s < \tau\} \leq \lim_{s\to\infty} qs\mathbb{P}^{B_{T'}}\{T_s < \tau_r\} = q_x \log 2.$$

Now, it suffices to show that $s\mathbb{P}^x\{T_s < \tau\}$ is monotone in s for large enough s. Take $s > N$ for N to be specified, and $\varepsilon > 0$. Then

$$\mathbb{P}\{T_{s+\varepsilon} < \tau\} = \mathbb{P}(T_s < \tau)\mathbb{P}(T_{s+\varepsilon} < \tau | T_s < \tau)$$

but from homework 3, question 5 we know that the probability of a Brownian motion starting on the circle of radius e^{-s} that does not encircles the closest point on circle of radius $e^{-s-\varepsilon}$ is larger than $1 - c(e^{-s} - e^{-s-\varepsilon})^\alpha$ for c, α fixed. In particular this tells us that

$$(s+\varepsilon)\mathbb{P}^x\{T_{s+\varepsilon} < \tau\} \geq (s+\varepsilon)\mathbb{P}^x\{T_s < \tau\} \cdot (1 - c(e^{-s} - e^{-s-\varepsilon})^\alpha)$$

since there are other way to reach $B_{e^{-s-\varepsilon}}$. Consider order

$$1 - c(e^{-s} - e^{-s-\varepsilon})^\alpha = 1 - ce^{-\alpha s}\varepsilon + o(\varepsilon)$$

for $s > N$ where we now pick N. Hence

$$\frac{s+\varepsilon}{s}(1 - O(\varepsilon e^{-s})) = 1 + \frac{\varepsilon}{s} - O(e^{-s}) \geq 0$$

we get that

$$(s+\varepsilon)\mathbb{P}^x\{T_{s+\varepsilon} < \tau\} \geq (s+\varepsilon)\mathbb{P}^x\{T_s < \tau\} \cdot (1 - c(e^{-s} - e^{-s-\varepsilon})^\alpha) \geq s\mathbb{P}^x\{T_s < \tau\}$$

which gives that the limit exists.

(2): This is actually the same as used in proof of Lemma 0.1: Define the conditional property up to when the path hits an ε ball, and

$$\phi(B_{s,\tau,\varepsilon}) = \mathbb{P}^x\{|B_{T_s\wedge\tau}| = R|\mathcal{F}_{s,\tau,\varepsilon}\}$$

Then we have

$$s\mathbb{P}^x\{T_s < \tau\} = \mathbb{E}^x[\mathbb{P}^x\{|B_{T_s\wedge\tau}| = R|\mathcal{F}_{s,\tau,\varepsilon}\}] = \mathbb{E}^x[\phi(B_{s,\tau,\varepsilon})] = MV(\phi_s, x, \varepsilon)$$

where ϕ_s is ϕ with parameter s. Now take $s \to \infty$ nothing is changed, so

$$G(x) = \lim_{s\to\infty} s\mathbb{P}^x\{T_s < \tau\} = MV(G, x, \varepsilon)$$

Thus $\phi(x)$ is harmonic on $\hat{D}$ since it's open.

(3):

From last question in Homework 3 question 5 we know that for $|x - z| < \varepsilon$, there esist c, α such that the probability of the path $B[0, T_\delta]$ that does not wind around z is smaller than $c\varepsilon^\alpha$ for $|x| > \delta > \varepsilon$ (well defined when ε small), where $T_\delta := \inf\{t : |B_t - z| = \delta\}$. Take $\varepsilon \to 0$ we know that $c\varepsilon^\alpha \to 0$ and hence the probability that $B[0, T_\delta]$ winds around z goes to 0. But D is simply connected so winding around z implies traversing outside of $\hat{D}$, so τ will be reached.

Thus, $G(x_n) \to 0$ since as $n \to \infty$, $\varepsilon \to 0$ and $c\varepsilon^\alpha \to 0$ and for large s, $\mathbb{P}^x\{\tau < T_s\} \to 1$.

(4):

Ito's formula says for $t \le \tau \wedge T$

$$dG(B_t) = \nabla G(B_t) \cdot dB_t + \frac{1}{2}\Delta G(B_t)dt = \nabla G(B_t) \cdot dB_t$$

by harmoniticity. But then $M_t := G(B_t)$ is a stochastic integral, hence a local Martingale.

(5):

Intuitively, if we want to reach τ we need B_t to go close enough (and eventually touches) ∂D, but then $G(B_t) \to 0$ and thus the probability under new measure is 0.

Formally, because $\mathbb{P}^*$ is well defined (that they agree on common part of domains for each t) we can take a rising sequence of $t \to \tau$:

$$\mathbb{P}^*\{\tau < \infty\} = \mathbb{P}^*\left\{\bigcup_{i=0}^{\infty} \tau \in [i, i+1)\right\} \le \sum_{i=0}^{\infty} \mathbb{P}^*\{\tau \in [i, i+1)\} = \sum_{i=0}^{\infty} \mathbb{E}[\mathbb{1}_{\tau\in[i,i+1)} G(B_{t<\tau})]$$

$$\le \sum_{i=0}^{\infty} \mathbb{E}\left[\lim_{n\to\infty;\tau - t_n < \frac{1}{n}} G(B_{t_n})\right] = \sum_{i=0}^{\infty} 0 = 0$$

and hence we only need to show $\mathbb{P}^*\{T < \infty\} = 1$.

In part 1, we have given already an upper bound for $G(x) \le q_x \log 2$ and when $x \to 0$ by this bound we know $|G(x)| < C$ for all $|x| < \delta$. This, plus the fact that G is smooth, and vanishes at ∂D gives us that $G(x)$ is bounded on $\hat{D}$. Thus

$$\mathbb{P}^*(T_s = \infty) = \lim_{n\to\infty} \mathbb{P}^*(T_s \geq n) = \lim_{n\to\infty} \mathbb{E}[\mathbb{1}_{T_s>n} G(B_n)] \leq C \lim_{n\to\infty} \mathbb{E}[\mathbb{1}_{T_s>n}] = 0$$

by neighborhood recurrence of 2d Brownian motion. Now take limit on s we get

$$\mathbb{P}^*(T = \infty) = \lim_{s\to\infty} \mathbb{P}^*(T_s = \infty) = 0.$$

(Note that the difference with pointwise non-recurrence is that we take limit in n first, then the limit in s.)

So we have shown $\mathbb{P}^*\{T < \infty\} = 1$, and $\mathbb{P}^*\{\tau = \infty\} = 1$. Combined we have

$$\mathbb{P}^*\{T < \infty, T < \tau\} = 1.$$

(6):

No. Assume M_t is a Martingale, then so is $M_{t\wedge T}$ by optional sampling. But $\mathbb{E}[M_0] = G(x) \in (0, \infty)$ where $\mathbb{E}[M_{n\wedge T}] \to 0$ as $n \to \infty$ by last part, contradiction.

□