APPLIED FUNCTIONAL ANALYSIS HOMEWORK 2

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Discussed with classmates.

Exercise 1. *(5.2) in book*

Proof.

(a):

$$x = \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} \left(x_i \cdot \sum_{j=1}^{n} \bar{L}_{ij} \bar{e}_j \right)$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \left(x_i^T \cdot \bar{L}_{ij} \right) \cdot \bar{e}_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \left(x_i^T \cdot \bar{L}_{ij} \right) \right) \bar{e}_j$$

where by definition we know

$$x = \sum_{j=1}^{n} \bar{x}_j \bar{e}_j$$

and thus

$$x_j = \sum_{i=1}^{n} \bar{L}_{ij} x_i$$

where by flipping j and i we get the wanted result.

(b): Discussed with Tim

We note that the only thing we need to prove is the relationship between ω_i and $\bar{\omega}_i$. If we can show

$$\omega_i = \sum_{j=1}^{n} L_{ij} \bar{\omega}_j$$

then the rest will be exactly the same as in part (a).

In order to do this, we assume $\omega_i = \sum_{k=1}^{n} Q_{i,k}\bar{\omega}_k$. Now since associated basis means that $\delta_{ji} = \omega_j e_i$, we get

$$\delta_{ji} = \omega_j e_i = \sum_{k=1}^{n} Q_{jk}\bar{\omega}_k \cdot \sum_{l=1}^{n} \bar{L}_{i,l}\bar{e}_l = \sum_{k=1}^{n}\sum_{l=1}^{n} Q_{jk}\bar{L}_{li}\bar{\omega}_k\bar{e}_l = \sum_{k=1}^{n}\sum_{l=1}^{n} Q_{jk}\bar{L}_{li}\delta_{kl}$$
$$= \sum_{k=1}^{n} Q_{jk}\bar{L}_{ki}$$

where the exchange of L and ω is not direct exchanging, but element wise computation's result combined.

The above means that $Q_{jk} = L_{ki}$ iff $i = j$, and hence $Q_{ik} = L_{ki}$. This by definition of Q means

$$\omega_i = \sum_{j=1}^{n} L_{ij}\bar{\omega}_j$$

and we are done by above reasoning.

$\square$

Exercise 2. *(5.6) in book.*

Proof.

(a):

For any $x \in X$, let $Y = \mathbb{R}x := \{tx | t \in \mathbb{R}\}$. Then Y is a subspace of X. Define

$$\psi(tx) := |t| \cdot ||x||$$

then we have $\psi(x) = ||x||$. Moreover, it's linear beacause

$$\psi((t+s)x) = (t+s)||x|| = t||x|| + s||x|| = \psi(tx) + \psi(sx)$$

and it's bounded and $||\psi|| = 1$ because

$$\inf\{c|\psi(tx) \leq ||tx||\} = 1$$

since for $tx \neq 0$, $\dfrac{|t| \cdot ||x||}{||tx||} = 1$ and when $tx = 0$ any $c = 1$ works as well.

But then by Hahn-Banach we get that there is an extension $\phi : X \to \mathbb{R}$ of ψ such that

$$\phi(x) = \psi(x) = ||x|| \text{ } and \text{ } ||\phi|| = ||\psi|| = 1.$$

(b):

If the condition is satisfied by all $\phi \in X^*$ it is satisfied by the one we've just constructed in (a). Thus

$$\phi(x) = \phi(y) \Rightarrow ||x - y|| = \phi(x - y) = 0$$

and hence $x = y$ by the first property of norms.

□

Exercise 3. *(5.10) in book.*

Proof.

$$Kf(x) := \int_0^1 k(x,y)f(y)dy$$

<u>K is compact:</u>

From question we know that k is a continuous function on a compact box $[0,1]^2$, thus k attains it's maximum and minimum on the box, i.e. we have $\forall x, y \in [0,1]$, $\exists l, u \in \mathbb{R}$ such that

$$l \leq k(x,y) \leq u$$

and in particular let

$$c = \max\{|l|, |u|\}.$$

Define $T \in X^*$ be the integral of $x \in X$, i.e.

$$T(f) := \int_0^1 f(x)dx$$

then for any sequence $\{f_n\}$ with $||f_n||_\infty < C$, we know that

$$|T(f_n)| \leq ||f_n||_\infty \leq C$$

so $\{T(f_n)\}$ is a sequence that is in the compact interval $[-C, C]$. This means that there is a subsequence $g_n := f_{\phi(n)}$ that $T(g_n)$ convergences, in particular the sequence $T(g_n)$ is Cauchy. So for any fixed $\varepsilon > 0$, $\exists N$ such that $\forall m, n \geq N$ we have

$$\left| \int_0^1 g_n(x) - g_m(x)dx \right| < \varepsilon.$$

Now back to K, we can just use the construction of subsequence g_n and large number N as above and get that $\forall n, m \geq N$

$$\begin{aligned}
|K(g_n) - K(g_m)| &= \left| \int_0^1 k(x,y) \left(g_n(y) - g_m(y)\right) dy \right| \\
&\leq \left| \int_0^1 |k(x,y)| \left(g_n(y) - g_m(y)\right) dy \right| \\
&\leq c \left| \int_0^1 \left(g_n(y) - g_m(y)\right) dy \right| \\
&\leq c\varepsilon
\end{aligned}$$

for any ε. Hence, $K(g_n)$ is a Cauchy subsequence of $K(f_n)$. But then since the space $C[0,1]$ is Banach, $K(g_n)$ converges to some point in $C[0,1]$. This shows that K is compact.

□

Exercise 4. *(5.14) in book.*

Proof.

If $T_n \to T$ uniformly, then $||T_n|| \to ||T||$:

Since

$$\sup(f+g) \leq \sup f + \sup g$$

letting $h = f + g$ we get

$$\sup h - \sup f \leq \sup(h - f).$$

Using this result we can compute that

$$\begin{aligned}
\Big| ||T_n|| - ||T|| \Big| &= \left| \sup_{||f||=1} ||T_n f|| - \sup_{||f||=1} ||Tf|| \right| \\
&\leq \left| \sup_{||f||=1} \big(||T_n f|| - ||Tf|| \big) \right| \\
&\leq \sup_{||f||=1} \Big| ||T_n f|| - ||Tf|| \Big| \\
\text{(trig)} \qquad &\leq \sup_{||f||=1} ||T_n f - Tf|| = ||T_n - T||
\end{aligned}$$

where we can take off the absolute value since in the use of triangle inequality, the only sign flipped is the sign inside $||T_n f - Tf||$.

Then since $||T_n - T|| \to 0$ we get $\Big| ||T_n|| - ||T|| \Big| \to 0$.

$\square$

Exercise 5. *(5.17) in book.*

Proof.

ker$(I-K)=\{0\}$:

Since K is linear, $K(0)=K(0)+K(0)=0$. More over, K is a contraction because for all $f,g\in X$

$$\frac{||Kf-Kg||}{||f-g||}=\frac{||K(f-g)||}{||f-g||}\le ||K||<1.$$

So since $K:X\to X$ is a self map on a Banach space by definition we get by Banach Contraction mapping theorem that K has a unique fixed point, which as we've shown is 0. Thus, the only $f\in X$ such that $f=Kf$ is 0.

But this means that

$$(I-K)f=0 \quad\Rightarrow\quad f=Kf \quad\Rightarrow\quad f=0$$

and hence $\ker(I-K)=\{0\}$. This means that there exists a partial inversion of $I-K\Big|_{\mathrm{Ran}(I-K)}$ defined on the range of this function.

$I+K+K^2+\dots$ is well defined and bounded (Series converge):

Now we show that the partial sum $S_n:=I+K+K^2+\cdots+K^n$ is Cauchy. Then by the completeness of X, we know that the limit exists.

We know that (WLOG assume $n>m$)

$$||S_n-S_m||\overset{linearity}{=}\left\|\sup_{||f||=1}\sum_{i=m+1}^{n}K^if\right\|\le\sup_{||f||=1}\frac{c^{m+1}(1-c^n)}{1-c}||f||=\frac{c^{m+1}(1-c^n)}{1-c}\le\frac{c^{m+1}}{1-c}$$

which we see that for all $\varepsilon>0$, letting $m>N(\varepsilon)$ big enough we can set $||S_n-S_m||\le\varepsilon$.

Hence S_n is a Cauchy sequence in $B(X)$, where since X is Banach so is $B(X)$, which means that $I+K+K^2+\cdots\in B(X)$ is well defined and bounded.

$(I-K)$ is invertible and it's inverse is the series above:

We now show that the operator $I-K$ is invertible by finding a $f\in X$ such that $(I-K)f=g$ for all $g\in X$.

For fixed $g\in X$, define

$$f:=(I+K+K^2+\dots)g$$

then

$$(I-K)f=(I-K)(I+K+K^2+\dots)g=\left(\sum_{i=0}^{\infty}K^i-\sum_{j=1}^{\infty}K^j\right)g=g$$

where we can just do the operation on bounded operators since the bounded operators form an algebra. But now we know $\mathrm{Ran}(K)=X$ and that $K^{-1}=(I+K+K^2+\dots)$ by above.

$\square$