PDE HOMEWORK 1

TOMMENIX YU
ID: 12370130
STAT 31220
DUE FRI MAR 31ST, 2023, 11PM

Discussed with classmates.

Exercise 1.

Proof.

For fixed x, let $g(t) := f(tx)$, then g is smooth since f is. By regular Taylor on $\mathbb{R}$ we have

$$g(t) = \sum_{l=0}^{k} \frac{1}{l!} g^{(l)}(0) t^l + O(|t|^{k+1}).$$

So let's investigate first what is $g^{(l)}$ and check that by plugging back to f and with a change of variable the multivariate Taylor follows.

Computing $g^{(l)}$:

By definition

$$g^{(0)}(t) = f(tx).$$

For the first derivative we compute

$$g'(t) = \frac{\partial f(tx)}{\partial t} = \frac{\partial f(tx_1, \dots, tx_n)}{\partial t} = x_1 \partial_1 f(tx) + \cdots + x_n \partial_n f(tx) = x \cdot Df(tx).$$

Using the above result we get that

$$g''(t) = \frac{\partial}{\partial t} \left(\sum_{i=1}^{n} x_i \partial_i f(tx) \right) = \sum_{1 \le i,j \le n} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}(tx)$$

and we can inductively prove this formula for $g^{(l)}$, which is just by directly taking derivative w.r.t. t. So we get the general formula:

$$g^{(l)}(t) = \sum_{1 \le \beta_1, \dots, \beta_l \le n} x_{\beta_1} \cdots x_{\beta_l} \frac{\partial^l f}{\partial x_{\beta_1} \cdots \partial x_{\beta_l}}(tx)$$

where $\beta_1, \dots, \beta_l$ are integers from 1 to n. So there's n^l summands in the summation. Now we rewrite this from into the α expression and get the coefficient. Let $\alpha := \alpha(\beta) := (\alpha_1, \dots, \alpha_l)$ be such that

$$x_{\beta_1} \cdots x_{\beta_l} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

so we only have to count how many times is this counted in the big summation. To count it we first note that there's α_1 among the l βs such that they are 1, and α_2 among the remaining $l-\alpha_1$ βs such that they are 2, and et cetera. Thus, the summation can be rewritten as

$$\begin{aligned}
g^{(l)}(t) &= \sum_{1\le\beta_1,\dots,\beta_l\le n} x_{\beta_1}\cdots x_{\beta_l}\frac{\partial^l f}{\partial x_{\beta_1}\cdots\partial x_{\beta_l}}(tx)\\
&= \sum_{|\alpha|=l}\binom{l}{\alpha_1}\binom{l-\alpha_1}{\alpha_2}\cdots\binom{l-\alpha_1-\cdots-\alpha_{n-1}}{\alpha_n}x^\alpha D^\alpha f(tx)
\end{aligned}$$

where $|\alpha| = \sum_i \alpha_i$ and $D^\alpha f$ is as it's definition in textbook. But note that

$$\begin{aligned}
&\binom{l}{\alpha_1}\binom{l-\alpha_1}{\alpha_2}\cdots\binom{l-\alpha_1-\cdots-\alpha_{n-1}}{\alpha_n}\\
=&\frac{l!}{\alpha_1!(l-\alpha_1)!}\frac{(l-\alpha_1)!}{\alpha_2!(l-\alpha_1-\alpha_2)!}\cdots\frac{(l-\alpha_1-\cdots-\alpha_{n-1})!}{\alpha_n!0!}=\frac{l!}{\alpha!}
\end{aligned}$$

for $\alpha!$ defined in problem.

Thus we have

$$g^{(l)}(t) = \sum_{|\alpha|=l}\frac{l!}{\alpha!}x^\alpha D^\alpha f(tx).$$

Plugging back to the Taylor expansion of g we get

$$\begin{aligned}
f(tx) = g(t) &= \sum_{l=0}^{k}\frac{1}{l!}g^{(l)}(0)t^l + O(|t|^{k+1}) = \sum_{l=0}^{k}\frac{1}{l!}\sum_{|\alpha|=l}\frac{l!}{\alpha!}x^\alpha D^\alpha f(0)t^l + O(|t|^{k+1})\\
&= \sum_{|\alpha|\le k}\frac{1}{\alpha!}D^\alpha f(0)(tx)^\alpha + O_{x\to 0}(|tx|^{k+1})
\end{aligned}$$

where $O(|t|^{k+1}) = O(|tx|^{k+1})$ since x is fixed. Now since $tx \in U$ is arbitrary we get what we want by letting $y = tx$.

$\square$

Exercise 2.

Proof.

The problem is

$$\begin{cases} u_t + b \cdot Du + cu = 0 & t > 0 \\ u(x,0) = g(x) & t = 0 \end{cases}$$

so we mimic the proof in class and define

$$z(s) := u(x + sb, t + s)$$

hence we have

$$\dot{z}(s) + cz(s) = 0$$

which by multiplying e^{cs} we have

$$\frac{d}{ds}\left(e^{cs}z(s)\right) = e^{cs}(\dot{z}(s) + cz(s)) = 0$$

and hence

$$z(s) = Ce^{-cs}$$

where as

$$Ce^{ct} = z(-t) = g(x - tb) \quad \Rightarrow \quad C = g(x - tb)e^{-ct}$$

and thus

$$u(x,t) = z(0) = e^0 g(x - tb)e^{-ct} = g(x - tb)e^{-ct}.$$

□

Exercise 3.

Proof.

Since $u = u(r\cos\theta, r\sin\theta)$, we compute the terms using chain rule.

$$u_r = \cos\theta \cdot u_x + \sin\theta \cdot u_y$$

$$u_{rr} = \cos\theta \cdot u_r + \sin\theta \cdot u_r = \cos^2\theta \cdot u_{xx} + \sin^2\theta \cdot u_{yy} + 2\sin\theta\cos\theta u_{xy}$$

$$u_\theta = -r\sin\theta \cdot u_x + r\cos\theta \cdot u_y$$

$$\begin{aligned} u_{\theta\theta} =& - r\cos\theta \cdot u_x + r^2\sin^2\theta \cdot u_{xx} - r^2\sin\theta\cos\theta u_{xy} \\ & - r\sin\theta \cdot u_y + r^2\cos^2\theta \cdot u_{yy} - r^2\sin\theta\cos\theta u_{xy} \end{aligned}$$

and hence

$$\begin{aligned} \frac{1}{r^2}u_{\theta\theta} &= -\frac{1}{r}\left(\cos\theta \cdot u_x + \sin\theta \cdot u_y\right) + \sin^2\theta \cdot u_{xx} + \cos^2\theta \cdot u_{yy} - 2\sin\theta\cos\theta u_{xy} \\ &= -\frac{1}{r}u_r + \sin^2\theta \cdot u_{xx} + \cos^2\theta \cdot u_{yy} - 2\sin\theta\cos\theta u_{xy} \end{aligned}$$

which means

$$\begin{aligned} \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r + u_{rr} &= u_{rr} + \sin^2\theta \cdot u_{xx} + \cos^2\theta \cdot u_{yy} - 2\sin\theta\cos\theta u_{xy} \\ &= \left(\sin^2\theta + \cos^2\theta\right)(u_{xx} + u_{yy}) = \Delta u. \end{aligned}$$

□

Exercise 4.

Proof.

We first compute

$$\begin{aligned}
\Delta uv &= \sum_{i=1}^{n} \partial_{ii} uv = \sum_{i=1}^{n} \partial_i \left(u_{x_i} v + u v_{x_i} \right) \\
&= \sum_{i=1}^{n} u_{x_i x_i} v + u v_{x_i x_i} + 2 u_{x_i} v_{x_i} \\
&= \Delta u \cdot v + u \cdot \Delta v + 2 Du \cdot Dv = 2 Du \cdot Dv
\end{aligned}$$

which means

$$\Delta uv = 0 \iff Du \cdot Dv = 0$$

which is what we want.

$\square$