APPLIED FUNCTIONAL ANALYSIS HOMEWORK 5

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Discussed with classmates.

Exercise 1. *(12.6) in book*

Proof.

the derivative is integrable:

Since $g_t(x) := \frac{d}{dt} f(x,t)$ (well defined) has absolutely value less than h, thus so is g^+ and g^-, which means g is integrable if it is measurable. But it is measurable since by definition

$$g_t(x) = \lim_{n\to\infty} \frac{f(x,t+1/n) - f(x,t)}{1/n}$$

which is a limit of measurable functions. Thus, g is integrable.

$\int_X f(t,x)d\mu$ is differentiable with $\frac{d}{dt}\int_X f(t,x)d\mu = \int_X \frac{d}{dt} f(t,x)d\mu$:

We show that it is differentiable by showing that the derivative quotient is well-defined, that the limit exists. This, as we'll see, gives us exactly the value of it's derivative.

The derivative quotient of $\int_X f(t,x)d\mu$ is (formally written as):

$$\frac{d}{dt}\int_X f(t,x)d\mu = \lim_{s\to 0} \frac{\int_X f(t+s,x)d\mu - \int_X f(t,x)d\mu}{s} = \lim_{s\to 0} \int_X \frac{f(t+s,x) - f(t,x)}{s} d\mu$$

by linearity. So we only need to define

$$h_s(t,x) := \frac{f(t+s,x) - f(t,x)}{s}$$

for $s > 0$, and prove that we can pass the limit when $s \to 0$. To do this we note that it is measurable because $f(t,x)$ is integrable, hence measurable, and $h_s(t,x)$ is a sum of measurable functions, times a constant, which is still measurable.

Now we show that there exists integrable function k such that $|h_s| \le k$. To do this we note that since we are eventually going to use $\frac{1}{n} \to 0$ to approximate the process of $s \to 0$, we only need to deal with $s \leq 1$. Thus, since we know that

$$\left|\frac{\partial f}{\partial t} f(x,t)\right| \leq g(x)$$

thus we can bound the derivative in t of $f(x,t_0)$ by $g(x)$. In other words, the function $f(x,t)$ is Lipschitz in t with Lipschitz constant $g(x)$. Thus, for $s \leq 1$

$$k(t,x) = g(x)$$

serves as an upper bound. Thus, since h_s are measurable and is dominated by integrable k, we get that

$$\lim_{s\to 0} \int_X h_s(t,x) d\mu = \int_X \lim_{s\to 0} h_s(t,x) d\mu$$

where we implicitly do the change of variable $s \to 1/n$ to fit the theorem. But observe that the above equality is nothing but

$$\frac{d}{dt} \int_X f(t,x) d\mu = \int_X \frac{d}{dt} f(t,x) d\mu$$

and we are done.

$\square$

Exercise 2. *(12.8) in book.*

Proof.

Since $|f_n| \leq g$, we know that $|f_n|^p \leq g^p$, and since $g \in L^p$, g^p is integrable, thus we can apply the DCT to get

$$\lim_{n\to\infty} \int_E |f_n| d\mu = \int_E |f|^p d\mu$$

where $E = \Omega \backslash N$ for $\mu(N) = 0$ where this exclusion of null set deals with both a.e. argument in problem. Now, we can bound

$$|f_n - f|^p \leq |f_n|^p + |f|^p \leq 2g$$

which again by DCT means

$$\lim_{n\to\infty} \int_E |f_n - f|^p d\mu = \int_E \lim_{n\to\infty} |f_n - f|^p d\mu \to 0$$

for $1 \leq p < \infty$. In this case since p is fixed we can just take degree $\frac{1}{p}$ to conclude convergent in L^p norm.

For $p = \infty$ the norm is just the supremum of a function, which point wise convergent is enough to conclude convergent.

$\square$

Exercise 3. *(12.12) in book.*

Proof.

Generalized Holder's inequality:

We first show that we can extend the inequality to 3 functions by first grouping the last 2. Then, for n functions, we can first group the last $n-1$ functions, then repeatedly separate one out until we are done, so it suffices us to prove that the inequality for 3 functions hold.

Note that by Holder's inequality we have

$$\begin{aligned}\int |g|^{\frac{p_2p_3}{p_2+p_3}}|h|^{\frac{p_2p_3}{p_2+p_3}}d\mu &\le \left|\left| \, |g|^{\frac{p_2p_3}{p_2+p_3}}\right|\right|_{\frac{p_2+p_3}{p_3}} \cdot \left|\left| \, |h|^{\frac{p_2p_3}{p_2+p_3}}\right|\right|_{\frac{p_2+p_3}{p_2}} \\ &= \left(\int |g|^{p_2}\right)^{\frac{p_3}{p_2+p_3}}\left(\int |h|^{p_3}\right)^{\frac{p_2}{p_2+p_3}}\end{aligned}$$

where everything is indeed well defined because $g \in L^{p_2} \Rightarrow |g|^{\frac{p_2p_3}{p_2+p_3}} \in L^{\frac{p_2+p_3}{p_3}}$, and a similar form for h. This plus Holder gives us that the function in the left most part above is in L^1, so everything is well defined.

Hence by Holder again we have

$$\begin{aligned}\int fghd\mu &\le ||f||_{p_1} \cdot ||gh||_{\frac{p_2p_3}{p_2+p_3}} \\ &= ||f||_{p_1} \cdot \left|\int |g|^{\frac{p_2p_3}{p_2+p_3}}|h|^{\frac{p_2p_3}{p_2+p_3}}d\mu\right|^{\frac{p_2+p_3}{p_2p_3}} \\ &\le ||f||_{p_1} \cdot ||g||_{p_2} \cdot ||h||_{p_3}\end{aligned}$$

which by argument above means we can prove for n functions, which is what we want.

□

Exercise 4. *(12.15) in book.*

Proof.

For $p < r < q$, if $f \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, then $f \in L^r(\mathbb{R}^n)$:

Since $f \in L^q(\mathbb{R}^n)$, we know that

$$\begin{aligned} \int |f|^r d\mu &= \int_{|f|\geq 1} |f|^r d\mu + \int_{|f|<1} |f|^r d\mu \\ &\leq \int_{|f|\geq 1} |f|^q d\mu + \int_{|f|<1} |f|^p d\mu \\ &\leq \int |f|^q d\mu + \int |f|^p d\mu < \infty \end{aligned}$$

and hence $|f|^r$ is integrable, which means $f \in L^r$.

$||f||_r \leq (||f||_p)^{\frac{1/r-1/q}{1/p-1/q}}(||f||_q)^{\frac{1/p-1/r}{1/p-1/q}}$:

This follows immediately from Holder's inequality. By computation we have

$$\begin{aligned} (\text{RHS})^r &= \left(\int |f|^p d\mu\right)^{\frac{p(q-r)}{q-p}} \left(\int |f|^q d\mu\right)^{\frac{q(r-p)}{q-p}} \\ &= \left(\int \left(|f|^{\frac{p(q-r)}{q-p}}\right)^{\frac{q-p}{q-r}} d\mu\right)^{\frac{q-r}{q-p}} \left(\int \left(|f|^{\frac{q(r-p)}{q-p}}\right)^{\frac{q-p}{r-p}} d\mu\right)^{\frac{r-p}{q-p}} \\ &= \left\| |f|^{\frac{q(q-r)}{q-p}} \right\|_{\frac{q-p}{q-r}} \cdot \left\| |f|^{\frac{q(r-p)}{q-p}} \right\|_{\frac{q-p}{r-p}} \end{aligned}$$

Where we note that

$$\frac{q(q-r)}{q-p} + \frac{q(r-p)}{q-p} = r$$

and

$$1\Big/\frac{q-p}{q-r} + 1\Big/\frac{q-p}{r-p} = 1$$

we can use Holder to get

$$(\text{LHS})^r = \int |f|^r d\mu = \int |f|^{\frac{q(q-r)}{q-p}} |f|^{\frac{q(r-p)}{q-p}} d\mu \leq \left\| |f|^{\frac{q(q-r)}{q-p}} \right\|_{\frac{q-p}{q-r}} \cdot \left\| |f|^{\frac{q(r-p)}{q-p}} \right\|_{\frac{q-p}{r-p}} = (\text{RHS})^r$$

where since $r > 1$ we know that the inequality below holds:

$$||f||_r \leq (||f||_p)^{\frac{1/r-1/q}{1/p-1/q}}(||f||_q)^{\frac{1/p-1/r}{1/p-1/q}}.$$

$\square$

Exercise 5. *(12.17) in book.*

Proof.

Let

$$f_n = \begin{cases} (n^2+n)^{\frac{1}{p}} & \frac{1}{n+1} \le x \le \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

then we can compute that

$$||f_n||_p = \left(\int_0^1 |f_n|^p dx\right)^{\frac{1}{p}} = \left(\frac{1}{n(n+1)}(n^2+n)\right)^{\frac{1}{p}} = 1$$

thus f_n is in the unit ball.

But then since the difference $f_n - f_m$ has norm 2, thus it's not Cauchy, thus no convergent subsequent.

$\square$

Exercise 6. *(12.18) in book.*

Proof. We choose the sequence

$$f_n = \begin{cases} n & 0 \le x \le \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

which is in the unit ball since the integral is 1.

We know (heuristically) that the limit of this function functions (as a function on the dual space) like the dirac delta δ_0.

Now for the proof. We know that $(L^1)^* = L^\infty$, so for convenience we just pick a bounded function g (as a representative of a essentially bounded function) and take the limit.

The dual space is represented as:

$$g \in L^\infty : f \mapsto \int fg dx$$

so by computation

$$\int_{\mathbb{R}} f_n g d\mu = n \int_0^{1/n} g dx$$

i.e. the average of g on $\left[0, \frac{1}{n}\right]$.

Let's choose only g continuous at 0, so that the limit tends to $g(0)$ (we can evalutate g since we've already specified it as a function that is continuous at that point). Now we choose g with $g(0) = 1$. Then if there is a weak limit $f_n \rightharpoonup f$ then

$$\int_{\mathbb{R}} gf dx = 1$$

for all bounded, continuous at 0 function g with $g(0) = 1$. Then we show that f cannot be supported on any S with $\mu(S) > 0$. We can WLOG assume $\int_S f dx \neq 0$ since f is measurable, which means the part on which f is negative is measurable. Also, we can WLOG assume $d(0, S) > \delta$, i.e. S is some distance away from 0.

If f is supported on S then we just pick

$$g_m = 1 - \mathbb{1}_S + m \cdot \mathbb{1}_S$$

and get that $\langle f, g_1 \rangle \neq \langle f, g_2 \rangle$, yet

$$\lim_{n\to\infty} \langle f_n, g_1 \rangle = 1 = \lim_{n\to\infty} \langle f, g_2 \rangle$$

so no such weak limit exists.

Now since no such limit exists, we can use a similar argument for any subsequence of f_n and the conclusion follows. $\square$

Exercise 7. *Prob (6.2) in book.*

Proof.

(a): $\underline{d(u, N) = |\bar{u}|}$:

(The definition in book should be $\bar{u} = \int_0^1 u dx$, since otherwise the inf of norm can be negative.)

First, we show that $d(u, N) \geq |\bar{u}|$ by contradiction. Assume there exists $g \in N$ such that $||u - g|| < |\bar{u}|$. Then we have

$$\left|\int_0^1 u dx\right| = \left|\int_0^1 (u - g) dx\right| \leq \int_0^1 |u - g| dx \leq ||u - g|| < \left|\int_0^1 u dx\right|$$

which is a contradiction. Thus $d(u, N) \geq |\bar{u}|$.

Then we show $d(u, N) \leq |\bar{u}|$ by picking $n = u - \bar{u}$ and check that $n \in N$ since

$$\int_0^1 n dx = \int_0^1 (u - \bar{u}) dx = \int_0^1 u dx - \bar{u} = 0$$

by definition of the mean of u. But then we have

$$||u - n|| = ||u - u + \bar{u}|| = |\bar{u}|$$

and hence $d(u, N) = \inf_{g \in N} d(u, g) \leq d(u, n) = |\bar{u}|$.

As for the infimum is attained, it's shown above.

(b): $\underline{d(x, M) = \frac{1}{2}}$:

Since $M \subset N$ we have $d(x, N) \leq d(x, M)$, i.e.

$$d(x, M) \geq d(x, N) = \left|\int_0^1 x dx\right| = \frac{1}{2}.$$

Moreover, let $k_n = \frac{n(2n-1)}{2(n-1)^2}$ for n large enough to make $g_n(1) < 3/2$ and

$$g_n := \begin{cases} -\frac{nx}{2} & 0 \leq x \leq \frac{1}{n} \\ k_n x - \frac{1}{2} - \frac{k_n}{n} & \text{otherwise} \end{cases}$$

be in $C([0, 1])$. Then

$$\int_0^1 g_n dx = \int_0^{\frac{1}{n}} -\frac{nx}{2} dx + \int_{\frac{1}{n}}^1 k_n x - \frac{1}{2} - \frac{k_n}{n} dx = -\frac{1}{4n} + \frac{(n-1)(k_n(n-1) - n)}{2n^2}$$
$$= -\frac{1}{4n} + \frac{(n-1)}{2n^2}\left(\frac{n(2n-1)}{2(n-1)} - n\right) = -\frac{1}{4n} + \frac{n}{4n^2} = 0$$

so $g_n \in M$.

Moreover, since the slope of g_n is greater than 1 for x larger than $\frac{1}{n}$, plus we've chosen n large, we know

$$\lim_{n\to\infty} ||u - g_n|| = (u - g)\left(\frac{1}{n}\right) = \frac{1}{2} + \frac{1}{n} \to \frac{1}{2}$$

as $n \to \infty$.

Thus, the $\inf_{m\in M} ||u - m|| \leq \frac{1}{2}$. Combined with above we get $d(u, M) = \frac{1}{2}$.

<u>The minimum is not attained:</u>

If the minimum is attained, then

$$x - \frac{1}{2} \leq f \leq x + \frac{1}{2}$$

for f being the point in M where the minimum is attained.

But since $\int_0^1 x - \frac{1}{2} dx = 0$ and $f - g \geq c > 0$ in $[0, \delta]$ since $f(0) > 0 - \frac{1}{2}$ and f is continuous. Thus

$$\int_0^1 f dx \geq \int_0^\delta c dx + \int_\delta^1 x - \frac{1}{2} > \int_0^1 x - \frac{1}{2} dx = 0$$

so $f \notin M$. Contradiction! So the minimum is not attained.

$\square$

Exercise 8. *Prob (6.5) in book.*

Proof.

Denote $S := \bigoplus_{n=1}^{\infty} H_n$.

For $x \in S$, denote $x = \sum_{n=1}^{\infty} x_n$, then $||x||^2 = \sum_{n=1}^{\infty} ||x_n||^2 < \infty$ since H_i is an orthogonal collection of subspaces, so the sum converges unconditionally by Lemma 6.23.

S is closed:

Let $x_n \in S$ be a converging sequence in H, i.e. $x_n \to x \in H$. Also, let $x_{n,i}$ be a sequence in i such that $x_n = \sum_{i=1}^{\infty} x_{n,i}$.

Since $x_n \to x$, we know that $\exists N$ such that $\forall n > N$, $||x - x_n|| < \varepsilon$. Assume, for contradiction, that $x \notin S$. Then we know $x = y + \sum_{i=1}^{\infty} y_i$ where $y_i \in H_i$ and $0 \neq y \perp H_i$ for all i, by projection theorem and the fact that H_i are closed subspaces. More specifically, we first decompose

$$x = x_1 + y_1$$

for $x_1 \in H_1$ and $y_1 \perp H_1$.

Now since $H_2 \perp H_1$ we know that if we write $y_1 = x_2 + y_2$ in the same manner for $x_2 \in H_2$, $y_2 \perp H_1 \oplus H_2$, and we write $x = x_2' + y_2'$ for $x_2' \in H_2$, $y_2' \perp H_2$, then $x_2 = x_2'$.

Thus, we can continue this process and decompose $x = y + \sum_{i=1}^{\infty} y_i$ where $y_i \in H_i$ and $0 \neq y \perp H_i$ for all i, as is mentioned above.

Now we compute

$$\begin{aligned}
||x - x_n|| = ||y + \sum_{i=1}^{\infty}(x_i - x_{n,i})|| &= \left(y + \sum_{i=1}^{\infty}(x_i - x_{n,i}), y + \sum_{i=1}^{\infty}(x_i - x_{n,i}) \right)^{\frac{1}{2}} \\
&= ||y||^2 + \left(y, \sum_{i=1}^{\infty}(x_i - x_{n,i}) \right) + \left(\sum_{i=1}^{\infty}(x_i - x_{n,i}), y \right) + \left\| \sum_{i=1}^{\infty}(x_i - x_{n,i}) \right\|^2 \\
\text{orthogonal } &= ||y||^2 + \left\| \sum_{i=1}^{\infty}(x_i - x_{n,i}) \right\|^2 \geq ||y||^2 > 0
\end{aligned}$$

This contradicts with our assumption that $||x - x_n|| \to 0$. Thus $x \in S$.

S is a linear subspace:

For $x, y \in S$, we need to show $\alpha x + \beta y \in S$ for $\alpha, \beta \in \mathbb{C}$, the base field.

Denote $x = \sum_{n=1}^{\infty} x_n$ and $y = \sum_{n=1}^{\infty} y_n$. Also, denote $S_N := \sum_{n=1}^{N} x_n$ and $T_N := \sum_{n=1}^{N} y_n$. First we have

$$\sum_{n=1}^{\infty} ||\alpha x_n + \beta y_n||^2 \leq \sum_{n=1}^{\infty} \alpha^2 ||x_n|| + \beta^2 ||y_n|| < \infty$$

hence also absolutely square summable.

But this also gives the unconditionally convergence of the tails $x - S_N$, and we use this to get

$$\begin{aligned}\alpha x + \beta y = \alpha S_N + \beta T_N + \alpha(x - S_N) + \beta(y - S_N) &= \sum_{n=1}^{N}(\alpha x_n + \beta y_n) + \sum_{n=N+1}^{\infty}(\alpha x_n + \beta y_n) \\ &:= P_N + R_N\end{aligned}$$

where $P_N \in S$ and R_N is the remainder, which we've shown above goes to 0 (using a triangle inequality). So the sequence is Cauchy, and by closeness of S we get that the limit is in S.

□

Exercise 9. *Prob (6.11) in book.*

Proof.

If $\mathcal{H}$ is separable then it has a countable orthonormal basis:

By theorem 6.29 in textbook, we know that $\mathcal{H}$ has orthonormal basis. For the purpose of contradiction, assume that the basis is uncountable.

Since the basis is orthonormal, we have that

$$||e_i - e_j|| = \left\langle e_i - e_j, e_i - e_j \right\rangle^{\frac{1}{2}} = \sqrt{||e_i||^2 + \langle e_i, e_j\rangle + \langle e_j, e_i\rangle + ||e_j||^2} = \sqrt{2}$$

and hence we can take the $\frac{1}{2}$ ball for each e_i where $i \in I$, such that $B\left(e_i, \frac{1}{2}\right) \cap B\left(e_j, \frac{1}{2}\right) = 0$ for $i \neq j$.

But since $\mathcal{H}$ separable, we can find a countable dense set $D \subset \mathcal{H}$. Since D dense and $e_i \in H$, there exists at least 1 element in the $\frac{1}{2}$ ball of each e_i. Since the balls are distinct, so $|D| \geq |I|$ (cardinal order), i.e. D is uncountable, contradiction! Thus, $\mathcal{H}$ has a countable orthonormal basis.

an orthonormal basis is contained in $\mathcal{M}$:

Since the basis is countable, we can let $I = \mathbb{N}$ via a bijection. Thus, for all $x \in \mathcal{H}$ we have

$$x = \sum_{i=1}^{\infty} x_i e_i$$

for $x_i \in \mathbb{C}$. Now, let $E = \{e_i | i \in \mathbb{N}\}$, then $\mathcal{H} = [E]$ and since we know $0 \in \mathcal{M}$ and some element close enough to e_1 is in $\mathcal{M}$. In particular, the element has non-zero e_1 component. So we can scale this vector to make it's norm exactly 1, since $\mathcal{M}$ is linear. Call this scaled vector

$$e_1' := a_1 e_1 + \sum_{i=2}^{\infty} a_i e_i$$

with $a_1 \neq 0$. Then we can find the space

$$\mathcal{H}_1 := (\mathbb{R} \cdot e_1')^{\perp} \cap \mathcal{M}$$

which is also separable and linear. So we can do the whole process of finding a new set of basis, and construct 1 vector with norm 1. We do this repeatedly to find a set of basis.

We cannot do this for any dense subset of $\mathcal{H}$:

Just take $\mathcal{H} = \mathbb{R}$ and $\mathcal{M}' = \mathbb{R} \backslash \{1, -1\}$. Then clearly there's no vector with norm 1, so no orthonormal basis.

$\square$

Exercise 10. *Prob (6.14) in book.*

Proof.

(a) ϕ_n are orthogonal:

For $m > n$, compute with integral by parts

$$\begin{aligned}\int_{\mathbb{R}} e^{x^2} H_n H_m dx &= (-1)^m \int_{\mathbb{R}} H_n \left(\frac{d}{dx}\right)^n e^{-x^2} dx = (-1)^m \int_{\mathbb{R}} H_n d\left(\frac{d}{dx}\right)^{m-1} e^{-x^2} \\ &= (-1)^m H_n \left(\frac{d}{dx}\right)^{m-1} e^{-x^2}\Big|_{-\infty}^{\infty} + (-1)^{m-1} \int_{\mathbb{R}} \left(\frac{d}{dx}\right)^{m-1} e^{-x^2} dH_n \\ &= 0 + (-1)^{m-1} \int_{\mathbb{R}} \frac{d}{dx} H_n \left(\frac{d}{dx}\right)^{m-1} e^{-x^2} dx \\ &\vdots \text{ (integral by parts m times)} \\ &= \int_{\mathbb{R}} \left[\left(\frac{d}{dx}\right)^m H_n\right] e^{-x^2} dx = 0\end{aligned}$$

since H_n is a polynomial of degree $n \leq m - 1$.

Thus, ϕ_n are orthogonal.

(b): ϕ_n is the eigen-function of H:

Let $A = \frac{d}{dx} + x$ and $A^* = -\frac{d}{dx} + x$. Then

$$\begin{aligned}A\phi_n &= \left(\frac{d}{dx} + x\right)\phi_n = \frac{d}{dx} e^{-\frac{x^2}{2}} H_n(x) + x\phi_n \\ &= -xe^{-\frac{x^2}{2}} H_n(x) + x\phi_n + e^{-\frac{x^2}{2}} \frac{d}{dx} H_n(x) = -x\phi_n + x\phi_n + e^{-\frac{x^2}{2}} \frac{d}{dx} H_n(x) \\ &= e^{-\frac{x^2}{2}} \frac{d}{dx} H_n(x)\end{aligned}$$

Thus we compute the derivative of H_n: on the one hand

$$\begin{aligned}\frac{d}{dx} H_n(x) &= \frac{d}{dx} (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2}\right) \\ &= (-1)^n 2xe^{x^2} \frac{d^n}{dx^n} \left(e^{-x^2}\right) + (-1)^n e^{x^2} \frac{d^{n+1}}{dx^{n+1}} \left(e^{-x^2}\right) \\ &= 2xH_n - H_{n+1}\end{aligned}$$

and on the other hand since

$$\frac{d^n}{dx^n} f \cdot g = \sum_{k=1}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

by chain rule, we have

$$\begin{aligned}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right) = \frac{d^n}{dx^n}\left(-2xe^{-x^2}\right) &= -2\sum_{k=1}^{n}\left(\binom{n}{k}x^{(k)}\left(e^{-x^2}\right)^{(n-k)}\right)\\ &= -2\left[x\frac{d^n}{dx^n}\left(e^{-x^2}\right) + n\cdot 1\cdot\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right)\right]\end{aligned}$$

since $x'' = 0$. Thus we plug in and get

$$\begin{aligned}\frac{d}{dx}H_n(x) &= 2xH_n + (-1)^n e^{x^2}\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right)\\ &= 2xH_n + (-1)^n e^{x^2}\cdot(-2)\left[x\frac{d^n}{dx^n}\left(e^{-x^2}\right) + n\cdot 1\cdot\frac{d^{n+1}}{dx^{n+1}}\left(e^{-x^2}\right)\right]\\ &= 2xH_n - 2xH_n + 2nH_{n-1} = 2nH_{n-1}\end{aligned}$$

and hence

$$A\phi_n = e^{-\frac{x^2}{2}}\frac{d}{dx}H_n(x) = e^{-\frac{x^2}{2}}2nH_{n-1} = 2n\phi_{n-1}.$$

Where as using the other implication above we get

$$\begin{aligned}A^*\phi_n &= -2x\phi_n - e^{-\frac{x^2}{2}}\frac{d}{dx}H_n(x)\\ &= -2x\phi_n - e^{-\frac{x^2}{2}}(2xH_n - H_{n+1}) = \phi_{n+1}.\end{aligned}$$

Moreover

$$\begin{aligned}(AA^* - 1)(y) &= \left(\frac{d}{dx} + x\right)\left[\left(-\frac{d}{dx} + x\right)(y)\right] - y\\ &= \left(\frac{d}{dx} + x\right)\left[-\frac{dy}{dx} + xy\right] - y\\ &= -\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y - x\frac{dy}{dx} + x^2y - y\\ &= -\frac{d^2y}{dx^2} + x^2y = Hy\end{aligned}$$

for all suitable y, thus $H = AA^* - 1$.

Now we compute

$$H(\phi_n) = (AA^* - 1)\phi_n = A(\phi_{n+1}) - \phi_n = 2(n+1)\phi_n - \phi_n = (2n+1)\phi_n$$

which means that indeed ϕ_n are eigen functions of H with eigenvalue $\lambda_n = 2n + 1$.

$\square$