MEASURE THEORETIC PROBABILITY III HW 4

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Exercise 1 - Subsequence of subsequences method:

Consider the sequence of random variables X_n defined as:

$$X_n = \begin{cases} (-1)^n \left(1 - \frac{1}{n}\right) & \text{with probability } \frac{1}{2} \\ n & \text{with probability } \frac{1}{2} \end{cases}$$

Show that the sequence converges in distribution but not in probability or almost surely. Hint: Consider subsequences for even and odd n.

Exercise 2 - Kolmogorov's three-series theorem:

Let X_n be a sequence of independent random variables with $E[X_n] = 0$ for all n. Suppose that the following conditions hold:

$\sum_{n=1}^{\infty} |E[X_n^2]| < \infty$ $\sum_{n=1}^{\infty} P(|X_n| > \varepsilon) < \infty$ for all $\varepsilon > 0$ $\sum_{n=1}^{\infty} E[|X_n|^3] < \infty$ Use Kolmogorov's three-series theorem to show that the series $\sum_{n=1}^{\infty} X_n$ converges almost surely.

Exercise 3 - Martingale convergence theorem:

Let X_n be a martingale with respect to a filtration F_n. Assume that there exists a constant K such that $|X_n| \leq K$ for all n. Prove that X_n converges almost surely. Hint: Apply the Doob's martingale convergence theorem.

Exercise 1.

Proof.

□

Exercise 2. *Ex 13.1*

Proof. Prove that every bounded martingale is uniformly integrable. Hint: Use the definition of a bounded martingale and the definition of uniformly integrable martingales.

Let X_n be a martingale. Show that if X_n is uniformly integrable, then $X_\infty = \lim_{n\to\infty} X_n$ exists almost surely, and $E[|X_\infty|] < \infty$. Hint: Use the martingale convergence theorem.

Let X_n be a martingale, and let T be a stopping time with respect to the filtration generated by X_n. Prove that if X_n is uniformly integrable, then X_T is integrable and $E[X_T] = E[X_0]$. Hint: Apply the optional stopping theorem to the stopped process $X_{n\wedge T}$.

Let Y_n be a sequence of i.i.d. random variables with $E[|Y_1|] < \infty$. Define the partial sum process $S_n = \sum_{k=1}^{n} Y_k$. Show that the martingale S_n is uniformly integrable if and only if $E[Y_1^2] < \infty$. Hint: Use the definition of uniformly integrable martingales and the properties of the partial sum process.

Let X_n be a martingale. Define a new martingale Y_n by $Y_n = X_n^2 - \sum_{k=1}^{n-1} E[X_k^2]$. Prove that if X_n is uniformly integrable, then Y_n is also uniformly integrable. Hint: Use the definition of uniformly integrable martingales and the properties of conditional expectation.

□

Exercise 3. *Ex 13.2.*

Proof.

Question 1: Let X_n be a martingale with respect to the filtration $\mathcal{F}_n$.

(a) Let X and Y be integrable random variables such that $E[X|\mathcal{F}_n] = Y$. Prove that $E[\phi(X)|\mathcal{F}_n] = \phi(Y)$ for any bounded, continuous function ϕ.

(b) Let $X_n = \sum_{k=1}^n Y_k$, where Y_k is a sequence of random variables with $E[Y_k|\mathcal{F}_{k-1}] = 0$. Prove that if $\sup_n E[|X_n|^2] < \infty$, then X_n is a uniformly integrable martingale.

Question 2: Consider a sequence of i.i.d. random variables X_n with $E[X_1] = 0$ and $Var(X_1) = \sigma^2 < \infty$. Define the partial sum process $S_n = \sum_{k=1}^n X_k$.

(a) Prove that the process $M_n = S_n^2 - n\sigma^2$ is a martingale.

(b) Let T be the stopping time defined as $T = \inf n \geq 1 : S_n = a$ or $S_n = -b$, where $a > 0$ and $b > 0$. Apply the optional stopping theorem to the martingale M_n and stopping time T to derive an expression for $E[S_T^2]$.

(c) Use the result from part (b) to compute the probability $P(S_T = a)$.

Question 3: Consider a continuous-time martingale X_t with respect to the filtration $\mathcal{F}_t$.

(a) State the Doob-Meyer decomposition theorem for continuous-time submartingales.

(b) Let X_t be a continuous-time submartingale. Prove that there exists an increasing, predictable process A_t such that $M_t = X_t - A_t$ is a continuous-time martingale.

(c) Let N_t be a Poisson process with intensity $\lambda > 0$. Show that the compensated Poisson process $M_t = N_t - \lambda t$ is a continuous-time martingale.

For each question, make sure to provide detailed proofs and arguments, using the definitions and properties of martingales, filtrations, stopping times, and any relevant theorems or lemmas.

Thus, we have shown that for any simple function g, $E[g(X)\mathbb{1}_A] = E[g(Y)\mathbb{1}_A]$ for all $A \in \mathcal{F}_n$. Now, let ϕ be a continuous bounded function. Since ϕ is Borel-measurable, we can approximate ϕ by a sequence of simple functions (g_n), such that $g_n \to \phi$ pointwise.

Question 1:

(a)Given that $E[X\mathbb{1}_A] = E[Y\mathbb{1}_A]$ for all $A \in \mathcal{F}_n$, we want to show that $E[\phi(X)|\mathcal{F}_n] = \phi(Y)$.

To show this, we will use the definition of conditional expectation. We need to show that for all $A \in \mathcal{F}_n$, we have $E[\phi(X)\mathbb{1}_A] = E[\phi(Y)\mathbb{1}_A]$.

First, let's establish that $\phi(Y)$ is $\mathcal{F}_n$-measurable. Since Y is $\mathcal{F}_n$-measurable, and ϕ is a continuous function, the composition $\phi(Y)$ is also $\mathcal{F}_n$-measurable.

Now, we cannot replace $\phi(X)$ with $\phi(Y)$ directly, as you correctly pointed out. Instead, we will use an approximation argument. Since ϕ is a continuous function on a compact set (because ϕ is bounded), it is uniformly continuous. This means that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x-y| < \delta$, then $|\phi(x) - \phi(y)| < \varepsilon$.

Let's approximate $\phi(X)$ by a simple function $s(X) = \sum_{i=1}^{m} c_i \mathbb{1} B_i(X)$, where B_i are Borel sets in the domain of X, and c_i are constants, such that $|s(X)-\phi(X)| < \varepsilon$ almost surely. Then, for each i, we have $E[s(X)\mathbb{1} A \cap B_i] = E[s(Y)\mathbb{1}_{A\cap B_i}]$. This implies $E[s(X)\mathbb{1}_A] = E[s(Y)\mathbb{1}_A]$ for all $A \in \mathcal{F}_n$.

Now, let's consider the difference $E[\phi(X)\mathbb{1}_A] - E[s(X)\mathbb{1}_A]$. We have:

$$\left|E[\phi(X)\mathbb{1}_A] - E[s(X)\mathbb{1}_A]\right| \leq E\left[|\phi(X) - s(X)|\mathbb{1}_A\right] \leq E\left[\varepsilon \mathbb{1}_A\right] = \varepsilon P(A)$$

where the inequality follows from the approximation we made. Similarly, we have $\left|E[\phi(Y)\mathbb{1}_A] - E[s(Y)\mathbb{1}_A]\right| \leq \varepsilon P(A)$. Hence,

$$\left|E[\phi(X)\mathbb{1}_A] - E[\phi(Y)\mathbb{1}_A]\right| \leq 2\varepsilon P(A).$$

Since this holds for any $\varepsilon > 0$, we have $E[\phi(X)\mathbb{1}_A] = E[\phi(Y)\mathbb{1}_A]$ for all $A \in \mathcal{F}_n$. Therefore,

Therefore, by the definition of conditional expectation, we have shown that $E[\phi(X)|\mathcal{F}_n] = \phi(Y)$. This completes the proof.

In summary, we approximated $\phi(X)$ by a simple function $s(X)$ and showed that the expectations of the approximations satisfy $E[s(X)\mathbb{1}_A] = E[s(Y)\mathbb{1}_A]$ for all $A \in \mathcal{F}_n$. Then, we used the uniform continuity of ϕ to argue that the difference between the expectations of $\phi(X)$ and $s(X)$, and $\phi(Y)$ and $s(Y)$, is arbitrarily small. This allowed us to conclude that $E[\phi(X)\mathbb{1}_A] = E[\phi(Y)\mathbb{1}_A]$ for all $A \in \mathcal{F}_n$, which implies that $E[\phi(X)|\mathcal{F}_n] = \phi(Y)$.

(b) From the given condition, $E[Y_k|\mathcal{F}_{k-1}] = 0$, so X_n is a martingale. Now we need to show that X_n is uniformly integrable. Note that:

$\sup_n E[|X_n|^2] = \sup_n E\left[\left|\sum_{k=1}^{n} Y_k\right|^2\right] \leq \sup_n \sum_{k=1}^{n} E[|Y_k|^2]$

By Cauchy-Schwarz inequality, we have:

$E[|X_n|^2] \leq \left(E\left[\sum_{k=1}^{n} |Y_k|^2\right]\right)^2 \leq n \sum_{k=1}^{n} E[|Y_k|^2]$

Since $\sup_n E[|X_n|^2] < \infty$, it follows that X_n is uniformly integrable.

Question 2:

(a) We want to show that $E[M_{n+1}|\mathcal{F}_n] = M_n$. Observe that:

$E[M_{n+1}|\mathcal{F}_n] = E[(S_{n+1}^2 - (n+1)\sigma^2)|\mathcal{F}_n] = E[((S_n + X_{n+1})^2 - (n+1)\sigma^2)|\mathcal{F}_n]$

Expanding, we get:

$E[M_{n+1}|\mathcal{F}n] = E[(S_n^2 + 2S_nX_{n+1} + X_{n+1}^2 - (n+1)\sigma^2)|\mathcal{F}_n]$

Using the linearity of conditional expectation:

$E[M_{n+1}|\mathcal{F}_n] = S_n^2 - n\sigma^2 + 2E[S_nX_{n+1}|\mathcal{F}_n] + E[X_{n+1}^2|\mathcal{F}_n] - \sigma^2$

As $E[X_{n+1}|\mathcal{F}_n] = 0$ and $E[X_{n+1}^2|\mathcal{F}_n] = \sigma^2$, we have:

$E[M_{n+1}|\mathcal{F}_n] = S_n^2 - n\sigma^2 = M_n$

Thus, M_n is a martingale.

(b) Since M_n is a martingale and T is a bounded stopping time, by the optional stopping theorem:

$E[M_T] = E[M_0]$

So, we have:

$E[M_T] = E[M_0]$

Substitute $M_T = S_T^2 - T\sigma^2$ and $M_0 = S_0^2 = 0$, we get:

$E[S_T^2] - E[T\sigma^2] = 0$

Thus,

$E[S_T^2] = E[T\sigma^2]$

(c) Using the result from part (b), we want to compute the probability $P(S_T = a)$. Note that:

$E[S_T^2] = a^2 P(S_T = a) + (-b)^2 P(S_T = -b)$

We also know from part (b) that $E[S_T^2] = E[T\sigma^2]$. Using the law of total expectation, we can write $E[T\sigma^2]$ as:

$E[T\sigma^2] = a^2 P(S_T = a) + (-b)^2 P(S_T = -b)$

Now, $P(S_T = a) + P(S_T = -b) = 1$, so we can solve for $P(S_T = a)$:

$P(S_T = a) = \frac{E[T\sigma^2]+b^2}{a^2+b^2}$

Question 3:

(a) The Doob-Meyer decomposition theorem states that for any continuous-time submartingale X_t, there exists a unique increasing, predictable process A_t such that $M_t = X_t - A_t$ is a continuous-time martingale.

(b) To prove the existence of such process A_t, we define the following process:

$A_t = \int_0^t E[dX_s|\mathcal{F}_{s-}]$

Since X_t is a submartingale, $E[dX_s|\mathcal{F}_{s-}] \geq 0$, and therefore, A_t is an increasing process. Furthermore, A_t is predictable as it is adapted to the filtration $\mathcal{F}_t$.

Now, we have:

$M_t = X_t - A_t = X_t - \int_0^t E[dX_s|\mathcal{F}_{s-}]$

We want to show that M_t is a martingale. Observe that:

$E[M_{t+h}|\mathcal{F}t] = E[Xt + h - \int_0^{t+h} E[dX_s|\mathcal{F}_{s-}]|\mathcal{F}_t]$

Using the linearity of conditional expectation and the fact that X_t is a submartingale, we have:

$E[M_{t+h}|\mathcal{F}t] = X_t + E[\int_t^{t+h} E[dX_s|\mathcal{F}s-]|\mathcal{F}t] - \int_0^t E[dX_s|\mathcal{F}s-]$

Since the integrand is predictable, we can apply the tower property:

$E[M_{t+h}|\mathcal{F}t] = X_t - \int_0^t E[dX_s|\mathcal{F}s-] = M_t$

Thus, M_t is a martingale, and the process A_t exists.

(c) For a Poisson process N_t with intensity $\lambda > 0$, we want to show that the compensated Poisson process $M_t = N_t - \lambda t$ is a continuous-time martingale.

First, we note that N_t has independent increments, so the process M_t also has independent increments. To show that M_t is a martingale, we need to verify that $E[M_{t+h}|\mathcal{F}_t] = M_t$ for all $t \geq 0$ and $h > 0$. We have:

$E[M_{t+h}|\mathcal{F}t] = E[Nt + h - \lambda(t + h)|\mathcal{F}t] = E[(Nt + h - N_t) - \lambda h|\mathcal{F}_t]$

Since $N_{t+h} - N_t$ is the number of events in the interval $(t, t + h]$ and has a Poisson distribution with parameter λh, its expectation is λh. Therefore:

$E[M_{t+h}|\mathcal{F}t] = E[(Nt + h - N_t) - \lambda h|\mathcal{F}_t] = \lambda h - \lambda h = M_t$

Thus, M_t is a continuous-time martingale.

$\square$

Exercise 4. *13.3*

Proof.

□

Exercise 5. *14.1*

Proof.

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