APPLIED FUNCTIONAL ANALYSIS HOMEWORK 8

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Discussed with classmates.

Exercise 1. *(9.1) in book*

Proof.

$\underline{\rho(A^*) \subset \overline{\rho(A)}}$:

For $\lambda \in \rho(A^*)$, we have that for all $x, y \in \mathcal{H}$

$$\langle (A^* - \lambda I)x, y\rangle = \langle A^* x, y\rangle - \overline{\lambda}\langle x, y\rangle = \langle x, (A - \overline{\lambda} I)y\rangle$$

and hence $(A^* - \lambda I)^* = (A - \overline{\lambda} I)$, which is also a bounded operator by open mapping theorem.

But since $(A^* - \lambda I)$ is invertible, so is it's adjoint since

$$I = I^* = \left((A^* - \lambda I)(A^* - \lambda I)^{-1}\right)^* = (A^* - \lambda I)^* \left((A^* - \lambda I)^{-1}\right)^*$$

hence $\overline{\lambda} \in \rho(A)$, and we are done with this direction.

$\underline{\rho(A^*) \supset \overline{\rho(A)}}$:

If $\overline{\lambda} \in \rho(A)$, then by above argument we know that $(A^* - \lambda I) = (A - \overline{\lambda} I)^*$, where the adjoint of a invertible operator is invertible is proven above, so $\lambda \in \rho(A^*)$.

$\square$

Exercise 2. *(9.3) in book.*

Proof.

If $\mu = \lambda$ then the result holds since both sides are 0.

If $\mu \neq \lambda$, since $(A - \lambda I)$ and $(A - \mu I)$ are invertible, we know that for any $y \in \mathcal{H}$, $\exists! x_1, x_2$ such that $(A - \lambda I)x_1 = (A - \mu I)x_2 = y$.

Now note that $A(A - \mu I) = (A - \mu I)A$ since $[A, A] = [A, I] = 0$ for lie brackets

$$\begin{aligned} R_\mu^{-1} R_\lambda^{-1}(x_1 - x_2) &= R_\mu^{-1}(A - \lambda I)(x_1 - x_2) = R_\mu^{-1}(y - Ax_2 + \lambda x_2) \\ &= (A - \mu I)(y - Ax_2 + \lambda x_2) = (Ay - \mu y) - (A - \mu I)Ax_2 + \lambda y \\ &= Ay - \mu y - Ay + \lambda y = (\lambda - \mu)y \end{aligned}$$

and since

$$(R_\lambda - R_\mu)y = x_1 - x_2$$

we know

$$R_\mu^{-1} R_\lambda^{-1}(\lambda - \mu)^{-1}(R_\lambda - R_\mu)y = y$$

hence

$$(R_\lambda - R_\mu)y = (\lambda - \mu)R_\lambda R_\mu y$$

where y is arbitrary, so the operators satisfy

$$R_\lambda - R_\mu = (\lambda - \mu)R_\lambda R_\mu.$$

□

Exercise 3. *(9.6) in book.*

Proof.

For all $\lambda \notin \overline{\{g(x), x \in \mathbb{R}\}}$, we know that $|g(x) - \lambda| \geq c > 0$ is bounded below and hence

$$\frac{f(x)}{g(x) - \lambda} \leq \frac{1}{c} f(x) \in L^2(\mathbb{R})$$

where we know that $(G - \lambda I)\dfrac{f(x)}{g(x) - \lambda} = f$ so

$$(G - \lambda I)^{-1} = \frac{1}{g(x) - \lambda}$$

which is well defined as we've justifed above.

Now, for any $\lambda = g(x_0)$ for some $x_0 \in \mathbb{R}$, we know that the function

$$f(x) = \frac{1}{\sqrt{\delta}} \mathbb{1}_{\{x_0 - \delta, x_0 + \delta\}}$$

is L^2 then as $\delta \to 0$ we know we're approaching an L^2 function, but the inverse under $G - \lambda I$ is not L^2 after the limit. Hence, $\lambda \in \sigma(G)$.

For $\lambda \in \overline{\{g(x), x \in \mathbb{R}\}}$ and $\lambda \neq g(x_0)$. Then since g is continuous bounded, $\lambda \neq \infty$ and $\lambda = \lim_{x \to \pm\infty} g(x)$. WLOG assume it's the limit at positive infinity.

Assume $\lambda \in \rho(G)$, but then we know that there's a sequence $g(n)$ that converges to λ where non of the terms is in $\rho(G)$, which means that $\rho(G)$ is not open, contradiction! Hence, $\lambda \in \sigma(G)$.

In conclusion, we know that $\sigma(G) = \overline{\{g(x), x \in \mathbb{R}\}}$.

Eigenvalues?

If G has eigenvalue λ corresponding to eigenvector f, then $Gf = \lambda f$, that is

$$(g(x) - \lambda) f(x) = 0$$

and since an eigenvector cannot be 0 it must follow that $g(x) - \lambda = 0$, hence only when $g(x) = c$ is a constant does the operator has eigenvalue c.

□

Exercise 4. *(9.7) in book.*

Proof.

Find K^*:

We use integral by parts and FTC to compute

$$\begin{aligned}\langle f, Kg\rangle &= \int_0^1 \overline{f}(x)\left(\int_0^x g(y)dy\right)dx = -\int_0^1 \left(\int_0^x g(y)dy\right) d\left(\int_x^1 \overline{f}(z)dz\right) \\ &= -\left(\int_0^x g(y)dy\right)\left(\int_x^1 \overline{f}(z)dz\right)\Big|_0^1 + \int_0^1 \left(\int_x^1 \overline{f}(z)dz\right) d\left(\int_0^x g(y)dy\right) \\ &= \int_0^1 \left(\int_x^1 \overline{f}(z)dz\right) g(x)dx = \left\langle \int_x^1 \overline{f}(z)dz, g\right\rangle = \langle K^* f, g\rangle\end{aligned}$$

which means

$$K^* f = \int_x^1 f(z)dz.$$

$||K|| = \dfrac{2}{\pi}$:

Source:

https://math.stackexchange.com/questions/155899/norm-of-integral-operator-in-l-2

We know

$$||K||^2 = \sup_{||f||=1} |\langle Kf, Kf\rangle| = \sup_{||f||=1} |\langle f, K^*Kf\rangle| = ||K^*K||$$

and thus we consider the operator K^*K.

Note that by example 9.23 if we let $k(x, y) = \mathbb{1}_{y\leq x}$ then

$$Kf(x) = \int_0^x f(y)dy = \int_0^1 k(x, y)f(y)dy$$

where since $k \leq 1$ we have that it is square integrable on the unit box, hence the operator K is a Hilbert-Schmidt operator, and hence compact. Similarly let $k(x, y) = \mathbb{1}_{y\geq x}$ we have that K^* is compact.

Thus, since K^*K is a combination of 2 compact operators (in fact one of them being compact is sufficient) we know that it is compact. Moreover, $(K^*K)^* = K^*K$ is self adjoint, and so we can use the result from the first part of the proof of theorem 9.16, that there is an eigenvalue $\lambda = \pm||K^*K||$.

Thus, we have

$$K^*Kf = \lambda f$$

for some eigenfunction f. But then note that by FTC we have

$$\frac{d^2}{dx^2}K^*Kf = -f(x); \qquad f(1) = 0, f'(1) = 0$$

and since this side is twice differentiable, we have that f is twice differtiable and

$$\lambda f'' = -\lambda f.$$

But we know the solution to this problem is $\alpha e^{iwx} + \beta e^{-iwx}$ for $w^2 = 1/\lambda$. (We used that K is nonzero.) But just plugging in to the original equation $K^*Kf = \lambda f$ we have

$$f = \cos\left(\frac{x}{\sqrt{|\lambda|}}\right)$$

and initial conditions gives us that

$$\frac{1}{\sqrt{|\lambda|}} = \frac{(2k+1)\pi}{2}; k \geq 0$$

and hence when $k = 0$ we get the upper bound

$$||K|| = \sqrt{||K^*K||} = \sqrt{|\lambda|} \leq \frac{2}{\pi}.$$

But we can just pick the above eigenfunction $f = \cos\left(\frac{\pi x}{2}\right)$ and get that the equality is attained, so $||K|| = \dfrac{2}{\pi}$.

<u>$r(K) = 0$:</u>

To show this we use the fact that $r(K) = \lim_{n\to\infty} ||K^n||^{1/n}$, for which we compute K^n.

$$\begin{aligned} K^2 f &= \int_0^x \int_0^y f(z)dzdy = -\int_0^x \int_0^y f(z)dzd(x-y) \\ &= (x-y)\left(\int_0^y f(z)dz\right)\Big|_{y=0}^x + \int_0^x f(y)(x-y)dx = \int_0^x f(y)(x-y)dx. \end{aligned}$$

So we now show by induction that

$$K^n f = \int_0^x f(y)\frac{(x-y)^{n-1}}{(n-1)!}dy$$

for which we know holds when $n = 1, 2$. So assuming that it holds for $n = m$, for $n = m + 1$ we use the fact that

$$\int_0^x \int_0^y 1dzdy = \int_0^x \int_z^x 1dydz$$

just by drawing out the lower right triangle in the 2d integration plane. So we get

$$\begin{aligned}
K^{m+1}f &= \int_0^x \int_0^y f(z)\frac{(y-z)^{m-1}}{(m-1)!}dzdy = \frac{1}{(m-1)!}\int_0^x \int_0^y f(z)(y-z)^{m-1}dzdy \\
&= \frac{1}{(m-1)!}\int_0^x \int_z^x f(z)(y-z)^{m-1}dydz = \frac{1}{(m-1)!}\int_0^x f(z)\frac{(x-z)^m}{m}dz \\
&= \int_0^x f(y)\frac{(x-y)^m}{m!}dy
\end{aligned}$$

as what we'd expected. So we get

$$K^n f = \int_0^x f(y)\frac{(x-y)^{n-1}}{(n-1)!}dy.$$

To bound the norm of K^n we have

$$||K^n f|| = \int_0^x f(y)\frac{(x-y)^{n-1}}{(n-1)!}dy \le \frac{1}{(n-1)!}||f||$$

so $||K^n|| \le \frac{1}{(n-1)!}$. So

$$r(K) = \lim_{n\to\infty} ||K^n||^{1/n} \le \lim_{n\to\infty}\left[\frac{1}{(n-1)!}\right]^{1/n} = 0$$

since $(n!)^{1/n} \to \infty$ (by Sterlings).

$\underline{0 \in \sigma_c(K)}$:

First, since $\sigma(K)$ is nonempty, and it cannot contain anything else other than 0, we know $0 \in \sigma(K)$.

We first show that $0 \neq \sigma_p(K)$, which is because that if $Kf = 0$ then we know that the integral from 0 to x of f^2 is 0 for all x. This means $f = 0$, but 0 is not an eigenvector, so we have $0 \neq \sigma_p(K)$.

If $0 \in \sigma_r(K)$, then proposition 9.12 tells us that $\overline{0} = 0$ is in the point spectrum of K^*, but for a similar argument as above we know that 0 cannot be an eigenvalue of K^*.

So $0 \in \sigma_c(K)$ since that's the only place left.

$\square$

Exercise 5. *(9.8) in book.*

Proof.

(a): point spectrum of S is empty:

If $\lambda \in \sigma_p(S)$, then it cannot be 0 since otherwise the corresponding x would be 0, not an eigenvector then. Assume $\lambda \neq 0$ we have

$$\cdots = \frac{1}{\lambda^2}x_3 = \frac{1}{\lambda}x_2 = x_1 = \lambda x_0 = \lambda^2 x_{-1} = \dots$$

and since each x_i is defined we must have

$$\lim_{n\to\infty} \frac{1}{\lambda^n} < \infty \quad and \quad \lim_{n\to\infty} \lambda^n < \infty$$

which means $|\lambda| = 1$. But this cannot be since then we have $|x_i| = c$ is a non-zero constant, which means x is not square summable, hence not in l^2.

So $\sigma_p(S) = \emptyset$.

(b): For $|\lambda| > 1$, $\text{Ran}(\lambda I - S) = l^2(\mathbb{Z})$:

First note that $\lambda I - S$ is well defined since λx is still square summable, with norm a $|\lambda|^2$ scaling of $||x||$.

Note that $||Sx|| = ||x||$ since it's just a change in indices in the series. Hence, $||S|| = 1$. But then we know for all $|\lambda| > 1$ outside the norm ball, $\lambda \in \rho(S)$, so $\lambda I - S$ is invertible, hence $\text{Ran}(\lambda I - S) = l^2(\mathbb{Z})$.

(c): For $|\lambda| < 1$, $\text{Ran}(\lambda I - S) = l^2(\mathbb{Z})$:

For $\lambda = 0$ we know it's true since S is invertible (inverse is shifting to the other direction).

For $\lambda \neq 0$, note that the norm $||S|| = 1$ so by triangle inequality we have

$$||S - \lambda I|| + ||\lambda I|| \geq ||S|| \quad \Rightarrow \quad ||S - \lambda I|| \geq ||S|| - |\lambda| = 1 - |\lambda| > 0$$

is bounded from below, thus by a priori estimate we know that the range is closed and we have the decomposition

$$l^2 = \text{Ran}(\lambda I - S) \oplus \ker(\lambda I - S)^*$$

where we can compute

$$\langle (\lambda I - S)x, y\rangle = \langle \lambda x, y\rangle - \langle Sx, y\rangle = \langle x, (\overline{\lambda} - S^*)y\rangle$$

where S^* is the shifting to left. Now we have

$$(\overline{\lambda} - S^*)x = 0 \Rightarrow \lambda x_i = x_{i+1} \quad \forall i$$

so either $x \notin l^2$ or $x = 0$, in which case we know the first is impossible, so $x = 0$ hence $\ker(\lambda I - S)^* = \{0\}$ and by orthogonal decomposition we have

$$\text{Ran}(\lambda I - S) = l^2(\mathbb{Z}).$$

(d): Unit circle is in the spectrum, and that's purely continuous:

We first show that $\sigma(S) = \sigma_c(S)$. This is because we know already that $\sigma_p(S) = \emptyset$ by (a). Moreover, by the decomposition

$$l^2 = \overline{\text{Ran}(\lambda I - S)} \oplus \ker(\lambda I - S)^*$$

where

$$(\bar{\lambda} - S^*)x = 0 \Rightarrow \lambda x_i = x_{i+1} \quad \forall i$$

which means that $|x_i| = c$ is a constant, hence impossible for $c \neq 0$. So $\ker(\lambda I - S)^* = \{0\}$ and thus $\text{Ran}(\lambda I - S)$ has closure l^2, so it's dense in l^2 and so $\lambda \notin \sigma_r(S)$ by definition.

Now it's sufficient to prove that for all $|\lambda| = 1$, there exists an element in l^2 that is not the image of any element under $\lambda I - S$. Let $\lambda = e^{i\theta}$. Assume every element as a preimage, then for all $x \in l^2$ we have

$$x_i = \{(\lambda I - S)y\}_i = e^{i\theta} y_n - y_{n+1}$$

is square summable. By triangle inequality we get

$$|e^{i\theta} y_n - y_{n+1}| \geq |y_n| - |y_{n+1}|$$

which means $|y_n| - |y_{n+1}|$ is square integrable.

But for $z_n = e^{i\theta n} \dfrac{1}{n^{1/3}}$ we know that $z_n \notin l^2$, while $|z_n| - |z_{n+1}|$ is square summable. But $\left|e^{i\theta} z_n - z_{n+1}\right| = \Big||z_n| - |z_{n+1}|\Big|$ since they have the same angle. Let

$$x_n := e^{i\theta} z_n - z_{n+1}$$

then $x \in l^2$ and it's inverse can only be $z_n \notin l^2$, so $x \notin \text{Ran}(\lambda I - S)$. Hence $\lambda \notin \rho(S)$ and we are done.

$\square$

Exercise 6. *(9.12) in book.*

Proof.

Hilbert Schmidt norms are independent of basis:

This is sort of done in one of the first few homeworks...

We can decompose $f_n = \sum_{m=1}^{\infty} c_{n,m} e_m$ in a well-defined way by Bessel's inequalities.

Thus we have

$$\begin{aligned}
\sum_{n=1}^{\infty} ||Af_n||^2 &= \sum_{n=1}^{\infty} \langle Af_n, Af_n \rangle = \sum_{n=1}^{\infty} \left\langle A \sum_{i=1}^{\infty} c_{n,i} e_i, A \sum_{j=1}^{\infty} c_{n,j} e_j \right\rangle \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\sum_{n=1}^{\infty} c_{n,i} c_{n,j} \right) \left\langle Ae_i, Ae_j \right\rangle = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} (C^T C)_{i,j} \left\langle Ae_i, Ae_j \right\rangle \\
&= \sum_{i=1}^{\infty} \langle Ae_i, Ae_i \rangle = \sum_{n=1}^{\infty} ||Ae_n||^2
\end{aligned}$$

since C is a basis to basis norm 1 operator hence $C^T C = I$ and thus only when $i = j$ we have 1, others vanish.

The Hilbert Schmidt norm of A and A^* are the same:

We can decompose $Ae_n = \sum_{m=1}^{\infty} \langle Ae_n, e_m \rangle e_m$ and hence we have

$$\sum_{n=1}^{\infty} ||Ae_n||^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle Ae_n, e_m \rangle|^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\langle e_n, A^* e_m \rangle|^2 = \sum_{m=1}^{\infty} ||A^* e_m||^2$$

and hence we are done since the Hilbert Schmidt norm is just taking square root on both sides.

$\square$