MEASURE THEORETIC PROBABILITY III HW 1

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Discussed with classmates.

Exercise 1.

Proof.

There's three parts to this problem:

(1) Show that this limit is well-defined;
(2) Show that converging from left has a limit;
(3) Show that converging from right has a limit.

<u>The limit is well-defined</u>:

If $s_n \to t$ and $u_n \to t$ from either above or below, and both limits exists, then we denote $\lim_{n\to\infty} X_{s_n} \to X_{t+}$ and $\lim_{n\to\infty} X_{u_n} \to X'_{t+}$, and we need to show that

$$X_{t+} = X'_{t+}.$$

But since both are coming from the same side we can join the sequence and make a new sequence a_n where we arrange terms so that a_n is monotone in the same direction. Thus for a.s. ω we know that in $\mathbb{R}$

$$X_{s_n}(\omega) \to X_{t+}(\omega)$$

and

$$X_{u_n}(\omega) \to X'_{t+}(\omega).$$

But by the second part that we'll show later that we also know $X_{a_n} \to X''_{t+}$ exists, then uniqueness of sequential limit in $\mathbb{R}$ implies

$$X_{t+}(\omega) = X'_{t+}(\omega) = X''_{t+}(\omega)$$

and this holds for a.s. $\omega \in \Omega$ so

$$X_{t+} = X'_{t+}.$$

<u>Converging from left has a limit:</u>

We just use the same proof in book. Here $s_n \uparrow t$.

Note that all is finite then by the fact that it is a Martingale we know they are L^1. Define (note we use extended real line)

$$\begin{aligned}\Lambda :&= \{\omega : X_{s_n}(\omega) \text{ does not converge to a limit in } [-\infty, \infty]\}\\ &= \{\omega : \liminf X_{s_n}(\omega) < \limsup X_{s_n}(\omega)\}\\ &= \bigcup_{a,b\in\mathbb{Q}:a<b} \{\omega : \liminf X_{s_n}(\omega) < a < b < \limsup X_{s_n}(\omega)\}\\ &=: \bigcup \Lambda_{a,b}\end{aligned}$$

but

$$\Lambda_{a,b} \subset \{\omega : U_{s_\infty}[a,b](\omega) = \infty\}$$

that is, if we view $Y_n := X_{s_n}$ as a martingale (which it is!) in its own right, then apply the corollary 11.4 we know $\mathbb{P}(U_{Y_n}[a,b] = \infty) = 0$. But this is exactly $\Lambda_{a,b}$. So taking countable union of measure 0 set we get that $\mathbb{P}(\Lambda) = 0$, hence concluding the proof.

What we have now is that there is an event A_1 such that all left rational limits are convergent with $\mathbb{P}(A_1) = 1$.

<u>Converging from right has a limit:</u>

We want to show that there is an event A_2 such that all right rational limits are convergent with $\mathbb{P}(A_2) = 1$. If we can do this just take $\Omega * = A_1 \cap A_2$ then since both has measure 1, so is $\Omega *$, and we are done.

But for this specific part we cannot use the same technique as above since X_{s_n} is no longer a Martingale. But in Durrett book chapter 4.7 there's detailed introduction to backward martingale, and in particular, Lemma 4.7.1 claims

Lemma 0.1. *If X_n for $n \le -1$ is a backward Martingale with decreasing Filtration (i.e. $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$), then $X_{-\infty} = \lim_{n\to-\infty} X_n$ exists a.s. and in L^1.*

Now we note that by the same deduction as in the last question, we have the result.

□

Exercise 2.

Proof.

Using Doob's decomposition we note that since Z_n is adapted, then there is a unique decomposition

$$Z_n = Z_0 + M_n + A_n = M_n + A_n$$

where M_n is a Martingale and A_n is predicable.

So our goal is to show $A_n \equiv 0$. We know already that $A_0 = 0$ so let's try work with that.

We know M_n is a martingale so $\mathbb{E}[M_T] = 0$ and hence take $\tau = k$ be the constant stopping time we have

$$\mathbb{E}[A_k] = \mathbb{E}[A_k] + \mathbb{E}[M_k] = \mathbb{E}[Z_k] = \mathbb{E}[Z_\tau] = 0.$$

Moreover, for any $F \in \mathcal{F}_0 \subset \mathcal{F}_1$ consider the stopping time

$$T_1 = \begin{cases} 1 & F \\ 0 & F^c \end{cases}$$

and it's a stopping time since $\{T = n\} \in \mathcal{F}_n$. Of course it's bounded and we can apply condition.

But then let's compute:

$$0 = \mathbb{E}[Z_{T_1}] = \mathbb{E}[A_{T_1}] = \int_F A_1 d\mathbb{P} + \int_{F^c} A_0 d\mathbb{P} = \int_F A_1 d\mathbb{P}.$$

So A_1 is 0 on all $\mathcal{F}_0$ measurable sets. By definition A_1 is $\mathcal{F}_0$ measurable so $A_1 \equiv 0$.

Now, for any $F \in \mathcal{F}_1 \subset \mathcal{F}_2$ consider the stopping time define

$$T_2 = \begin{cases} 2 & F \\ 1 & F^c \end{cases}$$

then apply the same argument above we'll get that

$$0 = \int_F A_2 d\mathbb{P}$$

and here $F \in \mathcal{F}_1$ is arbitrary. So $A_k = 0$ for all k by an induction argument, and we have

$$Z_n = M_n + 0 = M_n$$

is a Martingale.

$\square$

Exercise 3. *Ex 10.7.*

Proof.

Why T satisfies the condition:

For any n, we know that

$$\mathbb{P}(T \leq n + b|\mathcal{F}_n) \geq \min\{p, q\}^b =: \varepsilon$$

since even in the worst case there's some probability that it will reach the upper bar or lower bar. Thus by 10.5 we know $\mathbb{E}[T] < \infty$.

M is a Martingale:

M is integrable since $S_n \leq b$. M is adapted just because S_n is. To show the last property we note that for any $F \in \mathcal{F}_n$ we will have that $S_n = k$ for some k, thus

$$\int_F \mathbb{E}[M_{n+1}|\mathcal{F}_n]d\mathbb{P} = \left(\frac{q}{p}\right)^{k+1} p + \left(\frac{q}{p}\right)^{k-1} q = \frac{q^k}{p^k} = M_n(\omega \in F)$$

and so we are done since F is arbitrary.

N is a Martingale:

N is integrable since $S_n \leq b$. N is adapted just because S_n is. To show the last property we note that for any $F \in \mathcal{F}_n$ we will have that $S_n = k$ for some k, thus

$$\begin{aligned}\int_F \mathbb{E}[N_{n+1}|\mathcal{F}_n]d\mathbb{P} &= (k + 1 - (n + 1)(p - q))\, p + (k - 1 - (n + 1)(p - q))^{k-1}\, q \\ &= k + p - q - np^2 - p^2 + nq^2 + q^2 = k + (q - p)(n + 1)(p + q) - (q - p) \\ &= k - (p - q)n = N_n(\omega \in F)\end{aligned}$$

and so we are done since F is arbitrary.

Find $\mathbb{P}(S_T = 0)$ and $\mathbb{E}[S_T]$:

$$\mathbb{P}(S_T = 0) = \mathbb{P}(M_T) = 1$$

and we can use the fact that M is a Martingale to compute

$$\mathbb{E}[M_T] = \mathbb{E}[M_0] = \left(\frac{q}{p}\right)^a$$

and thus

$$\mathbb{P}(S_T = 0) + (1 - \mathbb{P}(S_T = 0)) * \left(\frac{q}{p}\right)^b = \left(\frac{q}{p}\right)^a$$

thus

$$\mathbb{P}(S_T = 0) = \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b}{1 - \left(\frac{q}{p}\right)^b}$$

and

$$\mathbb{E}[S_T] = \mathbb{P}(S_T = 0) * 0 + b * (1 - \mathbb{P}(S_T = 0)) = b - b \cdot \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^b}{1 - \left(\frac{q}{p}\right)^b}.$$

□

Exercise 4. *10.8*

Proof.

Let $\Theta(\omega) = p \in [0,1]$ be the probability of the head toss of the minted coin, then

$$\begin{aligned}
\mathbb{P}(B_n = k) &= \int_\Omega \mathbb{1}_{B_n=k} d\mathbb{P} = \int_\Omega \mathbb{P}(B_n = k|\sigma(\Theta))d\mathbb{P} \\
&= \int_0^1 \mathbb{P}(B_n = k|\Theta = p)dp = \int_0^1 \binom{n}{k} p^k(1-p)^{n-k}dp \\
&\overset{IBP}{=} \binom{n}{k} \left[\frac{1}{k+1} p^{k+1}(1-p)^{n-k-1}\Big|_{p=0}^1 + \frac{n-k-1}{k+1}\int_0^1 p^{k+1}(1-p)^{n-k-1}dp \right] \\
&\vdots \\
&\overset{IBP}{=} \binom{n}{k} \left[\prod_{i=1}^{n-k-1} \frac{n-k-i}{k+i} \right] \int_0^1 p^n (1-p)^0 dp \\
&= \frac{n!}{k!(n-k)!} \cdot \frac{(n-k)!k!}{n!} \cdot \frac{1}{n+1} = \frac{1}{n+1}
\end{aligned}$$

and thus is the same as in 10.2.

(Below is from discussion).

Let $B = (B_1, B_2, \dots, B_n)$ be the random vector, then for us to use Bayes rule we'll also have to compute first for b is a possible outcome

$$\mathbb{P}(B = b|\Theta = p) = \prod_{k=1}^n p^{\mathbb{1}_{b_k = b_{k-1}+1}} (1-p)^{\mathbb{1}_{b_k = b_{k-1}}} = p^{B_n}(1-p)^{n-B_n}$$

and thus taking marginal probability we have

$$\mathbb{P}(B = b) = \int_0^1 p^{B_n}(1-p)^{n-B_n}dp = \frac{B_n!(n-B_n)!}{(n+1)!}$$

and then Bayes rule we have

$$\mathbb{P}(\Theta = p|B = b) = \frac{\int_{\Theta=p} \mathbb{P}(B = b|\sigma(\Theta))d\mathbb{P}}{\int_\Omega \mathbb{P}(B = b|\sigma(\Theta))d\mathbb{P}} = \frac{(n+1)!}{B_n!(n-B_n)!} p^{B_n}(1-p)^{n-B_n}d\Theta$$

which means that it is the regular conditional pdf.

□

Exercise 5. *10.9*

Proof.

We have

$$\mathbb{E}[X_T; T < \infty] \overset{Non-negative}{\leq} \mathbb{E}[X_T] = \mathbb{E}[\liminf X_{T\wedge n}] \overset{Fatou}{\leq} \liminf \mathbb{E}[X_{T\wedge n}]$$
$$\overset{SuperMartingale}{\leq} \mathbb{E}[X_0].$$

Then we can loosen the bound and say $T \leq c$ to use Markov to get

$$c\mathbb{P}(\sup_n X_n \geq c) \leq \mathbb{E}[X_T; T < \infty] \leq \mathbb{E}[X_0].$$

□