SET THEORY

ABSTRACT. We'll cover character theory, non-commutative algebra, and some possibly advanced topics such as representation in $\mathbb{S}_n$

CONTENTS

1. 9/26: Representation and examples

Representation is about explicitly representing a group. For instance, the permutation of roots of polynomial degree 4 is a representation of $\mathbb{S}_4$. Also a rotation of tetrahedron would be a representation of $\mathbb{S}_4$.

We know that group actions are decided by the group structure, since it's decided by the disjoint orbits of the action.

Def 1.1. *For G a group, V a vector space over F, $GL(V)$ the set of linear maps $L : V \to V$ that are isomorphic. Let's say V is finite dimension because why not, and $e_1, \dots, e_n$ is a set of basis, and we sometimes also denote*

$$GL(V) = GL_n(F) = \{A_{n\times n} | |A| = 0\}.$$

Then, the representation of G on V is a group homomorphism $G \xrightarrow{\rho} GL(V)$ with $g \mapsto \rho(g) = A_g$.

Just to be more explicit, for ρ to be a homomorphism is to require

- $A_e = E_n$, the identity matrix.
- $A_{g_1 g_2} = A_{g_1} A_{g_2}$.
- $A_{g_1^{-1}} = A_{g_1}^{-1}$.

Let's see some examples of this.

Example 1.2. $\dim(\rho) = 1$.

In this case $GL(V)$ really is isomorphic to $\mathbb{C}^*$, the multiplicative operators. So we naturally find 2 representations:

- $\mathbb{S}_n \xrightarrow{\text{sign}} \mathbb{C}^*$ where $\sigma \mapsto \text{sgn}(\sigma) \in \{\pm 1\}$.
- $G \to \mathbb{C}^*$ is trivial, i.e. $g \mapsto 1$.

Example 1.3. *For $G = \mathbb{S}_3$.*

(1) The trivial representation is denoted $\begin{array}{|c|c|c|}\hline & & \\ \hline \end{array} =: (3)$.

(2) The alternating sign representation is denoted $\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} := (1,1,1)$.

(3) The standard representation, i.e. for $V := \{(x_1, x_2, x_3) \in \mathbb{C}^3 | x_1 + x_2 + x_3 = 0\}$ such that $\mathbb{S}_3 \circlearrowright V$ with

$$\sigma((x_1, x_2, x_3)) = (x_{\sigma(x_1)}, x_{\sigma(x_2)}, x_{\sigma(x_3)})$$

is denoted $\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} \end{array} := (2,1)$.

Example 1.4. *Regular representations.*

Say $G = \{g_1, \dots, g_n\}$ then the <u>regular representation</u> on $V = \mathbb{C}^n$ with basis $e_{g_1}, \dots, e_{g_n}$ is such that

$$\rho(g)e_{g_i} = e_{g \cdot g_i}.$$

For an example, let's say $G = \mathbb{Z}/2\mathbb{Z} = \{e, g\}$ and $V = \mathbb{C}^n$. Then we have $\rho : G \to GL_n(\mathbb{C})$ with $\rho(g) = A$.

Well one might wonder why we care about representing groups at all, since that's arbitrary. But really there's a complete structure of it.

In the above case, $A^2 = \rho(g^2) = E$ which implies $(A - E)(A + E) = 0$. Maybe not much can be said about general matrix equations, but we can for this case. First, eigenvalues are ± 1, and if we check $\ker(A - E)$ and $\operatorname{Im}(A - E)$ we'll see that they can't intersect, because if they do then we can assume $v \in \ker(A - E) \cap \operatorname{Im}(A - E)$ which means

$$v = (A - E)w = Aw - w$$

in kernel implies $Av = v$ and thus

$$Aw - w = A(Aw - w) = A^2 w - Aw = w - Aw \quad \Rightarrow \quad Aw = w$$

which means $v = 0$, so we are done.

Thus, $V = \ker(A - E) \oplus \operatorname{Im}(A - E)$ so after conjugation we know

$$A \sim \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array}\right)$$

where the first k dimensions are dimension of the kernel, and the rest are image.

Notably, we can do this for all finite groups.

1.1. **Frobenius determinant.** Let $G = \{g_1, \dots, g_n\}$ then we have a product table as the group. If we denote $x_1, \dots, x_n$ as variables, then we can map $g_i \mapsto x_i$ and compute the determinant of the "multiplication table", so for instance when $G = \mathbb{Z}/2\mathbb{Z}$ the table is and

$$\begin{array}{c|cc} \cdot & e & g \\ \hline e & e & g \\ g & g & e \end{array}$$

thus the determinant is

$$\det \begin{pmatrix} x_e & x_g \\ x_g & x_e \end{pmatrix} = (x_e - x_g)(x_e + x_g).$$

Ok, let's try for $G = \mathbb{Z}/3\mathbb{Z}$. If we use a, b, c to denote the variables then

$$\begin{aligned} \det &= 3abc - a^3 - b^3 - c^3 = -(a+b+c)(a^2+b^2+c^2-ab-bc-ac) \\ &= (a+b+c)(a+\zeta b+\zeta^2 c)(a+\zeta^2 b+\zeta c) \end{aligned}$$

where $\zeta^3 = 1$ is a generator of $\mathbb{Z}_3^*$. It's very surprising that we can do the factorization, but even more surprising that for a general finite group, the Frobenius determinant has

$$\det = P_1^{d_1} P_2^{d_2} \cdots P_k^{d_k}$$

where $P_1, \dots, P_k$ are irreducible polynomials in $x_g, \dots, x_{g_n}$ and, very shockingly, the degree of P_k is d_k and $k = \#$ conjugates.

As an illustration

$$\text{Fro } \det(\mathbb{S}_3) = (x_{g_1} + \cdots + x_{g_n})(x_{g_1} \pm \cdots \pm x_{g_n})P_3^2$$

where degree of P_3 is 2.

2. 9/29: Categories of representations

The goal is to understand representations of G, which is homomorphisms from $G \to GL(V)$.

Def 2.1. *We look at a* <u>*category*</u> $\underline{C}$*, which consists of a class of objects,* $Ob(\underline{C})$*, a set of* <u>*morphisms (arrows) that are*</u> $\mathrm{Hom}(a,b)$*, and* $\mathrm{Hom}(a,b)\times\mathrm{Hom}(b,c) = \mathrm{Hom}(a,c)$*. Also, they must satisfy*

- *Associativity.*
- *For any object a, there* $\exists\, \mathrm{id}_a \in \mathrm{Hom}(a,a)$ *such that* $id_b \circ f = f = f \circ \mathrm{id}_a$.

Examples of categories are:

- Sets;
- Vector spaces; Here, morphisms are linear maps.
- Groups, morphisms are homomorphisms.
- Category of Representations.

Now, let G be a set and V, W denote 2 particular representations from G to $GL(V)$ and $GL(W)$.

Def 2.2. *A morphism of G-representation is a map to and from vector spaces* $f : V \to W$ *such that*

(1) *f is linear;*

(2) *For all $g \in G$, we have*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_V(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "\rho_W(g)"] \\ V' \arrow[r, "f"] & W' \end{tikzcd}$$

An example can be that $V = \mathbb{C}^n$, $W = \mathbb{C}^m$, and $\rho_V(g) = A_g$ is a $n \times n$ matrix.

Another example is S_3. In example 1.3, we've seen all the representations of S_3. Now we use V_{perm} to denote the standard representation where $\sigma((x_1, x_2, x_3)) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$, and use $V_{(3)}$ to denote the trivial representation.

What is a morphism between those representations? We have $V_{(3)} \to V_{\text{perm}}$ by
$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "\rho_{(3)}"] & (x,x,x) \arrow[d, "\rho_{\text{perm}}"] \\ x \arrow[r, "f"] & (x,x,x) \end{tikzcd}$$
and for the other direction since we want eliminate symmetrical difference, we can use
$$\begin{tikzcd} (x_1,x_2,x_3) \arrow[r, "f"] \arrow[d, "\rho_{\text{perm}}"] & x_1+x_2+x_3 \arrow[d, "\rho_{(3))}"] \\ (x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}) \arrow[r, "f"] & x_1+x_2+x_3 \end{tikzcd}$$

Def 2.3. *An morphism of representations $f : V \to W$ is an* <u>*isomorphism*</u> *if it is an isomorphism of the corresponding vector spaces.*

For a general category isomorphism, we need for f, $\exists g$ such that $f \circ g = \mathrm{id}_W$ and $g \circ f = id_V$. For a map from V to V it might be that
$$\begin{tikzcd} V \arrow[r, "F"] \arrow[d, "A_g"] & V \arrow[d, "B_g"] \\ V \arrow[r, "F"] & V \end{tikzcd}$$
where we have $FA_g = B_g F$ so $FA_g F^{-1} = B_g$, namely they are conjugates.

Def 2.4. *For V a representation of group G and $\rho_v : G \to GL(V)$, a* <u>*subrepresentation of V*</u> *is a subspace $W \subset V$ such that $\forall w \in W, \forall g \in G, \rho_v(g)(w) \in W$, i.e. $\rho_v(g)(W) = W$ or in a simpler form we denote as $gW = W$.*

Examples are $V_{(3)} \subset V_{\text{perm}}$ so does $V_{(2,1)} \subset V_{\text{perm}}$.

Def 2.5. *A G representation is* <u>*simple*</u> *(irreducible) if it has only 2 sub-representation, i.e.* $(0), V$.

Def 2.6. *V_1, V_2 are 2 representations, then $V_1 \oplus V_2$ is a new representation where the space*
$$V_1 \oplus V_2 = \{(v_1, v_2) | v_1 \in V_1, v_2 \in V_2\}$$
where
$$\rho_{V_\oplus}(V_1, V_2) = (\rho_{V_1}(g)v_1, \rho_{V_2}(g)v_2).$$

As an example, we can decompose
$$\rho_{V_{\text{perm}}} = V_{(3)} \oplus V_{(2,1)}$$
where the subspaces are simple.

Next week we're going to prove the theorem:

Theorem 2.7. *For G a finite group, $F = \mathbb{C}$, then*

(1) There are finitely many simple representations $V_1, \ldots, V_s$ where $s = \#$ of conjugate classes.

(2) *For every G representation V,* $\exists! n_1, \dots, n_s \geq 0$ *such that*

$$V \cong V_1^{n_1} \oplus \cdots \oplus V_s^{n_s}.$$

Some examples for this:

Example 2.8.

(1) For $G = \mathbb{Z}/2\mathbb{Z} = S_2$, we've shown that we can decompose to

$$A \sim \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array}\right)$$

where the left part is V_1 and right is V_2.

(2) For $G = S_3$, we have $V_{(3)}, V_{(1,1,1)}, V_{(2,1)}$.

3. 10/2: Constructing new representations from old ones

Today we construct, from a representation of G, the spaces $V^*, V \otimes V, \mathbb{S}^n V, \Lambda^n V$. Recall from linear algebra that for V, W two vector spaces over F the set $\mathrm{Hom}_F(V, W)$, the vector spaces from $V \to W$ has dimension of the product of dimensions of V and W.

The intuition here is that all actions that's natural for linear algebra should be true for V, W as representations, and for $L \in \mathrm{Hom}_F(V, W)$ we can define gL by the relation
$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\rho_V(g)"] & V \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "gL"] & V \end{tikzcd}$$
which means we can construct it with

$$gL := \rho_w(G)L[\rho_V(g)]^{-1}$$

where we call the collection of gL for all g, L to be $\mathrm{Hom}_G(V, W)$ which we can guess that $\mathrm{Hom}_G(V, W) \subset \mathrm{Hom}_F(V, W)$ since the first is just the second with extra conditions.

3.1. **Dual Representation.**

Define $\mathrm{Hom}_F(V, F) := V^*$ to be the dual vactor space of V, i.e. the functionals on it. We denote the basis of V and V^* as $V : e_1, \dots, e_n$; $V^* : e^1, \dots, e^n$ where

$$e^i(x_1e_1 + \cdots + x_ne_n) = x_i.$$

Corollary 3.1. $\dim(V) = \dim(V^*)$ *by finite dimension.*

Proposition 3.2. *For V finite dimension, we have $(V^*)^* = V$.*

Proof.
$$v \sim \left[\phi_{V\to F} \mapsto \phi_{V\to F}(v)\right] \in (V^*)^*$$
and the function's always injective. Finite dimension solves else. □

3.2. **Tensor Products.**

Def 3.3. *The bilinear maps from $V \times W \to F$ is a function $f : V \times W \to F$ such that*

- $\lambda f(v, w) = f(\lambda v, w) = f(v, \lambda w)$;
- $\lambda f(v_1 + v_2, w) = f(v_1, w) + f(v_2, w)$;
- $\lambda f(v, w_1 + w_2) = f(v, w_1) + f(v, w_2)$.

Now we could define a tensor product using the above:

Def 3.4.
$$[\textit{Bilinear } V \times W \to F] \;=\; [V \otimes W]^*.$$

Def 3.5. *(Alternative) We can also define*

$$V \otimes W := \left\{ \text{Vector spaces of basis } (v, w) \middle/ \begin{cases} \lambda(v,w) = (\lambda v, w) = (v, \lambda w) \\ (v_1 + v_2, w) = (v_1, w) + (v_2, w) \\ (v, w_1 + w_2) = (v, w_1) + (v, w_2) \end{cases} \right\}$$

as quotient spaces.

Example 3.6. $V = \mathbb{C}e_1 + \mathbb{C}e_2$.

Then $V \otimes V$ is spanned by

$$(ae_1 + be_2) \otimes (ce_1 + de_2) = ac(e_1 \otimes e_1) + ad(e_1 \otimes e_2) + bc(e_2 \otimes e_1) + bd(e_2 \otimes e_2)$$

is 4-dimensional, where if we just pick $V \times W \to F$ it would be $|V| \cdot |W|$ dimension.

From above we also see that $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$.

Now, if $g \circlearrowright V$ and $g \circlearrowright W$ we want to ask in what way does g acts on their tensor product, and the answer would be

$$v \otimes w \mapsto (gv) \otimes (gw).$$

To be more explicit, using the same example 3.6 where $V = W = \mathbb{C}e_1 + \mathbb{C}e_2$, if $ge_1 = ae_1 + be_2$ and $ge_2 = ce_1 + de_2$ as the action on V, then the action on $V \otimes V$ is

$$g(e_1 \otimes e_1) = ge_1 \otimes ge_1 = a^2(e_1 \otimes e_1) + ab(e_1 \otimes e_2) + ba(e_2 \otimes e_1) + b^2(e_2 \otimes e_2)$$

and we write a table to make it explicit:

	$e_1 \otimes e_1$	$e_1 \otimes e_2$	$e_2 \otimes e_1$	$e_2 \otimes e_2$
$e_1 \otimes e_1$	a^2	ab	ba	b^2
$e_1 \otimes e_2$	ac	ad	bc	bd
$e_2 \otimes e_1$	ca	cb	da	db
$e_2 \otimes e_2$	c^2	cd	dc	b^2

and the matrix can be rewritten as

$$\begin{bmatrix} aA & bA \\ cA & dA \end{bmatrix} = A \otimes A.$$

Proposition 3.7.

$$\mathrm{Hom}_F(V, W) \cong V^* \otimes W.$$

For the direction, we take $V = F$ and $W = F$.

Proof. We find a linear $V^* \otimes W \mapsto \mathrm{Hom}_F(V, W)$, which we can do just by picking

$$\alpha_{V \to F} \otimes w \mapsto \alpha(v) \cdot w$$

and it's an bijection by finite dimension linear injection implies bijection. □

Now, by $\mathrm{Hom}(V, V) \cong V^* \otimes V \mapsto F$ where we can very naturally get $\alpha \otimes v \mapsto \alpha(v) = \mathrm{tr}$.
And $\mathrm{tr}(XAX^{-1}) = \mathrm{tr}(A)$ by above.

4. 10/4: Wedge powers and symmetric powers

Def 4.1. *A function is* <u>*symmetric*</u> *if*

$$f(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(n)}) = f(v_1, \dots, v_n)$$

and <u>*anti-symmetric*</u> *if*

$$f(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(n)}) = (-1)^\sigma f(v_1, \dots, v_n).$$

As an example, we note that if $V = \mathbb{C}e_1 \otimes \mathbb{C}e_2$ then $f(a, b, c) \to \mathbb{C}$ is symmetric, then the dimension is 4 since $f(e_1, e_2, e_1) = f(e_1, e_1, e_2)$ and the dimension of anti-symmetric functions are 0 because we can check that $f(e_1, e_1, e_1) = -f(e_1, e_1, e_1) = 0$.

Def 4.2. *Now we define the wedge powers and symmetric powers to be:*

$$(\mathbb{S}^n V)^* = \textit{ polynomial symmetrtic maps}$$

$$(\Lambda^n V)^* = \textit{ polynomial anti-symmetrtic maps}$$

For $\mathbb{S}^2 V$ it contains $e_1 \otimes e_1$, $e_2 \otimes e_2$,$e_1 \otimes e_2 + e_2 \otimes e_1$ and $\Lambda^2 V$ contains only $e_1 \otimes e_2 - e_2 \otimes e_1$.

Now there's some fact about wedge products: say V is dimension n, and then $V^* = f(x_1, \dots, x_n) = \sum a_i x_i$. Then $V = \Lambda^1 V$, and we have $\Lambda^k V = \begin{pmatrix} n \\ k \end{pmatrix}$.

Let's check this for $n = k$, then $(\Lambda^k V)^*$ is the anti-linear maps from $V^{\otimes n} \to F$, say

$$f(v_1, \dots, v_k) = \sum a_{1j_1} a_{2j_2} \dots a_{nj_n} f(e_{j_1} e_{j_2} \dots e_{j_n})$$

where the term $f(e_{j_1} e_{j_2} \dots e_{j_n})$ is non-zero iff they contain all different elements. Hence

$$f(e_{\sigma(1)}, \dots, e_{\sigma(n)}) = (-1)^\sigma f(e_1, \dots, e_n)$$

by anti-symmetry and thus we realize the expression of determinant:

$$f(v_1, \dots, v_k) = (-1)^\sigma f(e_1, \dots, e_n) \sum a_{1j_1} a_{2j_2} \dots a_{nj_n} = f(e_1, \dots, e_n) \det(v_1, \dots, v_n).$$

Now suppose $G \circlearrowright V$ and thus $g \mapsto A_g$. We want structures such that it is a G map, i.e. homomorphism. One thing we might use is

$$(A^T_{g_1 g_2})^{-1} = (A^T_{g_1})^{-1} (A^T_{g_2})^{-1}$$

and another is the determinant (which we notice is onto F).

5. 10/6: Complete reducibility

Theorem 5.1. *For any G finite group, it's representation can be decomposed into sums of irreducible representations:*

$$V \cong V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$$

where V_i are irreducible representations (irrepn), and $n_i \in \mathbb{N}$.

One example to this is the decomposition of $G := \mathbb{Z}/n\mathbb{Z}$ then all the decompositions are $1d$ representation, where the map ρ_s maps $k \mapsto e^{\frac{2\pi iks}{n}}$, which indeed is a repn.

A counterexample to this shows what could go wrong if we do not require finite. For instance we pick $\mathbb{Z} = G$ and since it's generated by 1 we can find all maps by finding $A \cong 1$. What we know is $n \cong A^n$. For $n = 2$ we pick $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ but then for the decomposition holds we need 2 eigenvectors, which it does not have. Note that for G finite $A^N = 1$ implies diagonalizable by character polynomial factorization.

Proof. Existence:

We use the following lemma:

Lemma 5.2. *For G finite, $F = \mathbb{C}$, if V is a repn of G and $W \subset V$, i.e. W is a subrepn, which means it's invariant under group action. Then there exists another invariant subrepn $W' \subset V$ such that $V = W \oplus W'$.*

We notice that this lemma directly proves existence, since our dimension is finite.

Proof. (Lemma) From homework or linear algebra we know that the image of a projector P is invariant. We can show that $\ker(P) \cap \mathrm{Im}(P) = \{0\}$ by the fact they are linear subspaces, and their definition. Moreover, we can decompose $V = \ker(P) \oplus \mathrm{Im}(P)$ since if $V = W \oplus W'$ for $W = \mathrm{Im}(P)$, then for $v = w + w'$, $Pv = w$ we get $W' = \ker(P)$.

Hence, the idea is to find a projector that in some ways corresponds to our subrepn W, which, to be specific, needs to be G-linear: $\forall g, P(gv) = gP(v)$. This will ensure that $\ker(P)$ is indeed a subrepresentation of G.

Now, take any projector P_0 from $V \to W$, then the operation gP_0 is naturally gP_0g^{-1} or at least that's what we want. Anyways we define

$$P := \frac{1}{|G|} \sum_{g \in G} gP_0g^{-1}$$

and can check that $\mathrm{Im}\, P \subset W$: just becuase $gP_0(v) \in W$ for any $v \in V$. Now the dimension of the image of P is nothing but the trace of the summation of P_0, which is the dimension of W.

Now, for $w \in W$

$$P(w) = \frac{1}{|G|} \sum_{g \in G} g P_0 g^{-1}(w) = \frac{|G|}{|G|} w$$

where in the middle part we've used the fact that $g^{-1}w \in W$ since $W \subset V$. And we can check that it is a projector (how). Given that, we check that it is G-linear, but this is easy because

$$P(hv) = \frac{1}{|G|} \sum_{g \in G} g P_0 g^{-1} w = \frac{1}{|G|} h \sum_{g \in G} (h^{-1}g) P_0 (h^{-1}g)^{-1} w = \frac{h}{|G|} \sum_{g \in G} g P_0 g^{-1} v = hP(v)$$

where the middle part is because we are summing over the same thing in different order.

Hence we are done. $\square$

We have used G-finite and that $\frac{1}{|G|}$ exist. Here, the proof works for $F = \mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{F}_q$ where $q \nmid |G|$.

Uniqueness:

Lemma 5.3. *(Schur's Lemma) for G finite and $F = \mathbb{C}$, V, W are irreducible representations, then the set of* $\operatorname{Hom}_G(V, W) := f : V \to W$ *such that* $f(gv) = gf(v)$ *can only be that*

$$\operatorname{Hom}_G(V, W) = \begin{cases} 0 & V \ncong W \\ \mathbb{C} & V \cong W \end{cases}$$

and if $V \xrightarrow{f} V$ *such that* $f \in \operatorname{Hom}_G(V, V)$ *then* $f(v) = \lambda v$.

Proof. Here the key is that we really can't do much for irreducible repns. If f is a G-morphism then $\ker(f)$ is a subrepn of V by definition, so is $\operatorname{Im}(f)$. But $\ker(f) = V \to f = 0$ and $\operatorname{Im}(f) = 0 \to f = 0$ since V is irreducible, thus if $f \neq 0$ then we need $\ker(f) = \{0\}$ and $\operatorname{Im}(f) = V$, so need f isomorphism, hence $V \cong W$.

For the second part, we need $F = \mathbb{C}$ since given $f : V \to V$ isomorphism, if the base field is $\mathbb{C}$ we can find $\exists v_0$ such that $fv_0 = \lambda v_0$ and hence $\ker(f - \lambda) \neq \{0\}$, so it is the whole space, so $f = \lambda \operatorname{id}$. $\square$

Now, for the last part suppose we have:

$$\begin{tikzcd} V_1^{n_1} \oplus \cdots \oplus V_s^{n_s} \arrow[r, "f"] & W_1^{m_1} \oplus \cdots \oplus W_s^{m_s} \arrow[d] \\ V_1 \arrow[u, "F"] & W_i \end{tikzcd}$$

where F is just id constrained on V_1 and we know there exists some W_i that is mapped to, since if there's no $W_i \cong V_1$ then $V_1 \in \ker(F)$ by linearity but f is isomorphism. Hence, we know $n_1 \le m_i$ by $\mathrm{Im}(V_1^{n_1}) \subset W_i^{m_i}$, and $m_i \le n_1$ with a backward argument so $n_1 = m_i$ and we are done.

$\square$

6. 10/9: CHARACTERS

Today we talk about characters, and the goal is to understand irreducible representations of G, $V_1, \dots, V_k$ where $K = \#$ of conjugate classes of G.

For V a G representation and $e_1, \dots, e_n$ a basis, we as usual assign $g \mapsto A_g \in GL_n$ and we want to find invariants: functions that has property $f(XAX^{-1}) = f(A)$. One example would be the determinant, but that would be just a representation on 1d space. The more interesting one we'll look at are these:

$$\mathrm{tr}(A), \mathrm{tr}(A^2), \dots, \mathrm{tr}(A^m)$$

and we assign ρ, our representation, a character function $\chi_\rho : G \to \mathbb{C}$ such that $\chi_\rho = \mathrm{tr}(A_g) = \mathrm{tr}(\rho(g))$.

Note that $\mathrm{tr}(A) = \sum \lambda_i$, $\mathrm{tr}(A^2) = \sum \lambda_i^2$, etc, and we can show that they generate the ring of symmetric polynomials. Moreover, it's not hard to see that knowing all those implies knowing all eigenvalues.

We primarily show properties of the character.

Property 6.1. $\chi_v(xgx^{-1}) = \chi_v(g)$ *for all* $x, g \in G$.

This is just by linear algebra. But let's see what this means: it means that χ_v is a "class function." To be explicit, let $\mathbb{C}[G]$ denote the vector space of all $G \mapsto \mathbb{C}$, then the dimension is obviously $\dim(\mathbb{C}[G]) = |G|$. Now we define $\mathbb{C}_{cl}[G] : G \to \mathbb{C}$ such that $f(xgx^{-1}) = f(g)$. Then we know $\dim(\mathbb{C}_{cl}[G]) = \#$ of conjugate classes.

This reduction means that we're dealing with a vector space with much smaller dimension.

Property 6.2. $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$.

Proof. Denote the basis of V_1 and V_2 by $e_1, \dots, e_n$ and $e_{n+1}, \dots, e_{n+m}$, then we can express the character function explicitly as

$$\chi_{V_1 \oplus V_2}(g) = \mathrm{tr} \begin{pmatrix} \rho_{V_1}(g) & 0 \\ 0 & \rho_{V_2}(g) \end{pmatrix} = \mathrm{tr}(\rho_{V_1}(g))\,\mathrm{tr}(\rho_{V_2}(g)).$$

□

We can extend this easily to

$$\chi_{V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}} = \sum n_i \chi_{V_i}.$$

Property 6.3. $\chi_V(g)$ *is a sum of roots of unity.*

A fact useful here is that if $A^n = 1$ then A is diagonalizable. One can see this easily by Jordan normal form.

Proof. To prove the property, we note $g^|G| = e$ and hence $\rho_V(g)^{|G|} = 1$ which means $\rho_V(g)$ diagonalizable. By evd we see that for each eigenvalue λ_i we have $\lambda_i^{|G|} = 1$, which means it is a root of unity, and of course $\chi_V(g) = \mathrm{tr}(\rho_V(g)) = \lambda_1 + \cdots + \lambda_n$. □

Property 6.4. $\chi_{V^*} = \overline{\chi_V}$.

Proof. Wait what? What does this has to do with complex conjugation? We'll see in a minute what's happening. We know that $\rho_{V^*} = (\rho_V(g))^T$ (check!) and for given V, g we choose basis such that

$$\rho_V(g) := \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

and hence

$$\chi_{V^*}(g) = \mathrm{tr} \begin{pmatrix} \lambda_1^{-1} & & \\ & \ddots & \\ & & \lambda_n^{-1} \end{pmatrix} = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\chi_V(g)}$$

where the last step is because for roots of unity $\lambda^{-1} = \overline{\lambda}$. Now do this for each g we see the relation holds. □

Property 6.5. $\chi_{V_1 \otimes V_2} = \chi_{V_1} \chi_{V_2}$.

Proof. Again, for each g we can find a basis such that $\rho_{V_1}(g)$ and $\rho_{V_2}(g)$ are diagonal, such that $\rho_{V_1}(g)e_i = \lambda_i e_i$ and $\rho_{V_2}(g)f_j = \mu_j e_j$.

$V_1 \otimes V_2$ has basis of the form $e_i \otimes f_j$ and thus

$$\rho_{V_1 \otimes V_2}(g)e_i \otimes f_j = (\lambda_i e_i) \otimes (\mu_j f_j) = \lambda_i \mu_j (e_i \otimes f_j)$$

thus

$$\mathrm{tr}(\rho_{V_1 \otimes V_2}(g)) = \sum \lambda_i \mu_j = (\lambda_1 + \cdots + \lambda_n)(\mu_1 + \cdots + \mu_m) = \mathrm{tr}(\rho_{V_1})\,\mathrm{tr}(\rho_{V_2}).$$

□

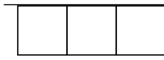
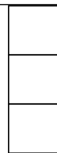
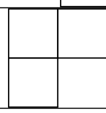
Note that this corresponds to a linear algebra fact that $\mathrm{tr}(A \otimes B) = \mathrm{tr}\, A\, \mathrm{tr}\, B$.

Example 6.6. *(for fun) Let $\chi : A \to \mathbb{C}^*$ be the character of each matrix, then we can write for fun*

$$\chi = \chi_\chi = \chi_{\chi_\chi} = \dots .$$

Example 6.7.

Let $G = S_3$, then we have the character table:

	e	$(12),(13),(23)$	$(123),(132)$
 trivial	1	1	1
 sign	1	-1	1
 standard	2	0	-1
$V^{\otimes 2}$	4	0	1
V^{perm}	3	1	0

where we compute the standard: we can always choose the most convenient basis here, so for one change we can make the change vertical and get $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ so trace is 0. For rotation it's by $\frac{2\pi}{3}$ so the rotation matrix is $\begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & \sin\left(\frac{2\pi}{3}\right) \\ -\sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{pmatrix}$ with trace -1.

For $V^{\otimes 2}$ we get it by property 5.

Example 6.8.

For general permutation representation, let $X = \{x_1, \dots, x_n\}$ and consider $V = \mathbb{C}e_{x_1} \oplus \cdots \oplus \mathbb{C}e_{x_n}$ where V is a G representation and so we see

$$g(a_1 e_1 + \cdots + a_n e_n) = a_1 e_{gx_1} + \cdots + a_n e_{gx_n}$$

and hence writing out $\chi_V(g)$ as a matrix it's full of 0 and 1 and is 1 only at (i, j) where $x_j = gx_i$, i.e. the matrix is roughly

$$\begin{array}{c|ccc} & e_{x_1} & \dots & e_{x_n} \\ \hline e_{x_1} & 0 & \dots & \\ \vdots & \vdots & & \\ e_{gx_1} & 1 & \vdots & \\ \vdots & \vdots & & \\ e_{x_n} & 0 & \dots & \end{array}$$

and so the trace is the number of elements fixed by g, denoted $\mathrm{Fix}(g)$. This also explains the last row of example above.

7. 10/11: IRREDUCIPLE REPRESENTATION AND CLASS FUNCTIONS

We have shown last time that operations on representations such as $\oplus, \otimes, ^*$ all have a counterpart for class functions such as $+, \times, \bar{}$. Today we show that irreducible representations is isomorphic to the basis of $C_{cl}[G]$.

But since we are talking of a basis, we should first define the basis and say they are orthogonal, thus we need an inner product.

Def 7.1. *For $f, g \in C[G]$ on $\mathbb{C}^n$, we define*

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) f_2(g).$$

Note that this is a Hermitian inner product on $\mathbb{C}$, which satisfies

$$\langle a_1 v_1 + a_2 v_2, w \rangle = a_1 \langle v_1, w \rangle + a_2 \langle v_2, w \rangle$$

$$\langle v, b_1 w_1 + b_2 w_2 \rangle = \bar{b_1} \langle v, w_1 \rangle + \bar{b_2} \langle v, w_2 \rangle$$

$$\langle v, v \rangle \in \mathbb{R}$$

and we might wonder why we've chosen the specific norm, which has a pretty obvious reason that it is G invariant. Note that the set of class functions is itself a representation of G: For $C[G] : \{f : G \to \mathbb{C}\}$, we can denote

$$g\phi(h) = \phi^g(h) := \phi(g^{-1}h)$$

and just by the definition, i.e. the summation there we have that $\langle f_1, f_2 \rangle = \langle f_1^g, f_2^g \rangle$.

Theorem 7.2. *For V_1, V_2 representations, we have*

$$\langle \chi_{V_1}, \chi_{V_2} \rangle = \begin{cases} 0 & V_1 \ncong V_2 \\ 1 & V_1 \cong V_2 \end{cases}$$

It's not hard to see a corollary:

Corollary 7.3. *# or representation $\leq$ # of conjugate classes.*

Proof. Define

$$V^G := \{v \in V | \forall g \in G, gv = v\}$$

then we consider the following defined operator:

$$P := \frac{1}{|G|} \sum_{g \in G} g \in \mathrm{Hom}_{\mathbb{C}}(V, V)$$

where the summation is actually on the action $\rho_v(g)$.

We claim that this P is a projector of V to V^G

We know $p(hv) = hp(v)$ as

$$\frac{1}{|G|}\sum_{g\in G}(gh)v = \frac{1}{|G|}\sum_{g} gv = \frac{1}{|G|}h\sum_{g} gv$$

and we can show $P^2 = P$ by the same argument:

$$\left(\frac{1}{|G|}\sum_{g} g\right)^2 = \frac{1}{|G|^2} = \sum_{g_1,g_2\in G} g_1g_2 = \frac{|G|}{|G|^2}\sum_{g\in G} g = P.$$

So we know it's an operator, but we don't yet know it's image. But we note

$$hP(v) = \frac{1}{|G|}\sum hgv = Pv$$

where since h is arbitrary we know $Pv \in V^G$ hence $\mathrm{Im}(P) \subset V^G$. For the other direction, we note that if $v \in V^G$ just apply we get

$$\frac{1}{|G|}\sum gv = v$$

and hence we've shown indeed P is a projector with image being V^G. But since P is a projector we know from homework

$$\dim(\mathrm{Im}\, P) = \mathrm{tr}\, P$$

which menas

$$\dim(V^G) = \mathrm{tr}\left(\frac{1}{|G|}\sum \rho_V(g)\right) = \frac{1}{|G|}\sum \mathrm{tr}(\rho_V g) = \frac{1}{|G|}\sum \chi_V(g)$$

where we note that by trace the χ_{trivial} is just 1 and hence

$$\frac{1}{|G|}\sum \chi_V(g) = 0 \iff \langle \chi_V, \chi_{\text{trivial}}\rangle = 0$$

and of course

$$\langle \chi_{\text{trivial}}, \chi_{\text{trivial}}\rangle = 1.$$

So we have proven the version of theorem for one of them being χ_{trivial}. Now to advance to the whole proof let V, W be 2 representations, then for V, W irreducible

$$\begin{aligned}\dim\left(\mathrm{Hom}_G(V,W)\right) &= \dim\left(\left(\mathrm{Hom}_F(V,W)\right)^*\right) = \dim\left((V^*\otimes W)^G\right)\\ &= \frac{1}{|G|}\sum_{g\in G}\chi_{V^*\otimes W}(g) = \frac{1}{|G|}\sum_{g\in G}\bar{\chi}_V(g)\chi_W(g) = \langle \chi_V, \chi_W\rangle\end{aligned}$$

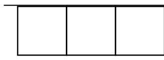
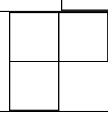
where the first equality is because if the function is not isomorphism then it's 0 by irreducible. But now we've shown the exact correspondence for irreducible representations. For the rest use Schur's lemma and we are done. □

One immediate consequence is that each χ has an orthogonality relation: say we have χ_1, χ_2, then

$$\sum \chi_1(g)\chi_2(g) = \begin{cases} 0 & \chi_1 \neq \chi_2 \\ 1 & \chi_1 = \chi_2. \end{cases}$$

Example 7.4.

We can now compute from the table from last time that they are orthogonal:

	e	$(12),(13),(23)$	$(123),(132)$
 trivial	1	1	1
 sign	1	-1	1
 standard	2	0	-1
$V^{\otimes 2}$	4	0	1
V^{perm}	3	1	0

Let's focus on the first and second row. We need to compute with repeating times so we compute

$$1(1 \times 1) + 3(1 \times (-1)) + 2(1 \times 1) = 0$$

so indeed.

8. 10/13: CLASS FUNCTION AND REPRESENTATIONS

Today we keep discussing the relation between representations and class functions. Remember we have the following correspondence:

representation		$C_{\text{cl}}[G]$
V	$\to$	χ_V
$\oplus$	$\to$	$+$
$\otimes$	$\to$	$\times$
V^*	$\to$	$\bar{f}$
$\dim(\mathrm{Hom}_G(V_1, V_2))$	$\to$	$\langle f, g\rangle$

and today we prove some more corollaries that are too easy to be true but that's just the powerful tool we have.

Corollary 8.1. *For V, W representations, then $\chi_V = \chi_W \iff V \cong W$.*

Proof. We can decompose $V = \oplus V_i^{n_i}$ then to know V it is enough to know n_i, but we know

$$\chi_V = \sum n_i \chi_{V_i} \quad \Rightarrow n_i \langle \chi_{V_i}, \chi_{V_j}\rangle = \langle \chi_V, \chi_{V_j}\rangle$$

and the rest follows what we've shown last time. □

Corollary 8.2. *(Criteria for irreducible)*

$$V \text{ is irreducible} \iff \langle \chi_V, \chi_V\rangle = 1 \iff \sum_{g\in G} |\chi_V(g)|^2 = |G|.$$

Proof. The second is just stating definition so we only proof the first equivalence.

$\Rightarrow$: $\langle \chi_V, \chi_V\rangle = \dim(\mathrm{Hom}(V, V)) = 1$ since irreducible. $\Leftarrow$: Say $V_2 = \oplus V_i^{n_i}$, then we compute

$$\langle \chi_V, \chi_V\rangle = \sum n_i^2 = 1$$

which means there is only one n_i and that value is 1, so we are done. □

There's two ways to view the regular representation V^R:

$$\begin{cases} \text{Functions on } G \\ \text{Permutational representation associated with the action of } G \text{ on itself.} \end{cases}$$

The first is more clear definition-wise while the second offers more insight.

Corollary 8.3. *For a regular representation, then*

$$\chi_{V^R}(g) = \#\,\textit{fixed points of premutation} \begin{pmatrix} g_1 & g_2 & g_3 & g_4 \\ gg_1 & gg_2 & gg_3 & gg_4 \end{pmatrix} = \begin{cases} 0 & g \neq e \\ |G| & g = e \end{cases}$$

Above is really just by simple conclusions of $gg_1 = g_1$.

Now, take any irreducible representation V_i where $V_R = \oplus_{i=1}^k V_i^{n_i}$, we have

$$n_i = \langle \chi_{V_R}, \chi_{V_i} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{V_R}(g) \overline{\chi_{V_i}(g)} = \frac{|G|}{|G|} \chi_{V_i}(e) = \dim(V_i)$$

by above computation. Thus

$$V_R = \oplus V_i^{\dim(V_i)}$$

which is of the same form of Frobenius determinant. Also, just taking the dimension of above we get

$$|G| = \sum (\dim(V_i))^2$$

and thus in particular the number of irreducible representations is finite. With this let's consider irreducible representations of S_4:

number	1	6	8	3	6
	e	(12)	(123)	(12)(34)	(1234)
trivial	1	1	1	1	1
sign	1	-1	1	1	-1
V_{perm}	4	2	1	0	0
V_{std}	3	1	0	-1	-1
$V_{\text{std}} \otimes V_{\text{sgn}}$	3	-1	0	-1	1
$V_{\text{exceptional}}$	2	0	-1	2	0

and now we explain:

- The first 2 lines are obvious;
- The V_{perm} row is because we've shown that is the number of fixed points. Now note $\langle \chi_{V_{perm}}, \chi_{V_{perm}} \rangle = 2$ so it's not irrepn.
- We try to decompose V_{perm}, for which we note $\langle \chi_{V_{perm}}, \chi_{V_{sgn}} \rangle = 0$ so it is perpendicular with the sgn representation, but $\langle \chi_{V_{perm}}, \chi_{V_{trivial}} \rangle = 1$ so one of the decomposition is V_{trivial}, and just use row of $V_{\text{perm}} - V_{\text{trivial}}$ we get the fourth row. One check that it is really the standard representation.
- Multiply V_{std} and V_{sgn} we get the fifth row.
- For the last row it's a bit tricky. One way to get this is that if we look at the first row we realize that there's already 4 irreducible representations that has value $1, 1, 3, 3$ and $1^2 + 1^2 + 3^2 + 3^2 + (?) = 24$ where inside the question mark can be four 1^2 or just one 2^2. Let's assume it's 2^2 and since

$$\sum \chi(g_1)\overline{\chi(g_2)} = \begin{cases} 0 & g_1, g_2 \text{ not conjugate} \\ \frac{|G|}{|\text{conj}|} & g_1, g_2 \text{ conjugate} \end{cases}$$

we use the fact that the first column of irreducible representations is orthogonal to each other columns. Thus we get last row.

The last construction might be a little unsatisfying, so let's see whether we can find a reason behind. Look at $S_n \xrightarrow{f} S_m$, then there's a lot of injections, lots of automorphisms such as conjugation, but for certain cases there are more than that, let's look at some special low dimension cases:

- $S_n \xrightarrow{\text{sgn}} S_2$.
- $S_4 \to S_3$ with $\ker = \{(12)(34), (13)(24), (14)(23), e\}$.
- $S_6 \to S_6$ where $(12) \to (12)(34)(56)$.
- $S_5 \to S_6$.

and we really get the last row because of this second morphism and through
$$\begin{tikzcd} S_4 \arrow[r] \arrow[dr] & S_3 \arrow[d] \\ & GL_n \end{tikzcd}$$

and indeed the first 3 rows are copied from the table of S_3, especially the row of (2,0,-1) generates the last row for S_4.

9. 10/16: Representation rings; number of irreducible representations

Def 9.1. *A* <u>*representation ring*</u> $R(G)$ *is a free abelian group on the isomorphism classes of representation defined by the quotient*

$$\sum n_i[V_i] \Big\backslash [V \oplus W] - [V] - [W]$$

then $R(G) \cong \mathbb{Z}^k$ *with basis* $[V_1], [V_2], \dots [V_k]$.

Moreover, we define $[V] \cdot [W] = [V \oplus W]$.

Example 9.2. *For an example,* $G = \mathbb{Z}/2\mathbb{Z}$ *then* $R(G) = \mathbb{Z}^2$ *containing* $[1]$ *and* $[-1]$ *where* $[-1][1] = [-1]$ *and* $[1]^2 = [-1]^2 = [1]$.

Once result that's remarkable is that $C_{cl}[G] = R(G) \otimes \mathbb{C}$. The amazing part is that $R(G)$ is very hard to specify, and $R(G) \otimes \mathbb{C}$ is very easy to deal with.

Now we prove a lemma for the later theorem.

Lemma 9.3. *For* $f \in C_{cl}[G]$, *we know*

$$F := \sum_{g \in G} f(g) \cdot g : V \to V$$

is a morphism of G-representation, i.e. $F \in \mathrm{Hom}_G(V, V)$.

We have shown before that for $f(g) \equiv e$ the result holds in former proofs.

Proof. Actually very straightforward! Compute and get:

$$F(xv) = \sum_{g \in G} f(g) g x v = \sum_{g \in G} f(g) x (x^{-1} g x) v = x \sum_{g \in G} f(x^{-1} g x)(x^{-1} g x) v = x F v$$

where the middle part is because $f(g)$ is just a number and it is a class function. □

Proposition 9.4. *For* f *a class function,* V *irrepn, then*

$$F := \sum f(g) g : V \to V$$

is a homothety: same as multiplication by a constant.

Proof. We only do half of the proof. Homothety means $Fv = \lambda V$ so $F = \lambda I$, and we just find λ, the rest is just check.

Note the dimension of F is $d := \dim(V)$ and suppose we have form

$$\sum f(g) \rho_v(g) = \lambda I_d$$

then we can take trace to get

$$\lambda d = \sum f(g) \operatorname{tr}(\rho_v(g)) = \sum f(g) \chi_v(g) = \sum f(g) \overline{\chi_{V^*}(g)} = |G| \langle f, \chi_{V^*} \rangle$$

and thus

$$\lambda = \frac{|G|}{dv}\langle f, \chi_{V^*}\rangle$$

the rest is just plug in and check. $\square$

Theorem 9.5. *# of irrepn = # of conjugate classes.*

Proof. Let $\chi_1, \dots, \chi_s$ be character of irrepn ,and $\chi_1, \dots, \chi_s$ are orthogonal vectors in $C_{cl}[G]$ from what we've shown before. We know $s \le k$ by injection.

Now assume $s < k$ strictly, then $\exists f \in C_{cl}[G]$ such that it is orthonormal to the rest basis by Gram-Schimidt. From homework, we know that the act of dual on irreducible representations is either flipping two elements or fixing one, so we know $\langle f \chi_{V^*}\rangle = 0$ since the first result says f is orthogonal to character of all irrepn.

Thus we know $\sum f(g)g$ acts by 0 for all irrepn V by proposition above, as the constant is 0. But we can find $V_{reg} = V_1^{d_1} \oplus \cdots \oplus V_s^{d_s}$ and act it by $\sum f(g)g$ which is 0: since we can decompose

$$C[G] = \oplus_{g\in G} C_{e_g}$$

we know that if we act on e_e we get

$$e \mapsto \sum f(g)g = 0$$

yet this really means $f(g) = 0$ since each g is independent for elements in End(G), which is what regular representations are. $\square$

The above shows that we can find a basis of $C_{cl}[G]$ is $\chi_1, \dots, \chi_k$ but if we denote the conjugate classes as $C_1, \dots, C_k$ then obviously another orthogonal basis is $\mathbb{1}_{C_k}$ with length:

$$\langle \mathbb{1}_{C_i}, \mathbb{1}_{C_j}\rangle = \frac{1}{|G|}\sum \mathbb{1}_{C_i}(g)\mathbb{1}_{C_j}(g) = \delta_{ij}\frac{|C_i|}{|G|}$$

and guess what this is: This is the character table!

10. 10/18: Orthogonal relation, Isotypical components, some associative algebra

Remember last time we introduced two orthogonal basis of the class functions, i.e. χ_{V_i} and $\mathbb{1}_{C_i} := \delta_{C_k}$. For a fixed C_i we can compute by basis decomposition that

$$\delta_{C_j}(g) = \sum_i \lambda_i \chi_{V_i}(g)$$

where

$$\lambda_i = (\delta_{C_j}, \chi_{V_i}) = \frac{1}{|G|} \sum_{g \in G} \mathbb{1}_{C_j} \overline{\chi_{V_i}(g)} = \frac{|C_j|}{|G|} \overline{\chi_{V_i}(g \in C_j)}$$

where the last part is because χ is a class function.

This means

$$\delta_{c_i}(g) = \frac{|C_j|}{|G|} \left(\sum_{v_i} \overline{\chi_{V_i}(c_j)} \chi_{V_i}(g) \right)$$

and this coefficient actually exactly expresses the number in the character table.

Proposition 10.1. *Two Orthogonal relations:*

1. *For χ_i, χ_j irreducible,*
$$\sum_{g \in G} \overline{\chi_j(g)} = |G| \delta_{ij}.$$
2. *For $g_1, g_2 \in G$,*
$$\sum_{i=1}^{k} \chi_i(g_1) \overline{\chi_i(g_2)} = \frac{|G|}{|C_i|} \delta_{g_1 \sim g_2}.$$

Now we proceed to isotypical components, for each V we can decompose it into $V = \oplus V_i^{n_i}$. For the trivial representation we know $V_1^{n_1} = V^G = \{v \in V | \forall g \in G, gv = v\}$, and in this case the project of course satisfies:

$$P = \frac{1}{|G|} \sum g; \quad \mathrm{Im}(P) = V^G$$

And we want to show such decomposition make sense. From last time we know that $\sum f(g) g \in \mathrm{Hom}_G(V, V)$ and if irreducible, then it is just multiplication by a constant. We can compute that for V irrepn we have

$$\frac{|G|}{d_V} (f, \chi_{V^*})$$

and look at

$$P_i := \frac{d_V}{|G|} \sum_{g \in G} \chi_{V_i^*}(g) g$$

then P_i acts on V_i by the multiplication of them which is 1 since constant cancel and inner product with itself, and acts on other where as 0. Thus

$$I = P_1 + \cdots + P_k$$

and we can decompose

$$v = P_1 v + \cdots + P_k v$$

as we usually do.

Now we give a preview of Associative algebra. It really is just a ring with identity such that the feild is a subset of A. So most importantly associative means $(\lambda a)b = \lambda(ab)$ associative with respect to the constant.

Example 10.2.

(1) $\mathbb{R}, \mathbb{C}$ over $\mathbb{R}$.
(2) M the ring with ijk, three 2nd degree extension.
(3) $M_{n\times n}(F)$.
(4) $A_1 \oplus A_2$.

What we'll eventually show is

Theorem 10.3.

$$C[G] \cong M_{d_1\times d_1}(\mathbb{C}) \oplus \dots \dots \cdots \oplus M_{d_k\times d_k}(\mathbb{C}).$$

11. 10/20: FROBENIUS DETERMINANT AND MORE ON ASSOCIATIVE ALGEBRA

Let $G := \{g_1, g_2, g_3, \dots, g_n\}$, and let $x_{g_1}, \dots, x_{g_n}$ be variables and let the Frobenius determinant be determined by the same as we did in the first class. In particular, $\det = F(x_{g_1}, \dots, x_{g_n}) \in \mathbb{Z}[x_{g_1}, \dots, x_{g_n}]$ which is a homogenous polynomial of degree n.

Theorem 11.1. *$\exists P_1, \dots, P_k \in \mathbb{Z}[x_{g_1}, \dots, x_{g_n}]$ such that*

$$F = P_1^{\deg P_1} \cdot P_k^{\deg P_k}$$

where P_i are irreducible.

Proof. Fact: $\chi_1(g) \approx x_e^{\deg p_1 - 1} \cdot x_g$.

Let $\rho : G \to GL_n$ be a representation and thus $\sum x_{g_i} \rho(g_i) \in GL_n$ and it's a polynomial as well as a matrix, so we can compute the determinant

$$\det \left(\sum x_{g_i} \rho(g_i) \in GL_n \right) = P_\rho$$

now take ρ to be the regular representation, i.e.

$$P_\rho(x_{g_1}, \dots, x_{g_n}) = \pm F(x_{g_1}, \dots, x_{g_n})$$

which is surprisingly just the case. Now let e_{g_i} be a basis then we have

$$\sum x_{g_i} \cdot g_i e_{s_{g_j}} = \sum \sum x_{g_i} \cdot e_{g_i g_j} = \sum \sum x_{g_i g_j^{-1}} \cdot e_{g_i}$$

but now looking at the decomposition

$$\mathbb{C}[G] \cong V_1^{d_1} \oplus \cdots \oplus V_k^{d_k}$$

and by results from before this is true, as we can view the F as a large matrix with blocks on the diagonal and thus

$$\det(F) = \det(\rho V_1)^{d_1} \cdot \det(\rho V_k)^{d_k}$$

and degrees match because of last time. □

Now we talk about associative Algebra over F.

Consider finite dimension case, $A = F^n$ as a vector space. where $e_1 \dots e_n$ is a basis, then we can decompose

$$e_i e_j = \sum_{k=1}^{n} \lambda_{ij}^k e_k.$$

Def 11.2. *A is a division algebra if $\forall x \in A, \exists y \in A$ s.t. $xy = 1$ (so not necessarily commutative).*

Example 11.3.

C - 2d algebra over real numbers.

H - 4-dim Algebra over R because for $g = a + bi + cj + dk \to \bar{g} = a - bi - cj - dk$ then

$$g\bar{g} = a^2 + b^2 + c^2 + d^2 \geq 0$$

which means $g^{-1} = \frac{\bar{g}}{g\bar{g}}$.

Fact: For $F = C$ then there is only one division algebra that is $\mathbb{C}$.

Proof. If A is a finite dimension division algebra over $\mathbb{C}$, then for $a \in A$ we can define $L_a \in GL_n(A)$ with $L_a x = ax$. Thus if we write out L_a explicitly it is

$$L_a = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

and hence is invertible. But we know it has an eigenvalue since $L_a x = \lambda x$ is just $\lambda = a$ by definition, but then every element is in C, and we are done. $\square$

Theorem 11.4. *Over $\mathbb{R}$, there are exactly 3 division algebra: $\mathbb{R}, \mathbb{C}, H$.*

Theorem 11.5. *Over finite field F_q every finite dimension algebra is a field F_{q^n}.*

Def 11.6. *For a finite-dimensional associative algebra over a field F, the <u>modulo</u> is just the algebra version of representation.*

An observation is that for G a group and F any field, we have

$$F[G] = \oplus_{i=1}^{n} F e_{g_i}$$

which is a group algebra. We have the following correspondence

$$\begin{array}{ccc} \text{modulo over } F[G] & \Longleftrightarrow & \text{G - representation} \\ F[G] \to M_{n\times n}[F] & \Longleftrightarrow & G \to GL_n(F) \end{array}$$

Theorem 11.7. *(generalized Schur's lemma) For A finite dimensional algebra, M_1 and M_2 are simple modules, then if $f : M_1 \to M_2$ be non-zero morphism of modules, then f is a isomorphism.*

Moreover, if M is simple then $\mathrm{Hom}_A(M, M)$ is an division algebra.

Theorem 11.8. *For A a finite dimension algebra such that whenever modules $M_1 \subset M_2$, there exists N such that $M_2 = M_1 \otimes N$. Then*

$$A \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k).$$

12. 10/23: Simple Algebras

We start to touch a little bit on Wedderburg A finite theory.

There's left modules as there are right modules. For a left modulo we have

$$A \times M \to M \quad : \quad (ab)m = a(bm)$$

and right modules are defined the same way:

$$A \times M \to M \quad : \quad (ab)m = b(am)$$

but that's a little bit confusing so we usually denote it with $m(ab) = (ma)b$.

Now, for $I \subset A$ a <u>submodule</u> is a subspace of A such that $\forall a \in A$, $aI \subset I$. In other words this is a left ideal.

Example 12.1.

For $M_{2\times2}(\mathbb{C})$ for $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and

$$I := \{X \in M_{2\times2} | Xv = 0\}$$

we know the left ideals are of the form $\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$ and right ideals are of the form $\begin{pmatrix} 0 & 0 \\ * & * \end{pmatrix}$.

Def 12.2. *A is <u>simple</u> if there's no non-trivial both-sided ideals.*

Def 12.3. *The <u>opposite algebra</u> A^{op} for A is such that the multiplication in A^{op} is $a * b = b \cdot a$.*

On matrices transpose is a opposite algebra, and even $\mathbb{C}[G]$ is self-opposite. But this is not in general true.

For left modules M, N we have $\mathrm{Hom}_A(M, N)$ (which of course as usual is defined by $f(am) = af(m)$)

We have the following:

- $\mathrm{Hom}_A(M_1 \oplus M_2, N) = \mathrm{Hom}_A(M_1, N) \oplus \mathrm{Hom}_A(M_2, N)$;
- $\mathrm{Hom}_A(M, N_1 \oplus N_2) = \mathrm{Hom}_A(M, N_1) \oplus \mathrm{Hom}_A(M, N_2)$;
- $\mathrm{Hom}_A(M_1 \oplus \cdots \oplus M_n, N_1 \oplus \cdots \oplus N_m) = \bigoplus \mathrm{Hom}_A(M_i, N_j)$.

But the last is a little bit hard to parse, and much less natural than the matrix representation:

$$\begin{pmatrix} \phi_{11} & \cdots \\ \vdots & \ddots \end{pmatrix}_{n\times m}$$

and ϕ_{ji} corresponds to the function in $\mathrm{Hom}_A(M_i, N_j)$.

Theorem 12.4. *For A algebra over F where M is a left module, the following are equal:*

(1) $M = \bigoplus S_i$ for S_i simple modules;

(2) $M = \sum_{i\in I} S_i$;
(3) $\forall N \subset M$, $\exists N'$ *with* $M = N \oplus N'$.

And for M *satisfying the above conditions we call it semi-simple.*

The essential part is that all S_i are simple so one element "generates" the space.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1): we first use AC to reduce the problem to finite dimension (since we'll need to find maximal ideal). Consider the maximal subset $J \subset I$ such that

$$\sum_{i\in J} S_i = \oplus_{i\in J} S_i$$

then we claim $\oplus_{i\in J} \in M$.

Reason: If $\exists m \in M$, $m \notin \oplus_{i\in J} S_i$ then we can decompose m such that $m = k + (m - k)$ where $k \in S_k$ for some $k \notin J$. But then $S_k \cap \oplus_{i\in J} S_i$ is either 0 or S_k by simple, hence it's 0 so $S_k + \oplus_{i\in J} S_i = S_k \oplus \left(\oplus_{i\in J} S_i\right)$ contradiction.

(1) $\Rightarrow$ (3): Say $M = \oplus S_i$ and we look at all subsets y such that $N + \sum_{j\in y} S_j = N \oplus (\sum S_j)$ and take $Y = \cup y$ and by (2) we just let $N' = \sum_{j\in Y} S_j$.

(3) $\Rightarrow$ (1): To show this we just decompose each into the smallest simple modules and then it's a direct product. $\square$

Corollary 12.5. *For A algebra and* M *a semi-simple module*

$$N \subset M \Rightarrow \begin{cases} N \textit{ is semi-simple} \\ M/N \textit{ is semi-simple} \end{cases}$$

Proof. For time we only show the first result. Say $L \subset N$ and $L' \oplus L = M$, then we want to show $L' \cap N$ is the complement of L in N. First note

$$(L' \cap N) \cap L = 0$$

is obvious and for $n \in N$ we can always decompose it with $n = e + e'$ for $e \in L$ and $e' \notin L$. But N is closed so $e' \in N \cap L'$. $\square$

13. 10/25: SEMI-SIMPLE ALGEBRAS

Theorem 13.1. *For G finite group and F any field, and $(G, \text{char } F) = 1$ where* char F *is q if $F = F_q$ and is 0 if infinite. Then every finite dimensional left module over $F[G]$ is semisimple.*

Proof. This looks daunting but really it's learned.

$$\text{Left module over } F[G] \iff F[G] \to \text{End}_F(M) \iff G - GL(M)$$

where M is nothing but a G representation. And use the same projection proof we've done will yield the result. The coprime condition is natural in this sense since we're summing with respect to G. □

Def 13.2. *A is* <u>*semi-simple*</u> *if every finite dimension module M is semi-simple.*

Intuitively, semi simple algebras are just like groups in the following sense.

Theorem 13.3. *For A finite dimensional associative algebra, the following are equivalent:*

(1) A is semi-simple.
(2) A is semi-simple as a left module over A. This is equivalent to saying as an A-module, $A \cong \bigoplus S_i^{n_i}$.
(3) $A \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$ for D_k division algebras.

We first show that for $\mathbb{C}[G]$, (3) holds.

Reason: Let $G := \bigoplus V_i$ are irrepn, then

$$\mathbb{C}[G] \xrightarrow{F} M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$$

since $\mathbb{C}$ is the only division ring over $\mathbb{C}$. But this is nothing but mapping

$$x \mapsto \text{diag}(\rho_{V_1}(x), \ldots, \rho_{V_n}(x))$$

in other words, a diagonal block matrix. And hence we can just make them direct sum.

Proof. (1) ⇒ (2) is obvious.

(2) ⇒ (1): For M finite dimensional, it has a basis $Fe_1 + \cdots + Fe_n$ and thus there exists map $A^n \to M$ by

$$(a_1, \ldots, a_n) \mapsto a_1 e_1 + \cdots + a_n e_n$$

where A semi-simple as a left module just means

$$A^n \cong \bigoplus S_i^{n_i}$$

just knowing what left module by itself means.

(3) ⇒ (2) is just check details, and it's homework.

(2) $\Rightarrow$ (3) is actually one line. Notice the fact:

$$\mathrm{End}_A(A) \cong A^{op}$$

becuase for $f \in \mathrm{End}_A(A)$ we can explicitly do

$$f(a) = af(1) \quad \Rightarrow \quad f(a) = ax$$

for $x = f(1)$. Hence we're just doing right module.

Thus we have

$$A^{op} \cong \mathrm{End}_A(A) = \mathrm{Hom}_A(A, A) = \mathrm{Hom}_A(\oplus S_i^{n_i}, \oplus S_i^{n_i})$$

and since S_i are simple homomorphisms are either identity to itself or zero.

In other words, $\mathrm{Hom}_A(S_i, S_j)$ for $i \neq j$ is 0, and since $\mathrm{Hom}_A(S_i, S_i)$ is an isomorphism, it is invertible. This gives the division ring (reverse element). So by discussion last time

$$\mathrm{Hom}_A(\oplus S_i^{n_i}, \oplus S_i^{n_i}) = \oplus \oplus \mathrm{Hom}_A(S_i^{n_i}, S_i^{n_i}) \cong M_{n_i}(D_i).$$

And now we have

$$A = M_{n_1}(D_1^{op}) \oplus \cdots \oplus M_{n_k}(D_k^{op})$$

. $\square$

Def 13.4. *For A finite dimension algebra,*

$$\mathrm{Rad}(A) = \{a \in A | aS = 0, S \text{ simple}\} \subset A.$$

A fact is that $\mathrm{Rad}(A)$ is a two sided ideal (obvious, just check). And what is expected is that once we rule out those bad terms, the remaining is good:

Theorem 13.5.

$$A \text{ semi-simple} \iff \mathrm{Rad}(A) = \{0\}.$$

14. 10/27: Radical of Associative Algebras

Today we finish out talk on associative algebras. We start by another fact:

Proposition 14.1.
$$\mathrm{Rad}(A) = \bigcap_{maximal\ \ ideals} L.$$

Proof. The fact of being an maximal ideal means A/L is a simple module if we view A as a left module. Then we know the correspondence of all ideals between M and N and the quotient M/N, this gives us the intuition that a maximal ideal A of A corresponds to A/L which has itself and 1 as subgroups.

Moreover, every simple module is obtained this way. Define
$$A \xrightarrow{f} S$$
where $a \mapsto av_0$ for $v_0 \in S$ that is non-zero. This is indeed the case.

Now, if $x \in \mathrm{Rad}(A)$ and L a maximal ideal, then $x(A/L) = 0$ but this means $xA/L = 0$ which in particular since $x \in xA$ so $x \in L$.

For the other direction, if $x \in \cap L$ and S simple, a map from A to S corresponds to an element in S so $xA = 0 \Rightarrow xv_0 = 0$ and the map has kernel $= L$, thus $xS = 0$. So we are done. □

Theorem 14.2. *A is semisimple* $\iff$ $\mathrm{Rad}(A) = 0$.

Proof. $\Rightarrow$: If A is semisimple, then $A \cong S_1 \oplus \cdots \oplus S_N$ then if $a \in \mathrm{Rad}(A)$ we know $a \cdot 1 = 0$ by linear composition.

$\Leftarrow$: For a finite dimensional, we know there exists a finite collection of maximal ideals such that
$$\bigcap_{maximal\ \ ideals} L = \bigcap_{i=1}^{n} L_i$$
which is because we can first find any L_1 then L_1 it has a dimension. Now we can always find an L_2 such that $\dim(L_1 \cap L_2) \leq \dim(L_1)$ and by finite dimension we are done.

Thus there is a function $A \to A/L_i$ for all i. Then we can construct
$$A \xrightarrow{f} A/L_1 \oplus A/L_2 \oplus \cdots \oplus A/L_n = 0$$
and thus
$$\ker(f) = \bigcap_{i=1}^{n} L_i = \mathrm{Rad}(A) = 0.$$
Thus f is injective and we are done since A is a subset of a semi-simple algebra, thus semi-simple. □

Now for S_1, S_2 simple modules, say M is a module of A with property $S_1 \subset M$ and $S_2 \cong M/S_2$, then if A is semisimple we know $M \cong S_1 \oplus S_2$, but that's not true in general.

In genreal we call the group of such M the group $\mathrm{Ext}^1(S_1, S_2)$ and so on.

Example 14.3.

An example is that $A = F_p[G]$ where $p||G|$.

We do have the criterion for $F = \mathbb{C}$ though. We know that any $x, y \in A$ has a corresponding left action which we denote by $L_a(x) := ax$, then define $\mathrm{tr}(L_x L_y) := \mathrm{tr}(xy)$, then we have:

Theorem 14.4. *For finite dimension algebra A over $\mathbb{C}$, A is semisimple iff $\mathrm{tr}(xy)$ is non-degenerate, i.e. if $\mathrm{tr}(xa) = 0$ for all x, then $a = 0$.*

15. 10/30: Center of Algebras

Today we consider the center of algebras, where

Def 15.1. *The center of a group is*

$$Z(G) := \{g \in G | xgx^{-1} = g, \forall x \in G\}$$

and the center of an algebra is the center of its multiplication.

Proposition 15.2. *For A an algebra, M an irreducible left A-module, $\rho : A \to \mathrm{End}(M)$, $x \in Z(A)$, then $\rho(X) = \lambda E$.*

Proof. For all $a \in A$, we know $\rho(a)\rho(x)v = \rho(x)\rho(a)v$ by the fact that it is a module and if we really consider this we realize $\rho(x) \in \mathrm{Hom}_A(M, M)$ and thus Schur's lemma directly give us the result. □

A consequence is that if G is an abelian group, then every irrepn is 1d, and we can decompose $V = \bigoplus V_i$ for each dimension 1.

Moreover, by homework we know (actually very obvious) that $Z(A_1 \oplus A_2) = Z(A_1) \oplus Z(A_2)$, and thus for G a finite group we can write

$$Z(\mathbb{C}[G]) = \mathbb{Z}(\bigoplus M_{n_i}(\mathbb{C})) = \mathbb{C} \oplus \cdots \oplus \mathbb{C}.$$

Now we want to know what is the radical center of $\mathbb{C}[G]$. The obvious one is e, but we note that the sum of all elements from a conjugate class, which is just because if we denote $e_i := \sum_{g \in C_k} g$, then

$$xe_ix^{-1} = \sum_{g \in C_i} xgx^{-1} = e_i.$$

As an example, we note then the e_i are $e, (12) + (23) + (31), (123) + (132)$.

Now we want to show that the e_i actually span $Z(\mathbb{C}[G])$.

Proof. If $a \in Z(\mathbb{C}[G])$, then $a := \sum_{g \in G} a_g g$ and for all $x \in G$ we can write

$$xax^{-1} = a \Rightarrow \sum a_g g = a = \sum a_g xgx^{-1} = \sum a_{xgx^{-1}} g = \sum_{i=1}^{k} \sum_{g \in C_i} a_i g = \sum a_i e_i$$

where really we can write this because $a_g = a_{xgx^{-1}}$. □

Now, we can check that $((12) + (23) + (31))^2 = 3((123) + (132))$, from which we really see that it's not an easy basis to use, and so we have another basis $f_{\chi_i} = (0, 0, \ldots, 1, 0, \ldots, 0)$ that is a projector, thus has property $f_{\chi_i}^2 = f_{\chi_i}$.

So we have an equivalence of vector space with respect to only direct sum structure that

$$Z(\mathbb{C}[G]) \sim \mathbb{C}_{cl}[G]$$

but that does not preserve the structure of characters.

As for what f_χ is we have proven before that it is a projector:

$$f_\chi = \frac{\dim(\chi)}{|G|} \sum \chi(g^{-1})g$$

and with regular ρ we have $\rho(f_\chi)e = f_\chi e = f_\chi$.

We'll discuss more of them when we discuss Hopf algebra structures.

16. 11/1: ALGEBRAIC INTEGERS

Def 16.1. *For $x \in \mathbb{C}$, x is algebraic if $\exists a_0, \dots, a_{n-1} \in \mathbb{Q}$ such that*

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

and denote $\overline{\mathbb{Q}}$. x is an algebraic integer if all those coefficients are integers, denote $\overline{\mathbb{Z}}$.

Some facts are:

Proposition 16.2. *(Fact 1) $\forall x \in \overline{\mathbb{Q}}, \exists d \in N, dx \in \overline{\mathbb{Z}}$.*

The proof is just letting d be the lcm of denominators.

Proposition 16.3. *(Fact 2) $\mathbb{Q} \cap \overline{\mathbb{Z}} = \mathbb{Z}$*

Proof. Assume $x = a/b$ with $(a, b) = 1$ then we just multiply till there's no denominator then

$$a^n + a_{n-1}a^{n-1}b + \cdots + a_0 b^n = 0$$

and if $p|b$, then $p|a$ we get contradiction. □

Proposition 16.4. *(Fact 3) For $A \in M_{n\times n}[\mathbb{Z}]$ the eigenvalues of A are in $\overline{\mathbb{Z}}$.*

This is actually very easy since writing out determinant we get $(-1)^n \lambda^n + \dots$.

Proposition 16.5. *(Fact 4) If $x, y \in \overline{\mathbb{Z}}$ then so is $x + y$ and xy. In other words $\overline{\mathbb{Z}}$ is a ring.*

Proof. We can show the reverse of fact 3, namely if $\lambda \in \overline{\mathbb{Z}}$ then there is a matrix with integer coefficients that has eigenvalue λ. Construction of matrix is an exercise. Now we can write $Av = xv$ and $Bw = yw$.

We know

$$(A \otimes B)(v \otimes w) = xy(v \otimes w)$$

and

$$(A \otimes I_{\dim(B)} +) I_{\dim(A) \otimes B}(v \otimes w) = (x + y)(v \otimes w).$$

□

Now the main observation before the proof is that for G a group and χ_V a character, $g \in G \Rightarrow \chi(g) \in \overline{\mathbb{Z}}$. The reason is that $g^n = e$ and $\chi(g) = \mathrm{tr}(\rho(g)) = \xi_1 + \cdots + \xi_n$ where ξ are roots of unity, then by closeness of $\overline{\mathbb{Z}}$ we get the result.

Now we can prove the main theorem:

Theorem 16.6. *For finite group G and V irrepn, we have* $\dim(v)||G|$.

Proof. For C a conjugate class, we have $\mathbb{Z}[G]$ the group ring and define $e_C = \sum_{g\in C} g$ then $e_C \in \mathbb{Z}[G]$.

A fact is that there exists integer coefficients such that

$$e_C^n + a_{n-1}e_C^{n-1} + \cdots + a_0 = 0.$$

The reason is that we can have the left multiplication matrix L_{e_C} which maps $\mathbb{Z}[G]$ to itself, i.e. an endomorphism for which we can of course write it as a matrix form and get it corresponds to a matrix with integer coefficients, and each row/column corresponds to an element in the group.

Here we note $L_{e_C}^n + a_{n-1}L_{e_C}^{n-1} + \cdots + a_0 = 0$ since what ever vector we apply we get the equation on left times the vector.

Now, for V irrepn, we know from before that $\rho(g_1 + \cdots + g_s) = \lambda I$ since the sum is in center, and hence

$$d_v\lambda = \sum_{g\in C} \chi_v(g_i) = |C| \cdot \chi_C$$

and hence $\dfrac{|C|\chi_C}{d_v} = \lambda$ thus $\lambda \in \overline{\mathbb{Z}}$ since

$$\rho(e_C^n + a_{n-1}e_C^n + \cdots + a_0)v = \rho(e_C^n)v + \cdots = 0$$

and then

$$\sum_{g\in G} \chi(G)\overline{\chi}(g) = \sum_{conj\ \ class} |C|\chi_C\overline{\chi}(g) = |G|$$

and then

$$\frac{|G|}{d_v} = \lambda\overline{\chi}(g) \in \overline{\mathbb{Z}}$$

since λ and $\overline{\chi}(g)$ are algebraic integers. Now $\dfrac{|G|}{d_v} \in \overline{\mathbb{Z}} \cap \mathbb{Q}$ hence it's an integer. $\square$

17. 11/3: Burnside Theorem

Theorem 17.1. *If G is a group and $|G| = p^a q^b$, p, q are primes, then G is not simple.*

Remark 17.2. *By simple we mean there is a non-trivial normal subgroup.*

Remark 17.3. *We can also say that G is solvable. In other words, there is a sequence of decreasing subgroups such that*

$$\{e\} \trianglelefteq G_1 \trianglelefteq G_2 \cdots \trianglelefteq G_n = G.$$

Note that this is equivalent since we can do this repeatedly.

This kind of structure is called matryoshka group.

One should read Galois thoery on a beach on a weekend. Anyways below is a quick version:

We get the Galois group of $\frac{\overline{\mathbb{Q}}}{\mathbb{Q}}$ is

$$G_{\overline{\mathbb{Q}}} := \left\{ \sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}; \sigma(x+y) = \sigma(x) + \sigma(y), \sigma(xy) = \sigma(x)\sigma(y), \sigma\left(\frac{m}{n}\right) = \frac{m}{n} \right\}$$

but note that we really cannot write out anything except the conjugate, which is indeed the case since we need AC. The picture is that this action of $G_{\overline{\mathbb{Q}}}$ on $\overline{\mathbb{Q}}$ is rotation in orbits, and we need to specify every orbit in order to characterize it.

We set out to prove Burnside theorem now. We use 2 theorems to achieve this.

Theorem 17.4. *(Theorem 1) For G finite group, V irrepn, C a conjugate class, assume $(|C|, \dim V) = 1$, then one of the following is true:*

- *$\chi_V(g) = 0$ for $g \in C$;*
- *or $\rho_v(g) = \lambda I_n$.*

Lemma 17.5. *For $\varepsilon_1, \dots, \varepsilon_n$ roots of unity, then if $\dfrac{\varepsilon_1 + \cdots + \varepsilon_n}{n} \in \overline{\mathbb{Z}}$ then one of the following is true:*

- *$\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n$*
- *or $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n = 0$.*

Proof. Assume it's not the case that all are the same, then by tiangle inequality we have

$$\left| \frac{\varepsilon_1 + \cdots + \varepsilon_n}{n} \right| < 1$$

and we denote $a_1 := a := \frac{\varepsilon_1 + \cdots + \varepsilon_n}{n}$, where we know that it is an algebraic integer, so there is a polynomial with integer coefficient such that

$$p(x) = (x - a_1)(x - a_2) \dots (x - a_n)$$

we claim that $a_i = \frac{\varepsilon_1^i + \cdots + \varepsilon_n^i}{n}$ where ε_j^i are also roots of unity. But that's easy because we can apply the orbit rotation of p to it and importantly $\sigma(a_1 + \cdots + a_n) = \sigma(a_1) + \cdots + \sigma(a_n)$ and since $|a_i| \leq 1$ we know since $a_1 < 1$ that

$$|\prod a_i| < 1$$

but we note that this is the last constant term of $p(x)$ so it's an integer, so $a = 0$. $\square$

Proof. We prove theorem 1. Recall from last time that $\dfrac{|C|\chi_V(g)}{\dim V} \in \overline{\mathbb{Z}}$ and thus there exists a, b such that $a|C| + b \dim V = 1$ and hence

$$\frac{\chi_v(g)}{\dim V} = a\frac{|C|\chi_v(g)}{\dim V} + b\chi_v(g)$$

and we know $\rho_v(g)$ has eigenvalue $\varepsilon_1 \ldots, \varepsilon_d$, hence $\frac{\varepsilon_1 \cdots + \cdots + \varepsilon_d}{d} \in \overline{\mathbb{Z}}$, which is because everything else in the equation is, so apply lemma we get the result, since all eigenvalues are the same means the character is scalar multiplication. $\square$

Theorem 17.6. *(Theorem 2) For G finite group, C conjugate class such that $|C| = p^k$ for $k > 0$, then G is not simple.*

Proof. We want to construct the kernel of $\rho : G \to GL_d(\mathbb{C})$ with $\ker(\rho)$ is non-trivial. Since we know the kernel is a normal subgroup, then we get the result.

Take $g \in C$, we have

$$0 = \sum_{V-irrpe} \dim V \chi_V(g)$$

and we rearrange to get

$$1 + \sum_{p|\dim V} \dim_v \chi_V(g) + \sum_{\mathcal{P} \not| \dim V} \dim V \chi_V(g) = 0$$

and so there exists V such that $p \not| \dim V$ and $\chi_v(g) \neq 0$ since otherwise we get $0 = 1 + p \sum \frac{\dim_V}{p} \chi_v(g)$ which means $\frac{1}{p} \in \overline{\mathbb{Z}}$ which is a contradiction.

Now we know $(|C|, \dim V) = 1$ and theorem 1 says that $\rho_V(g) = \lambda I$ andsof or all $a \in C$ $\rho_V(a) = \varepsilon I$ so we just pick $a_1 \neq a_2 \in C$ then we know $a_1 a_2^{-1} \in \ker(\rho_V)$ then we are done. $\square$

Proof. We prove Burnside theorem: For $|G| = p^a q^b$, we know

$$|G| = \sum |C| = \sum_{|C|=1} |C| + \sum_{|C|>1} |C| = |Z(G)| + \sum_{|C|>1} |C|$$

and thus if G is simple then $Z(G) = \{e\}$ so we know each $|C|$ has $pq|C|$, which then by a center (I'm not sure...?) equation we get contradiction. Really it's a group exercise. $\square$

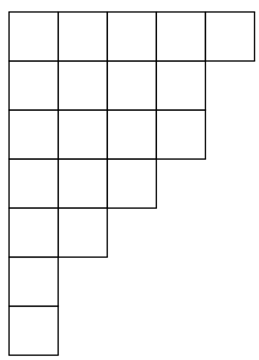
18. 11/6: Representation of Sn

S_n is order $n!$ and we happen to know well of its conjugacy classes, which is bijective to the cyclic structures of a permutation, which is bijective to partitions, i.e. we can decompose S_4 into $(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1)$.

One interesting thing of generating function is that we can write out the number of partitions: denote it by $p(n)$, then we have

$$\begin{aligned}\sum p(n)x^n &= 1 + x + 2x^2 + 3x^2 + 5x^4 + \dots \\ &= (1 + x + x^2 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots)\dots \\ &= \frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\cdots\end{aligned}$$

but that's just for fun. Now going back to young diagram, we can denote the partition $\lambda := (5, 4, 4, 3, 2, 1, 1)$ (randomly) by the block graph:



and we define the transposed diagram to be $\lambda' := (\lambda'_1, \dots, \lambda'_k)$ with $\lambda'_i :=$ # of λ_i such that $\lambda_i \geq i$, which is by itself a very hard to think of object, yet with the diagram we note that it is just the number of boxes in the column.

Now we start to prove the main thing:

Theorem 18.1. *There exists a natrual bijection between the conjugate classes and the irreducible representations.*

Consider the algebra of polynomials with S_n acting on it.

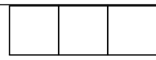
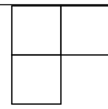
We can write $\mathbb{Q}[x_1, \dots, x_n]$ as the direct sum $A_0 \oplus A_1 \oplus \dots$ with A_i being the terms with total degree i. Then we have the action

$$\sigma p(x_1, \dots, x_n) = p(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$$

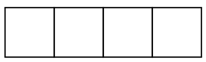
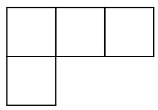
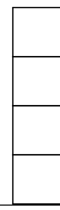
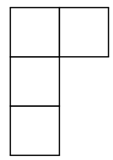
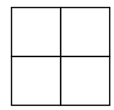
then $A_i = S^i V^*_{perm}$ and thus we write

$$\mathbb{Q}[x_1, \dots, x_n] = \oplus_{m\geq 0} S^m V^*_{perm}$$

and now we start to think of irreducible representations of S_3.

name	trivial	standard	alternating
diagram			
expression	1	$S_3(x_1 - x_2)$	$S_3(x_1 - x_2)(x_2 - x_3)(x_1 - x_3)$

and with the same spirit we just guess what's going to happen and it does happen: For S_4, let's denote $\Delta_{123} = (x_1 - x_2)(x_2 - x_3)(x_1 - x_3)$ and similarly denote all such symmetric functions

				
1	$S_4(x_1 - x_2)$	$S_4\Delta_{1234}$	$S_4\Delta_{123}$	$S_4\Delta_{12}\Delta_{34}$

so we do it by column. As for why, we'll see next time.

19. 11/8: Symmetric and anti symmetric polynomials

Def 19.1. *Define the elementary symmetric polynomial to be* $\sigma_m := \sum_{1\le i_1<\cdots<i_m\le n} x_{i_1}\dots x_{i_m}$

then we claim that the ring of symmetric polynomials has those as basis:

Theorem 19.2. *The symmetric polynomial ring of order n is* $\mathbb{Q}[\sigma_1,\dots,\sigma_n]$.

Def 19.3. *We define the* lexical graphic order *on monomials with*

$$x_1^{a_1}\dots x_n^{a_n} > x_1^{b_1}\dots x_n^{b_n}$$

if $a_1 > b_1$ *or* $a_1 = b_1$ *and* $a_1 > b_1$ *or* …

Def 19.4. *For polynomial p, define the* largest monomial $LM(p)$ *to be the largest term with respect to the lexical order. And denote its coefficient by* $CM(p)$.

Proof. Assume contrary, then we can find a polynomial with least degree that is not in the span, and we can pick the one term with largest $LM(p)$. Then we note that term $x_1^{a_1}\dots x_n^{a_n}$ must have $a_1 \ge \cdots \ge a_n$ because the polynomial is symmetric. Hence, we can use $\prod \sigma_i^{k_i}$ with $k_n = n$ and $k_i = a_i - a_{i+1}$ which contains the term. Then we know the result by the fact that two LM's product is the LM of the product.

Now we decompose result into sum of two vectors in the span, so we are done. □

Def 19.5. *An anti-symmetric polynomial is such that* $\sigma p = (-1)^\sigma p$.

Def 19.6. *The* Vandermonde determinant *is*

$$\Delta(x_1,\dots,x_n) = \prod_{1\le i<j\le n}(x_i - x_j).$$

Theorem 19.7. *For any* P *anti-symmetric,* $p = p'\Delta(x_1,\dots,x_n)$ *with* p' *symmetric.*

Proof. For P anti-symmetric, then $(12)p = -p$ and hence

$$p\Big|_{x_1=x_2} = 0$$

and thus $(x_1 - x_2)|p$. This means $\Delta|p$ with same argument, and $\frac{p}{\Delta}$ is symmetric because we just do sign check. □

Remark 19.8.

$$\Delta = \det\begin{pmatrix} 1 & 1 \\ x_1 & x_n \\ \vdots & \vdots \\ x_1^{n-1} & x_n^{n-1}\end{pmatrix}$$

by a change of columns revealing the symetricity, and really just an easy exercise.

Next time we establish the construction that the irrepns are in correspondence with the Vandermonde determinant of each rows of the Young diagram.

20. 11/13: PARTITION AND IRREPN

Today we conclude what we did last week. We see an example to recall what's happening.

Example 20.1. *For the partition $\lambda = (2, 2)$.*

$$V_\lambda = \left\{(x_1 - x_2)(x_3 - x_4), (x_1 - x_3)(x_2 - x_4), (x_1 - x_4)(x_2 - x_3)\right\} := \{a, b, b - a\}$$

which is just computation. For more information we try $(12)a = -a, (12)b = b - a, (12) = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ and we see the trace is 0. With similar computations we have the representation plot (from Day 8)

number	1 e	6 (12)	8 (123)	3 (12)(34)	6 (1234)
trivial	1	1	1	1	1
sign	1	-1	1	1	-1
V_{perm}	4	2	1	0	0
V_{std}	3	1	0	-1	-1
$V_{\text{std}} \otimes V_{\text{sgn}}$	3	-1	0	-1	1
$V_{\text{exceptional}}$	2	0	-1	2	0

where the detailed numbers corresponds to the trace.

Theorem 20.2. *V_λ is irreducible.*

Proof. Let the degree of polynomial be

$$d(\lambda) := \sum_{i=1}^{k'} \frac{\lambda_i'(\lambda_i' - 1)}{2}$$

$\square$

Let $R_d = \mathbb{C}[x_1, \dots, x_n] = S^d(V_{perm}^*)$ and by definition $V_r \subset R_d$ or we can write $V_r \hookrightarrow R_d$ embeds inside.

Proposition 20.3. $\mathrm{Hom}_{S_n}(V_\lambda, R_d) = \mathbb{C}$.

Corollary 20.4. *The proposition implies theorem.*

Proof. Write $V_\lambda = \oplus W_i^{n_i}$ and $R_d = \oplus W_i^{m_i}$ where W_i are irrepns, thus $\dim \mathrm{Hom} = \sum n_i m_i$ by Schur's lemma since W_i is either mapped to e or itself. But proposition says $\sum n_i m_i = 1$ hence both are 1 hence V_λ is irrepn. $\square$

Proof. (of proposition) Let $f \in \mathrm{Hom}_{S_n}(V_\lambda, R_d)$ and denote

$$p(x_1, \dots, x_n) = f(\Delta(x_1, \dots, x_{\lambda'_i}\Delta \dots))$$

and we note that f is linear so in particular it is antisymmetric within each partition $x_1, \dots, x_{\lambda'_i}$ and so on.

It is also the case that p is divisible by $\Delta(\lambda'_{i-1} + 1, \dots, \lambda'_i)$ since each term $(x_i - x_j)$ inside is divisible (again, by f homomorphism) and hence p is divisible by the whole polynomial inside, hence since p has the same degree we know f is just multiplication operation. $\square$

Using above method we can also prove that if $f \in \mathrm{Hom}_{V_\lambda, R_{d'}}$ with $d' < d(\lambda)$ then $f = 0$.

Theorem 20.5. *(Thoerem 2) Let λ_1, λ_2 be partitions of n then $V_{\lambda_1} \cong V_{\lambda_2}$ iff $\lambda_1 = \lambda_2$.*

This really says that different repn are not isomorphic, hence concludes our project.

Proof. Suppose $V_{\lambda_1} \cong V_{\lambda_2}$ then $d(\lambda_1) = d(\lambda_2)$. Note that this is the case since if not we WLOG assume $d(\lambda_1) > d(\lambda_2)$ but both embeds into $R_{d(\lambda_2)}$ thus one of them is 0, by composition we know either the homomorphism $V_{\lambda_1} \to V_{\lambda_2}$ is 0 or the embedding from $V_{\lambda_2} \hookrightarrow R_{d(\lambda_2)}$ is 0 and that's absurd.

Now we use the same argument to get $V_{\lambda_1} \hookrightarrow R_d$ and $V_{\lambda_2} \hookrightarrow R_d$ and hence $V_{\lambda_1} \equiv V_{\lambda_2}$ as subspaces of R_d.

Now we show that it can't happen that $\Delta() = \sum \Delta()$ where we fill in the () arbitrarily.

Claim: polynomials in V_{λ_1} and V_{λ_2} have no monomials in common if $\lambda_1 \neq \lambda_2$.

The reason is shown via an example, which is fascinating how things connects with each other. Suppose $\lambda = (5, 4, 2, 2)$ which is taken randomly, then we consider the terms $\Delta(1234)$ as the determinant of Vandermonde matrix it must look like $x_i x_j^2 x_z^3$ and etc. So the way to do this is to actually fill in the Young's diagram like this to get one monomial:

0	0	1	0	0
1	1	0	1	
2	3			
3	2			

which exhausts the structure of each nomomial. Thus we are done. $\square$

21. 11/15: Young Tableau

Question: Which Sp polynomials form a basis of $V_{\lambda'}$.

Def 21.1. *A Young Tableau is a Young diagram filled with integers* 1 *to n. A Standard Young Tableau is a Young Tableau such that the numbers increases right and down.*

Example 21.2.

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 6 & 9 \\ \hline 3 & 7 & 8 \\ \cline{1-3} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

Theorem 21.3. $\dim V_\lambda = \#$ *of standard Young Tableaus of shape* λ*.*

As an example, for a standard Young Tableau, consider the polynomial $Sp(T)$, which for instance

$$\begin{array}{ccc} \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} & \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \\ (x_1 - x_2) & (x_1 - x_3) & (x_1 - x_4) \end{array}$$

which we really see that that is the permutation.

Corollary 21.4. $\dim V_{\lambda'} = \dim V_\lambda$.

A fact is just that $V_{\lambda'} = V_\lambda \otimes \mathrm{sgn}$.

Lemma 21.5. *The space of Specht polynomials* $Sp(T)$ *for* T *being some standard Young Tableau are linearly independent.*

Proof. The mothod is to find a monomial not appearing there. We consider the smallest monomial in $Sp(T)$, where we use the lexical order defined last time and consider the smallest element, which will differ with respect to each different T. Note that in considering the Specht polynomials, each row has the same degree and each row's order is decided, so no two has the same lowest monomial. □

Let $f_\lambda = \#$ of SYT, where we obviously have $f_\lambda \leq \dim(V_\lambda)$ since they corresponds to some dimensions there.

Theorem 21.6. *There exists a bijection between permutations in* S_n *and pairs of SYT of same shape* λ*.*

Corollary 21.7. $f_\lambda = \dim(V_\lambda)$.

Proof.

$$\sum (\dim(V_\lambda))^2 = \sum_\lambda f_\lambda^2 = n!$$

and by $f_\lambda \leq \dim V_\lambda$ we have $f_\lambda = \dim V_\lambda$. □

Proof. (proof of bijection)

$$\sigma = \begin{pmatrix} 12345678 \\ 37124568 \end{pmatrix}$$

and we just construct

$$\begin{array}{|c|}\hline 3 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 3 & 7 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 7 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 7 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 7 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}$$

$$\vdots \qquad \vdots$$

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 4 & 5 & 6 & 8 \\ \hline 3 & 7 \\ \cline{1-2} \end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 5 & 6 & 7 & 8 \\ \hline 3 & 4 \\ \cline{1-2} \end{array}$$

where we note that we put it on right when we can, and if we cannot we find the first term on the right and squish the box down, and the one squished down will be put to the left.

What's left is to show bijection, but we note we can easily reverse the process since we know what is being done by right hand side. □

22. 11/17: BRANCHING: RESTRICTION AND INDUCTION

A good result is that the number of SYTs can be computed via hook length, which is how many things that is below and right of the box, including it self, so we can for instance mark the hook length:

7	4	3	1
5	2	1	
2			
1			

then the number of SYT is $\frac{n!}{\prod \text{hook\#}} = \frac{9!}{7 \cdot 5 \cdot 2 \cdot 4 \cdot 2 \cdot 3}$. Note that from the formula itself it's not obvious why this is even an integer.

We won't prove that but start to do branching.

Let $H < G$ be a subgroup then we can define restriction as Res_H^G which is V viewed as an H-representation.

Inductions are much complicated. We can decompose left cosets

$$G = g_1 H \cup \cdots \cup g_k H$$

and define

$$\mathrm{Ind}_H^G W := g_1 W \oplus g_2 W \cdots \oplus g_k W$$

where the difference is face level and this is in fact just kth degree of the same space.

$$g \cdot (g_i W) = g_{\sigma(i)} h_i W = g_{\sigma(i)} (h_i W)$$

is how we get g action using H action on W, where W is an H representation. Note that if W is the trivial Representation of H, then $\mathrm{Ind}_H^G(Id)$ is the premutation representation of G on left cosets.

We can in general get $\dim \mathrm{Ind}_H^G W = (\dim W)(G : H)$.

A slightly fancier way of defining is by functions: for $f \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[G], W)$ that are irreps from G to W we define

$$\mathrm{Ind}_H^G W = \mathrm{Hom}_H(\mathbb{C}[G], W) := \{f : G \to W | f(xh^{-1}) = hf(x)\}$$

and it's sufficient to know f at $g_1, \ldots, g_k$ to really know the full thing, so the dimension is $\dim W \cdot (G : H)$.

An even fancier way is just define $W \otimes_{\mathbb{C}[H]} \mathbb{C}[G]$ where in general we have $W \otimes_B A$, which is roughly the same thing as above, but more concise.

Now we compute the characters, which is actually easier.

Theorem 22.1.

$$\chi_{\mathrm{Ind}_H^G W}(g) = \frac{1}{|H|} \sum_{x \in G} \chi_W(xgx^{-1})$$

where $\chi_W \; := \mathbb{1}_H(g) \cdot \chi_W(g)$.

Proof.

$$\chi_{\mathrm{Ind}}(g) = \sum_{\text{diagonal block}} \chi_W(g_i^{-1} g g_i)$$

and the blocks we are noting that the actual matrix of g is a tensor of matrices where the first is where g sends H into, and the second is how that h acts on W. And the thing inside is so because $gg_i = g_i h$ and we want $\chi_W(h)$. Compute we get result. □

23. 11/27: FROBENIUS RECIPROCITY AND BRANCHING

Recall that the inner product of class functions are defined

$$(\chi_1, \chi_2)_G = \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \chi_2(g).$$

Theorem 23.1. *(Frobenius reciprocity) Let V be a G representation, W an H representation, and $H < G$.*

(1) (Baby version) For χ_1, χ_2 class functions of G, then

$$(V, \operatorname{Ind}_H^G W)_G = (\operatorname{Res}_H^G V, W)_H$$

in other words, Ind_H^G and Res_H^G are adjoint.

(2) (Version 2)

$$\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G V, W).$$

Note that the second version implies the first because, as we've shown before, the inner product yields the dimension of Hom. Now from first version to second, for each $\phi \in \operatorname{Hom}_H(\operatorname{Res}_H^G V, W)$ we need to find where v goes in Ind_H^G, establishing a correspondence. The idea is

$$g \rightsquigarrow gv \rightsquigarrow \phi(g, v) \in W$$

then check that this g is the corresponding one in Ind. We'll not bother about it too much though.

Proof. (of Baby version) Directly compute using notation and result from last time, where tilde means it's extended by zero onto G:

$$\begin{aligned}(\chi_V, \chi_{\operatorname{Ind}_H^G W})_G &= \frac{1}{|G|} \sum_{g_1 \in G} \chi_V(g_1) \left(\frac{1}{|H|} \sum_{g_2 \in G} \tilde{\chi}_W(g_2 g_1^{-1} g_2^{-1}) \right) = \frac{1}{|G||H|} \sum_{g_1, g_2 \in G} \chi_V(g_1) \tilde{\chi}_W(g_2 g_1^{-1} g_2^{-1}) \\ &= \frac{1}{|G||H|} \sum_{g_1, g_2 \in G} \chi_V(g_2 g_1^{-1} g_2^{-1}) \tilde{\chi}_W(g_2 g_1^{-1} g_2^{-1}) = \frac{|G|}{|G||H|} \sum_{h \in G} \chi_V(h) \tilde{\chi}_W(h)\end{aligned}$$

where χ_V is an actual class function so we can write out the above, and the last step is by counting how many repeat count of h: for each particular h, $g_1 = g_2 h g_2^{-1}$ so for each g_2 there is a corresponding g_1, and we have $|G|$ choices of g_2. Completing the computation we have

$$(\chi_V, \chi_{\operatorname{Ind}_H^G W})_G = \frac{1}{|H|} \sum_{h \in G} \chi_V(h) \tilde{\chi}_W(h) = (\operatorname{Res}_H^G V, W)_H$$

□

Theorem 23.2. *(Branching) We know $S_{n-1} < S_n$ and for λ a partition we have*

(1) $\operatorname{Res}_{S_{n-1}}^{S_n} V_\lambda = \oplus_{\mu \leq \lambda; |\mu| = n-1} V_\mu$.

(2) $\operatorname{Ind}_{S_{n-1}}^{S_n} V_\mu = \oplus_{\mu \leq \lambda; |\lambda| = n} V_\lambda$.

Note that if this is true, we can count the dimensions of representations by counting how many paths that generate the Young diagram from one box.

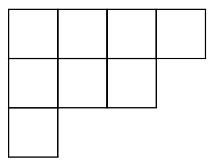
Proof. (idea) We first show (1) $\iff$ (2), then show (1). But

$$(1) \iff \left(\operatorname{Res}^{S_n}_{S_{n-1}} V_\lambda, V_\mu\right) = \begin{cases} 1 & \mu \leq \lambda \\ 0 & \text{otherwise} \end{cases}$$

by reciprocity

$$\iff \left(V_\lambda, \operatorname{Ind}^{S_n}_{S_{n-1}} V_\mu\right) = \begin{cases} 1 & \mu \leq \lambda \\ 0 & \text{otherwise} \end{cases} \iff (2)$$

and now we show (1). A rough argument is that, for instance we have



and we look at x_8. Since

$$V_\lambda = \operatorname{span}\left\{S_8 \Delta(123)\Delta(45)\Delta(67)\right\}$$

if we fix x_8 then we get x_8 has degree 0. But then we pick out all possible orders of x_8, and fix x_8 to do permutation by partition μ we see that we get everything, so we really have this intuitively. Checking things need to be done carefully. $\square$

24. 11/29: Frobenius Characterstic map

Let σ_i be the generator of symmetric functions, p_k be $\sum x_i^k$ and h_k be summation of all symmetric functions of an order, then we can write the symmetric polynomial as

$$\mathbb{Q}[x_1, \dots, x_n]^{S_n} = \mathbb{Q}[\sigma_1, \dots, \sigma_n] = \mathbb{Q}[p_1, \dots, p_n] = \mathbb{Q}[h_1, \dots, h_n]$$

One thing is that we note for degree k, if the number of variables is larger than k, then all such symmetric polynomials are of the same number of types, which allows us to define

$$\Lambda_k := [\mathbb{C}[X_1, \dots, x_k]]\Big|_{\deg=k} = [\mathbb{C}[X_1, \dots, x_{k+1}]]\Big|_{\deg=k} = \dots$$

and hence $\Lambda = \oplus_{k\geq 0}\Lambda_k$ is well defined. We also note that $\Lambda_k \otimes \Lambda_l \cong \Lambda_{k+l}$. An obvious observation is that $\dim_{\mathbb{Q}} \Lambda_n = \#$ conjugate classes since they have the same number of young diagrams.

Theorem 24.1. *(Frobenius Characteristic map) Let R_n be the $\mathbb{Q}$ vector space of functions from $S_n \xrightarrow{\chi} \mathbb{Q}$ with $\chi(x\sigma x^{-1}) = \chi(\sigma)$ is a class function. Then there exists an isomorphism*

$$\mathrm{ch} : R_n \cong \Lambda_n$$

or

$$\mathrm{ch} : \oplus_{n\geq 0} R_n \cong \Lambda.$$

Let's do some computation to make sense of the above. Let V be a S_k representation and W be a S_l repn. Then $S_k \times S_l \subset S_{k+l}$ because that's just permutations on two parts of $k+l$. We will finally show that

$$\mathrm{ind}_{S_k \times S_l}^{S_{k+l}} (V \otimes W) =: \chi_V \boxtimes \chi_W$$

being the product then the following is an isomorephism of rings:

$$\mathrm{ch}(\chi_W) = \frac{1}{n!} \sum_{\sigma \in S_l} \chi_W(\sigma) p_1^{\lambda_1(\sigma)} p_2^{\lambda_2(\sigma)} \dots p_k^{\lambda_k(\sigma)}$$

where $\lambda_i(\sigma)$ are the decomposition of cycles, which just corresponds to a row in the Young diagram.

Let's compute and convince us the result holds.

For S_1,

$$\square \mapsto p_1 = \sum x_i$$

and for S_2 we compute

$$\square\square \mapsto \frac{1}{2!}(p_1^2 + p_2) = x_1^2 + x_2^2 + x_1 x_2 = h_2$$

in particular we want them to have the same order, and the coefficients corresponds to the character table times the number of such kind of permulations. Also

$$\ydiagram{1,1} \mapsto \frac{1}{2!}(p_1^2 - p_2) = x_1x_2 = \sigma_2$$

where computation yields

$$\mathrm{Ind}_{S_1\times S_1}^{S_2} \ydiagram{1} \otimes \ydiagram{1} \sim p_1^2 = (x_1+x_2)^2 = h_2 + \sigma^2 \sim \ydiagram{2} \oplus \ydiagram{1,1}$$

which is absolutely amazing.

Do the same for S_3 we have

$$\ydiagram{3} \mapsto \frac{1}{3!}(p_1^3 + 3p_1p_2 + 2p_3) = (x_1^3 + x_2^3 + x_3^3) + \sum x_i^2x_j + x_1x_2x_3 = h_3$$

and we are sort of convinced that the regular permulation relates to h_3; further

$$\ydiagram{1,1,1} \mapsto \frac{1}{3!}(p_1^3 - 3p_1p_2 + 2p_3) = x_1x_2x_3 = \sigma_3$$

and

$$\ydiagram{2,1} \mapsto \frac{1}{3!}(2p_1^3 - 2p_3) = \sum x_i^2x_j + 2x_1x_2x_3 = h_3$$

since the character is $(2, 0, -1)$ and there are $(1, 3, 2)$ number of each type permutations.

and we can check that

$$\mathrm{Ind}_{S_2\otimes S_1}^{S_3} \ydiagram{1,1} \otimes \ydiagram{1} = \sigma_1\sigma_2$$

and

$$\mathrm{Ind}_{S_2\otimes S_1}^{S_3} \ydiagram{1,1} \otimes \ydiagram{1} = \mathrm{Ind}_{S_2\otimes S_1}^{S_3} \ydiagram{1,1} \sim \ydiagram{1,1,1} \oplus \ydiagram{2,1}$$

indeed holds by explicit computation.

Now what we do is we first give names, i.e. we call $ch(V_\lambda) = S_\lambda$ to be the Schur Polynomials, and we have a nice equality

$$S_\lambda = \sum_{\text{semi-Young Tableau}} x_1^{\#1}x_2^{\#2}\dots$$

where a semi-standard Young tableau is if the rows are less or equal to, while the columns still follow strictly less. An example is that $\boxed{}\boxed{}$ can be filled in by

$$\boxed{1}\boxed{1},\ \boxed{1}\boxed{2},\ \boxed{2}\boxed{2}$$

and hence those corresponds to $h_2 = x_1^2 + x_1x_2 + x_2^2$.

25. 12/1: Proof of Frobenius Characterstic map

Remember from last time that we have defined ch $: \oplus_{n\geq 0}\mathbb{Q})cl(S_n) \to \oplus\Lambda_n$ by

$$\mathrm{ch}(\chi) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi(\sigma)p_1^{\lambda_1(\sigma)}p_2^{\lambda_2(\sigma)}\dots p_k^{\lambda_k(\sigma)}$$

where $\lambda_i(\sigma)$ is the number of cycles of length i.

Theorem 25.1. *Such* ch *is an isomorphism of rings, where the product on class functions is defined to be*

$$\chi_V \boxtimes \chi_W \ \mathrm{ind}_{S_m\times S_n}^{S_{m+n}}(V\otimes W).$$

Proof. To prove this we need three things:

(1) $\boxtimes$ is indeed associative.
(2) $\mathrm{ch}(\chi_1\boxtimes\chi_2) = \mathrm{ch}(\chi_1)\cdot\mathrm{ch}(\chi_2)$.
(3) $\chi((n)) = h_n$.

where remember (n) refers to the trivial representation.

All those above implies theorem because 2,3 implies that ch is surjective since Λ_n has a $\mathbb{Q}$ basis and the numbers are the same, while

$$\mathrm{ch}([\lambda_1]\boxtimes[\lambda_2]\boxtimes\dots) = h_{\lambda_1}\dots h_{\lambda_k}$$

where we now refer to $\lambda_i$as $\lambda_1\leq\lambda_2\leq\dots$ and $\lambda_1+\lambda_2+\dots+\lambda_n = n$, i.e. the partitions.

To show 3, we have

$$\mathrm{ch}((n)) = \frac{1}{n!}\sum_{\sigma\in S_n}p_1^{\lambda_1(\sigma)}p_2^{\lambda_2(\sigma)}\dots p_k^{\lambda_k(\sigma)}$$

and we compute with generating functions that

$$\sum h_n t^n = \sum_{n\geq 0}\left(\sum_{i_1\leq\dots\leq i_n} x_{i_1}\dots x_{i_n}t^n\right) = \frac{1}{1-x_1t}\frac{1}{1-x_2t}\cdots\frac{1}{1-x_nt}$$

since we can write out each term as $1+x_1t+x_1^2t^2+\dots$ and thus the above expression is, by Taylor expansion, that

$$\begin{aligned}
&\exp\left\{\log\left(\prod\frac{1}{1-x_it}\right)\right\} = \exp\left\{\sum\log\frac{1}{1-x_it}\right\}\\
=&\exp\left\{x_1t+\frac{x_1^2t^2}{2}+\frac{x_1^3t^3}{3}+\dots+x_2t+\frac{x_2^2t^2}{2}+\frac{x_2^3t^3}{3}+\dots\right\}\\
=&\exp\left\{p_1t+\frac{p_1^2t^2}{2}+\frac{p_1^3t^3}{3}+\dots\right\} = \prod_{m\geq 1}\exp\left\{\frac{p_mt^m}{m}\right\}
\end{aligned}$$

and note that on the other hand

$$\sum \mathrm{ch}((n))t^n = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma\in S_n} p_1^{\lambda_1(\sigma)} p_2^{\lambda_2(\sigma)} \dots p_k^{\lambda_k(\sigma)}$$

and we can rewrite the permutation in terms of $a_1 + 2a_2 + \cdots + na_n = n$ and count how many repetitions: which is the number of conjugacy classes that we count to be total permutation over the number of total permutations with such partition:

$$z(a) = \frac{n!}{1^{a_1}a_1!2^{a_2}a_2!\dots n^{a_n}a_n!} = \sum_{\substack{n\geq 0 \\ a_1+2a_2+\cdots+na_n=n}} \frac{1}{a_1!}\left(\frac{p_1}{1}\right)^{a_1} t^{a_1} \frac{1}{a_2!}\left(\frac{p_2}{2}\right)^{a_2} t^{2a_2} \dots \frac{1}{a_n!}\left(\frac{p_n}{n}\right)^{a_n} t^{na_n}$$

$$= \prod_{a_k=1}^{n} \sum_{a_k=1}^{\infty} \frac{(p_k)^{a_k} t^{ka_k}}{a_k! k^{a_k}} = \prod \exp\left\{\frac{p_k t^k}{k}\right\}$$

so we have shown 3.

Now we show 1, that $\boxtimes$ is associative, and this really is to note that

$$\mathrm{Ind}_H^G \left(\mathrm{Ind}_K^H W\right) = \mathrm{Ind}_K^G W$$

and hence we could show associative after some toil. Again that's exercise.

Finally we show 2, we compute

$$\mathrm{ch}\left(\mathrm{Ind}_{S_n\times S_m}^{S_{m+n}}(\chi_1\otimes\chi_2)\right) = \frac{1}{n!}\sum_{\sigma\in S_n}\left(\mathrm{Ind}_{S_n\times S_m}^{S_{m+n}}(\chi_1\otimes\chi_2)\right)(\sigma) p_1^{\lambda_1(\sigma)} p_2^{\lambda_2(\sigma)} \dots p_k^{\lambda_k(\sigma)}$$

and we view

$$\phi := p_1^{\lambda_1(\sigma)} p_2^{\lambda_2(\sigma)} \dots p_k^{\lambda_k(\sigma)}$$

as a function that takes in σ and outputs a symmetric polynomial, and it satisfies

$$\phi(\tau\sigma\tau^{-1}) = \phi(\sigma)$$

and by recalling the definition of inner product we get

$$\mathrm{ch}\left(\mathrm{Ind}_{S_n\times S_m}^{S_{m+n}}(\chi_1\otimes\chi_2)\right) = \left\langle \mathrm{Ind}_{S_n\times S_m}^{S_{m+n}}(\chi_1\otimes\chi_2), \phi\right\rangle_{S_{m+n}} = \left\langle \chi_1\otimes\chi_2, \mathrm{Res}_{S_n\times S_m}^{S_{m+n}}\phi\right\rangle_{S_n\times S_m}$$

$$= \frac{1}{m!n!}\sum_{\substack{\sigma_1\in S_n \\ \sigma_2\in S_m}} \chi_1(\sigma_1)\chi_2(\sigma_2) p_1^{\lambda_1(\sigma_1)+\lambda_1(\sigma_2)} p_2^{\lambda_2(\sigma_1)+\lambda_2(\sigma_2)} \dots p_n^{\lambda_n(\sigma_1)+\lambda_n(\sigma_2)} = \mathrm{ch}(\chi_1)\cdot\mathrm{ch}(\chi_2).$$

hence we are done. $\square$

APPENDIX A. A

APPENDIX B. B

APPENDIX C. C

Acknowledgements.