PDE HOMEWORK 4

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Discussed with classmates.

Exercise 1.

Proof.

(1): Since U is bounded and u continuous so we can use DCT to pass limit, thus

$$\begin{aligned} e'(t) &\overset{DCT}{=} \int_U \partial_t u^2(x,t)dx = 2\int_U u\partial_t u = 2\int_U u\nabla\cdot(a(x)\nabla u) \\ &\overset{IBP}{=} 2\int_{\partial U} u(a(x)\nabla u) - 2\int_U a(x)|\nabla u|^2 = -2\int_U a|\nabla u|^2 \end{aligned}$$

since u vanishes on the boundary. One more derivative is similar:

$$\begin{aligned} e''(t) &\overset{above}{=} \partial_t\left[2\int_U u\nabla\cdot(a(x)\nabla u)\right] \overset{DCT}{=} 2\int_U \partial_t u\nabla\cdot(a(x)\nabla u) + u\nabla\cdot(a(x)\nabla u_t)dx \\ &= 2\int_U (\nabla\cdot a\nabla u)^2 dx + 2\int_U u\nabla\cdot(a\nabla(\nabla\cdot a\nabla u))dx \\ &\overset{IBP}{=} 2\int_U (\nabla\cdot a\nabla u)^2 dx - 2\int_U a\nabla u\cdot[\nabla(\nabla\cdot a\nabla u))]\,dx + 0 \\ &\overset{IBP}{=} 2\int_U (\nabla\cdot a\nabla u)^2 dx + 2\int_U (\nabla\cdot a\nabla u)^2 dx - \sum_{i=1}^n 2\int_{\partial U} a\partial_i u(\nabla\cdot a\nabla u) \\ &= 4\int_U (\nabla\cdot a\nabla u)^2 dx - \sum_{i=1}^n 2\int_{\partial U} a\partial_i u\partial_t u dx = 4\int_U (\nabla\cdot a\nabla u)^2 dx \end{aligned}$$

because $u(x,t) = 0$ on $\mathbb{R}^n\times[0,T]$ implies $\partial_t u = 0$ on ∂U.

(2):

$$
\begin{aligned}
(e'(t))^2 &= 4\left(\int_U u\nabla\cdot(a(x)\nabla u)\right)^2 = 4\left(\int_U udx\int_U \nabla\cdot(a(x)\nabla u)dx\right)^2\\
&\overset{C.S.}{\le} 4\left(\int_U udx\right)^2\left(\int_U \nabla\cdot(a(x)\nabla u)dx\right)^2 \le 4\left(\int_U\int_U u^2dx\right)\cdot\left(\int_U(\nabla\cdot a\nabla u)^2dx\right)\\
&= e(t)e''(t)
\end{aligned}
$$

(3):

Just take derivative and by (2)

$$f''(t) = \frac{e(t)e''(t) - [e'(t)]^2}{e^2} \ge 0$$

so f is convex.

Now we deduce the formula. Convexity gives (since everywhere defined)

$$f\left((1-\tau)t_1 + \tau t_2\right) \le (1-\tau)f(t_1) + \tau f(t_2)$$

now take exponential on both sides we have

$$e\left((1-\tau)t_1 + \tau t_2\right) \le \left[e(t_1)\right]^{1-\tau}\cdot\left[e(t_2)\right]^{\tau}$$

Let $\tau = \dfrac{t}{T}$ and $t_1 = 0, t_2 = T$, then this gives

$$e(t) \le e(T)^{1-t/T}e(0)^{t/T} = M^{1-t/T}\varepsilon^{t/T}.$$

(4):

$\varepsilon > 0$:

Either by what is the energy or the fact that there is a square so that it is non-negative. So at the place that it is not strictly positive, it is 0. But then since $e(T) = \varepsilon > 0$, by smoothness we know that at some point $e' > 0$, which is impossible by our computation in (1). Thus Even if we do not assume $e(t) > 0$ on the whole domain, it is implicitly implied, so proof in (3) works.

$\varepsilon = 0$:

This is a little bit trickier but fine. Since $e' \le 0$, if e reaches 0 at some time t, then it stays 0 after that point. Thus, we can define (due to completeness of $\mathbb{R}$)

$$s := \sup\{t | e(t) > 0\}.$$

We know by continuity that $e(s) = 0$ and $e(s-\delta) > 0$ for any δ. Now the only problem with our above argument is that f is not defined on $[s, T]$. But this does not matter since we can use approximation, i.e. let's say

$$\zeta = e(s-\delta) > 0$$

then we have that f is defined everywhere in $[0, s-\delta]$, so the above conclusion holds due to convexity:

$$e(t) \leq e(T)^{1-t/(s-\delta)} e(0)^{t/(s-\delta)} = M^{1-t/(s-\delta)} \zeta^{t/(s-\delta)} \leq C\zeta^{2t/s}$$

for δ small enough. Now we take $\delta \to 0$, then $\zeta \to 0$ by continuity, so we know for all $t \in (0, s)$, we have the property

$$e(t) \leq C\zeta^{2t/s} \to 0$$

thus $e(t) = 0$ everywhere on $(0, T]$. But e is continuous so $e(0) = M = 0$. Thus

$$0 = e(t) \leq e(T)^{1-t/T} e(0)^{t/T} = M^{1-t/T} \varepsilon^{t/T} = 0.$$

□

Exercise 2.

Proof.

(1): Using chain rule we have

$$\dot{e}(t) \overset{DCT}{=} \frac{1}{2}\int_{B(x_0),\alpha(t_0-t)} \partial_t \left(\frac{1}{c^2(x)}(\partial_t u)^2 + |\nabla u|^2\right) dx \\ - \frac{\alpha}{2}\int_{\partial B(x_0),\alpha(t_0-t)} \left(\frac{1}{c^2(x)}(\partial_t u)^2 + |\nabla u|^2\right) d\sigma$$

and the first term can be simplified by

$$\begin{aligned}
&\frac{1}{2}\int_{B(x_0),\alpha(t_0-t)} \partial_t \left(\frac{1}{c^2(x)}(\partial_t u)^2 + |\nabla u|^2\right) dx \\
&= \frac{1}{2}\int_{B(x_0),\alpha(t_0-t)} \frac{1}{c^2(x)}(2\partial_t u \cdot \partial_t^2 u) + 2\nabla u_t \cdot \nabla u dx \\
&= \int_{B(x_0),\alpha(t_0-t)} \frac{1}{c^2(x)}(\partial_t u \cdot c^2(x)\Delta u) + \nabla u_t \cdot \nabla u dx \\
&= \int_{B(x_0),\alpha(t_0-t)} \partial_t u \nabla \cdot \nabla u + \nabla u_t \cdot \nabla u dx \\
&= \int_{B(x_0),\alpha(t_0-t)} \nabla u_t \cdot \nabla u dx + \int_{\partial B(x_0),\alpha(t_0-t)} \partial_t u \nabla u d\sigma - \int_{B(x_0),\alpha(t_0-t)} \nabla u_t \cdot \nabla u dx \\
&= \int_{\partial B(x_0),\alpha(t_0-t)} u_t \frac{\partial u}{\partial \nu} d\sigma
\end{aligned}$$

so putting things together we have

$$\dot{e}(t) = \int_{\partial B(x_0),\alpha(t_0-t)} u_t \frac{\partial u}{\partial \nu} - \frac{\alpha}{2}\left(\frac{1}{c^2(x)}(\partial_t u)^2 + |\nabla u|^2\right) d\sigma.$$

(2):

We use Cauchy to get

$$|u_t \partial_n u| \overset{C.S.}{\leq} \frac{1}{2}\left(u_t^2 + |\partial_n u|^2\right) \leq \frac{1}{2}\left(u_t^2 + |\nabla u|^2\right)$$

so

$$\dot{e}(t) = \int_{\partial B(x_0),\alpha(t_0-t)} \frac{c^2(x) - \alpha}{2c^2(x)}(\partial_t u)^2 + \frac{1-\alpha}{2}|\nabla u|^2 d\sigma$$

and so as long as $\alpha \geq 1$ and $\alpha \geq c^2(x) \geq c_0^2$ then we have the desired result, thus a reasonable range is

$$\alpha \geq \max\{1, c_0^2\}.$$

(3):

When $t = 0$ we know that $\partial_t = 0$ and since $u = 0$ is constant we have $\nabla u = 0$. Thus $e(0) = 0$.

If we have $e(t) = 0$ for $0 \leq t \leq t_0$ then we know $\partial_t u = 0$ and $\nabla u = 0$ in the cone so there is no direction in the spacetime along which u will change in the cone, so u is constant there and hence is 0.

Thus this requires exactly that $\dot{e}(t) = 0$ for every t in the domain. If there is a point $z \in C$ the cone such that $c^2(z) - \alpha > 0$ then we can choose $c(x)$ to be the constant function $c(x) = c_0$ then we see that if we just choose to go only along the t direction and not at all the ν direction locally at z, then Cauchy Schwartz is attained at z and we get a strict positive derivative, which by continuity result in a strict change of u. Thus, we will need

$$c^2(x) - \alpha \leq 0$$

everywhere in $\partial B(x_0), \alpha(t_0 - t)$ for all t (even at endpoints due to continuity). So if we let

$$s := \inf\{c(x) : x \in C\}$$

then $\alpha = \max\{1, s^2\}$ is minimal.

$\square$