MEASURE THEORETIC PROBABILITY III HW 5

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Discussed with classmates.

Exercise 1.

Proof.

Theorem 13.7 in Williams says that we can show the sequence is UI if we can show the following:

(1) $X_n \in L^1$;
(2) $X \in L^1$;
(3) $X_n \to X$ in L^1.

So let's just try to solve this.

For (1) we get this because $E[X_n] \to EX < \infty$ as a sequence of real numbers, so $E[X_n] < M < \infty$ is uniformly bounded. But then it means each is integrable because $X_n \geq 0$.

For (2) we know $EX < \infty$, this means $EX^+ < \infty$ and $EX^- < \infty$, and thus $X \in L^1$.

For (3) note that first, $E[X_n] \to EX < \infty$ as a sequence of real numbers, so $E[X_n] < M < \infty$ is uniformly bounded. Then we break the integral into

$$\int_\Omega |X_n - X| d\mathbb{P} = \int_{\{|X_n - X| > \varepsilon\}} |X_n - X| d\mathbb{P} + \int_{\{|X_n - X| \leq \varepsilon\}} |X_n - X| d\mathbb{P}$$

for the second term we have

$$\int_{\{|X_n - X| \leq \varepsilon\}} |X_n - X| d\mathbb{P} \leq 1 \cdot \varepsilon = \varepsilon$$

and for the first we take limit and try to pass limit. Since $E[X_n] < M < \infty$ is uniformly bounded we know

$$|X_n - X| \leq 2M + \delta$$

is also uniformly bounded, so DCT means we can pass in the limit:

$$\begin{aligned}\lim_{n\to\infty} \int_{\{|X_n - X| > \varepsilon\}} |X_n - X| d\mathbb{P} &= \int_\Omega \lim_{n\to\infty} \mathbb{1}_{\{|X_n - X| > \varepsilon\}} (2M + \varepsilon) d\mathbb{P} \\ &\leq (2M + 1) \lim_{n\to\infty} \mathbb{P}\left(|X_n - X| > \varepsilon\right) = 0\end{aligned}$$

where the last equality is because $X_n \xrightarrow{as} X$ implies $X_n \xrightarrow{p} X$, and apply definition of convergence in probability.

$\square$

Exercise 2.

Proof.

(a):

Z_n is a martingale with respect to probability P:

We know Z_n is integrable because $\int_\Omega Z_n d\mathbb{P} = Q(\Omega) = 1 < \infty$.

We know it's measurable just by what it is (Radon- Nykodym derivatives).

For the crutial property we compute: For any $F \in \mathcal{F}_{n-1}$

$$\int_F Z_n d\mathbb{P} = Q(F) = \int_F Z_{n-1} d\mathbb{P}$$

since $F \in \mathcal{F}_{n-1}$ and $F \in \mathcal{F}_n$.

Now it's in general not true that it is also a Martingale with respect to Q. So we add some conditions.

A trivial condition is that $P = Q$, but that's too much to ask for. I have some guesses that are correct, but not sure whether equivalent:

Z_n is monotone a.s.:

Just because monotone we can do simple function approximation then get the c_i on each part, i.e. if $\sum_i c_i \mathbb{1}_{A_i} \uparrow Z_n$ then we can pass limit. Thus, by absolute continuous property means if $\sum_i c_i P(A_i \cap F) = 0$ then each is 0, so we can pass the limit to the simple function approximation to get to Z_n and get $\int_F Z_n dQ = \int_F Z_{n-1} dQ$.

$\int_F Z_n Z_{n-1} dP = \int_F Z_n dQ$:

This is indeed a new condition since we cannot use simple function approximation to pass Z_n since Z_{n-1} is not $\mathcal{F}_n$ measurable. If the condition hold then

$$\int_F Z_n dQ = \int_F Z_n Z_{n-1} dP = \int_F Z_{n-1} dQ$$

where the second is by simple function approximation, since Z_n is $\mathcal{F}_{n-1}$ measurable we can do this.

Z_n is invariant in n:

That's just obvious. But many of the conditions we thought of in study group is equivalent to this, say predictable, etc.

(b): If Z_n are UI then the result holds (even equivalent).

If Z_n is UI then there is a L^1 and a.s. limit Z because it is a Martingale. Note that we do not need UI to have a.s. limit Z^* since it is a martingale bounded above, hence supermartingale, hence a.s. limit exists. Now because every subsequence a.s. converge to Z, thus $Z^* = Z$ (subseq of subseq argument, not important point here though).

Let $F \in \mathcal{F}_N$ for arbitrary N, just by what L^1 converge means we have

$$\int_F Z d\mathbb{P} = \lim_{n\to\infty} \int_F Z_n d\mathbb{P} \overset{n>N}{=} \lim_{n\to\infty} Q(F) = Q(F)$$

thus if we define the left side to be

$$\mu(F) := \int_F Z d\mathbb{P}$$

then we know $\mu = Q$ on all the sets in $\bigcup_n^\infty F_n$. Then the two measures agree on the σ-algebra $\sigma\left(\bigcup_n^\infty F_n\right) = F_\infty$ by π - λ. Thus, if $\mathbb{P}(F) = 0$ then for all $F \in \mathcal{F}_\infty$

$$Q(F) = \mu(F) = \int \mathbb{1}_F Z d\mathbb{P} = 0$$

so $P \ll Q$ with respect to the set F_∞.

(To see equivalence just note if this holds then there is the Radon-Nykodym derivative Z that is the L^1 limit of Z_n, hence UI.)

(c):

First, we denote ∞ as a stopping time where $\{\infty \le n\} = \emptyset \in \mathcal{F}_n$ and $\{\infty = \infty\} = \Omega$. Then by appendix we have

$$\mathcal{F}_\tau \subset \mathcal{F}_{\tau\vee\infty} = \mathcal{F}_\infty$$

where we note the last step is because

$$\{s = \infty\} = \left(\bigcup_{i=1}^\infty \{s = i\}\right)^c \in \mathcal{F}_\infty.$$

Thus if we assume UI then since $Q \ll \mathbb{P}$ on $\mathcal{F}_\infty \supset \mathcal{F}_\tau$, we know $Q \ll \mathbb{P}$ on $\mathcal{F}_\tau$. As for the formula, by Doob's stopped optional stopping time theorem we have

$$\mathbb{E}[Z|\mathcal{F}_\tau] = Z_\tau$$

and being the left conditional expectation just means

$$Q(F) = \int_{F\in\mathcal{F}_\tau} Z d\mathbb{P} = \int_F Z_\tau d\mathbb{P}$$

since $F \in \mathcal{F}_\infty$. Thus Z_τ is indeed a formula for $\frac{dQ}{d\mathbb{P}}$ under the sigma-field $\mathcal{F}_\tau$.

(This is also equivalent as we just choose $\tau = n$ for every n and see we really need UI.)

$\square$

Exercise 3.

Proof.

Just bravely write out (first step taking out what is known) we have for all $F \in \mathcal{F}_{n-1}$

$$\int_F \mathbb{E}_{P_0}\left[X\frac{L_n(\theta)}{L_{n-1}(\theta)}|\mathcal{F}_{n-1}\right]dP_\theta \overset{P_\theta \gg P_0}{=} \int_F \mathbb{E}_{P_0}\left[XL_n(\theta)|\mathcal{F}_{n-1}\right]\frac{1}{L_{n-1}(\theta)}dP_\theta$$
$$= \int_F \mathbb{E}_{P_0}\left[XL_n(\theta)|\mathcal{F}_{n-1}\right]dP_0 = \int_F XL_n(\theta)dP_0 = \int_F XdP_\theta$$

and thus right hand side is an instance of the left hand side conditional expectation. We've implicitly used the fact $F \in \mathcal{F}_{n-1} \subset \mathcal{F}_n$ above.

$\square$

Exercise 4.

Proof.

We use the fact that a Gaussian measure is absolutely continuous with respect to the common Lebesgue measure. This is because, well, a Gaussian, say $N(0,1)$ is 0 on a set F iff the set has Lebesgue measure 0.

So we now argue why each X_n is a Gaussian. But this is obvious since under P_θ for all θ, we know X_n is a iterative addition of scaled and shifted guassians. Thus there is a definite density $\phi_{n,\theta}$ for each X_n under P_θ, which no where vanishes (it is a plane wave function).

For mutually absolutely continuous, say $P_{\theta_1}(A) = 0$, then

$$\int_A X_{n+1} d\mathbb{P}_{\theta_1} = \int_{X_{n+1}(A)} x\phi_{n+1,\theta_1} dx = 0$$

and this holds iff $\lambda(X_{n+1}(A)) = 0$ (argument from first paragraph). So do the same for P_{θ_1} and P_{θ_2} we get the result.

So we know

$$\frac{dP_\theta}{dP_0} = \frac{\phi_{n+1,\theta}}{\phi_{n+1,0}}$$

since we're considering on $\mathcal{F}_{n+1}$. Since both nominator and denominator does not vanish, this, and the inverse are both well-defined.

So we have offhand find the Radon-Nikodym derivative already. We'll find a more explicit expression later.

Now for an more explicit expression, we compute and get

$$\frac{\phi_{n+1,\theta}}{\phi_{n+1,0}} = \frac{\frac{1-\theta^{n+2}}{1-\theta}\phi(x - \theta^n X_1)}{\phi(x)}$$

where ϕ is the given density.

□