APPLIED DYNAMICAL SYSTEM HOMEWORK 3

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General ideas were discussed with many classmates in casual talks.

Exercise 1.

Solution:

α is just the right hand bound on the locations of the eigenvalues of the matrix A, i.e. $\mathrm{Re}(\lambda) \leq -\alpha$. This is the same α chosen in the proof of Lemma 2.29.

K is another constant chosen by lemma 2.29. This lemma yields equation (4.20) in textbook. This serves as a constant bound that takes A out of the game, so its easier to deal with. The existence of this K is just due to (as in proof of lemma 2.29) that the exponential decays much faster than polynomial.

Now since g is continuous and $g(0) = 0$ (see construction of g in proof), the constants ε and δ are as in usual definition in continuity: for any ε, exists corresponding δ such that the fluctuation of the function with in that δ ball cannot be larger than ε, etc. The only difference here is that we choose $\varepsilon \leq \dfrac{\alpha}{K}$, and the ball to be of radius $K\delta$. But that's really nothing since we've already fixed K.

So we've fixed δ above and we'll use this given $(K-1)\delta$ to function as if the "ε" in the continuity condition of the solution, where the "δ" is substituted by τ, and the center is shifted to $|y_0| \leq \delta$.

So we have for any small enough ε there exists δ satisfying the Lyapnov stable condition (since we need also this to conclude asymptotic stable). The fact that each close enough point will eventually goes to the equilibrium is encoded by implications of Gronwall's lemma, where, really, all else is constant except t. Thus it goes to 0.

Exercise 2.

Proof. (discussed with Ziang)

The Hartman-Grobman theorem says that there exist a neighborhood N of the equilibrium x^* such that there is a topological conjugate function h^{-1} such that ϕ_t is topologically conjugate to it's linearization ψ_t within N (choose it to be a ball centered at x^*).

We first show that linear asymptotic stability implies asymptotic stability. That is, we know

(1) The linearization is stable, i.e. for any ε, there exist a smaller δ such that for points starting in the δ ball, their orbit never exceeds the ε ball.
(2) There exists a neighbor U (choose it to be a ball centered at x^*) such that for all points $x \in U$, $\lim_{t\to\infty} \psi_t(x) = x^*$.

Note that we really need condition (1) to exclude cases like in example 4.17 or 4.18 (page 115-116).

We are justified to use the theorem since in linear asymptotic stable case, all eigenvalues of the linear matrix has real part smaller than 0, so its non-hyperbolic.

But then by Hartman-Grobman we know $h(\psi_t(x)) \to h(x^*)$ for $x \in M = N \cap U$, since h continuous we can exchange limit with h. But then $\phi_t(h(x)) \to h(x^*) = x^*$(equilibrium cannot change since remainder is of less order than the linear term) for all $h(x) \in W :=$ the largest ball contained in $h(N \cap U)$. And by going back and choose $x \in W' :=$ the corresponding ball(by Lyapunov condition) to the largest ball contained in $h^{-1}(W)$, we will be done since now no orbit of $\phi_t(x)$ will exceed W.

Thus the original system also has asymptotic stability at x^*.

The thing about the unstable case is that, for a linear unstable ODE, we don't even know that the equilibrium is hyperbolic, i.e. we can have a very large real eigenvalue that guarantees unstability with $\pm i$ that just appears there somehow. Then we cannot even apply the Hartman-Grobman theorem.

$\Big($ For hyperbolic cases though, proving linear unstability imply unstability is similar.

Since the system is unstable and at x^*, we can say that there exist ε such that there is a sequence $\{x_n\} \to x^*$ such that the orbit $\psi_t(x_n)$ will exceed $B_\varepsilon(x^*)$ at some time t.

We use the same h as above and then for $\varepsilon' = \min\{x | x \in h(B_\varepsilon(x^*))\}$, any orbit $\phi_t(h(x_n))$ will exceed $B_{\varepsilon'}(x^*)$ at some time t. Also, since h continuous $h(x_n) \to h(x^*) = x^*$ too. Thus we are done for the hyperbolic case. $\Big)$

□

Exercise 3.

The origin is the unique equilibrium and it is a saddle since on the x-axis, the flow is to the right on the right hand side of the origin and to the left on the left hand side.

The linear matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has the solution

$$\psi_t(x, y) = e^{tA} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e^t x \\ e^{-t} y \end{pmatrix}.$$

Where as the first term of the non-linear system should be the same due to same expression and no dependence on y, so $\phi_t(x, y) = (e^t x, \phi_{t,y}(x, y))$.

Solving the other differential equation we have:

$$\begin{aligned}
&\dot{y} = -y + x^2 \\
\Rightarrow &\frac{d}{dt}\left(e^t y\right) = e^t \dot{y} + e^t y = x^2 \\
\Rightarrow &\int_0^t \frac{d}{ds}\left(e^s y\right) ds = \int_0^t x(s)^2 ds = \int_0^t e^{2s} x_0^2 ds \\
\Rightarrow &e^t y(t) - y_0 = e^s y\Big|_0^t = x_0^2 \frac{1}{3} e^{3s}\Big|_0^t = x_0^2 \frac{1}{3}\left(e^{3t} - 1\right) \\
\Rightarrow &y(t) = e^{-t} y_0 + \frac{1}{3}(e^{2t} - e^{-t}) x_0^2
\end{aligned}$$

which implies by putting together

$$\phi_t(x, y) = \begin{pmatrix} e^t x \\ e^{-t} y + \frac{1}{3}(e^{2t} - e^{-t}) x^2 \end{pmatrix}.$$

Now we check that $\psi_t \circ H = H \circ \phi_t$. For left hand side we have

$$\psi_t(H(x, y)) = \psi_t(x, y - \frac{x^2}{3}) = \begin{pmatrix} e^t x \\ e^{-t}\left(y - \frac{x^2}{3}\right) \end{pmatrix}$$

and the right hand side

$$\begin{aligned}
H(\phi(x, y)) = H \begin{pmatrix} e^t x \\ e^{-t} y + \frac{1}{3}(e^{2t} - e^{-t}) x^2 \end{pmatrix} &= \begin{pmatrix} e^t x \\ e^{-t} y + \frac{1}{3}(e^{2t} - e^{-t}) x^2 - \frac{1}{3} e^{2t} x^2 \end{pmatrix} \\
&= \begin{pmatrix} e^t x \\ e^{-t} y - \frac{1}{3} e^{-t} x^2 \end{pmatrix} = \begin{pmatrix} e^t x \\ e^{-t}\left(y - \frac{x^2}{3}\right) \end{pmatrix}
\end{aligned}$$

so indeed they are the same.

Exercise 4.

I'll just use the example in class since it demonstrates things beautifully. (from notes)

Example 0.1. *For example, we have the case where an ODE system has its linearization with eigenvalues on the complex line:*

$$\begin{cases} \dot{x} = y + a(x^2 + y^2)x \\ \dot{y} = -x + a(x^2 + y^2)y \end{cases}$$

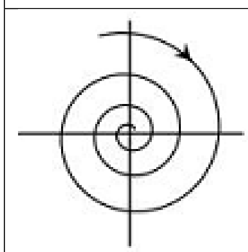
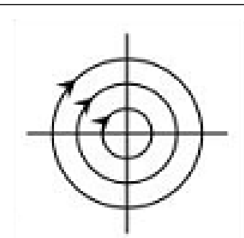
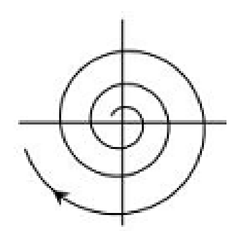
As we can see clearly, the linearization is $\dot{x} = Ax$ where $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and by its trace and determinant we know the eigenvalues are $\pm i$.

Note that in fact the ODE system is nothing but something that is converted from its polar form. So writing $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ we get (with convention) $\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$ (the clever thing to do is not think about arctan, since there's domain issue).

Solving the function we see:

$$\begin{aligned} & 2r\dot{r} = 2x\dot{x} + 2y\dot{y} \\ \Rightarrow & \sec^2\theta\dot{\theta} = \frac{\dot{y}x - \dot{x}y}{x^2} \\ \Rightarrow & \begin{cases} \dot{r} = ar^3 \\ \dot{\theta} = -1 \end{cases} \end{aligned}$$

Which then means that

$a < 0$	$a = 0$	$a > 0$
		
Asymptotic stable	Lyapunov stable	unstable

Now as we can see, the linear system is only Lyapunov stable (corresponding to $a = 0$). But when $a > 0$, the system is unstable and when $a < 0$, the system is actually asymptotic stable.

Exercise 5.

(Q5 in book).

The ODE system in question is

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y + x^2 z \\ \dot{z} = z \end{cases}$$

which has linearization $\dot{x} = Ax$ where $A = \mathrm{diag}(-1, -1, 1)$, which immediately gives us

$$\psi_t(x, y, z) = \begin{pmatrix} e^{-t}x \\ e^{-t}y \\ e^t z \end{pmatrix}.$$

Since the x and z term are linear already, we know $\phi_t(x, y, z) = (e^{-t}, \phi_{t,y}(x, y, z), e^t z)$. Now let's solve the middle term.

$$\begin{aligned}
&\dot{y} = -y + x^2 z \\
\Rightarrow &\frac{d}{dt}\left(e^t y\right) = e^t \dot{y} + e^t y = e^t x^2 z \\
\Rightarrow &\int_0^t \frac{d}{ds}(e^s y)\, ds = \int_0^t e^s x(s)^2 z(s) ds = \int_0^t e^s e^{-2s} x_0^2 e^s z_0 ds \\
\Rightarrow &e^t y(t) - y_0 = e^s y\Big|_0^t = x_0^2 z_0 s\Big|_0^t = t x_0^2 z_0 \\
\Rightarrow &y(t) = e^{-t} y_0 + t e^{-t} x_0^2 z_0
\end{aligned}$$

which means

$$\phi_t(x, y, z) = \begin{pmatrix} e^{-t}x \\ e^{-t}y + te^{-t}x^2 z \\ e^t z \end{pmatrix}.$$

The better way to go now is to add "bump" to the function and change it to $\dot{y} = -y + x^2 b(z)$ where $b(\xi) = \begin{cases} \xi & |\xi| < \varepsilon \\ 0 & |\xi| > \delta \end{cases}$ that is smooth, and the small values are really arbitrary. But for the question I guess the goal is to see why we need that (convergence issue), and thus we'll follow the example 4.37 first, then add the bump and follow example 4.38.

As is in the proof of Hartman-Grobman's theorem, we only need to compute for $t = 1$ the following equation to gain for the general H:

$$H(x, y, z) = e^{-A} H(\phi_1(x, y, z)) = \begin{pmatrix} e & & \\ & e & \\ & & 1/e \end{pmatrix} H(\frac{1}{e}x, \frac{1}{e}(y + x^2 z), ez).$$

Now we deal with it term by term, so we let $H = (K, L, M)^T$.

The iterative equation (4.38) in book for K, starting with $K^{(0)}(x, y, z) = x$, is

$$K^{(1)}(x, y, z) = eK^{(0)}\left(\frac{1}{e}x, \frac{1}{e}(y + x^2 z), ez\right) = e\frac{1}{e}x = x$$

so $K = x$.

Similarly, let $M^{(0)}(x, y, z) = z$,

$$M^{(1)}(x, y, z) = \frac{1}{e}M^{(0)}\left(\frac{1}{e}x, \frac{1}{e}(y + x^2 z), ez\right) = \frac{1}{e}ez = z$$

so $M = z$.

Solving for L is harder: let $M^{(0)}(x, y, z) = y$,

$$L^{(1)}(x, y, z) = eL^{(0)}\left(\frac{1}{e}x, \frac{1}{e}(y + x^2 z), ez\right) = y + x^2 z$$
$$L^{(2)}(x, y, z) = eL^{(1)}\left(\frac{1}{e}x, \frac{1}{e}(y + x^2 z), ez\right) = e\left(\frac{1}{e}y + \frac{1}{e}x^2 z + \frac{1}{e}x^2 z\right) = y + 2x^2 z$$
$$L^{(3)}(x, y, z) = eL^{(3)}\left(\frac{1}{e}x, \frac{1}{e}(y + x^2 z), ez\right) = e\left(\frac{1}{e}y + \frac{1}{e}x^2 z + 2\frac{1}{e}x^2 z\right) = y + 3x^2 z$$
$$\vdots$$
$$L^{(n)}(x, y, z) = y + nx^2 z \sim n \to \infty$$

as $n \to \infty$ when $x^2 z \neq 0$. So it's not globally convergent. But if we restrict ourselves on the yz-plane or xy-plane then $H = (x, y, z)$, which is what we'd expect since the system is linear itself.

Now we do with the bump function. The new

$$y(t) = e^{-t}\left(y_0 + \int_0^t e^{-2s}x_0^2 b(e^s z_0))ds\right) = e^{-t}\left(y_0 + x_0^2 B(z_0, t)\right).$$

Now if $z(s) < \varepsilon$ the result is the same as above, so $B(z, t) = tz$. To make this happen we have to have $z(s) < \varepsilon$ for all $0 < s < t$, which means $|z_0| < \varepsilon/t$. When $z(s) > \delta$ it is 0.This happens when $|z_0| > \delta/t$.

Letting $B(z) = B(z, 1)$ we have

$$B(z) = \begin{cases} z & |z_0| < \frac{\varepsilon}{t} \\ 0 & |z_0| > \frac{\delta}{t} \end{cases}$$

Doing the iteration now we have (K and M carries over from previous result) :

$$L^{(1)}(x, y, z) = eL^{(0)}\left(\frac{1}{e}x, \frac{1}{e}(y + x^2 B(z)), ez\right) = y + x^2 B(z)$$
$$L^{(2)}(x, y, z) = eL^{(1)}\left(\frac{1}{e}x, \frac{1}{e}(y + x^2 B(z)), ez\right) = y + x^2 B(z) + \frac{1}{e}x^2 B(ez)$$
$$L^{(3)}(x, y, z) = eL^{(2)}\left(\frac{1}{e}x, \frac{1}{e}(y + x^2 B(z)), ez\right) = y + x^2 B(z) + \frac{1}{e}x^2 B(ez) + \frac{1}{e^2}x^2 B\left(e^2 z\right)$$

$$\vdots$$

$$L^{(n)}(x, y, z) = y + \sum_{i=0}^{n-1} \frac{1}{e^i} x^2 B\left(e^i z\right)$$

Now that the sum vanishes when i is large since then $B(e^i z) > \delta$ for nonzero z, so we don't have a convergence issue. So for $n \geq N := \log(\delta/z)$, $B(e^n z) = 0$. So the locally convergent version of the solution is

$$H(x, y, z) = (x, L(x, y, z), z)$$

where

$$L(x, y, z) = y + \sum_{i=0}^{N-1} \frac{1}{e^i} x^2 B\left(e^i z\right)$$

which is unique up to our choice of b.

Exercise 6.

(1):

$$\begin{cases} \dot{x} = -x + y - y^2 - x^3 \\ \dot{y} = x - y + xy \end{cases}$$

Let $L = \frac{1}{2}(x^2 + y^2)$, then

$$\dot{L} = \frac{1}{2}(2x\dot{x} + 2y\dot{y}) = -x^2 + xy - xy^2 - x^4 + xy - y^2 + xy^2 = -(x-y)^2 - x^4 < 0$$

when $(x, y) \neq (0, 0)$. Therefore it is globally asymptotic stable at the origin.

(2):

$$\begin{cases} \dot{x} = y - x^2 + 3y^2 - 2xy \\ \dot{y} = -x - 3x^2 + y^2 + 2xy \end{cases}$$

(After an hour of tryouts, the Lyapunov function I found that I'm somehow trapped.)

Let $L = \dfrac{1}{2}(x^2 + y^2) + x^3 + y^3 - x^2y - xy^2$, then

$$\dot{L} = \dot{x}(x + 3x^2 - 2xy - y^2) + \dot{y}(y - x^2 + 3y^2 - 2xy) = \dot{x}(-\dot{y}) + \dot{y}\dot{x} = 0$$

which is a weak Lyapunov function, so at least the function is Lyapunov stable. (It really is a slap on the face when I checked the derivative...)

And as for whether it is only Lyapunov stable or it is asymptotic stable but I didn't find the correct function, I plotted this so it should be clear:

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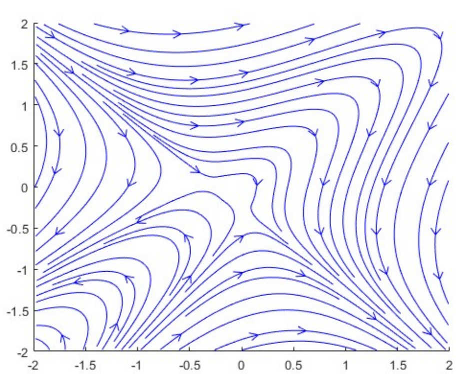
<td>
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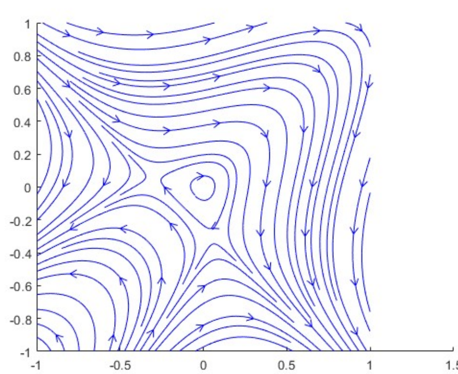
<td>
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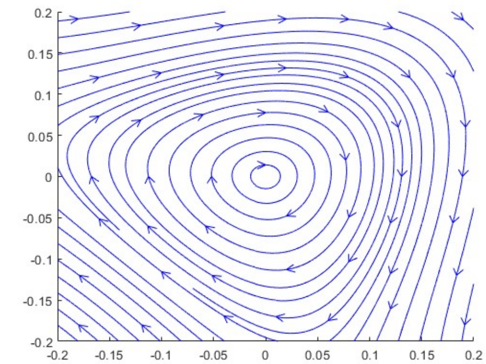
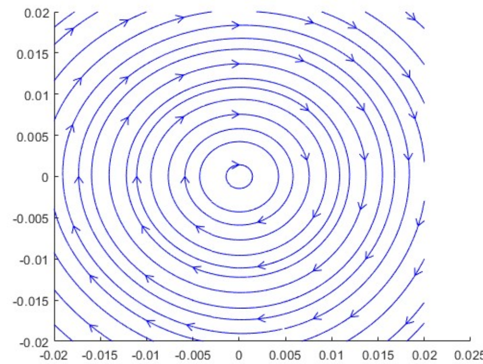
Nothing can yet be spotted form this scale, but when I zoom up

	
Scale: $[-0.2, 0.2] \times [-0.2, 0.2]$	Scale:$[-0.02, 0.02] \times [-0.02, 0.02]$

So indeed it looks like it is only Lyapunov stable, and our choice of Lyapunov function is as good as can be done.

Exercise 7.

(a):

The first thing to notice is that by definition

$$e^{tA} = I + tA + \frac{1}{2}t^2A^2 + \frac{1}{6}t^3A^3 + \dots$$

so

$$\frac{d}{dt}e^{tA} = A + tA^2 + \frac{1}{2}t^2A^3 + \frac{1}{6}t^3A^4 + \cdots = Ae^{tA} = e^{tA}A.$$

Now we can say the following starting with (4.24):

$$A^TS + SA = -I \Rightarrow e^{tA^T}(A^TS + SA)e^{tA} = -e^{tA^T}e^{tA}$$

Now since

$$\frac{d}{dt}\left(e^{tA^T}Se^{tA}\right) = \frac{d}{dt}\left(e^{tA^T}\right)Se^{tA} + e^{tA^T}S\frac{d}{dt}\left(e^{tA}\right) = e^{tA^T}(A^TS + SA)e^{tA}$$

we have by taking the integral

$$e^{sA^T}Se^{sA}\Big|_0^\infty = -\int_0^\infty e^{\tau A^T}e^{\tau A}d\tau$$

but we have to prove that the left hand side is well-defined at ∞ first. To see this, we note

$$\lim_{t\to\infty} e^{tA} = P\left(\lim_{t\to\infty} e^{t\Lambda}\right)P^{-1} = P\left(\lim_{t\to\infty}\begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n}\end{pmatrix}\right)P^{-1}$$

where since $\mathrm{Re}(\lambda_i) \le 0$ and $|e^ki| = 1$, the above limit goes to the zero matrix as $t \to \infty$. So we have

$$0 - S = -\int_0^\infty e^{\tau A^T}e^{\tau A}d\tau$$

$$\Rightarrow S = \int_0^\infty e^{\tau A^T}e^{\tau A}d\tau$$

(b):

Since

$$\tau A = \begin{pmatrix} -2\tau & 0 \\ 0 & -2\tau \end{pmatrix} + \begin{pmatrix} 0 & \tau \\ 0 & 0 \end{pmatrix} = D + N$$

we have

$$e^{\tau A} = e^D(I + N) = \begin{pmatrix} e^{-2\tau} & \tau e^{-2\tau} \\ 0 & e^{-2\tau} \end{pmatrix}$$

and since $e^{A^T} = (e^A)^T$ we know

$$e^{\tau A^t} = \begin{pmatrix} e^{-2\tau} & 0 \\ \tau e^{-2\tau} & e^{-2\tau} \end{pmatrix}$$

and thus

$$S = \int_0^\infty \begin{pmatrix} e^{-2\tau} & 0 \\ \tau e^{-2\tau} & e^{-2\tau} \end{pmatrix} \begin{pmatrix} e^{-2\tau} & \tau e^{-2\tau} \\ 0 & e^{-2\tau} \end{pmatrix} d\tau = \int_0^\infty \begin{pmatrix} e^{-4\tau} & \tau e^{-4\tau} \\ \tau e^{-4\tau} & \tau^2 e^{-4\tau} + e^{-4\tau} \end{pmatrix} d\tau.$$

Integrating term by term we have

$$S = \begin{pmatrix} 1/4 & 1/16 \\ 1/16 & 9/32 \end{pmatrix}.$$

So

$$L(x,y) = (x,y)S(x,y)^T = \frac{1}{4}x^2 + \frac{1}{8}xy + \frac{9}{32}y^2$$

and its derivative

$$\begin{aligned} \dot{L} &= \frac{1}{2}x\dot{x} + \frac{1}{8}\dot{x}y + \frac{1}{8}x\dot{y} + \frac{9}{16}y\dot{y} \\ &= \frac{1}{2}x(-2x+y) + \frac{1}{8}(-2x+y)y + \frac{1}{8}x(-2y) + \frac{9}{16}y(-2y) \\ &= -x^2 - y^2 < 0 \end{aligned}$$

whenever $(x,y) \neq (0,0)$. So indeed it is a strong Lyapunov function.

Exercise 8.

The most important thing is that since A is a linear operator, $Ax \sim x$ in scale, i.e. $Ax = O(x)$ and $x = O(Ax)$.

From last question we know for $x \neq 0$, in the linear case

$$\frac{d}{dt}\langle x, Sx\rangle = (Ax)^T Sx + x^T S(Ax) = 2\langle Ax, Sx\rangle < 0$$

since S symmetric. Moving on,

$$\langle Ax + g(x), Sx\rangle \leq \sup_{u\in\mathbb{D}^1}\langle Ax + u||g(x)||, Sx\rangle = \langle Ax, Sx\rangle + ||g(x)|| \sup_{u\in\mathbb{D}^1}\langle u, Sx\rangle$$

where $\mathbb{D}^1$ is the unit disc. Taking u to be the normalized vector of Sx we have

$$\sup_{u\in\mathbb{D}^1}\langle u, Sx\rangle = ||Sx||$$

which means we only need

$$||g(x)|| < \frac{\langle Ax, Sx\rangle}{||Sx||}$$

to get our result. Since the right hand side does not vanish due to last question, we know that the nominator is of order $\theta(x^2)$ (because S solely depends on A, so there's no such case where the inner product goes below $O(x^2)$), and the denominator $\theta(x)$. But since $g(x) = o(x)$, we know the condition is satisfied for small enough x.