CONVEX OPTIMIZATION HOMEWORK 1

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Exercise 1. *Find the dual cone of each K.*

Proof.

(1): $K = \{0\}$.

$$K^* := \{y | y^T x \geq 0, \forall x \in K\}$$

but $\forall y$, $y^T \cdot 0 = 0 \geq 0$, hence $K^* = \mathbb{R}^2$.

(2): $K = \mathbb{R}^2$.

Since for any $y \neq 0$, $y^T(-y) < 0$, so $y \notin K^*$. And $0 \in K^*$ for the same reason as in (1), so $K^* = \{0\}$.

(3): $K = \{(x_1, x_2) | |x_1| \leq x_2\}$.

First of all, K is nothing but the set between $y = x$ and $y = -x$ for $y \geq 0$, so it is a cone. We show that $K^* = K$.

$\underline{K \subset K^*}$:

For any $x, y \in K$ we have

$$y^T x = x_1 y_1 + x_2 y_2 \geq x_1 y_1 + |x_1||y_1| \geq x_1 y_1 + |x_1 y_1| \geq 0$$

since $x + |x| \geq 0$ for any $x \in \mathbb{R}$. Hence, $\forall y \in K$, we have that $\forall x \in K$ $y^T x \geq 0$, so $K \subset K^*$.

$\underline{K^* \subset K}$:

We prove this by proving that for any $y \notin K$ there exists $x \in K$ such that $y^T x < 0$. Say we have $y = (y_1, y_2)$ with $|y_1| > y_2$. Then, if $y_2 < 0$ we have

$$y^T(0, 1) = y_2 < 0$$

where as for $y_2 \geq 0$, by definition $y_1 \neq 0$. Now let $x = (-y_1, |y_1|)$ then we have

$$Y^T x = -y_1^2 + |y_1| y_2 \leq y_1^2 \left(-1 + \frac{y_2}{|y_1|}\right) < 0$$

since $y_1^2 > 0$ and $\dfrac{y_2}{|y_1|} < 1$ by assumption. Hence $y \notin K^*$. Which further means $K^* \subset K$.

(4): $K = \{(x_1, x_2)|x_1 + x_2 = 0\}$.

First, K is nothing but the line $y = -x$, so it's convex.

Then

$$y^T x \geq 0 \iff y_1 x_1 - y_2 x_1 \geq 0 \iff x_1(y_1 - y_2) \geq 0.$$

But if it is to hold for all $x_1 \in \mathbb{R}$, then $y_1 - y_2$ has to be both ≥ 0 ($x_1 = 1$) and ≤ 0 ($x_1 = -1$), so $y_1 - y_2 = 0$. That is, y is on the line $x = y$.

Since all reasoning above works both direction, $K^* = \{(x_1, x_2)|x_1 - x_2 = 0\}$.

$\square$

Exercise 2.

Proof.

I prove for the case when $\alpha \geq 0, \beta \geq 0$ and $\alpha + \beta > 0$, since if the sum is zero then we can find $C \neq 0$ with $\alpha x + \beta y \notin \{0\}$ for $x, y \in C$; if any is negative then we can find examples like $C = [1 - \varepsilon, 1 + \varepsilon]$ with $\alpha = -2$, $\beta = 1$, then $-1 + 3\varepsilon$ is in $\alpha C + \beta C$ but not in $(\alpha + \beta C)$.

$\underline{C \text{ convex} \Rightarrow \alpha C + \beta C = (\alpha + \beta)C:}$

$\underline{\alpha C + \beta C \subset (\alpha + \beta)C:}$

For $x \in \alpha C + \beta C$, we can write $x = \alpha c_1 + \beta c_2$ for $c_1, c_2 \in C$. Define

$$c_3 = \frac{\alpha}{\alpha + \beta} c_1 + \frac{\beta}{\alpha + \beta} c_2$$

then $c_3 \in C$ since $c_1, c_2 \in c$ and $\frac{\alpha}{\alpha + \beta} + \frac{\beta}{\alpha + \beta} = 1$ and they are both between 0 and 1. Yet then

$$x = (\alpha + \beta) \left(\frac{\alpha}{\alpha + \beta} c_1 + \frac{\beta}{\alpha + \beta} c_2 \right) = (\alpha + \beta) c_3 \in (\alpha + \beta)C.$$

$\underline{\alpha C + \beta C \supset (\alpha + \beta)C:}$

For $x \in (\alpha + \beta)C$ we have

$$x = (\alpha + \beta)c = \alpha c + \beta c$$

for some $c \in C$, thus $x \in \alpha C + \beta C$.

$\underline{C \text{ convex} \Leftarrow \alpha C + \beta C = (\alpha + \beta)C:}$

For $x, y \in C$ $\theta \in [0, 1]$. Let $\alpha = \theta$ and $\beta = 1 - \theta$, then we have by assumption

$$\theta x + (1 - \theta) y \in C$$

which proves what we want directly. □

Exercise 3.

Proof.

(a): $\bigcap_{\alpha \in \mathcal{A}} S_\alpha$ is convex for S_α convex:

For all $x, y \in \bigcap_{\alpha \in \mathcal{A}} S_\alpha$, $x, y \in S_\alpha$ for all $\alpha \in \mathcal{A}$. Yet then for any $\theta \in [0, 1]$

$$\theta x + (1 - \theta) y \in S_\alpha, \quad \forall \alpha \in \mathcal{A}$$

which means that

$$\theta x + (1 - \theta) y \in \bigcap_{\alpha \in \mathcal{A}} S_\alpha$$

hence it is convex.

(b):

Let

$$S := \left\{ a \in \mathbb{R}^k \middle| p(0) = 2, |p(t)| \leq 2, \alpha \leq t \leq \beta \right\}$$

where $p(t) = a_1 + a_2 t + \cdots + a_k t^{k-1}$.

Show S is convex using (a):

Let

$$S_x^+ := \left\{ a \in \mathbb{R}^k \middle| p(t) \leq 2, t = x \right\} = \left\{ a \in \mathbb{R}^k \middle| (1, x, x^2, \dots)^T a \leq 2 \right\}$$

and

$$S_x^- := \left\{ a \in \mathbb{R}^k \middle| -p(t) \leq 2, t = x \right\} = \left\{ a \in \mathbb{R}^k \middle| (1, x, x^2, \dots)^T a \geq -2 \right\}$$

and

$$S' := \left\{ a \in \mathbb{R}^k \middle| p(0) = 2 \right\}.$$

then S_x^+ and S_x^- are convex for all x since it's a halfspace. S' is convex since it's the hyperplane $\{a \in \mathbb{R}^k | a_1 = 2\}$.

Thus

$$S = S' \cap \left(\bigcap_{x \in [\alpha, \beta]} S_x^+ \cap S_x^- \right)$$

is an intersection of convex sets, so it's convex.

$\square$

Exercise 4.

Proof.

(1) the cone of positive semi-definite matrices of dimension n is proper:

Call that cone S. I just use the standard 2-norm here. But it really doesn't matter since all matrix norms are equivalent and the only thing I'm using is just that the norm of the difference tends to 0 (any norm tends to 0 by equivalence.)

Closed:

Suppose $A \in M \backslash S$, then it means that A has a negative eigenvalue. Then we can let that eigenvalue λ_i map to $\lambda_i + \varepsilon_i$ where

$$\varepsilon_i := \min\{\varepsilon, -\lambda_i/2\}$$

for any $\varepsilon > 0$. Thus we get a new matrix A' that is close enough to A since (with eigenvalue decomposition)

$$||A - A'|| \leq ||P|| \cdot ||P^{-1}|| \cdot ||\operatorname{diag}(0, \dots, \varepsilon_i, \dots, 0)|| \leq c\varepsilon_i \leq c\varepsilon.$$

And since A' is still not semi-definite positive, $M \backslash S$ is open, so S is closed.

solid (non-empty interior):

Any ε ball for small enough ε around I, the identity matrix, is semi-definite positive since for any I' in that ball

$$x^T I' x - x^T I x \leq ||x||^2 ||I - I'|| \leq c\varepsilon$$

and thus

$$x^T I' x \in [||x||^2 - c\varepsilon, ||x||^2 + c\varepsilon] \subset [0, \infty)$$

for any $x > 0$. If $x = 0$ then it automatically holds that $x^T I' x \geq 0$.

So at least I is in the interior of S, so it's interior is non-empty.

pointed (don't contain lines):

Assume that it contains a line cA for any $c \in \mathbb{R}$. Then for any x with $x^T A x \neq 0$ we know

$$(x^T A x) \cdot (x^T (-A) x) < 0$$

hence A and $-A$ cannot both be in S. So S contains no line.

(2) Show that the hyperbolic cone is proper:

Let

$$C := \{(x, y) \in \mathbb{R}^{n+1} | y^T y \leq x^2, x \geq 0\}.$$

First, it is convex because any $(x_1, y_1), (x_2, y_2) \in C, \theta \in [0, 1]$, we have $\theta x_1 + (1-\theta)x_2 > 0$ and

$$\begin{aligned}
[\theta y_1 + (1-\theta)y_2]^T[\theta y_1 + (1-\theta)y_2] &= \theta^2||y_1||^2 + (1-\theta)^2||y_2||^2 + 2\theta(1-\theta)y_1 \cdot y_2 \\
\text{(Cauchy-Schartz)} &\le \theta^2 x_1^2 + (1-\theta)^2 x_2^2 + 2\theta(1-\theta)||y_1|| \cdot ||y_2|| \\
&\le \theta^2 x_1^2 + (1-\theta)^2 x_2^2 + 2\theta(1-\theta)x_1 x_2 \\
&\le (\theta x_1 + (1-\theta)x_2)^2
\end{aligned}$$

Closed:

Suppose $(x, y) \in X \backslash C$, then it means $y^T y > x^2$. But since both $y^T y = ||y||$ and x^2 are continuous functions, so is $y^T y - x^2$. Thus any small change in x or y won't change the inequality. More specifically, let $y^T y - x^2 = c$, then we can find δ with $d((x', y'), (x, y)) < \delta$ such that

$$|c - (y'^T y' - x'^2)| \le \frac{c}{2}$$

hence $X \backslash C$ is open, so C is closed.

solid (non-empty interior):

Any ε ball for small enough ε around $(x, y) = (0, 1, 0, \dots, 0)$, has that $y^T y - y'^T y \le c\varepsilon$.

So at least $(x, y) = (0, 1, 0, \dots, 0)$ is in the interior of C, so it's interior is non-empty.

pointed (don't contain lines):

Since we are fixed in the halfplane $x \ge 0$, so if there is a line in C, on the line $x = 0$. But this requires that $||y||^2 \le 0$ thus $y = 0$ on every point in the line. But then there's no line, just a point. So C is pointed.

(3) The duals of cones in (a),(b) are themselves.

For the cone S, I use the same method as Example 2.24 in textbook. Here we used the trace as inner product, but since all norms are equivalent it really doesn't matter. Note that $\mathrm{tr}(XY) = \sum_{i.j=1}^{n} X_{i,j} Y_{i,j}$.

We now show that the cone is self-dual. Suppose $Y \neq S$, then $\exists q \in R^n$ with

$$q^T Y q = \mathrm{tr}(q q^T Y) < 0$$

which further means that for $X := qq^T$, $\mathrm{tr}(XY) < 0$, so $Y \notin S^*$.

So we only need to show that any $Y \in S$ is in S^*. For any $X \in S$ we can write by eigenvalue decomposition

$$X = \sum_{i=1}^{n} \lambda_i q_i q_i^T$$

for positive eigenvalues. Then

$$\mathrm{tr}(YX) = \mathrm{tr}\left(Y \sum_{i=1}^{n} \lambda_i q_i q_i^T\right) = \sum_{i=1}^{n} \lambda_i q_i Y q_i^T \geq 0$$

which means that $Y \in S^*$. Hence $S = S^*$.

As for C above, it is self dual since for any $(x, y) \notin C$, we have $||y||^2 > x^2$ or $x < 0$. If $x < 0$, then since $(1, 0) \in C$ thus

$$(x, y)^T(1, 0) = x < 0$$

so $(x, y) \notin C^*$.

If $x \geq 0$ and $||y||^2 > x^2$, since $(||y||, -y) \in C$ we get

$$(x, y)^T(||y||, -y) = x||y|| - ||y||^2 < 0$$

which means that $(x, y) \notin C^*$. Thus $C^* \subset C$.

Now, for $(x, y), (a, b) \in C$ we have

$$(x, y)^T(a, b) = ax + b \cdot y \geq ||b|| \cdot ||y|| + b \cdot y \geq |b \cdot y| + b \cdot y \geq 0$$

by Cauchy Schwartz. Therefore $C \subset C^*$.

In conclusion, $C = C^*$.

$\square$