MEASURE THEORETIC PROBABILITY III HW 1

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Discussed with classmates.

Exercise 1. *Ex 1 on pset.*

Proof.

Define $Y_m := \inf_{n\geq m} X_n$. Since X_n are measurable with respect to the background large σ-algebra , so is Y_n.

By definition of inf we have for $\forall A \in \mathcal{G}$

$$\int_A \inf_{m\leq n} X_n d\mathbb{P} \leq \inf_{n\geq m} \int_A X_n d\mathbb{P}$$

since $Y_m \leq X_n$ for all $n \geq m$ and $X_n \geq 0$. Now we take limit on m on both sides and get

$$\lim_{m\to\infty} \int_A Y_m d\mathbb{P} \leq \lim_{m\to\infty} \inf_{n\geq m} \int_A X_n d\mathbb{P} = \liminf_n \int_A X_n d\mathbb{P}$$

where for the left hand side we note that $Y_m \uparrow$ since the infimum is taken over less X_n, and by MCT we have

$$\int_A \liminf_n X_n d\mathbb{P} = \int_A \lim_{m\to\infty} Y_m d\mathbb{P} \overset{MCT}{=} \lim_{m\to\infty} \int_A Y_m d\mathbb{P} \leq \liminf_n \int_A X_n d\mathbb{P}$$

which is what we want since

$$\int_A \mathbb{E}\left[\liminf_n X_n|\mathcal{G}\right] d\mathbb{P} \leq \liminf_n \int_A X_n d\mathbb{P} = \liminf_n \int_A \mathbb{E}\left[X_n|\mathcal{G}\right] d\mathbb{P}$$

for all $A \in \mathcal{G}$ implies

$$\mathbb{E}\left[\liminf_n X_n|\mathcal{G}\right] \leq \liminf_n \mathbb{E}\left[X_n|\mathcal{G}\right]$$

as they are $\mathcal{G}$ measurable.

$\Bigg($A more detailed proof of the last point is because

$$S := \left\{\mathbb{E}\left[\liminf_n X_n|\mathcal{G}\right] - \liminf_n \mathbb{E}\left[X_n|\mathcal{G}\right] \geq \varepsilon > 0\right\}$$

is $\mathcal{G}$ measurable hence the integral of their difference is ≤ 0, which means $\mathbb{P}(S) = 0.\Bigg)$ $\square$

Exercise 2. *Ex 3 on pset.*

Proof.

Y is X measurable if $Y = f(X)$ for some f that is X measurable:

We do the usual approximation trick and note that for Y that are the indicator functions, $Y = \mathbb{1}_A$ for some $A \in \sigma(X)$. This means that $A = X^{-1}(B)$, and $Y = \mathbb{1}_B(X)$.

Now by linearity of integrals we get the result for simple Y. Using MCT we get for non-negative Y and separating into $Y = Y^+ - Y^-$ we get the result for integrable Y.

In particular, when doing MCT we have $\limsup f_n =: f$ is still measurable since measurable functions are closed under $\limsup$.

As for the problem...

Since $\mathbb{E}[Y|X]$ is X measurable it can be written as $h(X)$ for h measurable. Similar for $\mathbb{E}[Y'|X'] = h'(X')$.

If we can show that these h are the same then we are done due to

$$\mu_{h(X)} = \mathbb{P} \circ X^{-1} \circ h^{-1} = \mu_X \circ h^{-1} = \mu_{X'} \circ h^{-1} = \mu_{h(X')}.$$

But let's just directly compute $h(X')$, if it works then everything works.

We have on the one hand for $A' \in \sigma(X')$, define B be the corresponding borel set in $\mathbb{R}$ such that $(X')^{-1}(B) = A'$ and $A := X^{-1}(B)$

$$\int_A h(X) d\mathbb{P} = \int_{B\times\mathbb{R}} h(x) d\mu_{X,Y} = \int_{B\times\mathbb{R}} h(x) d\mu_{X',Y'} = \int_{A'} h(X') d\mathbb{P}$$

and on the other we have

$$\int_A h(X) d\mathbb{P} = \int_B y d\mu_Y = \int_B y d\mu_{Y'} = \int_{A'} Y' d\mathbb{P}$$

and hence $h(X') = \mathbb{E}[Y'|X']$. So we are done.

□

Exercise 3. *Ex 4 on pset.*

Proof.

Assume $X \in L^2$:

We have

$$\mathbb{E}[X^2] = \int_{\mathbb{R}} x^2 d\mu_X = \int_{\mathbb{R}} x^2 d\mu_{\mathbb{E}[X|\mathcal{G}]} = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]^2]$$

and for convenience we just let $Y := \mathbb{E}[X|\mathcal{G}]$. Thus we have

$$\mathbb{E}[(X-Y)^2] = \mathbb{E}[X^2] + \mathbb{E}[Y^2] - 2\mathbb{E}[XY]$$

where

$$\mathbb{E}[XY] = \mathbb{E}[\mathbb{E}[XY|\mathcal{G}]] = \mathbb{E}[Y\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[Y^2]$$

and hence

$$\mathbb{E}[(X-Y)^2] = \mathbb{E}[X^2] - \mathbb{E}[Y^2] = 0$$

which means for $S := \{X \neq Y\}$ we have $\mathbb{P}(S) = 0$, that is, X is a version of $\mathbb{E}[X|\mathcal{G}]$.

General case:

Let $Y_n = \min\{Y^+, n\} + \max\{Y^-, -n\}$ and $X_n = \min\{X^+, n\} + \max\{X^-, -n\}$ and WLOG we consider only the positive case since the other part is exactly the same (with a flipped Jensen's inequality).

By Jensen's inequality (for concave functions) we get

$$\mathbb{E}[\min\{X, n\}|\mathcal{G}] \leq \min\{\mathbb{E}[X|\mathcal{G}], n\} = Y_n$$

but since $\mathbb{E}[\min\{X, n\}] = \mathbb{E}[\min\{Y, n\}]$ so if one of them is strictly larger than the other, we know that the equality must hold a.s.. Hence we know

$$\mathbb{E}[\min\{X, n\}|\mathcal{G}] \overset{a.s.}{=} Y_n$$

but then $X_n = Y_n$ by the same second moment argument as they also have the same law, and since $X_n \uparrow X$ so is $Y_n \uparrow$ (result in class) so by MCT we can pass the limit and get $X \overset{a.s.}{=} Y$.

But just choosing different versions of the conditional expectation, we know it's measurable with respect to the completion of $\mathcal{G}$.

$\square$