BROWNIAN MOTION AND STOCHASTIC CALCULUS HW 5

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Discussed with classmates.

Exercise 1.

Proof.

(This page is more of a sanity check, computation on next page.)

We use Ito's formula to solve this with $Y_t = f(B_t)$ or $f(t, B_t)$.

We first justify the formal computation we'll use then just use it. In general, for

$$dX_t = R_t dt + A_t dB_t$$

we know that the drift term does not affect $\langle X_t \rangle$ because $\begin{cases} (dt)^2 \le ||\Pi_n|| \cdot t \to 0 \\ \sup |X_{\frac{i}{n}} - X_{\frac{i-1}{n}}| \to 0 \end{cases}$ where Π_n is the increasing mesh and the first line deals with dt^2 term and the second line the cross term (the full justification is similar to the one done in class for B_t, so I skip details). Now using orthogonal increment of B_t we have

$$\lim_{||\Pi||\to 0} \sum \left(\int_{t_i}^{t_{i+1}} A_t dB_t \right)^2 \overset{cts}{=} \lim_{||\Pi||\to 0} \sum A_{t_i}^2 \left(\int_{t_i}^{t_{i+1}} dB_t \right)^2 + O(\varepsilon)$$
$$\to \lim_{||\Pi||\to 0} \sum \int_{t_i}^{t_{i+1}} A_{t_i}^2 d\langle B_t \rangle + O(\varepsilon) \to \int_0^t A_s^2 ds$$

so we've justified the formal computation

$$d\langle X_t \rangle_t = A_t^2 dt.$$

For covariation the deduction is just the same and we get for

$$\begin{cases} dX_t = R_t dt + A_t dB_t \\ dY_t = S_t dt + C_t dB_t \end{cases}$$

we have

$$\langle X_t, Y_t \rangle_t = \int_0^t A_s C_s ds.$$

The rest is computation with Ito's formula.

(1): $f = x^2$ and hence

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)d\langle B\rangle_t = 2B_t dB_t + \frac{1}{2}2d\langle B\rangle_t = t + 2B_t dB_t$$

thus

$$A_t = (2B_t)^2 = 4B_t^2$$

and

$$C_t = (2B_t)X_t = 2B_t X_t.$$

(2): Similarly

$$\begin{aligned} df(X_t) &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X\rangle_t = 3X_t^2 dX_t + \frac{1}{2}6X_t d\langle X\rangle_t \\ &= 3X_t^2(X_t^2 dt + X_t dB_t) + 3X_t X_t^2 dt = 3X_t^3 dB_t + (3X_t^4 + 3X_3)dt \end{aligned}$$

thus

$$A_t = (3X_t^3)^2 = 9X_t^6$$

and

$$C_t = (3X_t^3)X_t = 3X_t^4.$$

(3): Denote $Z_t = \int_0^t X_s^2 + 1ds$ and we get

$$dZ_t = X_t^2 + 1dt$$

and note $\langle Z\rangle_t = 0$ by the same argument as above.

For $f = e^x$ we compute similarly

$$\begin{aligned} df(Z_t) &= f'(Z_t)dZ_t + \frac{1}{2}f''(Z_t)d\langle Z\rangle_t = e^{Z_t}dZ_t + \frac{1}{2}e^{Z_t}d\langle Z\rangle_t \\ &= e^{Z_t}(X_t^2 + 1)dt \end{aligned}$$

but then there's no variation term so

$$A_t = C_t = 0.$$

□

Exercise 2.

Proof.

(1):

We compute

$$\int_0^t B_s \circ d B_s = \lim_{n\to\infty} \sum_{j\le nt} \frac{B_{\frac{j-1}{n}} + B_{\frac{j}{n}}}{2} \left[B_{\frac{j}{n}} - B_{\frac{j-1}{n}} \right] = \frac{1}{2} \lim_{n\to\infty} \sum_{j\le nt} \left[B^2_{\frac{j}{n}} - B^2_{\frac{j-1}{n}} \right] = \frac{1}{2} B_t^2.$$

(2): First we assume that f has compact support. This gives boundedness of f, f', f''.

Let's write out according to definition that

$$\int_0^t f'(B_s) \circ d B_s = \lim_{n\to\infty} \sum_{j\le nt} \frac{f'(B_{\frac{j}{n}}) + f'(B_{\frac{j-1}{k}})}{2} \left[B_{\frac{j}{n}} - B_{\frac{j-1}{n}} \right]$$

and thus

$$\int_0^t f'(B_s) \circ d B_s - \int_0^t f'(B_s) d B_s = \lim_{n\to\infty} \sum_{j\le nt} \frac{f'(B_{\frac{j}{n}}) - f'(B_{\frac{j-1}{k}})}{2} \left[B_{\frac{j}{n}} - B_{\frac{j-1}{n}} \right]$$

and a naive bound gives, say $x < y$,

$$(y-x) \min_{x\le z\le y} f''(z) \le f'(y) - f'(x) \le (y-x) \max_{x \le z \le y} f''(z)$$

in particular from this we obtain the expression inside the limit:

$$B^- \le \sum_{j\le nt} \left[f'(B_{\frac{j}{n}}) - f'(B_{\frac{j-1}{k}}) \right] \left[B_{\frac{j}{n}} - B_{\frac{j-1}{n}} \right] \le B^+$$

where

$$B^- := \sum_{j\le nt} \left(B_{\frac{j}{n}} - B_{\frac{j-1}{n}} \right)^2 \min_{z\in\left[B_{\frac{j-1}{n}}, B_{\frac{j}{n}}\right]} f''(z); \quad B^+ := \sum_{j=1}^{n} \left(B_{\frac{j}{n}} - B_{\frac{j-1}{n}} \right)^2 \max_{z\in\left[B_{\frac{j-1}{n}}, B_{\frac{j}{n}}\right]} f''(z)$$

And as $n \to \infty$ (or to be careful, $n_j \to \infty$) we get $f''(z)^2 \to f''\left(B_{\frac{j-1}{n}}\right)$ by continuity and smooth, and we will get a fixed ϵ that works for this purpose, then we take $n \to \infty$ such that within the small slip $[s, s+\epsilon]$ the leftover sum goes to the variation on that slip, which is ϵ. Now we use Stieltjes integral and conclude:

$$B^{\pm} \to \frac{1}{2} \int_0^1 f''(B_t) dt$$

which then shows

$$\int_0^t f'(B_s) \circ d B_s - \int_0^t f'(B_s) d B_s = \frac{1}{2} \int_0^1 f''(B_t) dt$$

and using Ito's formula for regular stochastic integral we have

$$\int_0^t f'(B_s) \circ dB_s = f(B_t) - f(B_0) - \frac{1}{2}\int_0^1 f''(B_t)dt + \frac{1}{2}\int_0^1 f''(B_t)dt = f(B_t) - f(B_0)$$

analogous to Ito's formula.

(3): No. Calculation in part 1 says it is not since $\frac{1}{2}B_t^2$ is not a martingale, and the drift is $\frac{1}{2}t$.

$\square$

Exercise 3.

Proof.

(1): As we did in class, we first do formal computation to get an anzat, then veryfy that it satisfies the condition.

Formal computation:

Say we write out Ito's form:

$$F(X_t) - F(X_0) = \int_0^t F'(X_t)dX_t + \frac{1}{2}\int_0^t F''(X_t)d\langle X\rangle_t$$
$$= \int_0^t \left[F'(X_t)\cdot a\cot(X_t) + \frac{1}{2}F''(X_t)\right]dt + \int_0^t F'(X_t)dB_t$$

and since we want a Martingale, so we expect the drift term to vanish, which gives

$$F'(X_t)\cdot a\cot(X_t) + \frac{1}{2}F''(X_t) = 0$$

where it's an ODE in $F' := G$ which we solve to get

$$Ga\cot(x) + \frac{1}{2}G' = 0 \Rightarrow G = -2a\cot(x)G \Rightarrow G = c\sin^{-2a}(x)$$

and in particular we get

$$F = \int F' = \int_x^{\frac{\pi}{2}} c\sin^{-2a}(y)dy.$$

Verify:

F is positive as long as $c > 0$. $F(\pi/2) = 0$ just by integral. To show local Martingale, we've already defined a sequence of rising stopping times that $T_{\frac{1}{n}} \to T$ as $n \to \infty$ with probability 1. And we only need to show $M_t^{(n)}$ is a Martingale. We plug in Ito formula and using the fact that F' solves the PDE induced by the drift term to get

$$\mathbb{E}[M_t] - \mathbb{E}[M_0] = \mathbb{E}\left[\int_0^t F'(X_w)dB_w\right]$$

and since $F' = c\sin^{-2a}(x) \leq 1 < \infty$ we know the integral is a Martingale so the expectation is 0. Thus $\mathbb{E}[M_t] = \mathbb{E}[M_0]$. For a general Martingale property, for any $V \in \mathcal{F}_s$ we have

$$\mathbb{E}[M_t\mathbb{1}_V] - \mathbb{E}[M_s\mathbb{1}_V] = \mathbb{E}\left[\mathbb{1}_V\int_s^t F'(X_w)dB_w\right] = \mathbb{E}\left[\mathbb{1}_V\mathbb{E}\left[\int_s^t F'(X_w)dB_w\middle|\mathcal{F}_s\right]\right] = \mathbb{E}\left[\mathbb{1}_V\cdot 0\right] = 0$$

hence we're done showing that M_t is indeed a local Martingale for $c > 0$.

(2):

$\sin x \sim x$ around 0 and by p test if $0 < 2a < 1$ then we have the result. More precisely,

$$\sin(x)^{-2a} = \frac{c}{(x+O(x^3))^{2a}} = \frac{c}{(x)^{2a}} + O(\varepsilon)$$

as $x \to 0$. Thus $a \in (0, 1/2)$ is good. But notice that for $a \leq 0$ we are integrating a bounded function, so we are always good. Thus $a \in \left(-\infty, \frac{1}{2}\right)$ all satisfy the condition.

(3):

Since $M_{t\wedge T_\varepsilon}$ is a Martingale, so is $M_{t\wedge T_\varepsilon \wedge \tau}$ thus by Doob's optimal stopping theorem we have

$$F(x_0) = \mathbb{E}[M_0] = \mathbb{E}[M_{t\wedge T_\varepsilon\wedge\tau}]$$

and since X_t has variation term dB_t we know that with or without drift the 1d process has $\mathbb{P}\{\tau \wedge T_\varepsilon < \infty\} = 1$, and using $F(\pi/2) = 0$ we get

$$F(x_0) = F(\varepsilon)\mathbb{P}\{T_\varepsilon < \tau\} \Rightarrow \mathbb{P}\{T_\varepsilon < \tau\} = \frac{F(x_0)}{F(\varepsilon)}.$$

(4):

For this to be true we need $F(\varepsilon) \to \infty$ and by part (2) we know $a = \left[\frac{1}{2}, \infty\right)$ is the proper range.

□

Exercise 4.

Proof.

(1): Same thing as last question

Formal computation:

Say we write out Ito's form:

$$\begin{aligned} F(X_t) - F(X_0) &= \int_0^t F'(X_t) dX_t + \frac{1}{2}\int_0^t F''(X_t) d\langle X\rangle_t \\ &= \int_0^t \left[F'(X_t)\cdot mX_t + \frac{1}{2}F''(X_t)\right] dt + \int_0^t F'(X_t) dB_t \end{aligned}$$

and since we want a Martingale, so we expect the drift term to vanish, which gives

$$F'(X_t)\cdot mX_t + \frac{1}{2}F''(X_t) = 0$$

where it's an ODE in $F' := G$ which we solve to get

$$Gmx + \frac{1}{2}G' = 0 \Rightarrow G = ce^{-mx^2}$$

and in particular we get

$$F = \int F' = \int_0^x ce^{-my^2} dy.$$

Verify:

Denote $M_t = F(X_{t\wedge T})$. F is positive as long as $c > 0$. $F(0) = 0$ just by integral. To show Martingale, we plug in Ito formula and using the fact that F' solves the PDE induced by the drift term to get

$$\mathbb{E}[M_t] - \mathbb{E}[M_0] = \mathbb{E}\left[\int_0^t F'(X_w) dB_w\right]$$

and since $F' = ce^{-mx^2} \leq \max\{c, ce^{|mR^2|}\} < \infty$ we know the integral is a Martingale so the expectation is 0. Thus $\mathbb{E}[M_t] = \mathbb{E}[M_0]$. For a general Martingale property, for any $V \in \mathcal{F}_s$ we have

$$\mathbb{E}[M_t \mathbb{1}_V] - \mathbb{E}[M_s \mathbb{1}_V] = \mathbb{E}\left[\mathbb{1}_V \int_s^t F'(X_w) dB_w\right] = \mathbb{E}\left[\mathbb{1}_V \mathbb{E}\left[\int_s^t F'(X_w) dB_w \middle| \mathcal{F}_s\right]\right] = \mathbb{E}\left[\mathbb{1}_V \cdot 0\right] = 0$$

hence we're done showing that M_t is indeed a Martingale for $c > 0$.

(2):

By Doobs we have

$$F(1) = F(x_0) = \mathbb{E}[M_0] = \mathbb{E}[M_t] = \mathbb{E}[F(X_{t\wedge T})]$$

and since X_t has variation term dB_t we know that with or without drift the 1d process has $\mathbb{P}\{T < \infty\} = 1$, and using $F(0) = 0$ we get

$$F(1) = F(R)\mathbb{P}\{X_T = R\} \Rightarrow \mathbb{P}\{X_T = R\} = \frac{F(1)}{F(R)}.$$

(3):

This would require $\lim_{R\to\infty} F(R) = \infty$ or in other words

$$\lim_{R\to\infty} \int_0^R e^{-mx^2} dx$$

diverges. But exponential function decay speed tells us this diverges only when $m \leq 0$, i.e.

$$m \in (-\infty, 0].$$

□

Exercise 5.

Proof.

(1): By product rule

$$dX_tY_t = X_t dY_t + Y_t dX_t + d\langle X, Y\rangle_t = X_tY_t[\mu_2 dt + \sigma_2 dB_t] + X_tY_t[\mu_1 dt + \sigma_1 dB_t] + X_tY_t\sigma_1\sigma_2 dt$$
$$= Z_t\left[(\mu_1 + \mu_2 + \sigma_1\sigma_2)dt + (\sigma_1 + \sigma_2)dB_t\right]$$

(2): Ito's formula says

$$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X\rangle_t = \left(f'(X_t)X_t\mu_1 + \frac{1}{2}f''(X_t)X_t^2\sigma_1^2\right)dt + f'(X_t)X_t\sigma_2 dB_t$$

and we want RHS to be dB_t plus $f(1) = 0$ in order that $f(X_t) = B_t$, which requires that

$$f'(x)x\mu_1 + \frac{1}{2}f''(x)x^2\sigma_1^2 = 0$$

and

$$f'(x)x\sigma_2 = 1$$

where the second equation gives, denoting $h = f'$, gives

$$h = \frac{1}{\sigma_1 x} \Rightarrow h' = -\frac{1}{\sigma_1 x^2}$$

and also we get from first equation that

$$-\frac{1}{\sigma_1 x^2} = h' = -\frac{2\mu_1}{\sigma_1^2}\frac{1}{\sigma_1 x^2}$$

gives $\sigma_1^2 - 2\mu_1 = 0$ or no solution. If $\sigma_1^2 - 2\mu_1 = 0$ is satisfied, we get

$$f = \frac{\log x}{\sigma_1} + c$$

and by $f(1) = 0$ the initial condition we obtain $c = 0$ where the result is then (checked that it's C^2):

$$f = \begin{cases} \frac{\log x}{\sigma_1} & \sigma_1^2 - 2\mu_1 = 0 \\ DNE & \sigma_1^2 - 2\mu_1 \neq 0 \end{cases}$$

(3): Note that Z_t and X_t are of the same SDE for different parameters, so they have the same solution that we can plug in:

$$g = \begin{cases} \frac{\log x}{\sigma_1+\sigma_2} & \sigma_1^2 + \sigma_2^2 - 2\mu_1 - 2\mu_2 = 0 \\ DNE & \sigma_1^2 + \sigma_2^2 - 2\mu_1 - 2\mu_2 \neq 0 \end{cases}$$

(in fact I think that $f(t, X_t)$ has a general solution, but that's not what question asks)

□