PDE HOMEWORK 5

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Discussed with classmates.

Exercise 1.

Proof.

(1): From $\partial_x \partial_y u = 0$ we do integration in x to get

$$\partial_y u(x, y) - \partial_y u(0, y) = \int_0^x \partial_t \partial_y u(t, y) dt = \int_0^x 0 dt = 0$$

we note that $\partial_y u(0, y)$ is independent of x, hence only a function of y, and thus shifting terms we get

$$\partial_y u(x, y) = \partial_y u(0, y) =: C(y)$$

and again we have

$$u(x, y) - u(x, 0) = \int_0^y \partial_s u(x, s) ds = \int_0^y C(s) ds =: D(y) - G(0)$$

where again $u(x, 0)$ is independent of y so just a function of x. Shifting terms we get

$$u(x, y) = D(y) - D(0) + u(x, 0) =: G(y) + F(x)$$

and what we've shown is that for all u that satisfies $u_{xy} = 0$, there is some F, G such that $u = F(x) + G(y)$. We want to show that F, G can be arbitrary, and the way is to show equivalence between the following sets:

$$A := \{u | u_{xy} = 0\}; \quad B := \{F(x) + G(y) | \forall F, G\}$$

and what we've shown above is that we can find a corresponding $v \in B$ for each $u \in A$. Thus $A \subset B$. Now for $B \subset A$ this is obvious because for any $v = F(x) + G(y)$, $u_{xy} = 0$.

Thus $A = B$ and so $u = F(x) + G(y)$ where F and G are arbitrary.

(2):

Just use chain rule we have

$$\partial_\xi u\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}\right) = \frac{1}{2}\left(u_x + u_t\right)$$

and once more

$$u_{\xi\eta} = \partial_\eta \partial_\xi u\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) = \partial_\eta \frac{1}{2}\left(u_x + u_t\right) = \frac{1}{4}\left(u_{xx} - u_{xt} + u_{tx} - u_{tt}\right) = \frac{u_{xx} - u_{tt}}{4}$$

and hence

$$u_{\xi\eta} = 0 \iff u_{xx} - u_{tt} = 0.$$

(3):

By (2) we have $u_{\xi\eta} = 0$ for the wave equation problem. Here we view $u = u(\xi, \eta)$ as a function solely in these two variables. Thus, the initial condition becomes

$$u(\xi, \xi) =: u(\xi) = g(\xi)$$

and

$$u_t = \partial_t u(x+t, x-t) = u_\xi - u_\eta$$

so

$$(u_\xi - u_\eta)(\xi, \xi) = h(\xi).$$

I'll try to be clear by using $u_1 = u_\xi$ and $u_2 = u_\eta$ when ξ is used differently in a line (as both a variable integral bound and the direction in which we take derivative). We compute and get:

$$\partial_\xi u(\xi, \eta) - \partial_\xi u(\xi, \xi) = \int_\xi^\eta \partial_2 \partial_1 u(\xi, t) dt = \int_\xi^\eta 0 dt = 0$$

and hence

$$u(\xi, \eta) - u(\eta, \eta) = \int_\eta^\xi \partial_1 u(s, \eta) ds = \int_\eta^\xi \partial_1 u(s, s) ds.$$

Since $u_{\xi\eta} = u_{\eta\xi}$ we change the order of integration and get (exactly the same) the formula:

$$u(\xi, \eta) - u(\xi, \xi) = \int_\xi^\eta \partial_2 u(s, s) ds$$

and hence

$$u(\xi, \eta) = \frac{1}{2}\left(u(\xi, \xi) + u(\eta, \eta)\right) + \int_\eta^\xi (\partial_1 - \partial_2) u(s, s) ds$$

where by plugging in to the initial condition we've obtained we get

$$u(\xi, \eta) = \frac{1}{2} g(\xi) + \frac{1}{2} g(\eta) + \int_\eta^\xi h(s) ds$$

and plugging in x and t back we have

$$\tilde{u}(x, t) = u(\xi, \eta) = \frac{1}{2} g(x+t) + \frac{1}{2} g(x-t) + \int_{x-t}^{x+t} h(s) ds$$

where $\tilde{u}$ really is the original u in the wave equation written in regular form... but we've defined u as a function of ξ, η here. Anyway the formula holds.

$\square$

Exercise 2.

Proof.

(1):

By the hint we have

$$\left|t \fint_{\partial B(0,1)} h(x+t\zeta)dS(\zeta)\right| = \left|-t \fint_{\partial B(0,1)} \int_t^\infty \frac{\partial}{\partial s} h(x+s\zeta)dsdS(\zeta)\right|$$

$$= \left|-t \int_t^\infty \fint_{\partial B(0,1)} \zeta \cdot \nabla h(x+s\zeta)dsdS(\zeta)\right| \overset{y=x+s\zeta}{=} \left|-t \int_t^\infty \fint_{\partial B(x,s)} \frac{y-x}{s} \cdot \nabla h(y)dS(y)ds\right|$$

$$\leq \left|t \int_t^\infty \int_{\partial B(x,s)} \frac{1}{4\pi s^2}|1| \cdot |\nabla h| dS(y)ds\right| \overset{s\geq t}{\leq} \frac{1}{4\pi t}\int_{B(x,S)\setminus B(x,t)} |\nabla h| dy \leq \frac{1}{4\pi t}||\nabla h||_{L^1} = \frac{C}{t}||\nabla h||_{L^1}$$

where the S in the integral lower bound is the maximal radius of the ball centered at x that h is supported on.

(2):

Just plug in Kirkoff's formula we have

$$w = \frac{1}{4\pi t^2}\int_{\partial B(x,t)} [th(y) + g(y) + (y-x)\cdot \nabla g(y)]\, dS(y) = \fint_{\partial B(x,t)} th(y)dS(y)$$

and by above deduction we can bound

$$|w| \leq \frac{\tilde{C}}{t}||\nabla h||_{L^1} \leq \frac{C}{t}.$$

□

Exercise 3.

Proof.

(1):

Doing Fourier transform we have

$$(\partial_t^2 + (\xi^2 + 1))\hat{u} = 0$$

thus we write

$$\hat{u} = A\sin(t\sqrt{\xi^2+1}) + B\cos(t\sqrt{\xi^2+1})$$

and plugging in $\hat{u}|_{t=0} = \hat{f}$ we have $B = \hat{f}$ and $\hat{u}_t|_{t=0} = \hat{g}$ to get $A = \dfrac{\hat{g}}{\sqrt{\xi^2+1}}$

$$\hat{u} = \frac{\hat{g}}{\sqrt{\xi^2+1}}\sin(t\sqrt{\xi^2+1}) + \hat{f}\cos(t\sqrt{\xi^2+1}).$$

To find the inverse we first note that Fourier transform is linear so we do both part separately.

Thus if we define

$$G := \mathcal{F}^{-1}\left(\frac{\sin(t\sqrt{\xi^2+1})}{\sqrt{\xi^2+1}}\right)$$

and

$$F := \mathcal{F}^{-1}\left(\cos(t\sqrt{\xi^2+1})\right)$$

then we know

$$u = g * G + f * F.$$

$\square$

Exercise 4.

Proof.

Define energy

$$e(t) := \frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2 + |\nabla u|^2 + u^2 dx \geq 0$$

and take derivative to get

$$\begin{aligned}
\dot{e}(t) &= \int_{B(x_0,t_0-t)} u_t u_{tt} - \nabla u \nabla u_t + u_t u dx - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} u_t^2 + |\nabla u|^2 + u^2 dx \\
&\overset{IBP}{=} \int_{B(x_0,t_0-t)} u_t(\Box + u) dx + \int_{\partial B(x_0,t_0-t)} u_t \cdot \partial_n u - \frac{1}{2}\left[u_t^2 + |\nabla u|^2 + u^2\right] dx \\
&\overset{C.S.}{\leq} -\frac{1}{2}\int_{\partial B(x_0,t_0-t)} u^2 dx \leq 0
\end{aligned}$$

where the Cauchy Schwartz is

$$|u_t \partial_n u| \overset{C.S.}{\leq} \frac{1}{2}\left(u_t^2 + |\partial_n u|^2\right) \leq \frac{1}{2}\left(u_t^2 + |\nabla u|^2\right)$$

thus the energy has negative derivative.

But then since

$$e(0) = \frac{1}{2}\int_{B(x_0,t_0)} u_t^2 + |\nabla u|^2 + u^2 dx = 0$$

where the first and third term is by given condition, but knowing $u = 0$ everywhere on the boundary means $\nabla u = 0$ there, so we are done because $0 \leq e(t) \leq 0$, so it vanishes everywhere since the integral of energy over the whole solid cone is then 0, and everywhere's positive, so everywhere vanishes.

□