BROWNIAN MOTION AND STOCHASTIC CALCULUS HW 3

TOMMENIX YU
ID: 12370130
STAT 38500

Discussed with classmates.

Exercise 1.

Proof.

(1):

By definition for $x \geq 0$

$$\mathbb{P}(M_t \geq x) = \mathbb{P}\left(\max_{0\leq s\leq t} B_s \geq x\right) = 2\mathbb{P}\left(B_t \geq x\right)$$

and

$$\mathbb{P}\left(\sqrt{t}M_1 \geq x\right) = \mathbb{P}\left(\max_{0\leq s\leq 1} B_1 \geq \frac{x}{\sqrt{t}}\right) = 2\mathbb{P}\left(B_1 \geq \frac{x}{\sqrt{t}}\right) = 2\mathbb{P}\left(B_t \geq x\right)$$

and for $x < 0$ both are 1 so they have the same distribution.

(2):

From same logic as last part we have for $x \geq 0$:

$$\mathbb{P}(M_1 \leq x) = 1 - 2\mathbb{P}(B_1 \geq x) = 1 - \frac{2}{\sqrt{2\pi}}\int_x^\infty e^{-y^2/2}dy$$

and for $x < 0$

$$\mathbb{P}(M_1 \leq x) = 0.$$

So

$$F_M(x) = \mathbb{1}_{x\geq 0}\left(1 - \frac{2}{\sqrt{2\pi}}\int_x^\infty e^{-y^2/2}dy\right)$$

and differentiate we get

$$f_M = \mathbb{1}_{x\geq 0}\left(\frac{2}{\sqrt{2\pi}}\frac{d}{dx}\int_{-\infty}^x e^{-y^2/2}dy\right) = \begin{cases} \frac{2}{\sqrt{2\pi}}e^{-x^2/2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(3):

From last question we know by symmetry

$$\mathbb{E}[M_1] = \int_0^\infty x \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx = -\sqrt{\frac{4}{\pi}} \int_0^\infty e^{-x^2/2} d - x^2/2 = \sqrt{\frac{2}{\pi}}$$

and by scaling $\mathbb{E}[M_t] = \sqrt{\frac{2t}{\pi}}$.

(4):

For $a > 0$

$$\begin{aligned}\mathbb{E}[M_1 \mathbb{1}_{M_1 > a}] &= \int_a^\infty x \frac{2}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{-2}{\sqrt{2\pi}} \int_a^\infty e^{-x^2/2} d\frac{-x^2}{2} = \frac{-2}{\sqrt{2\pi}} \int_{-a^2/2}^\infty e^y dy \\ &= \frac{2}{\sqrt{2\pi}} e^{-a^2/2}\end{aligned}$$

(5):

Define function

$$S := \max\left\{B_1 + \max_{0\le s\le 1} B_s - M_1, 0\right\}$$

and in particular, this is the same random variable in distribution to (by Markov property) of "the increment of largest element from 0 to 2 than from 0 to 1, if any."

Denote $A := \{M_2 > M_1\}$, then

$$M_2 = \mathbb{1}_A M_2 + \mathbb{1}_{A^c} M_1 = M_1 + \mathbb{1}_A (M_2 - M_1) = M_1 + \mathbb{1}_A \left(B_1 + \max_{0\le s\le 1} B_s - M_1\right) = M_1 + S$$

and thus

$$\mathbb{E}[M_2 | \mathcal{F}_1] = \mathbb{E}[M_1 + S | \mathcal{F}_1] = M_1 + S = M_1 + \max\left\{B_1, 0\right\}.$$

□

Exercise 2.

Proof.

(1): For $\forall N > 0$, and WLOG denote a subsequence (that we'll specify later) $s_n := t_{\phi(n)}$ where ϕ is increasing. Now consider the event:

$$U^N := \left\{ \limsup_{n\to\infty} B_{s_n} \geq N\sqrt{s_n} \right\}$$

and note that if we consider the natural filtration $\mathcal{F}_n$ we get that $U^N \in \mathcal{T}_\infty$ the tail σ-algebra, and hence by Blumenthal's 0-1 law we know $\mathbb{P}(U^N) = 0$ or 1. But we note that

$$\mathbb{P}(U^N) = \mathbb{P}\left\{ \liminf_{n\to\infty} B_{s_n} \leq -N\sqrt{s_n} \right\}$$

and since both are 0 or 1 they are the same as their union:

$$\mathbb{P}(U^N) = \left\{ \limsup_{n\to\infty} |B_{s_n}| \geq N\sqrt{s_n} \right\}$$

Now we only have to use the stronger version of Second Borel Cantelli to show that the event $V_n^N := \left\{ |B_{s_n}| \geq N\sqrt{s_n} \right\}$ is not summable to conclude that this happens infinitely often. Using the fact that

$$\mathbb{P}\left(|B_{s_n}| \geq x | \mathcal{F}_{s_{n-1}} \right) \geq \mathbb{P}(B_{s_n} - B_{s_{n-1}} \geq x)$$

since if $B_{s_{n-1}} > 0$ the above holds and if $B_{s_{n-1}} \leq 0$ we have LHS $\geq \mathbb{P}(B_{s_n} - B_{s_{n-1}} \leq -x) = \mathbb{P}(B_{s_n} - B_{s_{n-1}} \geq x)$ by symmetry. Now we can bound

$$\mathbb{P}(V_n^N | V_{n-1}^N, \dots, V_1^N) \geq \mathbb{P}\left\{ \frac{B_{s_n} - B_{s_{n-1}}}{\sqrt{s_n - s_{n-1}}} \geq N \frac{\sqrt{s_n}}{\sqrt{s_n - s_{n-1}}} \right\} \geq c \exp\left\{ -\frac{N^2}{2} \frac{s_n}{s_n - s_{n-1}} \right\}$$

ans we want this to be unsummable, so for instance let's just require

$$e^{\frac{s_n}{s_n - s_{n-1}}} \leq n^{\frac{2}{N^2}} \quad \Rightarrow \quad \frac{s_n}{s_n - s_{n-1}} \leq \log n^{2/N^2}$$

but since $t_n \to \infty$ we can always pick $\phi(n)$ such that $s_n \gg s_{n-1}$ with $\dfrac{s_n}{s_n - s_{n-1}} < 1 + \varepsilon$ for any $\varepsilon > 0$, in particular $1 + \varepsilon = \log n^{2/N^2}$ for n large: this is the ϕ we pick for N. So we are done since we've shown that $\mathbb{P}(U^N) = 1$ by second Borel Cantelli, and N is arbitrary, so

$$\limsup_{n\to\infty} B_{t_n} \geq \limsup_{n\to\infty} B_{s_n} = \infty.$$

(2):

This is the easier direction, we just note

$$\mathbb{P}\left(B_{t_n} \geq \sqrt{t_n \log\log t_n} \right) = \mathbb{P}\left(B_1 \geq \sqrt{\log\log t_n} \right) \leq e^{-\frac{\log\log t_n}{2}} \left(\frac{1}{\log\log t_n} \right) \leq \frac{1}{\sqrt{\log t_n}}$$

and for it to be summable let's just let $t_n = e^{(n^4)}$ then we are done by Borel Cantelli.

$\square$

Exercise 3.

Proof.

(1):

- L^1: for any t
$$\int_\Omega M_t d\mathbb{P} = \int_\Omega |B_t|^2 - td\, d\mathbb{P}$$
and td is just a constant so $\int_\Omega |td| d\mathbb{P} = td$ is integrable, and $|B_t|^2$ is integrable because the second moment of a Multivariate Gaussian exists. Thus $M_t \in L^1$.
- Adapted: $B_t \in \mathcal{F}_t$ and thus $M_t \in \mathcal{F}_t$.
- $\mathbb{E}[M_n|\mathcal{F}_m] = M_m$ for $n \geq m$: we know

$$|B_n|^2 = [B_m + (B_n - B_m)] \cdot [B_m + (B_n - B_m)] = |B_m|^2 + |B_n - B_m|^2 + 2\langle B_m, B_n - B_m\rangle$$

and given a particular path, described by $A \in \mathcal{F}_m$, we know that B_m is fixed and $B_n - B_m$ is in dependent of A, so

$$\int_A \langle B_m, B_n - B_m\rangle d\mathbb{P} = \int_\Omega \sum_i (B_m)_i (B_n - B_m)_i d\mathbb{P} = \sum_i \int_\Omega (B_m)_i (B_n - B_m)_i d\mathbb{P}_i = 0$$

where $\mathbb{P}_i$ is the corresponding marginal distributions. As for the square term we note

$$\int_A |B_n - B_m|^2 d\mathbb{P} = \sum_i \int_\Omega (B_n - B_m)_i^2 d\mathbb{P}_i = d(n-m)$$

and thus

$$\begin{aligned}\int_A M_n d\mathbb{P} &= \int_A |B_n|^2 - tnd\mathbb{P} \\ &= \int_A |B_m|^2 - tm + |B_n - B_m|^2 - t(n-m) + 2\langle B_m, B_n - B_m\rangle d\mathbb{P} \\ &= \int_A |B_m|^2 - tm\; d\mathbb{P} = \int_A M_m d\mathbb{P}\end{aligned}$$

thus M_n is a Martingale.

(2):

T_R is a stopping time since $\{T_R < s\} \in \mathcal{F}_s$. Now by Doob's stopping time theorem we know M_{T_R} is a martingale so $\mathbb{E}[M_{T_R}] = \mathbb{E}[M_0] = 0$ but on the other hand we have

$$\mathbb{E}[M_{T_R}] = \mathbb{E}[|B_{T_R}|^2 - dT_R] = \mathbb{E}[|B_{T_R}|^2] - d\mathbb{E}[T_R] = 0$$

so

$$\mathbb{E}[T_R] = \frac{1}{d}\mathbb{E}[|B_{T_R}|^2] = \frac{1}{d}\mathbb{E}[R^2] = \frac{R^2}{d}.$$

$\square$

Exercise 4.

Proof.

(1):

For $\varepsilon < 1$ fixed, denote the collection of all covers by C, then we know that there is a net (since can be uncountable) of covers such that the inf within the expression of $H^\alpha_\varepsilon(V)$ is taken upon. Call this net N. Now we can compare:

$$\begin{aligned} H^\alpha_\varepsilon(V) &= \inf_{U\in C}\sum_j \left(\operatorname{diam}(U_j)\right)^\alpha = \inf_{U\in N}\sum_j \left(\operatorname{diam}(U_j)\right)^\alpha \\ &\geq \varepsilon^{\alpha-\beta}\inf_{U\in N}\sum_j \left(\operatorname{diam}(U_j)\right)^\beta \geq \varepsilon^{\alpha-\beta}\inf_{U\in C}\sum_j \left(\operatorname{diam}(U_j)\right)^\beta = \varepsilon^{\alpha-\beta}H^\beta_\varepsilon(V) \end{aligned}$$

thus

$$H^\beta_\varepsilon(V) \leq \varepsilon^{\beta-\alpha}H^\alpha_\varepsilon(V)$$

and taking limit on both sides yields (since $\beta-\alpha>0$):

$$H^\beta(V) \leq \lim_{\alpha\to 0}\varepsilon^{\beta-\alpha}H^\alpha_\varepsilon(V) = 0\cdot H^\alpha(V) = 0$$

and by definition it's a positive value so $H^\beta(V)=0$.

(2):

This is only a contrapositive. First we fix $\beta>\alpha$, under this condition (1) is:

$$\begin{aligned} \left\{\{H^\alpha(V)<\infty\}\to\{H^\beta(V)=0\}\right\} &\iff \left\{\sim\{H^\beta(V)=0\}\to\sim\{H^\alpha(V)<\infty\}\right\} \\ &\iff \left\{\{H^\beta(V)>0\}\to\{H^\alpha(V)=\infty\}\right\} \end{aligned}$$

where the last equality is by the fact that the definition of $H^\alpha(V)\in[0,\infty]$, and note the third statement is exactly our goal (with α,β exchanged in notation), since we can do it for all pairs (α,β) with $\beta>\alpha$ and β fixed to move the condition inside.

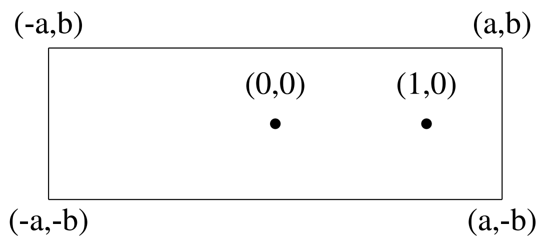
$\square$

Exercise 5.

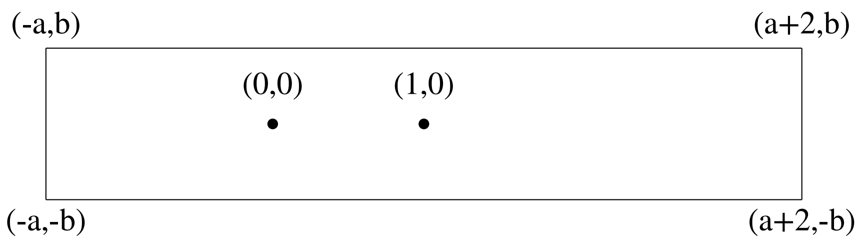
Proof.

(1):

For $a \leq 1$ the result is obvious since $\tau_a \equiv 0$. Below only consider $a > 1, b > 0$.



The question asks us to show that the probability that a Brownian motion starting at the point $(1,0)$ will touch the verticel sides first than the horizontal is strictly positive, i.e. $\mathbb{P}\{\tau_a < T_b\} \geq \delta_{a,b} > 0$. To begin, define stopping time $T_a := \inf\{t : B_t^1 \leq -a \vee B_t^1 \geq a+2\}$, i.e. I made the box symmetric with respect to starting point:



then we know $\mathbb{P}\{\tau_a < T_b\} \geq \mathbb{P}\{T_a < T_b\}$ since in order to touch the new right side $[(a+2,b),(a+2,-b)]$ we'd have to touch the section $[(a,b),(a,-b)]$. Now we can simply compute, since B^1 and B^2 is independent we compute that for a fixed time t

$$\begin{aligned}\mathbb{P}\{\tau_a < T_b\} \geq \mathbb{P}\{T_a < t < T_b\} &= \mathbb{P}\{T_a < t\}\mathbb{P}\{t < T_b\} \\ &= \mathbb{P}\left\{\max_{0\leq s\leq t}|B_s| \geq a\right\}\left(1 - 2\mathbb{P}\left\{\max_{0\leq s\leq t}|B_s| \geq b\right\}\right) \\ &= 2\mathbb{P}(B_t \geq a)\left(1 - 4\mathbb{P}(B_t \geq b)\right) =: \delta_{a,b} > 0\end{aligned}$$

so we are done.

(2):

Here we repeatedly use the last result. But first we show one simple lemma:

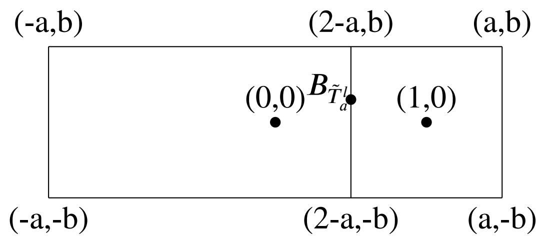
Lemma 0.1. *Let* $T_a^l := \inf\{t : B_t^1 \leq -a\}$ *and* $T_a^r := \inf\{t : B_t^1 \geq a\}$. *Then there exists* $\delta > 0$ *such that* $\mathbb{P}(T_a^l < \min\{T_a^r, T_b\})$.

Proof. What the lemma says is that the probability that the path touches the left boundary and not the other 3 sides of the rectangle is strictly positive.

To show this, we first note that apply result to rectangle $[(a,b),(a,-b),(2-a,-b),(2-a,b)]$ then touching left and touching left is symmetric, so there's a half of the probability of touching vertical line that touches the left end, in particular if define $\tilde{T}_a^l := \min\{t : B_t^1 \leq 2-a\}$ then $\mathbb{P}(\tilde{T}_a^l < \min\{T_a^r, T_b\}) > 0$. So we can pick corresponding $(a_1, b_1) = (a,b)$ and $(a_2,b_2) = (a-(2-a), \min\{|b - B^2_{\tilde{T}_a^l}|, |b + B^2_{\tilde{T}_a^l}|\})$, i.e. we find a smaller box which horizontally touches the line $x = a$ and vertically does not touch $y = \pm b$. Since $a_k = 2a_{k-1}$, i.e. each time we double the length moving leftward, we will eventually touch $x = -a$ as $a > 1$ before we touch other 3 sides with positive probability.

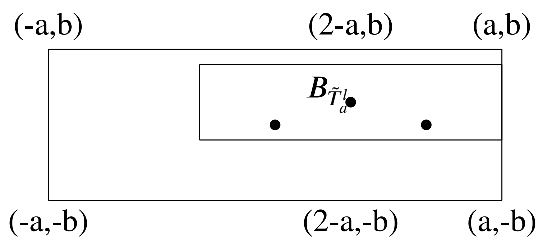
Piture-wise we are doing this:

Step 1:



and for the second step we pick the new rectangle:

Step 2:

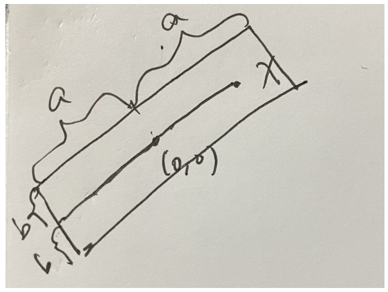


$\square$

For convience, define the event of touching the left side first as $E^x_{l,a,b}$ with starting point at $x \in (\mathbb{R}, 0)$. Note that this event is rotational invariant, so it's appropriate to define

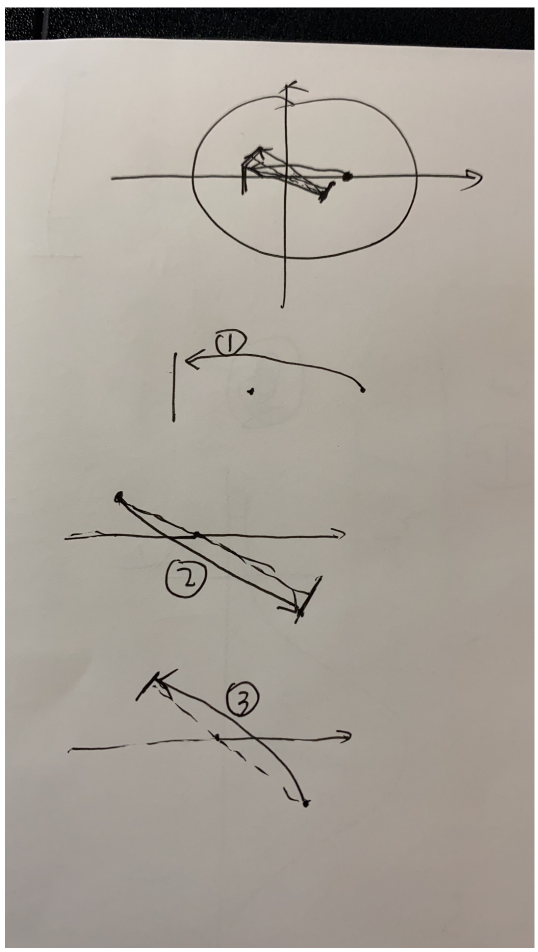
$E^{|x|}_{l,a,b}$:= {starting at point x, find coordinate where x is on the first axis, then for new coordinate}

where the picture really is simple:



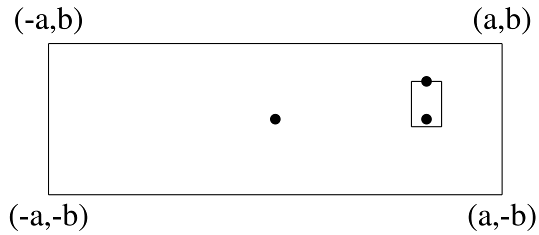
thus rotational invariance also tells us $\mathbb{P}(E^{|x|}_{l,a,b}) = \mathbb{P}(E^{(|x|,0)}_{l,a,b}) = \delta_{l,a,b} > 0$

Now what we do is a little hard to describe with words but very obvious with the illustration:

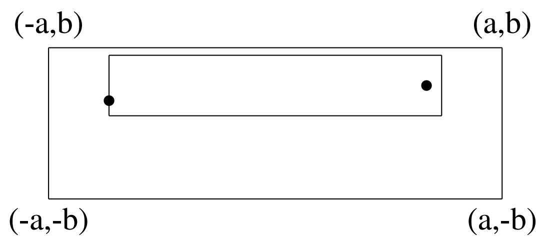


I'll explain this. First, we find $1 = |x_1| < a_1 < e$ and b_1 such that $a_1^2 + b_2^2 < e$. In particular we want a left side of a rectangle that is left of origin and contained in the circle. For the path generated, we want it to be homotopy with the straight line segment $[x_1, x_2]$ where x_1 is the starting point and x_2 is the ending point. WLOG we pick x_2 above x-axis To do this we consider the event (by lemma it has positive probability) that go straight up first, then go left:

Step 1: go up:



Step 2: go left:

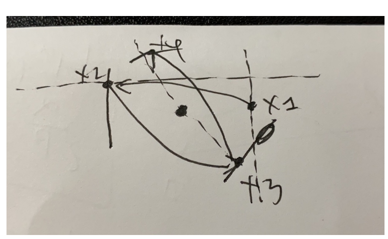


So anyway we have constructed an event with positive probability that has path homotopy to $[x_1, x_2]$, which is in particular does not loop around the origin.

Now with rotational invariant there exists constants a_2, b_2 such that the ending segment is wholy below the x-axis, and wholy left of $x = 1$, and we know there's positive probability that the path is homotopy to $[x_2, x_3]$, where we choose x_3 to be the lower half of the "end-side" of the rectangle. That is, we've almost looped around the origin.

For the next step we do the same except that we pick our a_3, b_3 such that the "end-side" is wholy above the line $y = (x_2)_2$, i.e. the y value of x_2. Then we pick the upper side to contain our endpoint then we have already looped around the origin. The 3 events are independent and all with positive probability, call it $1 - \rho$, so we are done.

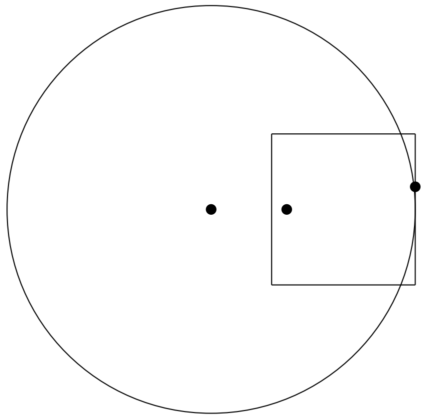
For a much clearer picture of the above description, we only did this:



(I hope this is clearer)

So we have shown that it is with positive probability that the unbounded component does not contain the origin. Let's show that it is not 1.

This is much easier as we just find the box



and thus there's positive probability it reaches the right first.

(3):

If we can show that the probabilty shown in part (1) is invariant under same scaling of both a and b we can confidently apply the scaled version of the probability space n times since each time we start from value $|x_k| = e_k$ (with rotational invariant) and want to touch $|x_{k+1}| = e^{k+1}$ before touching the origin. And the quotient is always e.

So let's do it: Since the 2d scaling factor on B_t^1 is the same as in 1d, we have

$$\mathbb{P}\{eB_1^1 \geq ea\} = c\int_{e^2a^2}^{\infty} \exp\left\{-\frac{e^4t^2}{2}\right\} dt = c\int_{a^2}^{\infty} \exp\left\{-\frac{t^2}{2}\right\} dt = \mathbb{P}\{B_1^1 \geq a\}$$

then everything from part (1) through the first part of part (2) passes through. In particular, if we pick the same-to-scale box each time we get a multiplication of ρ each time we extend outwards. Thus, the desired probability is at most ρ^n.

Explanation to last sentence: In (2) we've chosen the probability $1-\rho$ to be the probability that the path winds around 0. As long as there is one time (rather than each time) we winds around 0 we will wind around it, so that ρ^n is indeed what we need.

(4):

This is just rescaling with the last part with a bit more computation.

Now, we can explicitly find n such that

$$\frac{1}{e^{n+1}} < \varepsilon \leq \frac{1}{e^n}$$

and for such an ε we know we can rescale to starting at $(1, 0)$ and ending when touching the circle with radius $\dfrac{1}{\varepsilon} \in [e^n, e^{n+1})$. The trick is that for this particular ball, the probability it does not disconnect the origin from the unbounded component is smaller than a particular number q with $q \in (\rho^{n+1}, \rho^n]$ by part (3). In particular $q \le c\rho^n$ for some constant c.

We also know that, after scaling the starting point to $(1, 0)$, the larger the radius is, the more probability we have to disconnect. In other words:

$$\mathbb{P}(\text{ non-disconnect starting at } \varepsilon) \le \mathbb{P}\left(\text{ non-disconnect starting at } \frac{1}{e^n}\right)$$

So what's left is just that we want $c\varepsilon^\alpha < q = c\rho^n$, i.e.

$$-\log \rho = \log_{e^{-n}}(\rho^n) \le \alpha \le \log_{e^{-n-1}}(\rho^n) = -\frac{n}{n+1}\log \rho$$

so at least $-\log \rho > 0$ is a possible choice of α, where c is chosen along the way. Note here that c, ρ are all independent of ε.

For a very very rough estimate, in our construction of $1 - \rho$, probability of sure disconnection, we've at least chosen 3 less than half probability selections. So $(1 - \rho) \ll 1/8$ and hence $1 > \rho \gg 7/8$. So the probability at least $< -\log 7/8 = 0.57$. (no where accurate)

Of course, for us to find the best α we need to find the best ρ, which is terrible toil.

$\square$