SET THEORY HW 4

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MATH 300

Exercise 1. *$M[G]$ satisfies:*

(1) Pairing.
(2) Extensionality.
(3) Union of A and B.

Proof.

(1): Pairing.

We know 1, the largest element in $\mathbb{P}$ is always in G, any ultrafilter, so given a, b the name

$$\tau := \{\langle a, 1\rangle, \langle b, 1\rangle\}$$

evaluates to $\{a, b\}$, hence $\{a, b\} \in M[G]$.

(2): Extensionality.

What we can use is that M is a ZF system. Note that M is transitive, so for X, Y distinct evaluation of names, which are themselves names by definition of names, since $\mathbb{P}$ names in $M^{\mathbb{P}}$ are elements of M, we get that $X \cap M \neq Y \cap M$ by extentionality of M. Hence, $M[G]$ is extensional.

(3): Union of two elements.

Say $A = \tau_G^A$ and $B = \tau_G^B$ for names τ^A and τ^B. Then just take

$$\tau := \tau^A \cup \tau^B$$

we get $\tau_G = A \cup B$ since for each element x in either A or B it is a evaluation from some names somewhere down the "tower" of names (or at some step of the recursive process it is evaluated out), so we have $x \in \tau_G$. The other direction is just because $A \cup B$ exhausts all possible x that can be evaluated. □

Exercise 2. *In the example of partial functions, $\forall p \in \mathbb{P}$, p forces that f_G has infinite domain. In other words*

$$p \Vdash \forall_\alpha x \exists_\alpha y (y > x \wedge y \in \operatorname{dom} f_G)$$

where we adopted Cohen's notation of α-labelling.

Proof. By definition, for $p \in \mathbb{P}$, $p \Vdash \forall_\alpha x \theta(x)$ if for all $q \leq p$ and for all $c \in S_\beta$, $\beta \leq \alpha$, q does not force $\sim \theta(c)$. In the above and the following, we use

$$\theta(c) := \exists_\alpha y (y > c \wedge y \in \operatorname{dom} f_G)$$

as an abbreviation.

Now we ask a few yes-no questions, and answer them from the last to the first, which will then get us to our results:

(1) Does p force $\forall x \theta(c)$?
(2) For q defined as above, does q force $\sim \theta(c)$?
(3) For all $r \leq q$, does r not force $\theta(c)$?

Note that $r \leq q$ really is saying that $\operatorname{dom}(r) \supset \operatorname{dom}(q)$ and since c fixed, there will eventually be some $n > c$ for some r' with $n \in \operatorname{dom}(r')$ since otherwise $\operatorname{dom}(r) \subset \{0, 1, 2, \dots, c\}$ for all $r \leq q$. Thus, $r' \Vdash \theta(c)$.

So the answer to (3) is No; Answer to (2) is No; Answer to (1) is Yes by argument in the beginning. Note we are done by this since c is arbitrary. $\square$

Exercise 3. *Let G be an upward closed subset of $\mathbb{P}$, then the following are equivalent:*

- *G is generic for $\mathbb{P}$ over M;*
- *For $\forall$ maximal antichain $I \in M$ of $\mathbb{P}$ we have $|G \cap I| = 1$.*

Proof. ($\Rightarrow$:)

Let I be a maximal antichain, G generic, and let

$$J = \{p \in \mathbb{P} : \exists q \in I : q \leq p\}$$

which is the downward closure of elements of I. Now we need to show J is dense:

Consider $r \in \mathbb{P}$, then it must be compatible with some $i \in I$ by maximality of antichain, so we let s be such that $s \leq i$ and $s \leq r$ since they are compatible, and by definition of J we know $s \in J$.

But J dense means G meets J and by upward closure we know G meets I. But G cannot meet two elements of I since that would contradict consistency, so $|G \cap I| = 1$.

($\Leftarrow$:)

Suppose G is upward closed and $|G \cap I| = 1$, There are 3 things to show in order to show G generic:

(a) upward closed.
(b) Directed ($\forall p, q \in G$, $\exists r \in G$ such that $r \leq p$ and $r \leq q$).
(c) Meet all dense sets.

(a) is given, now we show (b) and with (b) we show (c).

Let J be a dense set and let's prove G meets it. Consider the antichain of elements of J and by Zorn's lemma there is a maximal one I^*, we claim that it is really a maximal antichain of $\mathbb{P}$, but this by assumption means G meets it.

To see this, suppose $r \in \mathbb{P}$, since J dense $\exists q \in J$ with $q \leq r$ and necessarily q is compatible with some elements of I^*, so r must also be compatible with that element, this means we cannot add r to the antichain, hence maximality.

So we have (c). Now to show (b), given $p, q \in G$, then define

$$X := \{r : (r \leq q, r \leq p) \vee (r \perp q) \vee (r \perp p)\}$$

and X is dense because for any $p' \in G$ either p' is consistent with p and q, in which case they have a common lower bound (i.e. $r \leq q \vee p \vee p'$) that is in X, or it is not consistent with p or q, which is captured by $(r \perp q) \vee (r \perp p)$ for $r \leq p'$ (or r stronger than p', hence implies p').

Since X is dense we know G meets X and so if we pick $t \in G \cap X$ then either $t \leq p \vee q$, in which case we are done, or $t \perp q$, but then we can extend $\{t, q\}$ to a maximal chain in I that G meets twice, contradiction, so this case cannot happen. So G is indeed directed.

So we conclude that G is generic. $\square$