SET THEORY HW 3

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MATH 300

Theorem 0.1. *(from p82) Let $A(x_1, \dots, x_n)$ be a formula in ZF. One can prove in ZF that for any set S there is a set $S' \supset S$ such that $|S'| = \max(\aleph_0, |S|)$ and for all $\bar{x}_i \in S'$, $A(\bar{x}_1, \dots, \bar{x}_n) \iff A_{S'}(\bar{x}_1, \dots, \bar{x}_n)$ where $A_{S'}$ is A with all quantifiers restricted to S'.*

Proof.

Let $A := Q_1 y_1, Q_2 y_2, \dots, Q_m y_m B(x_1, \dots, x_n, y_1, \dots, y_m)$ where we WLOG assume Q_1 is existential, since for universal quantifier we just make it $\sim \exists \sim$, and B has no quantifiers.

Now let T be any set (which we'll use just to define T^*), define function $f_r : T^n \times T^{r-1} \to V$ such that by AC $f_r(\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_{r-1})$ is mapped to an element for which the statement:

$$Q_{r+1} y_{r+1}, \dots, Q_m y_m B(\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_{r-1}, y_r, y_{r+1}, \dots, y_m)$$

holds. If there is no such y_r then we define $f_r(\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_{r-n}) = 0$.

Cohen here validifies the usage of AC by looking at all V_α, the sets defined by ZFC and thus for all y_r (if there are any) there is a least α such that there are some in V_α. Thus we use axiom(schema) of separation to get a set containing only suitable candidates of y_r, and f_r is well defined.

Now we define

$$T^* := T \cup \mathrm{Ran}(f_r)$$

and we note that for $|T|$ infinite, $|T^*| = |T|$ since $|\mathrm{Ran}(f_r)| \le |T|^{n+r-1}$. Now we define $S_0 := S$ the given set in theorem, and define $S_{i+1} := S^*$ and $S' := \cup_i S_i$ so $|S'| = \max(\aleph_0, |S|)$ by above.

Now we claim that for $\bar{x}_i \in S'$, $A(\bar{x}_n, \dots, \bar{x}_n) \iff A_{S'}(\bar{x}_1, \dots, \bar{x}_n)$.

To show this we use induction on r. We use notation

$$\begin{aligned} C(\bar{x}_i, \bar{y}_j) &:= C(\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_{r-1}, \bar{y}_r) \\ &:= Q_{r+1} y_{r+1}, \dots, Q_m y_m B(\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_{r-1}, y_r, y_{r+1}, \dots, y_m) \end{aligned}$$

this is just to easier denote $C_{S'}(\bar{x}_1, \dots, \bar{x}_n, \bar{y}_1, \dots, \bar{y}_{r-1}, y_r)$.

For $r = m$, A has no quantifier of course the claim holds since all is in S' already.

Now assume we have for $r > r_0$ (we're actually doing normal upward induction on number of quantifiers, but we're counting Q_r to Q_m so this is indeed a $>$ sign), then for the case $r = r_0$, with a given $\bar{x}_i, \bar{y}_j \in S'$, they are all in S_k for some k. But then by the way we've defined S_{k+1}, if $C(\bar{x}_i, \bar{y}_j)$ is true then $f_{r+1}(\bar{x}_i, \bar{y}_j) \in S_{k+1}$ and by induction

$$C(\bar{x}_1, \ldots, \bar{x}_n, \bar{y}_1, \ldots, \bar{y}_{r-1}, \bar{y}_r, f_{r+1}(\bar{x}_i, \bar{y}_j))$$

is true in S' since it has less quantifiers.

If $C(\bar{x}_i, \bar{y}_j)$ is false then for no $y_{r+1} \in S'$ is it true, so it is false in S'.

Since m finite by induction we are done.

$\square$