MEASURE-THEORETIC PROBABILITY I

ABSTRACT.

CONTENTS

1. 9/26: Construction of Brownian motion

For fixed t, B_t is a random variable, and hence B is a collection of random variables $\{B_t : t\}$ indexed by time. We assume $B_0 = 0$.

Here are a few properties of Brownian motion:

- Independent increments: if $s < t$, the random variable $B_t - B_s$ is independent of $\sigma\{B_r : r \leq s\}$.
- Identically distributed (stationary) increments: if $s < t$, then $B_t - B_s$ has the same distribution as $B_{t-s} - B_0 = B_{t-s}$.

Above are lists of characters that we want to have, but let's ask a few questions on whether they are necessary or sufficient.

One might think that independent increments and identically distributed increments are enough, but an uncontinuous process would also satisfy them. For instance the Poisson process. And thus we want to rule that out. We take the stupidest way of making the process continuous, namely asking it to be so.

- Continuity: with probability 1, $t \mapsto B_t$ is continuous in t.

They are actually sufficient to capture a Brownian motion.

Theorem 1.1. *If a process satisfies above, then for each t, there $\exists m, \sigma^2$ such that $B_t \sim N(mt, \sigma^2 t)$.*

For notation, we denote m as the drift and σ^2 as the variance parameter of the Brownian motion.

With the characteristic in mind, let's define the process now.

Def 1.2. *A stochastic process is a collection of random variables indexed by time.*

Def 1.3. *A stochastic process $\{B_t, t \geq 0\}$ is called a (1d) Brownian motion (Wiener process) (starting at 0) with drift m and variance parameter σ^2 if*

- $B_0 = 0$.
- *Has independent increments.*
- *If $s < t$, then $B_t - B_s \sim N(m(t-s), \sigma^2(t-s))$.*
- *Has continuous path.*

Note that we cannot drop continuous path, since if we do then for each ω we can find a measure 0 set of discontinuity, but then Ω can be uncountable.

Proposition 1.4. *If B_t is a Brownian motion satisfying the above definition with $0, 1$, then $Y_t = \sigma B_t + mt$ satisfy the above with m, σ^2.*

We call B_t with $m = 0, \sigma^2 = 1$ the standard Brownian motion, $N(0, 1)$ the standard normal, and in finance, we sometimes call σ the volatility.

We have a definition, good. But definitions do not create, and we start to show existence of Brownian motions by construction.

What we need for this purpose is a large enough probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that there is a countable collection of independent standard normals on it. The usual Lebesgue $[0, 1]$ is enough (homework).

Curiously enough, we hardly can deal with larger spaces since functional analysis can't deal with non-separable spaces.

Our construction will contain the following steps:

(1) Define B_t for a countable dense set of time (Dyadic rationals) using $\{N_j\}$.
(2) Show that with probability 1, $t \mapsto B_t$ is uniformly continuous in t.
(3) Extend B_t to $t \in [0, 1]$ by continuity.
(4) Check that this works.

We choose uniform continuity because that makes sense for a dense set. We'll show step 1 and 2 today and others next time.

Def 1.5. *Dyadic rationals are defined by $D := \bigcup_{n=0}^{\infty} \mathcal{D}_n$ where*

$$\mathcal{D}_n := \left\{ \frac{k}{2^n} | k = 0, 1, 2, \dots, 2^n \right\}.$$

Note that $\mathcal{D}_n$ is increasing with respect to set ordering. Now let the set of standard normals be denoted $\{N_q : q \in D\}$. We first construct $B_0 = 0$ and $B_1 = N_1$. We do want $B_{\frac{1}{2}}$ to be of the form $B_{\frac{1}{2}} := \dfrac{B_1 - B_0}{2} + cN_{\frac{1}{2}}$ and we can compute that $c = \frac{1}{2}$ by variance:

$$\mathrm{Var}(B_{\frac{1}{2}}) = \frac{1}{2}; \quad \mathrm{Var}\left(\frac{B_1}{2}\right) = \frac{1}{4}$$

which gives us $c = \frac{1}{2}$. This gives us $B_1 - B_{\frac{1}{2}} = \frac{1}{2}B_1 - \frac{1}{2}N_{\frac{1}{2}}$.

Proposition 1.6. *$B_{\frac{1}{2}}$ and $B_1 - B_{\frac{1}{2}}$ are independent.*

Proof. By construction, $(B_{\frac{1}{2}}, B_1 - B_{\frac{1}{2}})$ has a joint centered Gaussian distribution, which means covariance $= 0$ actually means they are independent (not true in general!). So we compute

$$\mathrm{Cov}(B_{\frac{1}{2}}, B_1 - B_{\frac{1}{2}}) = \mathbb{E}[B_{\frac{1}{2}}(B_1 - B_{\frac{1}{2}})] = \mathbb{E}\left[\left(\frac{B_1}{2} + \frac{N_{\frac{1}{2}}}{2}\right)\left(\frac{B_1}{2} + \frac{N_{\frac{1}{2}}}{2}\right)\right] = 0.$$

The fact that uncorrelated implies independence is because the covariance matrix is diagonal, and we have $f_{XY} = f_X f_Y$. Alternatively, you can compute density to verify this. $\square$

Now this process can be applied repeatedly, so we've finished step 1.

Theorem 1.7. *With probability 1, $t \mapsto B_t$ for $t \in D$ is uniformly continuous.*

First let's restate what to prove here. For

$$K_n := \sup\{|B_s - B_t| : s, t \in D, |s - t| \leq 2^{-n}\}$$

uniformly continuous means with probability 1, $\lim_{n\to\infty} K_n = 0$.

So in order to show this, we show a stronger statement.

Theorem 1.8. *With probability 1, if $\alpha < \dfrac{1}{2}$,*

$$\lim_{n\to\infty} 2^{\alpha n} K_n = 0.$$

Or you can intuitively think this as $K_n \lesssim 2^{-\frac{n}{2}}$, i.e. $|B_t - B_s| \approx |t - s|^{\frac{1}{2}}$, which reflects its standard deviation.

Proof. Define

$$Y_n := \max\left\{|B_{\frac{1}{2^n}} - B_0|, |B_{\frac{2}{2^n}} - B_{\frac{1}{2^n}}|, \ldots, |B_{\frac{2^n}{2^n}} - B_{\frac{2^n-1}{2^n}}|\right\}$$

then

$$\mathbb{P}\{Y_n > x_n\} = \mathbb{P}(\max\{\dots\} \geq x_n) \leq \sum_{j=1}^{2^n} \mathbb{P}\left(\left|B_{\frac{j}{2^n}} - B_{\frac{j-1}{2^n}}\right| \geq x_n\right)$$

$$= 2^n\mathbb{P}\left(\left|B_{\frac{1}{2^n}} - B_0\right| \geq x_n\right) = 2^n\mathbb{P}\left(\left|\frac{1}{2^n}B_1\right| \geq x_n\right) = 2^{n+1}\mathbb{P}(B_1 \geq 2^{\frac{n}{2}}x_n)$$

where if we choose x_n such that

$$\sum_{n=1}^{\infty} 2^{n+1}\mathbb{P}(B_1 \geq 2^{\frac{n}{2}}x_n) < \infty$$

then by Borel-Cantelli we know with probability 1 for all n large we'd have $Y_n \leq x_n$.

So we need to find a more exact x_n for the proposition. We have

$$P(N_1 \geq x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \leq \int_x^{\infty} e^{-\frac{xy}{2}}dy = \frac{2}{\sqrt{2\pi}}\frac{1}{x}e^{-\frac{x^2}{2}} \leq Ce^{-\frac{x^2}{2}}$$

since x is bounded below. Thus, let's say $x_n := 2^{-n/2}a_n$ and $a_n = \sqrt{b_n \log n}$ to simplify and get

$$\mathbb{P}\left(B_1 \geq 2^{\frac{n}{2}}x_n\right) \leq C\exp\left(-\frac{a_n^2}{2}\right) \leq n^{-\frac{b_n}{2}}.$$

Anyways, $x_n = c2^{n/2}2\sqrt{\log n}$ works here for all n sufficiently large. Note that using independent rather than subadditivity only saves a multiplicative difference.

We'll continue next time.

□

2. 9/28: Construction of Brownian motion; No where differentiable; Markov process

2.1. **Construction.**

First we continue the proof from last time. Remember

$$Y_n = \max\left\{\left|\frac{B_j}{2^n} - \frac{B_{j-1}}{2^n}\right| : j = 1, 2, 3..., 2^n\right\}$$

and what we want to show is the $\forall \alpha < \frac{1}{2}$, with probability 1 we have $2^{\alpha n} Y_n \to 0$.

Proposition 2.1. *With probability 1*

$$\limsup_{n\to\infty} \frac{2^{n/2}}{\sqrt{n}} Y_n \leq \sqrt{2\log 2}.$$

So we add that $\sqrt{n}$ to balance the infinities.

Proof. The proposition is equivalent to saying that $\forall \varepsilon > 0$, with probability 1, for all n sufficiently large

$$Y_n \leq \frac{\sqrt{n}}{2^{n/2}} \sqrt{2(\log 2)(1+\varepsilon)}.$$

Now we don't really have much to use when we want to prove arguments with probability 1, so we use Borel Cantelli. To show the above, it's sufficient to show that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(Y_n > \sqrt{n}2^{-n/2}\sqrt{2(\log 2)(1+\varepsilon)}\right) < \infty$$

now we note that with union bound, and denote $x_n := 2^{-n/2}\sqrt{2(\log 2)(1+\varepsilon)}$ for convenience we have

$$\begin{aligned}
\mathbb{P}(Y_n > X_n) &\leq \sum_{j=1}^{2^n} \mathbb{P}\left(\left|\frac{B_j}{2^n} - \frac{B_{j-1}}{2^n}\right| > x_n\right) = 2^n \mathbb{P}\left(|B_{\frac{1}{2}}| > x_n\right) \\
&= 2^{n+1}\mathbb{P}\left(B_1 > \sqrt{n}\sqrt{2(\log 2)(1+\varepsilon)}\right) \\
&\leq C 2^n e^{-(\sqrt{n}\sqrt{2(\log 2)(1+\varepsilon)})^2/2} \leq C \cdot e^{n\log 2} \cdot e^{-n\log 2(1+\varepsilon)} \leq C e^{-n\varepsilon}
\end{aligned}$$

where the middle parts are explained last time, and that explains where the log 2 come from: to balance the 2^n. □

Now we prove Theorem 1.7 about uniform continuity. Remember

$$K_n := \sup\{|B_s - B_t| : s, t \in D, |s - t| \leq 2^{-n}\}.$$

Corollary 2.2. *There exists C such that with probability 1*

$$\limsup_{n\to\infty} \frac{2^{n/2}}{\sqrt{n}} K_n \leq C.$$

Proof. We use triangle inequality to show the following:

$$K_n \leq 2 \sum_{j=n+1}^{\infty} Y_j.$$

Now by definition of K_n we have $\frac{k-1}{2^n} < s < t < \frac{k}{2^n}$ or $\frac{k-1}{2^n} < s < \frac{k}{2^n} < t$. Both cases uses the same argument so we only use the first case. Since $s \in D$ we can go along D with lower order points to $\frac{k}{2^n}$, i.e. we can write

$$\frac{k}{2^n} = s + s_{n+1} + s_{n+2} + \ldots$$

where $S_m = \frac{1}{2^m}$ or 0. Thus triangle inequality implies

$$\left| s - \frac{k}{2^n} \right| < \sum_{j=n+1}^{\infty} Y_j$$

do the same for t we've shown our bound for K_n. Now we pick $2 \geq \sqrt{2\log 2}$ to bound the sum:

$$K_n \leq 2 \cdot 2 \sum_{j=n+1}^{\infty} 2^{-j/2}\sqrt{j}$$

which converges. □

Using above, we have

$$\sup_{s,t\in D, s<t} \frac{|B_t - B_s|}{\sqrt{(t-s)(|\log(t-s)|+1)}} < \infty$$

which I think is not very obvious. To break it down, we first note that with probability 1

$$\limsup_{n\to\infty} \frac{2^{n/2}}{\sqrt{n}} \sup_{|t-s|<2^{-n}} |B_t - B_s| \leq C$$

where for any n and $|t-s| < 2^{-n}$ we have

$$\log(t-s) < -n \quad \Rightarrow \quad |\log(t-s)| > n \quad \Rightarrow \quad \frac{1}{\sqrt{|\log(t-s)|+1}} < \frac{1}{\sqrt{n}}$$

hence

$$\frac{1}{\sqrt{n}} \sup_{|t-s|<2^{-n}} |B_t - B_s| \geq \sup_{|t-s|<2^{-n}} \frac{|B_t - B_s|}{\sqrt{|\log(t-s)|+1}}$$

but for the other term we had to use the other direction so for any n and $2^{-n-1} < t - s < 2^{-n}$ we have

$$\frac{1}{\sqrt{t-s}} \geq \sqrt{2} \cdot 2^{n/2}$$

and hence

$$C > \limsup_{n\to\infty} \frac{2^{n/2}}{\sqrt{n}} \sup_{|t-s|<2^{-n}} |B_t - B_s| \geq \frac{1}{\sqrt{2}} \limsup_{n\to\infty} \sup_{\substack{2^{-n-1}<t-s<2^{-n}\\ s,t\in D, s<t}} \frac{|B_t - B_s|}{\sqrt{(t-s)(|\log(t-s)|+1)}}$$

where we note that n appears only in the sup, and hence if we denote

$$\sup_{\substack{2^{-n-1}<t-s<2^{-n}\\ s,t\in D, s<t}} \frac{|B_t - B_s|}{\sqrt{(t-s)(|\log(t-s)|+1)}} =: d_n$$

then

$$\sup_{s,t\in D, s<t} \frac{|B_t - B_s|}{\sqrt{(t-s)(|\log(t-s)|+1)}} \leq \max\left\{d_1, d_2, \dots, d_N, \sqrt{2}C\right\} < \infty$$

where N is chosen by limsup such that for all $m > N$ no term exceeds C.

Step 3: taking limit to construct B_t for all $t \in [0, 1]$.

We simply define here: for $t \in [0, 1]$

$$B_t := \lim_{s\to t, s\in D} B_s.$$

This is well defined up to probability 0 (it's true but I'll not say it in every sentence): first we find a sequence $s_n \to t$, then $\lim_n B_{s_n}$ exists because we can choose s_n close enough to make B_{s_n} Cauchy. Then, for any two sequences, $x_n \to t$ and $y_n \to t$ they both have a limit, and the limits are close to each other since $\lim |B_{x_n} - B_{y_n}| \to 0$ and by bound on it on D. Thus the limit is well-defined. In the same manner the limit has value B_s for $s \in D$.

Step 4: Checking it's a Brownian motion.

This is more or less an exercise but let's check:

- $B_0 = 0$.
- Independent increment: if $s < t$, then

  $$\mathcal{F}_s := \sigma\left\{B_r : r \leq s\right\} = \sigma\left\{B_r : r \leq s, r \in D\right\}$$

  where the second is because limit of X_n is still $\sigma(X_1, X_2, \dots)$ measurable. Also, let

  $$\mathcal{G}_s := \sigma\left\{B_t - B_s : t \geq s\right\} = \sigma\left\{B_{\tilde{t}} - B_{\tilde{s}} : \tilde{t} \geq \tilde{s} \geq s, \tilde{s}, \tilde{t} \in D\right\}$$

  then they are independent because we've shown for those in D.
- $B_t - B_s \sim N(0, t-s)$: this is just because

  $$B_t - B_s = \lim_{\substack{\tilde{s}\to s, \tilde{t}\to t\\ \tilde{s},\tilde{t}\in D}} (B_{\tilde{t}} - B_{\tilde{s}}) \xrightarrow{d} N(0, t-s).$$

- Continuous path is obvious by our definition via continuity.

2.2. **Properties of Brownian motion.**

A definition from analysis is the following.

Def 2.3. $f : [0,1] \to \mathbb{R}$ *is Holder continuous of order* $\beta \geq 0$ *if* $\exists C < \infty$ *such that for all* s, t

$$|f(s) - f(t)| \leq C|s-t|^{\beta}.$$

Some facts are:

- If $\beta > 1$ then the function is constant.
- A function is weakly Holder-β continuous if it's Holder for all $\alpha < \beta$.
- The following are listed from strongest to weakest:
  (1) C_1.
  (2) Lipschitz.
  (3) Holder α for $0 < \alpha < 1$.
  (4) Holder β for $\beta < \alpha$.
  (5) Uniformly continuous.
  (6) Continuous.

During our construction we've shown already that B_t is weakly Holder $\frac{1}{2}$ continuous, and we'll prove in homework that it is not Holder $\frac{1}{2}$.

Theorem 2.4. *With probability 1 the function* $t \mapsto B_t$ *is nowhere differentiable.*

Before we march into the proof, let's note the difference between the following two statements:

- For each t, with probability 1, B_t is not differentiable at t.
- With probability 1, for each t, B_t is not differentiable at t.

The first roughly claims the function is not differentiable at a point, where as the second says it's not differentiable at any point. As an illustration, we use the following event to get some intuition: Let $A_t := \{B_t \neq 1\}$, then

- $\{\forall t, \mathbb{P}(A_t) = 1\}$ is of course correct since each B_t is a normal,
- but $\mathbb{P}(\forall t, A_t) = \mathbb{P}(\cap_t A_t) < 1$ since as long as the function has some point above 1, then IVT says it must cross 1, hence the probability is not 1.

Proof. Assume $\exists t$ such that $f'(t) = k$. Then $\exists \delta > 0$ such that if $|s-t| \leq \delta$ then $|f(t) - f(s)| \leq 2k|s-t|$, which is equivalently saying that $\exists N$ such that for all $|s-t| < \frac{1}{n}, |r-t| < \frac{1}{n}$ where $\delta > \frac{1}{n}$ for all $n \geq N$ we have

$$|f(s) - f(t)| \leq 4k\frac{1}{n}.$$

Now, define the maximum of 3 consecutive portions in the motion:

$$Z(k,n) := \max\left\{\left|B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right)\right|, \left|B\left(\frac{k+1}{n}\right) - B\left(\frac{k}{n}\right)\right|, \left|B\left(\frac{k+2}{n}\right) - B\left(\frac{k+1}{n}\right)\right|\right\}$$

and the minimum over those as

$$Z_n := \min\left\{ Z(k,n) \middle| k = 1, 2, \dots, n-2 \right\}.$$

If B is somewhere differentiable, then $\exists M$ such that $Z_n \leq \dfrac{M}{n}$ for all n, where the maximum is neglected as we choose $\delta > \frac{3}{n}$.

In order to contradict, we define an event

$$E_M := \left\{ Z_n \leq \frac{M}{n} \text{ for } \forall n \text{ large} \right\}$$

and if $\forall M$, $\mathbb{P}(E_M) = 0$ we know the function cannot be differentiable at any point. We do this by showing $\forall M$, $\lim_{n\to\infty} \mathbb{P}\left\{ Z_n \leq \dfrac{M}{n} \right\} = 0$. Again, we estimate by union bound and each $Z(n,k)$ are identically distributed:

$$\mathbb{P}\left\{ Z_n \leq \frac{M}{n} \right\} \leq \sum_{k=1}^{n} \mathbb{P}\left\{ Z(n,k) \leq \frac{M}{n} \right\} \leq n\mathbb{P}\left\{ \max\{|B_{\frac{1}{n}}|, |B_{\frac{2}{n}} - B_{\frac{1}{n}}|, |B_{\frac{3}{n}} - B_{\frac{2}{n}}|\} \leq \frac{M}{n} \right\}$$

$$\leq n\left[\mathbb{P}\left(|B_{\frac{1}{n}}| \leq \frac{M}{n}\right)\right]^3 \leq n\left[\mathbb{P}\left(|B_1| \leq \frac{M}{\sqrt{n}}\right)\right]^3$$

$$\leq n\left[2\frac{M}{\sqrt{n}}\frac{1}{\sqrt{2\pi}}\right]^3 \leq C \cdot \frac{1}{\sqrt{n}} \to 0$$

and we see why we've picked the maximum of three things. Note here the bound is very loose. □

One question is are we doing enough to eliminate all other possibilities of B.M. via our definition. But what do we really mean? In the sense of the joint distribution of an uncountable collection of processes, we know B.M. is unique.

Def 2.5. *A process X_t is called a (centered) Gaussian process if for each $t_1, \dots, t_n$, the vector $(X_{t_1}, \dots, X_{t_n})$ has a (centered) multivariate Gaussian distribution.*

Now we show this for B.M.. First assume $t_1 < t_2 < \cdots < t_n$, then $(B_{t_1}, B_{t_2}, \dots, B_{t_n})$ has such that

$$B_{t_1} = B_{t_1}; \quad B_{t_2} = B_{t_1} + (B_{t_2} - B_{t_1})$$

where each one is a linear combination of independent normals. And the covariance matrix looks like, for $s < t$

$$\Sigma_{st} = \mathbb{E}[X_s X_t] = \mathbb{E}[B_s(B_s + (B_t - B_s))] = \mathbb{E}[B_s^2] + \mathbb{E}[B_s(B_t - B_s)] = \mathbb{E}[B_S^2] = s = \min\{s, t\}.$$

Now we show the Markov property for B.M.. Let's say we look at a B.M. at time t, then the process starting at this time would also be a Brownian motion. In other words, if $Y_s = B_{t+s} - B_t$ then Y_s is a Brownian motion as well that is independent of $\mathcal{F}_t = \sigma\{B_s : s \leq t\}$. But that's obvious.

In general, Markov property means that if the conditional distribution of future given past and present only depends on the present value.

Proposition 2.6. *(Reflection principle) Given t, define* $T_a := \min\{s : B_s = a\}$*, then* $\{T_a \leq t\}$ *is* $\{B_s \geq a$ *for some* $s \leq t\}$ *by definition. The reflection principle says that*

$$\mathbb{P}(T_a \leq t) = 2\mathbb{P}(B_t \geq a)$$

or equivalently

$$\mathbb{P}(B_t \geq a | T_a \leq t) = \frac{1}{2}.$$

This leads to stopping times and note it relates a random time to an event of a known time.

3. 10/3: Markov and Martingales

Def 3.1.
- *A Filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *is an increasing collection of sub* σ*-algebras of* $\mathcal{F}_\infty := \sigma\left(\bigcup_{t\geq 0}\mathcal{F}_t\right)$.
- *A stochastic process* X_t, $t \geq 0$ *is adapted with respect to the filtration if* $\forall t$, X_t *is* $\mathcal{F}_t$ *measurable.*

Def 3.2. *A process* B_t *is called a Standard (1d) Brownian motion adapted to* $\{\mathcal{F}_t\}$ *if*

- $B_0 = 0$.
- B_t *is adapted to* $\{\mathcal{F}_t\}$.
- *Has independent increments.*
- *If* $s < t$*, then* $B_t - B_s \sim N(m(t-s), \sigma^2(t-s))$.
- *Has continuous path.*

Note that there's a slight issue with completeness of σ-algebra: $\{\omega : B_t(\omega) \text{ is continuous in } t\}$ may or may not be measurable but contained in A with $\mathbb{P}(A) = 1$. But let's pretend we're over that.

Def 3.3. *A random variable* T *taking values in* $[0, \infty]$ *is called a stopping time with respect to* $\{\mathcal{F}_t\}$ *if* $\forall t$*, the event*

$$\{t \leq T\} \in \mathcal{F}_t.$$

Some examples are:

- Constant;
- If V is a closed set, then let $T_V = \min\{t : B_t \in V\}$. Note that there's more subtlety if the sets are open.
- If S, T are stopping times, then so are $S \wedge T$, $S \vee T$.

Def 3.4. *If* T *is a stopping time, then* $\{\mathcal{F}_t\}$ *is the* σ*-algebra corresponding to "information available at time* t*", i.e. the collection of events* A *such that* $\forall t$*,*

$$A \cap \{T \leq t\} \in \mathcal{F}_t.$$

We'll show in homework that it is indeed a σ-algebra.

So we can define a stopping time of the first time a Brownian motion that touches a:

$$T_a = \min\{t : B_t = a\}.$$

Def 3.5. *Strong Markov Property: Suppose* B_t *is a Brownian motion with any drift and variance,* T *is a stopping time with respect tot he same filtration* $\{\mathcal{F}_t\}$*. Assume that* $\mathbb{P}\{T < \infty\} = 1$*, let* $Y_s = B_{s+T} - B_T$*, then* Y_s *is a Brownian Motion independent of* $\mathcal{F}_T$*.*

Proof. Since the proof is fairly straightforward, i.e. it is exactly what we'd expect it to be, we only give a sketch of it.

Case 1: If T can only take a finite number of values, then the partition

$$\{T = t_1\}, \{T = t_2\}, \dots, \{T = t_n\}$$

and thus we can check the property with regular Markov property.

Case 2: Approximate T by stopping times taking only finite number of values: define

$$T_n : \Omega \to \left\{ \frac{1}{2^n}, \frac{2}{2^n}, \dots, \frac{n2^n}{2^n} \right\}$$

and

$$T_n = \begin{cases} \frac{k}{2^n} & \frac{k-1}{2^n} \le T < \frac{k}{2^n} \\ \frac{n2^n}{2^n} & T > n \end{cases}$$

and check that it's a stopping time. Then take limit and we are done. The key is that the stopping time we've defined only stops after we know about T, which is allowed in this scenario. □

Def 3.6. *Reflection Principle: If B_t with drift 0 and $a > 0$, then*

$$2\mathbb{P}(B_t \ge a) = \mathbb{P}(\max_{0 \le s \le t} B_s \ge a) = \mathbb{P}(T_a \le t)$$

and also since $\mathbb{P}\{T_a \le t\} = \mathbb{P}\{T_a < t\}$ so $\mathbb{P}\{B_t \ge a | T_a < t\} = \frac{1}{2}$.

One of the important facts here is that If B_t is a standard B.M. and $a > 0$, then $Y_t := \frac{1}{a} B_{a^2 t}$ is also a standard B.M..

Example 3.7.

If $0 < r < s < \infty$ and B_t standard, then how do we find

$$q(r, s) := \mathbb{P}\left\{ B_t = 0 \text{ for some } r \le t \le s \right\}.$$

The idea is to scale it to $[1, 1 + t]$, then estimate. Define

$$q(t) := \mathbb{P}\left\{ B_s = 0 \text{ for some } 1 \le s \le 1 + t \right\} := \mathbb{P}(A)$$

then

$$q(t) = \int_{-\infty}^{\infty} \mathbb{P}(A | B_1 = x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 2 \int_0^{\infty} \mathbb{P}(A | B_1 = x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

and

$$\mathbb{P}(A | B_1 = x) = \mathbb{P}\{\max_{0 \le s \le t} B_s \ge x\} = \mathbb{P}\{\min_{0 \le s \le t} B_s \le -x\}$$

thus plugging in we get

$$q(t) = 4 \int_0^{\infty} \left[\int_{x/\sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \right] \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1 - \frac{2}{\pi} \arctan\left(\frac{1}{\sqrt{t}} \right)$$

by polar substitution.

Note that as $t \to \infty$, $q(t) \to 1$.

Proposition 3.8. *One-dimensional standard Brownian Motion is pointwise recurrent, that is, for each N we have*

$$\mathbb{P}\{\exists t \geq N : B_t = 0\} = 1.$$

In other words, the set $z := \{t : B_t = 0\}$ is unbounded.

In homework, we'll show that $Y_t = tB_{\frac{1}{t}}$ is a standard B.M., given that then with probability 1 for $\forall \varepsilon > 0$ the set $Z_\varepsilon \backslash \{0\}$ is not empty, here $Z_\varepsilon := \{s \leq t : B_s = 0\}$.

Def 3.9. *For $\{\mathcal{F}_n\}_{n \in \mathbb{N}^*}$ a discrete filtration, the set $\{M_n\}$ is a Martingale with respect to $\{\mathcal{F}_n\}$ if*

- *$M_n \in L^1$;*
- *Adapted;*
- *For $m < n$, we have $\mathbb{E}[M_n | \mathcal{F}_m] = M_m$.*

Def 3.10. *A process M_t is a Martingale with respect to $\{\mathcal{F}_t\}$ if*

- *$M_n \in L^1$;*
- *Adapted;*
- *For $s < t$, we have $\mathbb{E}[M_t | \mathcal{F}_s] = M_s$.*

A worry here is that if there's uncountable collection it might cause trouble. The good thing is that actualy doesn't matter. We'll not talk about that though.

Def 3.11. *A process is a continuous Martingale if M_t is also a continuous function of t.*

Note the term continuous refer to the path, not the filtration.

We have the result that a Brownian motion with drift 0 on any filtration is an L^2 continuous Martingale.

Another observation is that if $s < t$, then

$$\mathbb{E}(B_t | \mathcal{F}_s) = \mathbb{E}(B_s | \mathcal{F}_s) + \mathbb{E}(B_t - B_s | \mathcal{F}_s) = B_s + \mathbb{E}[B_t - B_s] = B_s.$$

Now we do the Blumenthal $0-1$ law. But first let's recall the Kolmogorov $0-1$ law.

Theorem 3.12. *Suppose $X_1, X_2, \dots$ are independent and let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ increasing and $\mathcal{G}_n := \sigma(X_{n+1}, \dots)$ decreasing. Then we know $\mathcal{F}_n \perp \mathcal{G}_n$ and we define $\mathcal{F}_\infty := \sigma\left(\bigcup_{n=1}^\infty \mathcal{F}_n\right)$ and define the tail σ-algebra as $\mathcal{T} = \bigcap_{n=1}^\infty \mathcal{G}_n$. Then*

$$\left\{\omega : \lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = 0\right\} \in \mathcal{T}.$$

The 0-1 law states that if $A \in \mathcal{T}$ then $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

A sketch of proof would need the following lemmas.

Lemma 3.13. *If $A \in \mathcal{F}_\infty$ and $\varepsilon > 0$, then $\exists A_n \in \mathcal{F}_n$ such that $\mathbb{P}(A_n \triangle A) < \varepsilon$.*

Corollary 3.14. *If $A \in \mathcal{T}$ then $\exists A_n$ independent of A with $\mathbb{P}(A \triangle A_n) < \infty$.*

From those we note that anything in the tail is independent with itself so the σ-algebra is trivial thus we solve the equation $\mathbb{P}(A) = \mathbb{P}(A)\mathbb{P}(A)$ to get our result.

Theorem 3.15. *(Blumenthal 0-1 law) Let B_t be B.M. and $\{\mathcal{F}_t\}$ be the standard filtration where $\mathcal{F}_t := \sigma\{B_s : s \le t\}$, and from $B_0 = 0$ we know $\mathcal{F}_0$ is trivial. Now we define*

$$\mathcal{F}_{0+} := \bigcap_{\varepsilon>0} \mathcal{F}_\varepsilon$$

then if $A \in \mathcal{F}_{0+}$ we get $\mathbb{P}(A) = 0$ or 1.

The idea is just the same! We note $\bigcap_{\varepsilon>0} \mathcal{F}_\varepsilon = \sigma(\cup \mathcal{G}_n)$.

4. 10/5: Quadratic variation, iterated logarithm

4.1. Quadratic Variation.

Def 4.1. *For B_t a standard Brownian motion, let Π be a partition and define its mesh by*

$$||\Pi|| := \max_j \{t_j - t_{j-1}\}.$$

Now let Π_n be a sequence of partitions which we denote terms by

$$0 = t_{0,n} < t_{1,n} < \cdots < t_{k_n,n} = 0.$$

Then we define

$$Q(1,\Pi) := \sum_{j=1}^{k_n} \left(B_{t_j} - B_{t_{j-1}} \right)^2$$

and denote $Q_n := Q(1, \Pi_n)$, $Q_n(t) := Q(t, \Pi_n)$.

First we note that the quadratic variation is a random variable, so we can compute it's expectation and variance.

$$\mathbb{E}[Q_n] = \sum_{j=1}^{k_n} \mathbb{E}\left[\left(B_{t_j} - B_{t_{j-1}} \right)^2 \right] = \sum_{j=1}^{k_n} (t_j - t_{j-1}) = 1$$

using variance. The variance is

$$\begin{aligned}\operatorname{Var}(Q_n) &= \sum_{j=1}^{k_n} \operatorname{Var}\left(\left(B_{t_j} - B_{t_{j-1}} \right)^2 \right) = \sum_{j=1}^{k_n} \operatorname{Var}\left(\left(\sqrt{t_j - t_{j-1}} B_1 \right)^2 \right) \\ &= \sum_{j=1}^{k_n} (t_j - t_{j-1})^2 \operatorname{Var}(B_1^2) = 2 \sum_{j=1}^{k_n} (t_j - t_{j-1})^2 \le ||\Pi_n|| 2 \sum_{j=1}^{k_n} (t_j - t_{j-1}) = 2||\Pi_n||\end{aligned}$$

where the second equality comes from same distribution, the variance is computed with formula (remember, even moments are of the formula $(n-1)!!$).

Now, with Chebyshev's inequality we get

$$\mathbb{P}\left\{ |Q_n - 1| \ge \varepsilon \right\} \le \frac{\operatorname{Var}(Q_n)}{\varepsilon^2} \le \frac{2||\Pi_n||}{\varepsilon^2}.$$

Above is almost a direct proof of the following theorem:

Theorem 4.2. *If $||\Pi_n|| \to 0$, then $Q_n \to 1$ in probability;*
If $\sum_{n=1}^{\infty} ||\Pi_n|| < \infty$, then $\lim_{n\to\infty} Q_n = 1$.

We see that the first statement follows immediately, and the second follows with Borel Cantelli.

Note that when we say "if $\sum_{n=1}^{\infty} ||\Pi_n|| < \infty$, then $\lim_{n\to\infty} Q_n = 1$" we are not saying "with probability 1, $\sum_{n=1}^{\infty} ||\Pi_n|| < \infty$, then $\lim_{n\to\infty} Q_n = 1$."

The second is actually wrong since we can have uncountable choice and we'll choose partition for a fixed path to violate this.

Theorem 4.3. *If $\sum_{n=1}^{\infty} ||\Pi_n|| < \infty$, then with probability 1, for all $0 \le t \le 1$ we have*

$$Q_n(t) \to t.$$

This does not follow immediately since we only have the result for countable collection of t, but here the collection is uncountable. But not really far away!

Proof. We use above to do for all t rational. Then we notice that if $s \le t$ we have $Q_n(s) \le Q_n(t)$ because we're just adding more positive things, the rest follows from function limit. $\square$

Def 4.4. *(Formal definition) If X_t is a process, then its quadratic variation $\langle X \rangle_t$ is*

$$\langle X_t \rangle = \lim_{n\to\infty} \sum_{t_{j,n} \le t} \left(X_{t_{j,n}} - X_{t_{j-1,n}} \right)^2$$

if the limit exists.

As we've shown above, for a standard B.M., $\langle B_t \rangle = t$.

Proposition 4.5. *If B_t is a standard B.M. and $Y_t = mt + \sigma B_t$ then $\langle Y_t \rangle = \sigma^2 t$.*

Proof. This means a drift doesn't really affect the quadratic variation. We compute

$$(Y_{t_j} - Y_{t_{j-1}})^2 = \sigma^2 (B_{t_j} - B_{t_{j-1}})^2 + 2\sigma (B_{t_j} - B_{t_{j-1}})(t_j - t_{j-1}) m + (t_j - t_{j-1})^2 m^2$$

and hence

$$\sum_{t_j \le t} (Y_{t_j} - Y_{t_{j-1}})^2 = \sigma^2 \sum_{t_j \le t} (B_{t_j} - B_{t_{j-1}})^2 + 2\sigma m \sum_{t_j \le t} (B_{t_j} - B_{t_{j-1}})(t_j - t_{j-1}) + m^2 \sum_{t_j \le t} (t_j - t_{j-1})^2$$

and let's compute term by term. As $||\Pi_n|| \to 0$ we know

$$\sigma^2 \sum_{t_j \le t} (B_{t_j} - B_{t_{j-1}})^2 \to \sigma^2 t$$

$$m^2 \sum_{t_j \le t} (t_j - t_{j-1})^2 \le m^2 ||\pi_n|| t \to 0$$

and the middle part

$$\left| 2\sigma m \sum_{t_j \le t} (B_{t_j} - B_{t_{j-1}})(t_j - t_{j-1}) \right| \le 2\sigma m \sup_j |B_{t_j} - B_{t_{j-1}}| \sum_{t_j \le t} (t_j - t_{j-1}) \to 0$$

since by continuity

$$\sup_j |B_{t_j} - B_{t_{j-1}}| = 0.$$

□

Now we give an alternative definition of quadratic variation.

Def 4.6. *For continuous L^2 martingale M_t, we know M_t^2 is a sub martingale and there exists unique $\langle M_t \rangle_t$ such that $M_t^2 - \langle M \rangle_t$ is a Martingale. This we define to be the quadratic variation.*

For each X_t we can(can we ?) make it martingale and then apply the above definition.

4.2. **Law of the Iterated logarithm.**

Theorem 4.7. *If B_t is a standard B.M. and $\phi(t)$ is deterministic, then with probability* 1

$$\limsup_{t\to\infty} \frac{B_t}{\phi(t)} = 1$$

if $\phi(t) = \sqrt{2t \log\log t}$.

Proof. First, we build towards 0-1 law by finding a tail σ-algebra. Define $\mathcal{G}_t := \sigma\{B_{s+t} - B_t, s \geq 0\}$ then we can define $\mathcal{T}_\infty := \bigcap_t \mathcal{G}_t$. By 0-1 law, $A \in \mathcal{T}_\infty$ implies $\mathbb{P}(A) = 0$ or 1.

Now we find the event A in the tail σ-algebra. Define

$$A_\varepsilon := \left\{ \omega : \limsup_{t\to\infty} \frac{B_t}{\sqrt{2(1\pm\varepsilon)t\log\log t}} \leq 1 \right\}$$

where we'll use the $\pm\varepsilon$ for both directions in the proof. We can see that $A_\varepsilon \in \mathcal{T}_\infty$ since it's about all the tail events, so $\mathbb{P}(A_\varepsilon) = 0$ or 1.

Note that we also have a symmetry argument:

$$\mathbb{P}\left\{ \limsup_{t\to\infty} \frac{B_t}{\sqrt{2(1\pm\varepsilon)t\log\log t}} \geq 1 \right\} = \mathbb{P}\left\{ \liminf_{t\to\infty} \frac{B_t}{\sqrt{2(1\pm\varepsilon)t\log\log t}} \leq -1 \right\}$$

and thus reflection principle tells us

$$\mathbb{P}(A_\varepsilon) = \mathbb{P}\left\{ \limsup_{t\to\infty} \frac{|B_t|}{\sqrt{2(1\pm\varepsilon)t\log\log t}} \leq 1 \right\}.$$

The next step is to squeeze time by a factor of ρ^n for some $\rho > 1$ which we'll pick later on.

Now we show one side of the equality, so we use $-\varepsilon$ here. Define event

$$V_n := \left\{ \left| B_{\rho^n} \right| \geq \sqrt{2\rho^n(1-\varepsilon)\log\log\rho^n} \right\}$$

then we want to use Borel Cantelli to show that V_n occurs infinitely often, since the sets are "almost independent." More precisely, we want to use the following form of BC:

Lemma 4.8. *Let $\mathcal{F}_n := \sigma\{A_1, A_2, \dots\}$, then if there exists q_n with $\sum q_n = \infty$ such that for all n, $\mathbb{P}(A_n | \mathcal{F}_{n-1}) \geq q_n$, we can conclude that with probability 1, A_n occurs infinitely often.*

Proof. We need $\mathbb{P}(A_n^c \cap A_{n+1}^c \cap \dots) = 0$ for all n, but we really can get from assumption that

$$\mathbb{P}(A_n^c \cap A_{n+1}^c \cap \dots) \leq (1-q_n)(1-q_{n+1})\cdots \to 0$$

where the last convergence uses the fact that $\sum q_n$ diverge. □

Now, the claim is $\mathbb{P}(V_{n+1}|V_1,\dots,V_n)$ need to be summable. We note that

$$\mathbb{P}(V_{n+1}|V_1,\dots,V_n) \geq \mathbb{P}\left(B_{\rho^{n+1}} - B_{\rho^n} \geq \sqrt{2\rho^n(1-\varepsilon)\log\log\rho^n}\right)$$

by a symmetry argument since

$$\mathbb{P}\left(B_{\rho^{n+1}} - B_{\rho^n} \geq \sqrt{2\rho^n(1-\varepsilon)\log\log\rho^n}\right) = \frac{1}{2}\mathbb{P}\left(\left|B_{\rho^{n+1}} - B_{\rho^n}\right| \geq \sqrt{2\rho^n(1-\varepsilon)\log\log\rho^n}\right)$$

and if we discuss whether B_{ρ^n} is positive we'd note that the latter exactly corresponds to what we need. Now we further simplify to get

$$\mathbb{P}\left(B_{\rho^{n+1}} - B_{\rho^n} \geq \sqrt{2\rho^n(1-\varepsilon)\log\log\rho^n}\right) = \mathbb{P}\left(\frac{B_{\rho^{n+1}} - B_{\rho^n}}{\sqrt{\rho^{n+1}-\rho^n}} \geq \frac{\sqrt{2\rho^n(1-\varepsilon)\log\log\rho^n}}{\sqrt{\rho^{n+1}-\rho^n}}\right)$$

where now the right hand side is easy to manipulate as:

$$\frac{\sqrt{2\rho^n(1-\varepsilon)\log\log\rho^n}}{\sqrt{\rho^{n+1}-\rho^n}} = \sqrt{2\frac{\rho}{\rho-1}(1-\varepsilon)[\log n + \log\log\rho]]}$$

and we want $\dfrac{\rho}{\rho-1}(1-\varepsilon) \leq 1$, which we can do if we choose ρ large enough. Now for the above chosen δ we have

$$\sqrt{2\frac{\rho}{\rho-1}(1-\varepsilon)[\log n + \log\log\rho]]} = \sqrt{2(1-\delta)[\log n + \log\log\rho]}$$

and hence by asymptotic behavior of normal distribution we get

$$\begin{aligned}\mathbb{P}\left(\frac{B_{\rho^{n+1}} - B_{\rho^n}}{\sqrt{\rho^{n+1}-\rho^n}} \geq \frac{\sqrt{2\rho^n(1-\varepsilon)\log\log\rho^n}}{\sqrt{\rho^{n+1}-\rho^n}}\right) &\sim \exp\left\{\frac{-2(1-\delta)[\log n + \log\log\rho]}{2}\right\} \\ &\gtrsim Cn^{\delta-1}\frac{1}{\sqrt{2(1-\delta)[\log n + \log\log\rho]}}\end{aligned}$$

which is not summable. Now B.C. proves that V_n occurs infinitly often, which means as $\varepsilon \to 0$ we have $\limsup_{t\to\infty} \dfrac{B_t}{\sqrt{2t\log\log t}} \geq 1$, as a sequence of 1 converges to 1.

For the other direction, we can just use the original Borel Cantelli. We define

$$U_n := \left\{\exists \rho^n \leq t \leq \rho^{n+1} \text{ with } |B_t| \geq \sqrt{2t(1+\varepsilon)\log\log t}\right\}$$

and it suffice to show the other direction that $\sum \mathbb{P}(U_n) < \infty$. We notice that

$$
\begin{aligned}
U_n &\subset \left\{ \max_{0 \le t \le \rho^{n+1}} B_t \ge \sqrt{2\rho^n (1+\varepsilon) \log\log \rho^n} \right\} \le 2\mathbb{P}\left\{ B_{\rho^{n+1}} \ge \sqrt{2\rho^n (1+\varepsilon) \log\log \rho^n} \right\} \\
&\le 2\mathbb{P}\left\{ \frac{B_{\rho^{n+1}}}{\sqrt{\rho^{n+1}}} \ge \sqrt{2\frac{1}{\rho}(1+\varepsilon)[\log n + \log\log \rho]} \right\} \\
&\le 2\exp\left\{-(1+\delta)[\log n + \log\log \rho]\right\} \le Cn^{-1-\delta}
\end{aligned}
$$

is summable. The middle part requires ρ small enough. Hence we are done. $\square$

5. 10/10: More properties of Brownian motion; Multidimensional Brownian motion

We first consider the zero set of 1d Brownian motion: let B_t be the standard Brownian motion, $Z := \{t : B_t = 0\}$ and define $Z_t := Z \cap (0, t]$. Then with probability 1 Z is closed by continuity.

Def 5.1. *$t \in Z$ is right-isolated if $\exists \varepsilon > 0$ such that $(t, t+\varepsilon) \cap Z = \emptyset$, and its left-isolated if $\exists \varepsilon > 0$ such that $(t-\varepsilon, t) \cap Z = \emptyset$. And t is isolated if it is both right-isolated and left-isolated.*

What we have shown last time and in homework is that with probability 1 0 is not right isolated.

Proposition 5.2. *With probability 1*

- *The set of left-isolated points is countable;*
- *The set of right-isolated points is countable;*
- *There's no isolated points;*
- $\mu(Z) = 0$.

Proof. We first find all left-isolated points, then show they are countable, and not right-isolated. Then we show similarly(though with difference) that right-isolated points is countable, then we show measure 0.

For $q \in \mathbb{Q}^+$ we note $\mathbb{P}(B_q = 0) = 0$ just by countable union. Then with probability 1 $Z \cap \mathbb{Q}^+ = \emptyset$. We can thus define

$$T_q := \min\{t \geq q : B_t = 0\}$$

then $\{T_q\}$ is a countable family of stopping time. But we know for each isolated point it is the value of T_q for some q so left-isolated points are countable. Now we use strong Markov property to note that T_q is not right-isolated because 0 is not. Thus we've shown there's no isolated points.

Now, we look backwards and define

$$\sigma_q := \max\{t < q : B_t = 0\}$$

which is well defined since Z is closed, and we see for each q we can assign an interval (σ_q, T_q) and thus right-isolated points are countable since

$$Z = [0, \infty) \setminus \bigcup_q (\sigma_q, T_q).$$

Not only have we shown that left- or right-isolated zero are countable, we can also compute

$$\mathbb{E}[\mu(Z_1)] = \mathbb{E}\left[\int_0^1 \mathbb{1}_{B_s=0} ds\right] = \int_0^1 \mathbb{P}(B_s = 0) ds = 0.$$

□

Note that there's something peculiar happening: it really feels like the Cantor set in that we're deleting intervals one by one.

Not only so, we can actually show that the zero set of B.M. is homeomorphic to the Cantor set, and it is also uncountable. For a closer look at this topological property we define fractal dimensions.

Def 5.3. *Suppose we have a subset of* $[1, 2]$*, then we can cover* $[1, 2]$ *with n intervals of length* $1/n$ *and denote* $X_n :=$ # *intervals that intersects* Z*. Then*

$$\mathbb{E}[X_n] = \sum_{j=1}^{n} \mathbb{P}\left\{ S \cap \left[\frac{j-1}{n} + 1, \frac{j}{n} + 1\right] \neq 0 \right\} \sim cn^D$$

where D *is defined to be the* <u>*box dimension*</u>*.*

For the cantor set, its box dimension is $\frac{\log 2}{\log 3}$, and the zero set is $\frac{1}{2}$. This also means that box dimension is not invariant under homeomorphism.

Def 5.4. *For* α, ε*, define*

$$H_{\varepsilon}^{\alpha}(V) = \inf \sum_{j=1}^{\infty} (\mathrm{diam}(U_j))^{\alpha}$$

where the inf *is taken over all covering of* V*, i.e.* $\bigcup_{j=1}^{\infty} U_j \supset V$ *with* $\mathrm{diam}(U_j) < \varepsilon$*. Then the* <u>*Hausdorff measure*</u> *is*

$$H^{\alpha}(V) := \lim_{\varepsilon \to 0} H_{\varepsilon}^{\alpha}(V)$$

where the limit exists by MCT.

Proposition 5.5. H^{α} *is a Borel measure whereas* H_{ε}^{α} *is not.*

Def 5.6. *For a set* V*, there exists* d *such that*

$$H^{\alpha}(V) = \begin{cases} \infty & \alpha < D \\ 0 & \alpha > D \end{cases}$$

and we define the above D *to be the* <u>*Hausdorff dimension*</u> *of* V*.*

Now, for the zero set, we have the conclusion that it's Hausdorff dimension $\frac{1}{2}$, and the cantor set also has $\frac{\log 2}{\log 3}$. Note that these two sets all have same Hausdorff dimension as their Box dimension. They are generally different but for our purpose they are the same. In general, Hdim $\leq$ box dimension, since for box we choose non-overlapping sets, and for the Hausdorff dimension, they can overlap, so naturally smaller.

Now we define the cantor function, which is constant on each part the cantor set excludes. We know it's continuous and if $a < b$, then $f(b) - f(a) > 0 \iff C \cap (a, b) \neq \emptyset$.

We define the Cantor measure as $\mu_{\text{can}}[a,b] = f(b) - f(a)$. Thus we know the Cantor measure of the Cantor set is 1, which makes sense.

We now define local time, which put in a non-reasonable way is: "L_t is the amount of time that Brownian Motion has spent at 0 by time t."

This is not reasonable because if you take it to be the actual time it meets 0 then its infinite, but if you take it to be the measure it's 0. But we define it in the following way.

If $s < t$, then $L_t - L_s > 0 \iff (s,t) \cap Z \neq \emptyset$ and we can look at

$$L_{t,\varepsilon} := \frac{1}{2\varepsilon} \int_0^1 \mathbb{1}_{|B_s| \leq \varepsilon} ds$$

and define the limit $L_t := \lim_{\varepsilon \to 0} L_{t,\varepsilon}$ which really means the density of Brownian motion around 0, taken limit. So we can compute

$$\mathbb{E}[L_{t,\varepsilon}] = \frac{1}{2\varepsilon} \int_0^t \mathbb{P}\left\{|B_s| < \varepsilon\right\} ds \sim \frac{1}{2\varepsilon} \int_0^t 2\varepsilon \frac{1}{\sqrt{2\pi s}} ds = \int_0^t \frac{1}{\sqrt{2\pi s}} = \sqrt{\frac{2}{\pi}} t^{1/2}$$

and of course if you try harder you can show that the limit exists.

Theorem 5.7. *With probability 1, the limit $L_t = \lim\limits_{t \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{B_s \leq \varepsilon} ds$ exists for all t and the convergence is also in L^2.*

We restate the fact that If $s < t$, then $L_t - L_s > 0 \iff (s,t) \cap Z \neq \emptyset$, and hopefully it makes more sense now. Another fact is that L_t is weakly Holder $\frac{1}{2}$ continuous, and to show this we can fix t, then do for dyatic t, then use MCT to pass to all t.

We can also define local time at x by

$$L_t := \lim_{t \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{|B_s - x| \leq \varepsilon} ds.$$

We still have scaling rule: L_t has same distribution as $t^{\frac{1}{2}} L_1$. Yet another fact is that M_t has the same distribution as L_t, where

$$M_t := \max_{0 \leq s \leq t} \{B_s\}.$$

5.1. **Brownian motion in several dimension.**

Def 5.8. *If $B_t^1, B_t^2, \ldots, B_t^d$ are independent standard Brownian Motion and*

$$B_t := (B_t^1, B_t^2, \ldots, B_t^d)$$

then B_t is a <u>standard Brownian motion in $\mathbb{R}^d$</u>.

Proposition 5.9. *B_t has the following properties:*

- $B_0 = 0$*;*

- *Independent increments with respect to* $\mathcal{F}_t = \sigma\{\mathcal{F}_t^1, \ldots, \mathcal{F}_t^d\}$*;*
- *If* $s < t$, $B_t - B_s \sim N(0, (t-s)I)$ *where the latter is the Covariance matrix;*
- *Continuous path.*

We also know that B_t has density

$$\left[\frac{1}{\sqrt{2\pi t}e^{-\frac{x_1^2}{2t}}}\right] \cdots \left[\frac{1}{\sqrt{2\pi t}e^{-\frac{x_d^2}{2t}}}\right] = \frac{1}{(2\pi t)^{d/2}} e^{\frac{-|x|^2}{2t}}.$$

We can also define Brownian Motion in $\mathbb{R}^d$ with drift $\mu \in \mathbb{R}^d$ and covariance matrix Γ, which has $B_t - B_s \sim N((t-s)\mu, (t-s)\Gamma)$.

Now, if $A^{d\times d}$ is invertible, then $Y_t = AB_t + t\mu$ gives a brownian motion of drift μ and covariance AA^T.

Question: if we start a standard Brownian motion at $x \in \mathbb{R}^d \backslash \{0\}$ and let $D := \{y \in \mathbb{R}^d : r < |y| < R\} =: D(r, R)$ and $x \in D$, then let $\tau = T(r, R)$ be the first time such that $|B_t| = r$ or $|B_t| = R$, then what is the probability that $|B_\tau| = R$?

We first start with 1d case, but that's an easy martingale exercise: First note that if T is a stopping time then $\tilde{M}_t = M_{t\wedge T}$ is a martingale, and we know $\mathbb{P}(\{T = \infty\}) = 0$, and

$$\mathbb{E}[B_{t\wedge T}] = \mathbb{E}[B_0] = x$$

by martingale property, and the first is bounded by property. Let $B_T := \lim_{t\to\infty} B_{t\wedge T}$ which exist by DCT plus bounded. Now classic problem gives

$$\mathbb{P}(B_T = R) = \frac{x - r}{R - r}.$$

6. 10/12: HARMONIC FUNCTIONS AND BROWNIAN MOTION

First we only consider our domain to be open connected subset of $\mathbb{R}^d$.

Denote

$$MV(f;x,\varepsilon) := \int_{|x-y|=\varepsilon} f(y)ds(y)$$

where the measure is the surface measure scaled such that

$$\int_{|x-y|=\varepsilon} ds(y) = 1.$$

Def 6.1. *The mean value property is: if $x \in D$ and $\varepsilon < \operatorname{dist}(x, \partial D)$, then $f(x)$ is the same thing as the mean value of f, which is $MV(f;x,\varepsilon)$ for any ε.*

Def 6.2. *A function $f : D \to \mathbb{R}$ is harmonic if it's continuous (in fact locally-integrable is enough) and satisfies the mean value property (spherical).*

Now, for B_t the standard Brownian motion with d dimension starting at x, define $\tau := \min\{t : |B_t - x| = \varepsilon\}$ (well-defined because closed) then

$$MV(f;x,\varepsilon) = \mathbb{E}[f(B_\tau)].$$

Def 6.3. *The Laplacian of f is $\Delta f = \displaystyle\sum_{j=1}^{d} \frac{\partial_2 f}{\partial x_j^2}$.*

A fact is that if f is C^2 in D, then just by taylor we can get

$$\frac{1}{2d}\Delta f(x) = \lim_{\varepsilon \to 0} \frac{MV(f;x,\varepsilon) - f(x)}{\varepsilon}.$$

Now we have the theorem.

Theorem 6.4. *f is harmonic D iff f is C^2 and $\Delta f = 0$.*

This is easily proved by calculus.

Now we return to the problem last time, we define

$$T_{r,R} := \min\{|B_t| = r \text{ or } R\} = \min\{t : B_t \in \partial D\}$$

and

$$\phi(x) := \mathbb{P}^x\{|B_T| = R\}$$

where the upper index refers to the starting point. By rotational invariance, $\phi(x) = \phi(|x|)$ and if $\phi(x) = 0$ we know $|x| = r$, while if $\phi(x) = 1$ we know $|x| = R$, and the boundary is continuous. Most importantly, ϕ satisfies MVP by strong Markov property.

We define the conditional property up to when the path hits an ε ball, and

$$\phi(B_{T,\varepsilon}) = \mathbb{P}^x\{|B_T| = R | \mathcal{F}_{T,\varepsilon}\}$$

Then we have

$$\mathbb{P}^x\{|B_T| = R\} = \mathbb{E}^x[\mathbb{P}^x\{|B_T| = R|\mathcal{F}_{T,\varepsilon}\}] = \mathbb{E}^x[\phi(B_{T,\varepsilon})] = MV(\phi, x, \varepsilon)$$

Thus $\phi(x)$ is harmonic on $D_{r,R}$ with boundary value 0 on $\{|y| = r\}$ and 1 on $\{|y| = R\}$ is rotational invariant. Now let's just say we solve PDE and get the unique solution:

$$\phi_{r,R}(x) = \begin{cases} \frac{|x|^{2-d} - r^{2-d}}{R^{2-d} - r^{2-d}} & d > 2 \\ \frac{\log|x| - \log r}{\log R - \log r} & d = 2 \end{cases}$$

and we note that they are all harmonic in the space without the origin. This is unique up to a multiplicative and addition constant.

6.1. **Recurrence and Transience of Brownian motion.**

For $d \geq 3$, let $T_r = \min\{t : |B_t| = r\}$ and

$$\mathbb{P}^x\{T_r < \infty\} = \lim_{R\to\infty} \mathbb{P}^x\{|B_{T_{r,R}}| = R\} = \lim_{R\to\infty} \frac{|R|^P 2 - d - |x|^{2-d}}{R^{2-d} - r^{2-d}} = \left(\frac{r}{|x|}\right)^{d-2} < 1$$

thus we know there's always possibility that the point will not compact again to r. This means that eventually it will not comback infinitely many times as each time it is multiplied by a constant less than 1. This proves the following theorem.

Theorem 6.5. *With probability 1, Brownian motion is transient for $d \geq 3$.*

Theorem 6.6. *With probabilty 1, Brownian motion in 2d is*

- *Neighborhood recurrent: for every $z \in \mathbb{R}^2$, $\varepsilon > 0$ the Brownian motion visits the disk $D_\varepsilon(z)$ at arbitrary large times.*
- *Not pointwise recurrent: for all x, with probability 1, for all $t > 0$, $B_t \neq x$.*

Proof. With the same method, we have

$$\mathbb{P}^x\{T_r < \infty\} = \lim_{R\to\infty} \mathbb{P}^x\{|B_{T_{r,R}}| = R\} = \lim_{R\to\infty} \frac{\log|x| - \log r}{\log R - \log r} = \mathbb{1}$$

so it will return. But for pointwise return, say the point is 0 and define $T := \min\{t : B_t = 0\}$ then

$$\mathbb{P}^x\{T < \infty\} = \lim_{R\to\infty} \mathbb{P}^x\{\text{reach 0 before } R\}$$

and for each little r we consider the event inside and get

$$\lim_{R\to\infty} \mathbb{P}^x\{\text{reach 0 before } R\} \leq \lim_{R\to\infty} \lim_{r\to 0} \mathbb{P}^x\{|B_{T_{r,R}} = r|\} = \lim_{R\to\infty} \lim_{r\to 0} \frac{\log|x| - \log r}{\log R - \log r} = 0.$$

$\square$

This means the B.M. has a dense path but not $\mathbb{R}^2$.

Some facts are: if $d \geq 2$, the path $\{B_t : t \geq 0\}$ has Hausdorff dimension 2 but has 0 in the Hausdorff 2 measure, which in our case is just the Lebesgue measure.

Now we consider the case for boundaries. Suppose $f : \partial D \to \mathbb{R}$ is continuous then we try to find the unique continuous function $f : \bar{D} \to \mathbb{R}$ satisfying $f \equiv F$ on ∂D that is harmonic in D.

This is a similar argument of Maximal principle, i.e.

$$\max_{x \in \bar{D}} f(x) = \max_{x \in \partial D} f(x)$$

and the proof is regular. This also shows that two different solutions must be the same, so unique. Now let $T = \min\{t : B_t \in \partial D\}$ then

$$f(x) = \mathbb{E}^x[F(B_T)]$$

is MVP implies f harmonic, for F suitable boundary condition. But this is not an exact inverse to harmoniticity implies MVP, since we don't yet know continuity on boundary. In fact that is not always true!

Let D be the set of $D = \{x \in \mathbb{R}^2, 0 < |x| < 1\}$ and define F to be equivalent to 1 for $|x| = 1$ and $F(0) = 0$. But we know

$$f(x) = \mathbb{P}^x\{|B_T| = 1\} = 1$$

since there's probability 0 that the motion will return to a point 0, and hence f is 1 all over thus not continuous at the origin.

What might we add to make a suitable condition?

Def 6.7. *If $x \in \partial D$, then let $\sigma = \inf\{t > 0, B_t \in \partial D\}$ then x is <u>regular</u> if $\mathbb{P}^x\{\sigma = 0\} = 1$.*

To illustrate, a regular point must be "smooth" enough.

Example 6.8.

The following are examples of regular points:

- smooth;
- Has a cone in the complement of D.
- If the dimension is 2 then connected is enough.
- For 3d domain, we have a counterexample where if we remove a line from a sphere, it's not regular at the endpoint.
- Even more, if we remove a curvy cone which is pointy at the origin, it's also not regular.

Proposition 6.9. $f(x) = \mathbb{E}^x[F(B_T)]$ *is continuous at every regular boundary point.*

The idea for all this is that the Dirichelet problem on bounded domain has a solution for every continuous $F \iff$ every point on ∂D is regular.

7. 10/17: DIRICHLET PROBLEMS

Suppose, same as last time, that we have bounded domain $D \subset \mathbb{R}^d$ and continuous function $F : \partial D \to \mathbb{R}$. Also let B_t be standard Brownian motion with stopping time $T := \min\{t : B_t \in \partial D\}$ and $z \in D$ we have defined

$$f(z) := \mathbb{E}^z[F(B_T)]$$

which is harmonic, and if $f \equiv F$ on ∂D then it is continuous at each regular point.

By maximum principle, there is only one such function up to harmonic uniqueness.

Now, if we extend F to only bounded measurable then everything above holds, and the only difference is that f is continuous at regular points where F is continuous, but that's obvious.

It's also possible to express $f(z)$ with out the usage of probability, but that requires more work and we'd develop toward it, but not complete it.

Def 7.1. *For D domain, then harmonic measure on D(or ∂D) at $z \in D$ is the hitting measure of Brownian motion starting at z stopped at ∂D. That is, if $V \subset \partial D$, then*

$$\mu(V) := \mathrm{hm}_D(V; z) := \mathbb{P}^z\{B_T \in V\}$$

where it reads "harmonic measure in D starting at z."

Note that for any harmonic measure

$$\mathrm{hm}_D(\partial D; z) = \mathbb{P}_z\{T < \infty\}$$

thus D bounded implies that hm_D is a probability measure.

We can rewrite things:

$$\mathbb{E}^z[F(B_T)] = \int_{\partial D} F(w)\, \mathrm{hm}_D(dw; z)$$

where the choice of notation on measure is just personal preference. The above expresses the meaning "mean value of F as seen from z."

As one can or cannot tell, what we are going to do is nothing but complex analysis applied to harmonic situations.

A fact: if ∂D is smooth, then $\mathrm{hm}_D(\cdot; z)$ is absolutely continuous with respect to the surface measure for $d-1$ dimension, i.e.

$$\mathrm{hm}_D(V; z) = \int_V H_D(z, w) s(dw)$$

where H_D is roughly the derivative of hm_D, or the density of B_T with respect to the surface measure:

$$H_d(z, w) s(dw) = \mathrm{hm}_D(dw; z)$$

and when the above holds we call H_D the Poisson kernel.

For the special case of $D = \{z \in \mathbb{R}^d : |z| < 1\}$, we can compute explicitly

$$H_D(z, w) = \frac{1}{c_d} \frac{1 - |z|^2}{|z - w|^d}$$

where c_d is the surface measure of the unit sphere.

If $d = 2$, then

$$H_D(z, w) = \frac{1}{2\pi} \frac{1 - |z|^2}{|z - w|^2}.$$

Now we introduce some properties of harmonic measure: if $V \subset \partial D$

$$h(z) = \mathrm{hm}_D(v; z) = \mathbb{P}^z\{B_T \in V\}$$

then h is the unique harmonic function in D with boundary condition

$$F(w) = \begin{cases} 1 & w \in V \\ 0 & w \notin V \end{cases}$$

and we can define $\mathrm{hm}_D(\cdot; z)$ with the above formula. But so long with that.

In particular, if D contains the closed unit ball (can scale and shift) B and f is harmonic in D, then for $\forall |z| < 1$

$$f(z) = \int_{|w|=1} f(w) H_B(z, w) s(dw)$$

from last time's work.

Proposition 7.2. *(Harnack's inequality): $\forall r \in (0, 1)$, $\exists C = C(r, d)$ such that if $f : B \to (0, \infty)$ is positive and harmonic, then for all $|z|, |\tilde{z}| \le r$ we have*

$$f(z) \le C f(\tilde{z}).$$

We can also show that

$$C = \max_{|z|,|\tilde{z}| \le r; |w|=1} \frac{H_B(z, w)}{H_B(\tilde{z}, w)}.$$

Notice that the above is just complex analysis, and non-negativity really excludes all bad cases.

A generalized version of above is:

Proposition 7.3. *For every k, there exist $C = C(k, d)$ such that if $f : B \to \mathbb{R}$ is harmonic, then for any k-th order partial derivative (sum of partial orders is k, so notation is in vector form):*

$$|\partial^k f(0)| \le C ||f||_\infty.$$

Proposition 7.4. *Even more generally, for each $z \in D$ let $r_z := \frac{1}{2}\operatorname{dist}(z, \partial D)$, then there exists $C = C(k, d)$ such that if $f : D \to \mathbb{R}$ is harmonic, then any kth derivative we have*

$$|\partial^k f(0)| \le C \cdot (r_z)^k \cdot \max_{|z-w| \le r_z} ||f(w)||.$$

Above, the term $(r_z)^k$ really comes out from differentiating $f(r_z x)$ k times.

Proposition 7.5. *(Harnack Principle): If D is a domain, K a compact subset, then $\exists C = C(k, D)$ such that if $f : D \to (0, \infty)$ positive is harmonic and $z, w \in K$, then $f(z) \le Cf(w)$.*

Proof. The really essential part is that a domain is connected. Since K is compact, we can use a path to create a connected compact set which we WLOG call K. Now we just find an open cover of K and find a finite subcover, then multiply those constants within each ball that connects z and w, then we are done. □

Now we consider unbounded domains. Some results are much easier to show with baby stochastic calculus, so we just state them for now.

For D unbounded, we can write $K := \mathbb{R}^d \backslash D$ then K is closed.

Suppose $F : K \to \mathbb{R}$ is continuous, then our goal is to find $f : \mathbb{R}^d \to \mathbb{R}$ such that

- f is harmonic on D;
- $f \equiv F$ on K;
- f is continuous on $\mathbb{R}^d$.

Now the uniqueness argument fails.

Example 7.6. *Counterexample to above.*

Let $d = 1$ and $K = (-\infty, 0]$, $D = (0, \infty)$, then define $F \equiv 0$ and we have

$$f(x) = \begin{cases} 0 & x \le 0 \\ cx & x > 0 \end{cases}$$

for any c will be a solution.

Now we add a condition that K is compact, i.e. D being cofinite, then if F is bounded we do have uniqueness for the above 3 conditions.

Now we try to see uniqueness of the solution to the dirichlet problem:

Theorem 7.7. *Let $T := \min\{t \ge 0 : B_t \in K\}$, then assume $\forall z \in D$,*

$$\mathbb{P}^z\{T < \infty\} = \mathbb{P}^z\{B(0, \infty) \cap K\} > 0$$

then if $d = 1$ or 2, there exist unique solution to the bounded dirichlet problem

- *f is harmonic on D;*
- *$f \equiv F$ on K;*
- *f is continuous on $\mathbb{R}^d$.*

given by $f(z) = \mathbb{E}^z[f(B_T)]$.

For $d \ge 3$ and K bounded, we know $\mathbb{P}^z\{T < \infty\} < 1$. Define the complement $g(z) := \mathbb{P}^z\{T = \infty\} > 0$ then we have g is harmonic in D, g is bounded, and if $g \equiv 0$ on K then g is continuous. Then the solution to

- *f is harmonic on D;*
- *$f \equiv F$ on K;*
- *f is continuous on $\mathbb{R}^d$.*

are of the following form:

$$f(z) = \mathbb{E}^z[F(B_T)\mathbb{1}_{T<\infty}] + c \cdot \mathbb{P}^z\{T = \infty\}.$$

Remark 7.8.

There are a few remarks on the above theorem.

(1) The condition
$$\mathbb{P}^z\{T < \infty\} = \mathbb{P}^z\{B(0,\infty) \cap K\} > 0$$
for $d = 1$ or 2 is just saying we don't want K to be a point of a countable collection of points.

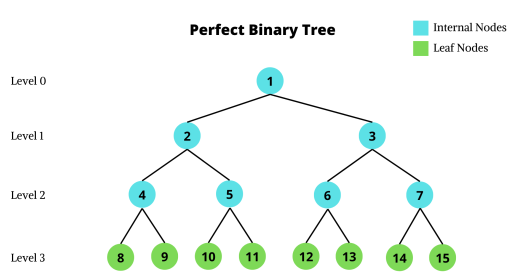
(2) To proof the first part, we use neighborhood recurrent to show $\mathbb{P}^z\{T < \infty\} = 1$ then we use a Martingale argument, for which we'll skip now and leave to the last few weeks.

(3) For the last expression, we can view it as if we are considering $\partial D = \partial K \cup \{\infty\}$. This is natural because we do so on the complex plane, and harmonic is just holomorphic if we stood far enough. But still we need to justify that.

The rough justification is that for $\mathbb{R}^d$ we can always draw larger and larger balls of radius r^n and as we go further and further, we starts at the same point, roughly, and so each point on the circle is treated the same. In other words, $\{\infty\}$ is really an equivalence class on its own.

That's really sloppy and needs justification that we'll not do.

On the last point, what we can do is to give an example that the infinity is not a single point. Consider the binary tree with random walk on the tree:



each point is connected to 3 branches(except the initial point and we ignore) so

$$\mathbb{P}(\text{increase}) = \frac{2}{3}$$

each step and the process is transcient. But the size of the infinities is continuum and structures on $[0, 1]$ is really not isomorphic with that on a point.

8. 10/19: Brownian motion and Green's Function

The semigroup of operators is defined for standard brownian motion B_t with

$$P_t(x,y) = \text{"transition probability"} = \text{density of } B_t \text{ in } y$$

which is the probability of time t at y if starts at x.

If $B_0 = x$ then

$$P_t(x,y) = \frac{1}{(2\pi t)^{d/2}} e^{-|y-x|^2/2t}$$

and define operator

$$P_t f(x) = \mathbb{E}^x[f(B_t)] = \int_{\mathbb{R}^d} f(y) P_t(x,y) dy$$

and this forms a semigroup of operators since

$$P_{t+s}(x,y) = \int_{\mathbb{R}^d} P_t(x,z) P_s(z,y) dz$$

in the sense that $P_{t+s} f = P_t[P_s f](x)$ and indeed we have $P_0 = e$.

Def 8.1. *The infinitesmal generator is defined by*

$$Af(x) := \lim_{t \downarrow 0} \frac{P_t f(x) - f(x)}{t}.$$

Proposition 8.2. *If f is C^2, then for standard Brownian Motion we have*

$$Af(x) = \frac{1}{2} \Delta f(x).$$

Note that the above implies

$$f(t,x) = P_t(x,y) \Rightarrow f(t,x) = P_t(y,x) \Rightarrow \frac{\partial}{\partial t} f(t,x) = \frac{1}{2} \Delta f(t,x)$$

Proof. WLOG let $x = 0, f(0) = 0$ with shift. Then Taylor yields

$$f(x) = \sum_{j=1}^d b_j x_j + \frac{1}{2} \sum_{j=1}^d \sum_{k=1}^d a_{jk} x_j x_k + o(|x|^2)$$

where $b_j = \partial_j f(0), a_{jk} = \partial^2_{jk} f(0)$. For $B_t = (B_t^1, \dots, B_t^d)$ we have

$$f(B_t) = \sum_{j=1}^d b_j B_t^j + \frac{1}{2} \sum_{j=1}^d \sum_{k=1}^d a_{jk} B_t^j B_t^k + o(|B_t|^2)$$

drift $\mu : (\mu_1, \dots, \mu_d) \in \mathbb{R}^d$

$$\mathbb{E}[f(B_t)] = \sum_{j=1}^d b_j \mathbb{E}[B_t^j] + \frac{1}{2} \sum_j \sum_k a_{jk} \mathbb{E}[B_t^j B_t^k] + o(|B_t|^2) = \sum_{j=1}^d b_j t \mu_j + \frac{1}{2} \sum_j \sum_k a_{jk} \Gamma_{jk} t + o(t^2)$$

thus

$$Af(x) = \sum_{j=1}^{d} \frac{\partial f}{\partial x_j}(0)\mu_j + \frac{1}{2}\sum_j \sum_k \frac{\partial^2 f}{\partial_j \partial_k}(0)\Gamma_{jk} + o\left(\frac{t^2}{t}\right)$$

and in particular if $\Gamma_{ij} = Id$, then and the above is just $\frac{1}{2}\Delta f(0)$. And use dot notation for ∂_t we have

$$\dot{f}(t,x) = \mu \cdot \nabla f(x) + \frac{1}{2}\sum \frac{\partial^2 f}{\partial_j \partial_k}(x)\Gamma_{jk}.$$

$\square$

Working harder we get

$$P_t(x,y) = \frac{1}{\sqrt{2\pi \det \Gamma}} \exp\left\{\frac{(y-x-t\mu)^T \Gamma^{-1}(y-x-t\mu)}{2t}\right\}.$$

Note that if we fix y, and view it as a function of x then it's the same but $\mu \mapsto -\mu$.

Heat equation in $\mathbb{R}^d$: we want to find function satisfying

- $\dot{u}(t,x) = \frac{1}{2}\Delta u(t,x)$, $t > 0$.
- $u(0,x) = F(x)$.
- u is continuous on $[0,\infty) \times \mathbb{R}^d$.

And we can understand the solution as

$$u(t,x) = \mathbb{E}^x[F(B_0)] = \int_{\mathbb{R}^d} P_t(x,y)F(y)dy.$$

The easier proof is just check. The particular point is to need F good enough to exchange ∂_t and $\int$.

Now, for a bounded domain, define $P_t^D(x,y)$ is the density of B_t (in y) for Brownian motion starting at x, and killed when it hit ∂D. In particular

$$\int_D P_t^D(x,y)dy = P^x\{T > t\}$$

where $T = T_D = \min\{s : B_s \in \partial D\}$. This could be understood as

$$\begin{aligned} P_t^D(x,y) &= P_t(x,y) - \text{"probability at y at time t and left D"} \\ &= P_t(x,y) - \mathbb{E}^x\left[\mathbb{1}_{T<t} P_{t-T}(B_T, y)\right] \end{aligned}$$

Now we consider Heat equation in D:

- Initial function $u_0 : D \to \mathbb{R}$;
- Boundary condition: F on ∂D, $u(t,x), t \geq 0, x \in \bar{D}$.

in other words

- $u(t,x) = F(x), x \in \partial D$.

- $\dot{u}(t,x) = \dfrac{1}{2}\Delta u(t,x), x \in D, t > 0.$
- $u(0,x) = u_0(x), x \in D.$

Then

$$u_t(x) = \mathbb{E}^x\left[\mathbb{1}_{T\le t}F(B_t) + u_0(B_t)\mathbb{1}_{T>t}\right]$$

as $t \to 0$ and $u_t \to f$ where f is the harmonic equation $f = \mathbb{E}^x[F(B_T)]$.

Now, green's function Brownian Motion in $\mathbb{R}^d$ for $d \ge 3$ (since we need transient) is such that

$$\begin{aligned} G(x,y) &= \text{"expected amount of time spent at y starting x"} \\ &= \int_0^\infty P_t(x,y)dt = G(y,x) = G(0,y-x) \end{aligned}$$

Now

$$G(x) := G(0,x) = \int_0^\infty \frac{1}{(2\pi t)^{d/2}} e^{\frac{-|x|^2}{2t}} dt$$

as $t \to \infty$ the above term goes to $\sim \frac{1}{t^{d/2}}$ is integrable at infinity, and as $t \to 0$ it diverges if $x = 0$ but it's fine if $x \neq 0$. And do integral we get

$$G(x) = C_d|x|^{2-d}$$

where $C_d := \dfrac{\Gamma\left(\frac{d}{2}-1\right)}{2\pi^{d/2}}$.

Here $G(x,y)$ is harmonic in x for $x \in \mathbb{R}^d \setminus \{y\}$.

Now for $d = 2$, we can define the potential kernel (Green's function) by

$$a(x) = \lim_{T\to\infty}\left[-\int_0^T P_t(0,x)dt + \int_0^T P_t(0,e_1)dt\right]$$

for $x \neq 0$ the above

$$= \lim_{T\to\infty}\left[\int_0^T -P_t(0,x) + P_t(0,e_1)dt\right] = \int_0^\infty -P_t(0,x) + P_t(0,e_1)dt$$

and to check that its well defined

$$\frac{1}{2\pi t}\left[e^{-\frac{|e_1|^2}{2t}} - e^{-\frac{|x|^2}{2t}}\right]$$

and fix x then let $t \to \infty$ then taylor cancels the first order term and the order is $\frac{1}{t^2}$ and we can compute $a(x) = \dfrac{1}{\pi}\log|x|$.

Green's function for such D:

$$G_D(x,y) = \int_0^\infty P_t^D(x,y)dt$$

for $d \geq 3$

$$G_D(x, y) = G(x, y) - \mathbb{E}^x[G(B_t, y)]$$

and for $d = 2$

$$G_d(x, y) = \mathbb{E}^x[a(B_T, y)] - a(x, y)$$

where $a(x, y) = a(y - x)$..

For fixed y, the function $h(x) = G_D(x, y)$ is harmonic in $D\backslash\{y\}$ as $x \to y$ and

$$G_D(x, y) \sim \begin{cases} G(x, y) & d \geq 3 \\ G(x, y) - a(x, y) & d = 2 \end{cases}$$

and $G_D(x, y) = G_D(y, x)$, which is a simple graph proof.

9. 10/24: Stochastic Integration

To begin, we are interested in the following SDE:

$$\frac{d}{dt}F(t) = C(t, F(t))$$

which we can of course formally move the dt to right side and see what we get. To do so we try to clarify things.

We want to say that at time t, X_t acts like a Brownian motion with drift R_t and variance parameter A_t^2, which is

$$dX_t = R_t dt + A_t dB_t \Rightarrow X_t = \int_0^t R_s ds + \int_0^t A_s dB_s.$$

But what is dB_t really? To start make sense of things we start from simple processes, just like we start measurability from simple functions.

Def 9.1. *A_t is a simple process with repsect to $\{\mathcal{F}_t\}$ if there exists a finite number of times $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = \infty$ and random variables $Y_0, Y_1, \ldots, Y_n$ where $Y_j \in \mathcal{F}_{t_j}$, and $A_t = Y_j$ for $t_j \leq t < t_{j+1}$.*

We call a simple process L^2 or bounded if Y_j has the corresponding property.

Now if A_t is a simple process, then we can define

$$Z_t = \int_0^t A_s dB_s$$

for $t_j \leq t < t_{j+1}$. And in particular we can write so

$$Z_t = \sum_{k=0}^{j-1} Y_k[B_{t_{k+1}} - B_t] + Y_j[B_t - B_{t_j}]$$

in the same interval.

We now introduce some properties of stochastic integral.

Linearity: if A_t, C_t are simple and $a, b \in R$, then $K_t := aA_t + bC_t$ is simple and

$$\int_0^t aA_t + bC_t(dB_t) = a\int_0^t A_t dB_t + b\int_0^t C_t dB_t.$$

Z_t is a Martingale (note that here we've implicitly have bounded): we need to show $\mathbb{E}(Z_t|\mathcal{F}_s) = Z_s$ and we can WLOG assume $s = t_j, t = t_k, j < k$. Then

$$\mathbb{E}[Z_{t_k}|\mathcal{F}_{t_j}] = \mathbb{E}\left[\sum_{i=0}^{k-1} Y_i(B_{t_{i+1}} - B_{t_i})\middle|\mathcal{F}_{t_j}\right] = Z_{t_j} + \sum_{i=j}^{k-1}\mathbb{E}\left[Y_i(B_{t_{i+1}} - B_{t_i})\middle|\mathcal{F}_{t_j}\right]$$

$$= Z_{t_j} + \sum_{i=j}^{k-1}\mathbb{E}\left[\mathbb{E}\left[Y_i(B_{t_{i+1}} - B_{t_i})\middle|\mathcal{F}_{t_i}\right]\middle|\mathcal{F}_{t_j}\right] = Z_{t_j} + \sum_{i=j}^{k-1}\mathbb{E}\left[Y_i \cdot 0\middle|\mathcal{F}_{t_j}\right] = Z_{t_j}$$

is indeed a martingale.

Proposition 9.2. *(Variance rule/Ito isometry) If A_t is a L^2 simple process, then*

$$\mathrm{Var}(Z_t) = \mathbb{E}[Z_t^2] = \int_0^t \mathbb{E}[A_s^2]ds = \mathbb{E}\left[\int_0^t A_s^2 ds\right].$$

Proof. WLOG $t = t_k$, just compute we have

$$Z_t^2 = \sum_{j=0}^{k-1} Y_j^2 (B_{t_{j+1}} - B_{t_j})^2 + 2\sum_{j<i} Y_i Y_j (B_{t_{j+1}} - B_{t_j})(B_{t_{i+1}} - B_{t_i})$$

and take expectation on both sides, the second term is (say $j < i$)

$$\begin{aligned}\mathbb{E}\left[Y_i Y_j (B_{t_{j+1}} - B_{t_j})(B_{t_{i+1}} - B_{t_i})\right] &= \mathbb{E}\left[\mathbb{E}\left[Y_i Y_j (B_{t_{j+1}} - B_{t_j})(B_{t_{i+1}} - B_{t_i})\middle|\mathcal{F}_{t_i}\right]\right] \\ &= \mathbb{E}\left[Y_i Y_j (B_{t_{j+1}} - B_{t_j})\mathbb{E}\left[(B_{t_{i+1}} - B_{t_i})\middle|\mathcal{F}_{t_i}\right]\right] = 0\end{aligned}$$

and for the other term

$$\mathbb{E}\left[Y_j^2 (B_{t_{j+1}} - B_{t_j})^2\right] = \mathbb{E}\left[Y_j^2 \mathbb{E}\left[(B_{t_{j+1}} - B_{t_j})^2\middle|\mathcal{F}_{t_i}\right]\right] = \mathbb{E}\left[Y_j^2\right](t_{j+1} - t_j) = 0$$

and hence

$$\mathbb{E}[Z_t^2] = \sum_{j=0}^{k-1} \mathbb{E}[Y_j^2](t_{j+1} - t_j) = \int_0^{t_k} \mathbb{E}[A_s^2]ds.$$

□

Now we start to do the limit thing. Define A_t which is bounded continuous process adapted to the filtration $\{\mathcal{F}_t\}$ and $|A_t| \le K$. We want to define

$$\int_0^k A_s dB_s \sim \lim_{n\to\infty} \int_0^t A_s^{(n)} dB_s$$

for $A_s^{(n)}$ simple and approaching A_s. But we need to really agree on what approaching means.

Lemma 9.3. $\forall t$*, we can find a sequence of simple process $A_t^{(n)}$ with $|A_t^{(n)}| \le K$ such that*

$$\lim_{n\to\infty} \int_0^t \mathbb{E}\left[(A_s^{(n)} - A_s)^2\right] ds = 0.$$

That is, the integral of L^2 converges.

Proof. For ease take $t = 1$ and let

$$A_t^{(n)} := n\int_{\frac{k-1}{n}}^{\frac{k}{n}} A_s ds, \qquad \frac{k}{n} \le t \le \frac{k+1}{n}$$

thus $A_t^{(n)}$ is simple and we can see that $A_t^{(n)} \to A_t$ pointwise.

Now, by bounded convergence and the fact that both A_s and $A_t^{(n)}$ are bounded we can get the limit

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^1 (A_s^{(n)} - A_s)^2 ds\right] = 0.$$

Change the limit and we are done. □

Since $A_t^{(n)} \to A_t$ in the above sense, let's see what happen to $Z_t^{(n)}$. We of course define

$$Z_t^{(n)} = \int_0^t A_s^{(n)} dB_s$$

and hence

$$Z_t^{(n)} - Z_t^{(m)} = \int_0^t A_s^{(n)} - A_s^{(n)} dB_s$$

where we take expectation and apply variance rule to get

$$\mathbb{E}[Z_t^{(n)} - Z_t^{(m)}] = \int_0^t \mathbb{E}[(A_s^{(n)} - A_s^{(n)})^2] ds$$

By lemma $\{Z_t^{(n)}\}$ is Cauchy and so there exists a Z_t as L^2 limit of $Z_t^{(n)}$. To get over the only L^2 convergence (not continuous) we define $Z_t^{(n)}$ for the dyatic t as above and prove the estimate (homework):

$$\mathbb{E}[|Z_t - Z_s|^4] \leq C|t-s|^2$$

and we'll show that this gives on the dyatic points, and the whole continuity follows. From L^2 limit, the variance rule is passed easily.

It satisfies

- Linearity;
- Martingale;
- Variance estimate;
- Continuity.

And to ease up on continuity proof we note the progressive measurability such that $A_s(w)$ is measurable on the product measure $\Omega \times [0,t]$. And we basically repeat the above argument once again.

Now what we've done is for a bounded function, what about just any?

Well, Let $T := \min\{|A_s| \geq n\}$ and let $A_s^{(n)} = A_{s\wedge T_n}$ and now everything's bounded we use above argument. Moreover, if $n > \max_{0\leq s\leq t} |A_s|$ we have already that

$$Z_t^{(n)} := \int_0^t A_s^{(n)} dB_s = \int_0^t A_s dB_s$$

and for others we can always take limit:

$$\lim_{n\to\infty} \int_0^t A_s^{(n)} dB_s = \int_0^t A_s dB_s$$

where the limit is actually trivial since it's eventually a constant.

Here it has:

- Linearity;
- NOT necessarily a martingale.
- Variance estimate: $\mathrm{Var}(Z_t) = \int_0^t \mathbb{E}[A_s^2]ds$ with posibility that it being ∞.
- Continuity.

10. 10/26: QUADRATIC VARIATION AND ITO'S FOMULA

10.1. Quadratic Variation.

Proposition 10.1. *(Maximal L^2 inequality for Martingale) If M_t is a continuous L^2 Martingale with $M_0 = 0$, then* $\forall \varepsilon > 0$

$$\mathbb{P}\left(\max_{0\le s\le t} |M_s| \ge \varepsilon\right) \le \frac{\mathbb{E}[M_t^2]}{\varepsilon^2}.$$

Some facts are:

- M_t^2 is a submartingale by Jensen's inequality:

$$\mathbb{E}[M_t^2|\mathcal{F}_s] \ge \left[\mathbb{E}(M_t|\mathcal{F}_s)\right]^2 = M_s^2.$$

- To prove the above, it suffices to prove that $\forall n$

$$\mathbb{P}\left(\max_{k=0,1,\dots,n} \left|M_{\frac{k}{n}t}\right| \ge \varepsilon\right) \le \frac{\mathbb{E}[M_t^2]}{\varepsilon^2}$$

  and if we can do that, then for fixed n, let σ be the first k with $\left|M_{\frac{k}{n}t}\right| \ge \varepsilon$ which is the same as $M_{\frac{k}{n}t}^2 \ge \varepsilon^2$ and

$$\mathbb{E}[M_t^2 \mathbb{1}_{\sigma=k}] \ge \mathbb{E}\left[\left|M_{\frac{k}{n}t}\right|^2 \mathbb{1}_{\sigma=k}\right] \ge \varepsilon^2 \mathbb{P}(\sigma = k)$$

  where adding up we get

$$\mathbb{E}[M_t^2] \ge \mathbb{E}[M_t^2 \mathbb{1}_{\sigma\le n}] \ge \varepsilon^2 \mathbb{P}(\sigma \le n) = \varepsilon^2 \mathbb{P}\left(\max_{0\le s\le t} |M_s| \ge \varepsilon\right).$$

In the following we fix $t = 1$, and let A_s for $0 \le s \le 1$ be a continuous and adapted process with partition $\Pi^{(n)} := 0 = t_0 < t_1 < \cdots < t_n = 1$ and we define (here we use continuity to simplify the averaging from last time) $A_s^n := A_{t_n}$ for $t_n \le t < t_{n+1}$. Let $Z_t^n := \int_0^t A_s^n dB_s$ and $Z^n := \int_0^t A_s dB_s$, then say $|A_s| \le K$ then both are continuous Martingales, from which we know their difference $M_t := Z_t - Z_t^n$ is also a continuous Martingale which we use above proposition to get

$$\mathbb{P}\left(\max_{0\le s\le t} |M_s| \ge \varepsilon\right) \le \frac{\mathbb{E}[M_1^2]}{\varepsilon^2} = \frac{\int_0^1 \mathbb{E}[(A_t - A_t^n)^2]dt}{\varepsilon^2}.$$

The above could be formulated in the following:

Theorem 10.2. *For A_s continuous with $\int_0^1 \mathbb{E}[A_s^2]ds < \infty$ and $\Pi^{(n)}$ a partition of $[0,1]$ such that*

$$\sum_{n=1}^{\infty} \int_0^1 \mathbb{E}[|A_t - A_t^n|^2]dt < \infty$$

If $Y^{(n)} = \max_{0\leq t\leq 1} |Z_t - Z_t^{(n)}|$, then with probability 1 we know $T^{(n)} \to 0$.

Now, define $Z_t = \int_0^t A_s dB_s$, the quadratic variation

$$\langle Z_t \rangle = \int_0^t A_s^2 ds$$

which intuitively says that the total amount of randomness is decided by A_s. For an example, let's see for a specific case: $A_s = \sigma, A_s^2 = \sigma^2$, and $Z_t \sim \sigma B_s \Rightarrow \langle Z \rangle_t = \sigma^2 t$ and with a orthogonal argument for non-overlapping increments, we have

$$Z_t = Z_s + (Z_t - Z_s)$$

and we can separate Z_t into two pieces, then take product, then cancel terms.

But now we claim that $\langle Z \rangle_t$ is the unique increasing process such that $Z_t^2 - \langle Z \rangle$ is a Martingale, and it is also

$$\langle Z \rangle = \lim_{n\to\infty} \sum_{k<t2^n} \left[Z_{\frac{k}{2^n}} - Z_{\frac{k-1}{2^n}} \right]^n .$$

10.2. **Ito's formula.** For a start let's consider $\int_0^t B_s dB_s$. We might just guess it's $\frac{1}{2}(B_t^2 - B_0^2) = \frac{1}{2}B_t^2$. This is of course wrong because right hand side is not even a Martingale.

Ok maybe we can guess that if we minus the drift and make it a Martingale everything would work. Well that yields in $\frac{1}{2}B_t^2 - \frac{t}{2}$, and that is indeed the result, but that was sort of just luck. We'll see what's really going on now.

Theorem 10.3. *Suppose $f \in C^2$ function $f : \mathbb{R} \to \mathbb{R}$ and B_t is standard Brownian Motion, then*

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds.$$

Proof. First we assume that f has compact support. This gives boundedness of f, f', f''. Now, say $x < y$, then

$$\frac{1}{2}(y-x)^2 \min_{x\leq z\leq y} f''(z) \leq f(y) - f(x) - (y-x)f'(x) \leq \frac{1}{2}(y-x)^2 \max_{x\leq z\leq y} f''(z)$$

Now for ease, we assume that $t = 1$ and $f(0) = 0$ WLOG, then we see that

$$f(B_1) = \sum_{j=1}^{n} \left[f\left(B_{\frac{j}{n}}\right) - f\left(B_{\frac{j-1}{n}}\right) \right]$$

is just a oscillating sum. And applying the Taylor approximation above to each intermediate term we get that for each n

$$A + B^- \le \sum_{j=1}^{n} \left[f\left(B_{\frac{j}{n}}\right) - f\left(B_{\frac{j-1}{n}}\right) \right] \le A + B^+$$

where

$$A := \sum_{j=1}^{n} f'\left(B_{\frac{j-1}{n}}\right) \left[B_{\frac{j}{n}} - B_{\frac{j-1}{n}} \right]$$

and

$$B^- := \frac{1}{2} \sum_{j=1}^{n} \left(B_{\frac{j}{n}} - B_{\frac{j-1}{n}} \right)^2 \min_{z \in \left[B_{\frac{j-1}{n}}, B_{\frac{j}{n}} \right]} f''(z); \quad B^+ := \frac{1}{2} \sum_{j=1}^{n} \left(B_{\frac{j}{n}} - B_{\frac{j-1}{n}} \right)^2 \max_{z \in \left[B_{\frac{j-1}{n}}, B_{\frac{j}{n}} \right]} f''(z)$$

And as $n \to \infty$ (or to be careful, $n_j \to \infty$) we have

$$\lim_{n \to \infty} A = \int_0^t f'(B_s) dB_s$$

since A is essentially a simple process approximation, and limit follows from last time this converges to exactly the right hand side.

For $B^{\pm}$, we first get $f''(z)^2 \to f''\left(B_{\frac{j-1}{n}}\right)$ again by continuity, and we will get a fixed ε that works for this purpose, then we take $n \to \infty$ such that within the small slip $[s, s+\varepsilon]$ the leftover sum goes to the variation on that slip, which is ε. Now we use Stieltjes integral and conclude:

$$B^{\pm} \to \frac{1}{2} \int_0^1 f''(B_t) dt.$$

Now we extend to non-compact f: we just take $f_n \in C^2$ supported on $[-n, n]$, then

$$f_n(B_t) - f_n(B_0) = \int_0^t f_n'(B_s) dB_s + \frac{1}{2} \int_0^t f_n''(B_s) ds$$

We know by continuity that for each ω, $\exists n$ such that $|B_s(\omega)| \le n$, $\forall s \le t$, and hence for all large n there's no matter at all. But we're only doing it till time t whatever, and this means what happens outside does not matter at all since we won't apply f to those points, thus the limit is carried over and we truly have

$$f(B_t) - f(B_0) = \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t f''(B_s) ds.$$

$\square$

Now for convenience we don't really write the whole formula, but rather use the following notation to denote the exact formula above. For the equation

$$X_t = X_0 + \int_0^t R_s ds + \int_0^t A_s dB_s$$

we denote it by

$$dX_t = R_t dt + A_t dB_t$$

and hence the above theorem says

$$d[f(B_t)] = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt.$$

Now we state and show one version of Ito's formula:

Theorem 10.4. *Suppose $f(t,x)$ is a function $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, which is C^1 in t and C_2 in x, then*

$$f(t,B_t) - f(0,B_0) = \int_0^t \partial_s f(s,B_s)ds + \int_0^t \partial_x f(s,B_s)dB_s + \frac{1}{2}\int_0^t \partial^2_{xx} f(s,B_s)ds$$

that is

$$d[f(t,B_t)] = \left[\dot{f}(t,B_t) + \frac{1}{2}f''(t,B_t)\right]dt + f'(t,B_t)dB_t.$$

Here of course we use dot for derivative in time and prime in space.

Proof. The rest is actually very simple. We let $t = 1$ for ease again and write

$$f(1,B_1) - f(0,B_0) = \sum_{j=1}^n \left(f\left(\frac{j}{n}, B_{\frac{j}{n}}\right) - f\left(\frac{j-1}{n}, B_{\frac{j-1}{n}}\right)\right)$$

$$= \sum_{j=1}^n \left(f\left(\frac{j}{n}, B_{\frac{j}{n}}\right) - f\left(\frac{j-1}{n}, B_{\frac{j}{n}}\right)\right) + \sum_{j=1}^n \left(f\left(\frac{j-1}{n}, B_{\frac{j}{n}}\right) - f\left(\frac{j-1}{n}, B_{\frac{j-1}{n}}\right)\right)$$

where using the above theorem we already get two terms in x derivative from the second summation.

For the first summation, we note by Taylor that

$$\sum_{j=1}^n \left(f\left(\frac{j}{n}, B_{\frac{j}{n}}\right) - f\left(\frac{j-1}{n}, B_{\frac{j}{n}}\right)\right) = \sum_{j=1}^n \dot{f}\left(\frac{j-1}{n}, B_{\frac{j}{n}}\right)\frac{1}{n} + O(n \cdot n^{-2}) \to \int_0^1 \dot{f}(s,B_s)ds.$$

□

Example 10.5.

We look at a simple case of $f(s,x) = e^{as}e^{bx}$, then we have

- $\dot{f}(s,x) = af(s,x)$;
- $f'(s,x) = bf(s,x)$;

- $f''(s,x) = b^2 f(s,x)$.

Plugging in we have

$$df(t,B_t) = \left[af(t,B_t) + \frac{b^2}{2}f(t,B_t)\right]dt + bf(t,B_t)dB_t.$$

This means that if we have the SDE for X_t with $X_0 = 1$ of the form

$$dX_t = \left(a + \frac{b^2}{2}\right)X_t dt + bX_t dB_t$$

and in more particular we want to solve

$$dX_t = mX_t dB_t$$

then we just let $b = m$ and $a + \frac{b^2}{2} = 0$, then the solution is just

$$X_t = e^{mB_t - \frac{m^2}{2}t}.$$

Example 10.6.

An easier example is such that $f(x) = x^2$ then $f' = 2x$ and $f'' = 2$ then

$$B_t^2 - B_0^2 = \int_0^t 2B_s dB_s + \int_0^t 2ds = 2\int_0^t B_s dB_s + 2t$$

and this explains why it's lucky: it's just a matter of integration.

Now we generalize a little more.

Theorem 10.7. *Suppose f is C^2 on (a,b), B_t is Brownian motion with $a < B_0 < b$ and*

$$T = \min\{t : B_t = a \text{ or } b\}$$

then if $t < T$ we have

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds.$$

The proof is just taking f_n to be smoothened down from $a + \frac{1}{n}$ and also on right side. The reason we need to do it is f might blow up at endpoints.

11. 10/31: Diffusions, generator of solutions, multivariate stochastic integral

We will use the natural expression from Ito's formula and say

$$\int_0^t Y_s dX_s = \int_0^t Y_s R_s ds + \int_0^t Y_s A_s dB_s$$

and

$$\langle x \rangle_t := \lim_{n\to\infty} \sum_{j<tn} \left[X\left(\frac{j}{n}\right) - X\left(\frac{j-1}{n}\right) \right]^2$$

And Heuristically we have

$$X\left(\frac{j}{n}\right) - X\left(\frac{j-1}{n}\right) \approx R\left(\frac{j}{n}\right)\frac{1}{n} + A\left(\frac{j}{n}\right)\left[B\left(\frac{j}{n}\right) - B\left(\frac{j-1}{n}\right)\right]$$

taking squares that correspond to

$$\left[X\left(\frac{j}{n}\right) - X\left(\frac{j-1}{n}\right)\right]^2 \approx R^2\frac{1}{n^2} + 2RA\frac{1}{n}dB + A^2(dB)^2$$

which sums up, with $n \to \infty$, to $\int_0^t A_s^2 ds$ since the first two terms goes to 0.

And so we have the following expression of Ito, with proof exactly the same as that presented last time:

Proposition 11.1. *If $f(t,x)$ is C^1 in t and C^2 in x, then X_t satisfies*

$$f(t,X_t) - f(0,X_0) = \int_0^t \hat{f}(s,X_s)ds + \int_0^t f'(s,X_s)dX_s + \frac{1}{2}\int_0^t f''(s,X_s)A_s^2 ds$$

or alternatively we write

$$df(t,X_t) = \hat{f}(t,X_t)dt + f'(t,X_t)dX_t + \frac{1}{2}f''(t,X_t)d\langle X\rangle_t.$$

11.1. Diffusion and Generator.

Theorem 11.2. *Say we have*

$$dX_t = m(t,X_t)dt + \sigma(t,X_t)dB_t$$

where m, σ are deterministic, and let $x_0 = y_0$, and they are both uniformly Lipschitz in (t,x) with Lipschitz constant β, then there is a solution.

Proof. As usual we use Picard's iteration. Here assume $t = 1$ and for ease let's write $m(x,X_s) := m(X_s)$ and same for σ.

We start with initial guess $X_t^{(0)} = y_0$. Then we recursively guess with

$$X_t^{(k+1)} = y_0 + \int_0^t m(X_s)ds + \int_0^t \sigma(s,X_s)dB_s$$

and our goal is to show $X_t := \lim_{n\to\infty} X_t^{(n)}$ exists in L^2 sense.

We compute

$$\mathbb{E}\left[|X_t^{(k+1)} - X_t^{(k+1)}|^2\right] = \mathbb{E}\left[\left|\int_0^t m(X_s^{(k)}) - m(X_s^{(k-1)})ds + \int_0^t \sigma(X_s^{(k)}) - \sigma(X_s^{(k-1)})dB_s\right|\right]$$

$$\leq 2\left[\mathbb{E}\left[\left(\int_0^t m(X_s^{(k)}) - m(X_s^{(k-1)})ds\right)^2\right] + \mathbb{E}\left[\left(\int_0^t \sigma(X_s^{(k)}) - \sigma(X_s^{(k-1)})ds\right)^2\right]\right] =: A + B$$

where we bound one by one. For B we use the variance rule to get

$$\begin{aligned} B &= \int_0^t \mathbb{E}\left[\left(\sigma(X_s^{(k)}) - \sigma(X_s^{(k-1)})ds\right)^2\right] ds \\ &= \int_0^t \mathbb{E}\left[\beta^2(X_s^{(k)} - X_s^{(k-1)})^2\right] ds \leq \beta^2 \int_0^t \mathbb{E}\left[(X_s^{(k)} - X_s^{(k-1)})^2\right] ds \end{aligned}$$

and

$$A \leq \mathbb{E}\left[\left(\int_0^t \beta|X_s^{(k)} - X_s^{(k-1)}|ds\right)^2\right] \leq \beta^2\mathbb{E}\left[\int_0^t |X_s^{(k)} - X_s^{(k-1)}|^2 ds\right]$$

by Jensen's. So now we skip the hard work and claim that there is a constant C such that

$$\mathbb{E}[|X_t^{(k+1)} - X_t^{(k)}|^2] \leq C\int_0^t \mathbb{E}(|X_s^{(k)} - X_s^{(k-1)}|)$$

and hence there exist a $\lambda < \infty$ such that

$$\mathbb{E}[|X_t^{(k+1)} - X_t^{(k)}|^2] \leq \frac{\lambda^{k+1}t^{k+1}}{(k+1)!}$$

which gives a solution X_t in L^2 for each t that establishes continuity. $\square$

Now we introduce generators, we define

$$Lf(x) := \lim_{t\downarrow 0} \frac{\mathbb{E}^x[f(X_t)] - \mathbb{E}^x[f(X_0)]}{t} = \lim_{t\downarrow 0} \frac{\mathbb{E}^x[f(X_t) - f(x_0)]}{t}$$

or in other words the derivative of $\mathbb{E}^x[f(X_t)]$ to t at 0. Hence, Ito's formula yields

$$\begin{aligned} f(X_t) &= \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f'(X_s)d\langle x\rangle_s + f(x_0) \\ &= f(x_0) + \int_0^t f'(X_s)\sigma(X_s)dB_s + \int_0^t \left(f'(X_s)m(X_s) + \frac{1}{2}f''(X_s)\sigma^2(X_s)\right) ds \end{aligned}$$

taking expectation and assuming m, σ are locally bounded we have

$$\mathbb{E}^x\left[\int_0^t f'(X_s)\sigma(X_s)dB_t\right] = 0$$

because the inside is a stochastic integral, hence a Martingale, and the rest in $Lf(x)$ with a Taylor is

$$0 + t\left[f'(x_0)m(x_0) + \frac{1}{2}f''(x_0)\sigma^2(x_0)\right] + O(t^2)$$

and thus

$$Lf(x) = m(x)f'(x) + \frac{\sigma^2(x)}{2}f''(x)$$

which in other words say

$$\frac{d}{dt}u(t,x) = Lu(t,x).$$

Now we do product rule:

$$\begin{cases} dX_t = R_t dt + A_t dB_t \\ dY_t = S_t dt + C_t dB_t \end{cases}$$

and as normal we do calculation

$$\begin{aligned} &X(t+\Delta t)Y(t+\Delta t) - X(t)Y(t) \\ =&[X(t+\Delta t) - X(t)][Y(t+\Delta t) - Y(t)] + X(t)[Y(t+\Delta t) - Y(t)] + Y(t)[X(t+\Delta t) - X(t)] \end{aligned}$$

where the first term on right would disappear if X, Y are deterministic. For stochastic processes that is not the case, instead we keep that and define

Def 11.3. *The* <u>*covariation*</u> *of* X_t, Y_t *is*

$$\langle X_t, Y_t\rangle_t := \lim_{n\to\infty}\sum_{j\le tn}\left[X\left(\frac{j}{n}\right) - X\left(\frac{j-1}{n}\right)\right]\left[Y\left(\frac{j}{n}\right) - Y\left(\frac{j-1}{n}\right)\right].$$

and if we follow assumption we have

$$\langle X_t, Y_t\rangle_t = \int_0^t A_s C_s ds$$

and $d(X_tY_t) = X_t dY_t + Y_t dX_t + d\langle X, Y\rangle_t$.

11.2. **multi-dimension Stochastic integral.** Claim: if $j \neq k$, then $\langle B_t^j, B_t^k\rangle = 0$ whose reason is that for two random variables

$$\sum_{m<tn}\left[B^j_{\frac{m}{n}} - B^j_{\frac{m-1}{n}}\right]\left[B^k_{\frac{m}{n}} - B^k_{\frac{m-1}{n}}\right]$$

and as long as they are independent it is 0.

So we have

$$\begin{aligned} dX_t^{(1)} &= R_t^{(1)}dt + A_t^{(1,1)}dB_t^1 + \cdots + A_t^{(1,d)}dB_t^d \\ &\vdots \\ dX_t^{(n)} &= R_t^{(n)}dt + A_t^{(n,1)}dB_t^1 + \cdots + A_t^{(n,d)}dB_t^d \end{aligned}$$

and here

$$d\langle X_t^{(j)}, X_t^{(k)}\rangle = \sum_{l=1}^{d} \langle A_t^{(j,l)}, A_t^{(k,l)}\rangle$$

which really gives the general Ito formula for which the proof is the same so we'd not do it:

Theorem 11.4. *Suppose $f(T, \mathbb{R}^d) \to \mathbb{R}$ is C^1 in t and C^2 in x, then for $X_t = (X_t^{(1)}, \dots, X_t^{(n)})$*

$$df(t, X_t) = \dot{f}(t, X_t)dt + \sum_{j=1}^{n} \left[\partial_{x_j} f(t, X_t)\right] dX_t^{(j)} + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \partial_{jk}^2 f(t, X_t) d\langle X_t^{(j)}, X_t^{(k)}\rangle$$

We usually denote

$$\sum_{j=1}^{n} \left[\partial_{x_j} f(t, X_t)\right] dX_t^{(j)} = \nabla f(t, X_t) \cdot dX_t$$

And an example is when X_t is a d-dimensional Brownian Motion itself with $n = d$. Then

$$df(t, B_t) = \dot{f}(t, B_t)dt + \nabla f(t, X_t) \cdot dX_t + \frac{1}{2}\Delta f(t, B_t)dt$$

since the cross terms are gone.

Now we tell a little bit about continuous Martingale.

We know already a Stochastic integral might not be a Martingale (homework).

Def 11.5. *A process Z_t is a continuous local Martingale on $[0, T)$ open in $[0, \infty]$ (where T can be random) if $\exists$ sequence of stopping time τ_n which are increasing and such that with probability 1 $\lim_{n\to\infty} \tau_n = \tau$ and for each n, $M_t^{(n)} = Z_{t\wedge\tau_n}$ is a continuous Martingale.*

Note that Stochastic integrals are local Martingales with

$$\tau_n := \min\{t < T : |Z_t| \geq n\}.$$

12. 11/2: Bessel's Process and European Option

For $Z_t = \int_0^t A_s dB_s$, we know it is a local Martingale. We offer a counterexample that it is not a Martingale, which funnily is the Martingale strategy.

Example 12.1.

We let $A_t = 1$ for $0 \le t \le \frac{1}{2}$, and we stop when $Z_{1/2} \ge 1$, i.e. A_s afterwards, then we can compute $\mathbb{P}\left(Z_{\frac{1}{2}} \ge 1\right) = \rho$, and now for each later time that is of the form $1 - 1/2^k$ we can find some A_s such that

$$\mathbb{P} Z_{3/4} \ge 1 | Z_{1/2} = \rho$$

so we see that the stopping time has probability 1 less than 1, and hence Z_s is not a Martingale.

In any case, the relation

$$\mathbb{E}[Z_t^2] = \int_0^t \mathbb{E}[A_s^2] ds$$

holds, but it could be infinity. If it is indeed less than ∞ then we know Z is a L^2 Martingale. We can't conclude for it diverge.

Example 12.2. *Bessel's Process*

Suppose

$$dX_t = \frac{a}{X_t} dt + dB_t$$

or in other words this is a diffusion for $m(x) = \frac{a}{x}, \sigma(x) = 1$.

Since the equation blows up at 0 we assume $X_0 = x_0 > 0$, and define

$$T := \min\{t : X_t = 0\}; \quad T_\varepsilon := \min\{t : X_t = \varepsilon\}$$

but note that m, σ are then uniformly Lipschitz for $t \le T_\varepsilon$, hence has a solution and thus as $\varepsilon \to 0$ we know for each $0 \le t < T$ has a solution.

The usual setting is $a = \dfrac{d-1}{2}$ for the system a d dimensional Bessel process. And we sometimes call $\nu = a - \frac{1}{2}$ the index.

Now consider the process of trying to stop when we meet r or R, i.e. $\tau := \min\{t : X_t \in \{r, R\}\}$ and we define $\phi(x) := \mathbb{P}^x\{X_\tau = R\}$, and

$$M_t = \mathbb{P}^x\{X_\tau = R | \mathcal{F}_t\} = \mathbb{E}^x[\mathbb{1}_{\{X_\tau = R\} | \mathcal{F}_t}]$$

and observe that M_t is a bounded Martingale, since it is just an expectation conditioned to a σ-algebra which is even UI. We also note that by definition $M_t = \phi(M_{t \cap \tau})$ and note that

$$dX_t = m(X_t)dt + \sigma(X_t)dB_t$$

already implies that X_t has Markov property.

Now we want to use Ito's formula, but we need to be careful since we need C^2. So let's first assume C^2 and find a solution. For $r < x < R$ we have

$$d\phi(X_t) = \phi'(X_t)dX_t + \frac{1}{2}\phi''(X_t)d\langle X\rangle_t = \phi'(X_t)\left[\frac{a}{X_t}dt + dB_t\right] + \frac{1}{2}\phi''(X_t)dB_t$$
$$= \left[\phi'(X_t)\frac{a}{X_t} + \frac{1}{2}\phi''(X_t)\right]dt + \phi'(X_t)dB_t$$

And if we think for a bit, for the equation

$$dX_t = R_t dt + A_t dB_t$$

we notice that if R_t is non-zero, then there is a drift, then not martingale, so we need $R_t \equiv 0$ and hence the equation we must satisfy is

$$\phi'(x)\frac{a}{x} + \frac{1}{2}\phi''(x) = 0$$

and to solve which is elementary, we let $g(x) = \phi'$ and get $g(x) = cx^{-2a}$ and solving everything yields

$$\phi(x) = \begin{cases} \frac{x^{1-2a}-r^{1-2a}}{R^{1-2a}-r^{1-2a}} & a \neq \frac{1}{2} \\ \frac{\log x - \log r}{\log R - \log r} & a = \frac{1}{2} \end{cases}$$

which is C^2.

Now we start from beginning and see that this is indeed the solution. Just for the above ϕ, we can use Ito's formula and get that

$$d\phi(X_t) = \phi'(X_t)dB_t, \quad t < T$$

from which we know that $\phi(X_{t\wedge\tau})$ is a continuous local Martingale since it is the stochastic integral of

$$A_t = \begin{cases} \phi'(X_t) & t < T \\ 0 & t \geq T \end{cases}$$

and by definition the function $\phi(X_{t\wedge T})$ is a probability so it is bounded, now we can use optimal stopping theorem and get

$$M_0 = \phi(x) = \mathbb{E}^x[\phi(X_\tau)] = \mathbb{P}^x\{X_\tau = R\}\phi(R) + \mathbb{P}^x\{X_\tau = r\}\phi(r) = \mathbb{P}^x\{X_t = R\}$$

which now shows that the solution can only be this ϕ, since we've written out its distribution.

Corollary 12.3. *If X satisfies*

$$dX_t = \frac{a}{X_t}dt + dB_t$$

and $T := \min\{t : X_t = 0\}$, then with probability 1

$$\begin{cases} T = \infty & a \geq \frac{1}{2} \\ T < \infty & a < \frac{1}{2} \end{cases}$$

Another fact is that suppose $B_t = (B_t^1, \dots, B_t^d)$ is d dimensional standard Brownian Motion and

$$X_t = |B_t| = \sqrt{(B_t^1)^2 + \cdots + (B_t^d)^2}$$

then there exists a standard Brownian motion W_t such that in fact

$$dX_t = \frac{a}{X_t} dt + dW_t$$

with $a = \frac{d-1}{2}$ and thus we see this for $d = 1$ is correct.

If $a = \dfrac{1}{2}$ then with probability 1, $\inf_{t>0} X_t = 0$ and for $a \geq \frac{1}{2}$ with probability 1 $X_t \to \infty$.

Example 12.4. *Feymann-Kac Formula*

Suppose

$$dX_t = mX_t dt + \sigma X_t dB_t$$

or in other words we let $m(x) = mx$ and $\sigma(x) = \sigma x$.

Now we consider European Option. For fixed time $T > 0$ and fixed price S. Option is at time T we can by a share of stock at price S so the value of option at T is $F(X_t)$ where

$$F(x) = (x - s)_+ = \begin{cases} x - s & x > s \\ 0 & s \geq x \end{cases}$$

and $\phi(t, x) =$ value of option at time $t < T$ for which we define it to be

$$\phi(t, x) = \mathbb{E}^x \left[e^{-r(T-t)} F(X_T) | X_t = x \right]$$

where the r term is the inflation rate. Now we want to know what PDE does $\phi(t, x)$ satisfy?

We assume

$$dX_t = m(t, X_t)dt + \sigma(t, X_t)dB_t$$

$$dR_t = r(t, X_t)R_t dt$$

and in other words

$$R_t = R_0 \exp\left\{ \int_0^t r(s, X_s)ds \right\}$$

$$\phi(t, x) = \mathbb{E}\left[\exp\left\{ \int_0^t r(s, X_s)ds \right\} F(X_T) \middle| X_t = x \right]$$

and we can compute

$$M_t := \mathbb{E}[R_T^{-1} F(X_T) | \mathcal{F}_t] = R_t^{-1} \mathbb{E}\left[\exp\left\{ \int_0^t r(s, X_s)ds \right\} F(X_T) \middle| \mathcal{F}_t \right] = R + t^{-1}\phi(t, X_t)$$

and note that the key is the martingale, since it is just a thing conditioned on $\mathcal{F}_t$.

Now we try to use Ito, for which we need to assume ϕ to be C^1 in t and C^2 in x. We use Ito to get

$$\begin{aligned} d\phi &= \dot{\phi}dt + \phi' dX_t + \frac{1}{2}\phi'' d\langle X\rangle_t = \dot{\phi}dt + \phi'[mdt + \sigma dB_t] + \frac{1}{2}\phi''\sigma dt \\ &= \left[\dot{\phi} + m\phi' + \frac{1}{2}\phi''\sigma^2\right] dt + \sigma\phi' dB_t \end{aligned}$$

and since $dR_t = rR_t dt$ and elementary calculus reveals $dR_t^{-1} = -rR_t^{-1}dt$ and hence product rule says

$$dM_t = dR_t^{-1}\phi(t, X_t) + R_t^{-1}d\phi + d\langle R_t^{-1}, \phi\rangle$$

where the last term here is 0 since $dR_t^{-1} \sim dt$. SO we directly compute

$$dM_t = R_t^{-1}\left[\left(-r\phi + \dot{\phi} + m\phi' + \frac{1}{2}\sigma^2\phi''\right) dt + \sigma\phi' dB_t\right]$$

which we get that if this is a solution the equation it satisfies is

$$\dot{\phi}(t, X_t) = r\phi(t, X_t) - m(t, X_t)\phi'(t, X_t) - \frac{1}{2}\sigma^2(t, X_t)\phi''(t, X_t).$$

We state the theorem (we can do better than this but for now let's just state regularity).

Theorem 12.5. *Suppose X_t satisfies*

$$dX_t = m(t, X_t)dt + \sigma(t, X_t)dB_t$$

and $X_0 = x_0$, suppose the payoff F exists at time $T > 0$ with $\mathbb{E}^{x_0}[|F(X_T)|] < \infty$, suppose

$$\phi(t, x) = \mathbb{E}\left[\frac{R_t}{R_T}F(X_t)\middle| X_t = x\right]$$

suppose $\phi(t, x)$ is C^1 in t and C^2 in x, then

$$\dot{\phi} = r\phi - m\phi' - \frac{1}{2}\sigma^2\phi''$$

with terminal condition $\phi(T, x) = F(x)$.

13. 11/7: Integration with respect to Martingale, Wiener Measure

We start by a conclusion from last time. We can solve initial condition problem

$$\begin{cases} \dot{u}(t,x) = L_x u(t,x) \\ u(0,x) = F(x) \end{cases}$$

to be

$$u(t,x) = \mathbb{E}^x[F(X_t)].$$

And for terminal condition

$$\begin{cases} \dot{\phi}(t,x) = L_x u(t,x) \\ \phi(T,x) = F(x) \end{cases}$$

then we can say that if $\phi(t,x) = u(T-t,x)$ then $\dot{\phi}(t,x) = -L_x\phi(t,x)$.

Example 13.1.

Say we have a geometric Brownian motion with $m(x) = mx$ and $\sigma(x) = \sigma x$, and say we have the interest rate r then the equation from last time would be

$$\dot{\phi}(t,x) = r\phi(t,x) - mx\phi'(t,x) - \frac{1}{2}\sigma^2 x^2 \phi''(t,x).$$

13.1. Integration with respect to Martingales.

Let's consider what it means to integrate with respect to a continuous Martingale. Suppose M_t is a continuous Martingale with $\{\mathcal{F}_t\}$ and $M_0 = 0$ in the following.

Def 13.2. *If $f : [0,1] \to \mathbb{R}$ is a function, then its variation is*

$$V_f(1) := \sup_{\Pi} \sum_{j=1}^{n} |f(t_j) - f(t_{j-1})|.$$

We say that f is bounded variation if $V_f < \infty$.

Proposition 13.3. *If f is bounded variation, then $f = f_+ + f_-$, where f_+ and f_- are non-decreasing.*

Another fact is

Proposition 13.4. *If f has bounded variation and is continuous, then $V_f(0+) = 0$.*

Proposition 13.5. *If a continuous Martingale with $M_0 = 0$ has paths of bounded variation, then w.p.1 $M_t = 0$ for all t.*

This we'll prove. Note that it really is 0 since the drift term is killed by the Martingale property.

Proof. WLOG let $t = 1$ then we want to show that $\mathbb{E}[M_1^2] = 0$.

Case 1: For $V_M(1) \leq K < \infty$ uniformly bounded, we have

$$\mathbb{E}[M_1^2] = \mathbb{E}\left(\sum_{j=1}^{n}\left(M_{\frac{j}{n}} - M_{\frac{j-1}{n}}\right)^2\right)$$

and focusing on the inside we get

$$\sum_{j=1}^{n}\left(M_{\frac{j}{n}} - M_{\frac{j-1}{n}}\right)^2 \leq \sum_{j=1}^{n}\delta_n\left|M_{\frac{j}{n}} - M_{\frac{j-1}{n}}\right| \leq V(M)\delta_n \leq \delta_n K$$

where

$$\delta_n := \max\left\{\left|M_{\frac{j}{n}} - M_{\frac{j-1}{n}}\right|\right\}$$

and since M_t has continuous path we get that $\delta_n K \to 0$ with probability 1.

Then we have the expectation goes to 0 and we are done.

General case: We just use $\tau_k := \min\{t : V_M(t) = k\}$ which is well defined since $V_f(t)$ is continuous in t if f is bounded variation and is continuous. Now we have for any k

$$\mathbb{E}\left[\left(M_1^{(k)}\right)^2\right] = 0$$

and using MCT we get that $\mathbb{E}[M_1^2] = 0$. □

Now remember the definition that for M_t continuous, the quardratic variation M_t, namely $\langle M\rangle_t$ is the unique increasing process such that $\langle M\rangle_t$ is a Martingale. This makes more sense now.

Now let's just pick a second easiest one $M_t = Z_t = \int_0^t A_s dB_s$ then we can compute (in homework) that

$$\langle M\rangle_t = \int_0^t A_s^2 ds.$$

Now, suppose we want to define $Z_t = \int_0^t A_s dM_s$, and we assume $\langle M\rangle_t = t$, which means it's a constant bet, then we could use the same method to define everything, and this is called Ito integral.

Theorem 13.6. *If M_t is a continuous Martingale with respect to $\mathcal{F}_t$ and $\langle M\rangle_t = t$ with $M_0 = 0$, then M_t is a standard Brownian.*

Basically, what walks like a duck quacks like a duck.

Proof. $M_0 = 0$ is given and continuity is immediate. We want to show that, if $s < t$, then $M_t - M_s$ is a normal random variable with variance $t - s$ independent of $\mathcal{F}_s$, which it suffices to show that $\forall \lambda$,

$$\mathbb{E}\left[e^{i\lambda(M_t - M_s)}\middle|\mathcal{F}_s\right] = e^{\frac{-\lambda^2(t-s)}{2}}.$$

For ease we assume $s = 0$ and use Ito's formula to get for $f = e^{i\lambda x}$ (note that we really can use the moment generating function, but the character is much easier due to the reason we'll see right now):

$$f(M_t) - f(M_0) = \int_0^t f'(M_s)dM_s + \frac{1}{2}\int_0^t f''(M_s)d\langle M\rangle_s$$

where for the first term we know it is a Martingale because f' is a bounded function (exactly why adding an i) and we use the condition to get

$$\mathbb{E}\left[f(M_t) - f(M_0)\right] = -\frac{\lambda^2}{2}\int_0^t \mathbb{E}[f(M_s)]ds$$

by direct derivative calculation. Now taking derivative we get

$$\frac{d}{dt}\mathbb{E}[f(M_t)] = -\frac{\lambda^2}{2}E[f(M_t)]$$

and solving this ODE we get $\mathbb{E}[f(M_t)] = e^{-\frac{\lambda^2}{2}t}$, the desired result. □

13.2. **Wiener Measure.**

We take the space $C[0,1]$, i.e. continuous function with sup norm and we take the Borel σ-algebra of $C[0,1]$ because it's a well defined topology. We denote $B(f,\varepsilon)$ to be the ε open ball around f.

Now we can define the probability measure

$$\mathcal{W}[B(f,\varepsilon)] = \mathbb{P}\{|f(t) - B_t| < \varepsilon, 0 \le t \le 1\}$$

and this is called the Wiener measure. We define similarly $\mathcal{W}_{m,\sigma^2}$ with obvious generalizations.

Question: Are the measures $\{\mathcal{W}_{m,\sigma^2}\}$ absolutely continuous or singular with respect to each other.

This question is, in other words, can we see the difference of two brownian motions by looking at their finite path, can we distinguish difference in drift and variance?

We can show that for $\sigma^2 \neq 1$ we get $\mathcal{W}_{0,1} \perp \mathcal{W}_{0,\sigma^2}$. To show this, we only need to find a set S with $\mathcal{W}(S) = 1$ and $\mathcal{W}_{0,\sigma^2}(S) = 0$.

But we just define

$$A := \left\{ f : \lim_{n\to\infty} \sum_{j=1}^{2^n} \left(f\left(\frac{j}{2^n}\right) - f\left(\frac{j-1}{2^n}\right)\right)^2 = 1 \right\}$$

but of course $\mathcal{W}(A) = 1$ and $\mathcal{W}_{0,\sigma^2}(A) = 0$.

For the drift difference, the deduction is harder. The result is that they are absolutely continuous. We'll first do a discrete version of this now, which is essentially not needed in the continuous case, but it does give intuition.

We start from a standard simple random walk S_n with $\{\pm 1\}$ for each step. Then if we define the scaling

$$W^{(1)}_{\frac{k}{n}} = \frac{S_k}{\sqrt{n}}$$

then path each will have probability $\frac{1}{2^n}$ because it's essentially discrete, and it's not hard to show that $\mathcal{W}^n \rightharpoonup \mathcal{W}$ weakly. But what is not so obvious is how to put a drift in.

In general, there are two ways we can do this, one is to change the path so it's with a drift, or we can adjust the probability, which we'll do here. So we just let

$$\mathbb{E}\left[W^{(n)}_{\frac{k+1}{n}} - W^{(n)}_{\frac{k}{n}} \right]$$

which requires us to take the step $\pm\frac{1}{\sqrt{n}}$ with probability $\dfrac{1}{2}(1 \pm \delta)$. And we get $\delta = \frac{m}{\sqrt{n}}$.

So we get that for a specific path, i.e. $\omega \in \Omega$, let J be the number of going up or heads, then

$$p^*(\omega) = \left[\frac{1}{2}\left(1 + \frac{m}{\sqrt{n}}\right)\right]^J \left[\frac{1}{2}\left(1 + \frac{m}{\sqrt{n}}\right)\right]^{n-J}$$

and the normal probability is

$$p(\omega) = \frac{1}{2^n}$$

and we rewrite $J = \frac{n}{2} + r\sqrt{n}$ where r is roughly the total skew. Then

$$\frac{p^*(\omega)}{p(\omega)} = \left(1 - \frac{m^2}{n}\right)^{\frac{n}{2}} \left[\left(1 + \frac{m}{\sqrt{n}}\right)\right]^{r\sqrt{n}} \left[\left(1 + \frac{m}{\sqrt{n}}\right)\right]^{-r\sqrt{n}}$$

and as $n \to \infty$ we get that the quotient goes to $e^{2rm - m^2/2}$ and since $W^{(n)}_1 = 2r = \dfrac{n/2 + r\sqrt{n} - (n/2 - r\sqrt{n})}{\sqrt{n}}$.

So the quotient is $e^{mW_1 - \frac{m^2}{2}}$ as $n \to \infty$.

14. 11/9: Non-negative Martingales and Change of measures

Last time, we've seen how we construct the change of measure in the discrete case. Today, we see how continuous cases works.

$\mathcal{W}$ and $\mathcal{W}_{m,1}$ are mutually absolutely continuous (Sometimes it's called equivalent) with

$$\frac{d\mathcal{W}_{m,1}}{\mathcal{W}} = e^{mB_1 - \frac{1}{2}m^2}$$

being the Radon-Nycodym derivative.

For B_t the standard Brownian motion, we define $e^{mB_t - \frac{m^2}{2}t}$ and since its a Martingale, we get

$$dM_t = mM_t dB_t$$

with $M_0 = 1$. Note that this is a non-negative Martingale, and for $t > 0$, on $\mathcal{F}_t$, define a probability measure Q_t via

$$Q_t[v] = \mathbb{E}[\mathbb{1}_v M_t], v \in \mathcal{F}_t$$

then Q_t is a well defined probability measure because

$$\nu(V) = \int_V f \, d\mu$$

is a measure. And we can do

$$\frac{Q_t}{d\mathbb{P}} = M_t$$

A fact is that if $s < t$, $V \in \mathcal{F}_s$ then $Q_s(V) = Q_t(V)$, the proof is just checking:

$$Q_t(V) = \mathbb{E}[\mathbb{1}_V M_t] = \mathbb{E}[\mathbb{E}[\mathbb{1}_V M_t | \mathcal{F}_s]] = \mathbb{E}[\mathbb{1}_V M_s] = Q_s(V)$$

and we can define Q as a measure on $\mathcal{F}_\infty$ such that if $V \in \mathcal{F}_t$ then $Q(V) = \mathbb{E}[\mathbb{1}_V M_t]$ if X is $\exists t$ measurable then $\mathbb{E}_Q[X] = \mathbb{E}[XM_t]$.

Note that the fact that Q is a measure is actually the same as Q_t, and is not defined by a limit... One subtle point here is that $Q \perp P$ as measures on $\mathcal{F}_\infty$ since $M_t \to 0$ with probability 1 in $\mathbb{P}$.

Theorem 14.1. *With respect to Q, B_t is a Brownian motion with variance parameter* 1 *and drift m.*

Proof. $B_0 = 0$ and continuity is obvious.

We want to show that if $s, t > 0$ then the conditional distribution of $B_{t+s} - B_s$ given $\mathcal{F}_s$ is $N(mt, t)$. And we need to show

$$\mathbb{E}_Q \left[e^{\lambda(B_{t+s} - B_s)} \middle| \mathcal{F}_s \right] = e^{\lambda mt + \frac{\lambda^2 m^2}{3} t}$$

which means that for every $v \in \mathcal{F}_s$ we need to show

$$\mathbb{E}_Q \left[e^{\lambda(B_{t+s} - B_s)} \mathbb{1}_V \right] = e^{\lambda tm + \frac{\lambda^2 m^2}{2} t} \mathbb{E}_Q[\mathbb{1}_V] = e^{\lambda mt + \frac{\lambda^2 m^2}{2} t} \mathbb{E}[\mathbb{1}_V M_s]$$

and now we show it:

$$\mathbb{E}\left[e^{\lambda(B_{t+s}-B_s)}\mathbb{1}_V\right] = \mathbb{E}\left[M_{t+s}e^{\lambda(B_{t+s}-B_s)}\mathbb{1}_V\right] = \mathbb{E}\left[\mathbb{1}_V\mathbb{E}\left[M_{t+s}e^{\lambda(B_{t+s}-B_s)}\middle|\mathcal{F}_s\right]\right]$$
$$= \mathbb{E}\left[\mathbb{1}_V M_s \mathbb{E}\left[e^{mY_s-\frac{m^2}{2}t}e^{\lambda Y_s}\middle|\mathcal{F}_s\right]\right] = \cdots = e^{\lambda mt+\frac{\lambda^2 m^2}{2}t}\mathbb{E}_Q[\mathbb{1}_V]$$

where $Y_s := B_{t+s} - B_s$ and hence $M_{t+s} = M_s e^{mY_s - \frac{m^2}{2}t}$ and the dots are left as an exercise. $\square$

14.1. **Girsanov Theorem.** We have done with a specific example of this above. In general, we care about the non-negative Martingale satisfying SDE

$$dM_t = A_t M_t dB_t$$

where $M_0 = 1$ and $M_t = e^{Y_t}$ where Y_t is an integral

$$\int_0^t A_s dB_s - \frac{1}{2}\int_0^{t^2} A_s^2 ds$$

and

$$dY_t = A_t dB_t - \frac{1}{2}A_t^2 dt$$

and by Ito:

$$d\left[e^{Y_t}\right] = e^{Y_t}dY_t + \frac{1}{2}e^{Y_t}d\langle Y\rangle_t = e^{Y_t}\left[A_t dB_t - \frac{1}{2}A_t^2 dt + \frac{1}{2}A_t^2 dt\right] = A_t M_t dB_t$$

where we will show on Tuesday that if the above holds, then we get a Martingale. But today we do the forward direction.

Def 14.2. *We define a measure Q_t on $\mathcal{F}_t$ by $Q_t(V) = \mathbb{E}[M_t \mathbb{1}_V]$ with $Q_t \ll \mathbb{P}$, if $s < t$ and $V \in \mathcal{F}_s$, then $Q_s(V) = Q_t(V)$ then we can define Q just as we did before.*

Theorem 14.3. *For the above Q, let $W_t = B_t - \int_0^t A_s ds$, then W_t is a standard Brownian motion with respect to Q, i.e.*

$$dB_t = A_t dt + dW_t$$

where W_t is a Q standard Brownian motion.

We first give a physicist proof, then give a real one.

For the physicists' one, we have $dM_t = A_t M_t dB_t$ and

$$B_{t+\Delta t} - B_t = \begin{cases} \sqrt{\Delta t} & p = \frac{1}{2} \\ -\sqrt{\Delta t} & p = \frac{1}{2} \end{cases}$$

and with a random walk approximation we "morally" get

$$M_{t+\Delta t} = M_t[1 + A_t\sqrt{\Delta t}]$$

and we get

$$\frac{M_{t+\Delta t}}{M_t} = 1 + A_t\sqrt{\Delta t}$$

and similarly for the new measure we have

$$\begin{cases} \sqrt{\Delta t} & p = \frac{1}{2}\left(1 + A_t\sqrt{\Delta t}\right) \\ -\sqrt{\Delta t} & p = \frac{1}{2}\left(1 - A_t\sqrt{\Delta t}\right) \end{cases}$$

where if we compute $EE_Q\left[B_{t+\Delta t} - B_t\right] = A_t\Delta t$.

Sort of... doesn't really matter since we'll do an actual proof.

Proof. Assume A_t is bounded with $A_t \leq K < \infty$. We have continuity easily, and we wnat to show that with respect to Q, W_t is a Martingale, with quadratic variation t by our result last time. In particular

$$W_t = B_t - \int_0^t A_s ds$$

and we have with probability 1 $\langle B\rangle_t\Big|_{\mathbb{P}} = t$ yet this carries on to Q by absolute continuous, and since drift term does not affect the variation, we get our result that $\langle W\rangle_t\Big|_Q = t$.

So all that's left to show is W_t is a Martingale, i.e. we need to show that if $s < t$ then $\mathbb{E}_Q[W_t|\mathcal{F}_s] = W_s$, and if $v \in \mathcal{F}_s$ then

$$\mathbb{E}_Q[\mathbb{1}_V W_t] = \mathbb{E}_Q[\mathbb{1}_V W_s]$$

and we observe the form

$$\mathbb{E}_Q[\mathbb{1}_V W_t] = \mathbb{E}[\mathbb{1}_V W_t M_t]$$

so we only need to show that $W_t M_t$ is a Martingale. Now what's left is calculus: We have

$$dW_t = dB_t - A_t dt, \qquad dM_t = A_t M_t dB_t$$

by product rule we get

$$d[W_t B_t] = M_t dW_t + W_t dM_t + d\langle W, M\rangle_t = M_t[(1 + A_t W_t)]dB_t$$

by plugging in so in particular

$$W_t M_t = \int_0^t M_s[1 + W_s A_s]ds$$

now we claim that

$$\int_0^t \mathbb{E}\left[M_s^2(1 + W_s A_s)^2\right] < \infty$$

for $A_t < K$ and in general what we need is to take limits when $A_{t\wedge K}$ we need assumption that M_t is a Martingale then by UI Martingale $M_{t\wedge\tau}$ we get the result. (And we skip!) $\square$

15. 11/14: GIRSANOV THOEREM CONTINUED AND EXAMPLES

Recall that in the setting of Girsanov Theorem, we required the process to be a non-negative Martingale, which is not in general true. What we have is the fact that M_t is a supermartingale. Let T_n be a sequence of stopping times $\tau_n \uparrow t$ such that if $M_r^{(n)} = M_{r\cap\tau_n}$ then $M_r^{(n)}$ is a Martingale since M_t is a local Martingale.

Now we need to show that $\mathbb{E}[M_t|\mathcal{F}_s] \leq M_s$. Suppose $V \in \mathcal{F}_s$, let $V_k = V \cap \{\tau_k \geq s\}$ then

$$\mathbb{E}\left[\mathbb{1}_{V_k} \cdot M_t^{(n)}\right] = \mathbb{E}\left[\mathbb{1}_{V_k} \cdot M_s^{(n)}\right] = \mathbb{E}[\mathbb{1}_{V_k} M_s]$$

where the first equality is Martingale and the second is definition of V_k.

Suppose $V \in \mathcal{F}_s$, now Fatou's says

$$\mathbb{E}[\mathbb{1}_{V_k} M_t] = \mathbb{E}\left[\lim_{n\to\infty} \mathbb{1}_{V_k} M_t^{(n)}\right] \leq \lim_{n\to\infty} \mathbb{E}\left[\mathbb{1}_{V_k} M_t^{(n)}\right] = \mathbb{E}[\mathbb{1}_{V_k} M_s]$$

and since when $k \to \infty$ we have $\mathbb{1}_{V_k} \to \mathbb{1}_V$ and by MCT, we get

$$\mathbb{E}[M_t \mathbb{1}_V] \leq \mathbb{E}[M_s \mathbb{1}_V] \Rightarrow \mathbb{E}[M_t|\mathcal{F}_s] \leq M_s$$

and in particular if $\mathbb{P}(U) > 0$ where $U := \{\mathbb{E}[M_t|\mathcal{F}_s] < M_s\}$ then M_t is a strict supermartingale.

An obvious fact by supermartingale is

Proposition 15.1. *If M_t is a continuous supermartingale that satisfies non-negative and has*

$$dM_t = A_t M_t dB_t, M_0 = 1$$

and if $\mathbb{E}[M_t] = 1$, then M_s for $0 \leq s \leq t$ is a Martingale.

Now the reason for possible non-Martingale-ness is that Fatou's lemma can be a strict inequality, i.e. some mass goes to ∞.

Let

$$T_n := \min\{t | M_t + |A_t|^2 \leq n\}$$

and let

$$A_t^{(n)} := \begin{cases} A_t & t < T_n \\ 0 & t \geq T_n \end{cases}$$

$M_t^{(n)} := M_{t\cap T_n}$ and then we know

$$dM_t^{(n)} = A_t^{(n)} M_t^{(n)} dB_t$$

and $M_t^{(n)}$ is a Martingale. Then we can apply last week's Girsanov to this and define $Q_t^{(n)}$ for the measure

$$E_{Q_t^{(n)}} = \mathbb{E}_p[Y M_t^{(n)}]$$

for $Y \in \mathcal{F}_t$ then with respect to $Q_t^{(n)}$ the standard B.M. is

$$W_t^{(n)} = B_t - \int_0^t A_s^{(n)} ds$$

as $n \to \infty$ we have

$$W_t^{(n)} \to W_t := B_t - \int_0^t A_s ds$$

and thus we can define $W_t := \mathbb{P}^*$ on the paths. But note we don't know whether $Q_t \ll \mathbb{P}$. We maybe can see this with criterion: Let $T := \lim_{n\to\infty} T_n$, with probability 1 T is finite, but not necessarily for Q.

Theorem 15.2. *Suppose $dM_t = A_t M_t dB_t$ and $M_0 = 1$, then if any of the following holds, then for $0 \le s \le t$, M_s is a Martingale:*

(1) $\mathbb{E}[M_t] = 1$.
(2) $\mathbb{P}^*\{T \ge t\} = 1$.
(3) $\mathbb{E}\left[\exp\left\{\frac{\langle Y\rangle_t}{2}\right\}\right] < \infty$ *where* $M_t = e^{Y_t}$.

TO see why we check

$$\mathbb{E}[M_{t\wedge T_n}\mathbb{1}_{\{T_n\ge t\}}] = \mathbb{E}[M_t\mathbb{1}_{\{T_n\ge t\}}] = \mathbb{P}^*\{T_n \ge t\}$$

if as $n \to \infty$

$$\lim_{n\to\infty} \mathbb{E}[M_{t\wedge T_n}] \ge \lim_{n\to\infty} \mathbb{E}[M_t\mathbb{1}_{\{T_n\ge t\}}] \ge \mathbb{P}^*\{T_n \ge t\}$$

and hence if right hand side is 1 we don't lose things.

One thing unsatisfying about this is we have to first find a t then do for s below. What is a condition that might help us get to any s? For discrete Martingale, UI might help, but here we look at measure $\mathbb{P}_n^*$ on $\mathcal{F}_n$ measurable sets, and $\mathbb{P}_n^*(V) = \mathbb{E}[\mathbb{1}_V M_n]$.

In general, a good condition is that $M_\infty < \infty$ w.p.1 with respect to $\mathbb{P}^*$, and we might use examples to see some special cases of this condition.

Now we do some examples to illustrate in some special cases where we can get the condition.

Example 15.3.

Let B_t be B.M. with $B_0 = 1$ and $T := \min\{t : B_t = 0\}$ and $M_t := B_{t\wedge T}$ is a non-negative martingale, for $t < T$ we have $dM_t = dB_t = \frac{1}{B_t} M_t dB_t$ thus

$$dB_t = A_t dt + dW_t = \frac{1}{B_t}dt + dW_t$$

which is a Bessel process with $a = 1$, $d = 3$ and hence in the new measure

$$\mathbb{P}^*\{T < \infty\} = 0$$

since it's 3d and does not hit the origin again.

Example 15.4.

In the same setting, this time we tilt by B_t^r and stop also when we reach 0. Now Ito's formula gives

$$dB_t^r = rB_t^{r-1}dB_t = \frac{r(r-1)}{2}B_t^{r-2}dt$$

which has geometric form

$$B_t^r\left[\frac{r}{B_t}dB_t + \frac{r(r-1)}{2B_t^2}dt\right]$$

where we let

$$C_t = \exp\left\{-\int_0^t \frac{r(r-1)}{2B_s^2}ds\right\}$$

and

$$dC_t = -C_t\frac{r(r-1)}{2B_t^2}dt$$

using this and let $M_t = C_tB_t^r$ we get $A_t = \frac{r}{B_t}$ and hence in $\mathbb{P}^*$ we ge the same result, except that we know when $r \geq \frac{1}{2}$ the Bessel process has $\mathbb{P}^*\{T < \infty\} = 0$.

Example 15.5. *Several Brownian motions.*

Suppose we have $B_t^1, \ldots, B_t^d$ and M_t is non-negative Martingale with

$$dM_t = M_t\left[A_t^1dB_t^1 + \cdots + A_t^ddB_t^d\right] = M_t[\vec{A}_td\vec{B}_t]$$

and in the new measure

$$dB_t^i = A_t^idt + dW_t^i.$$

Example 15.6.

Suppose bounded domain $D \subset \mathbb{R}^d$ and h is a non-negative harmonic function on D. Then B_t is a standard d dimensional BM with $B_0 = x \in D$ and $T := \min\{t|B_t \in \partial D\}$ with $M_t = h(B_t)$ for $0 \leq t \leq T$, then M_t is a Local martingale with

$$dh(B_t) = \nabla hB_t \cdot dB_t + \frac{1}{2}\Delta h(B_t)dt$$

but second order term vanishes because h is harmonic and hence we have

$$dh(B_t) = h(B_t)\frac{\nabla h(B_t)}{h(B_t)}dB_t$$

and now let's say $h \geq 0$, which is actually implied by harmonic plus non-negative, then if we tilted by $h(B_t)$ we get

$$dB_t = \frac{\nabla h(B_t)}{h(B_t)}dt + dW_t.$$

16. 11/16: BLACK-SCHOLES FORMULA; COMPLEX BROWNIAN MOTION

16.1. **Black-Scholes Formula.**

Let S_t be the price of stocks and R_t be bonds. Assume

$$dS_t = S_t[mdt + \sigma dB_t]$$

and

$$dR_t = rR_t dt$$

and consider European option where T is fixed time in future and K is strike price. Option is the option to buy one share of stock at time T at price K. Define value

$$f(t,x) = v(t,x) = F(S_t) := (S_t - K)_+$$

i.e. find $v(t,x)$ price of option at time t given $S_t = x$. So we have

$$v(0,x) = \mathbb{E}^x\left[R_T^{-1}F(S_T)\right]$$

$$v(t,x) = \mathbb{E}\left[R_T^{-1}F(S_t)|S_t = x\right]$$

We also have "no arbitrage" condition, which is a method that gaurantees not losing money. $v(t,x)$ is the amount of money needed at time t, assuming current $S_t = x$ to invest in stocks and bonds we can pay off the option at time t. A portfolio is a_t units of stock and b_t units of bond. Then $V_t = a_t S_t + b_t R_t$ and we want

$$V_T = F(S_T).$$

Assume self-financing portfolio, i.e.

$$dV_t = a_t dS_t + b_t dR_t$$

and

$$dV_t = a_t S_t[mdt + \sigma dB_t] + b_t[rR_t dt] = [a_t m S_t + r(V_t - a_t S_t)]dt + \sigma a_t S_t dB_t$$

and since $V_t = f(t, S_t)$ and thus

$$dV_t = (\dot{f} + mS_t f' + \frac{1}{2}f''\sigma^2 S_t^2)dt + \sigma S_t f' dB_t$$

and we want them to be the same, from which we solve and get $a_t = f'$ and plugging in we get (still, $V_t = f$) the equation:

$$\dot{f}(t,x) = rf(t,x) - rf'(t,x)x - \frac{1}{2}\sigma^2 x^2 f''(t,x)$$

and we ask the question that why is there no m dependence?

The reason is that we can find a probability $\mathbb{P}^*$ tilted by $e^{\frac{(r-m)}{\sigma}B_t - \frac{1}{2}\left(\frac{r-m}{\sigma}\right)^2 t}$ such that

$$dB_t = \frac{r-m}{\sigma}dt + dW_t$$

and $dS_t = S_t[rdt + \sigma dW_t]$.

The solution is

$$f(t,x) = R_t \mathbb{E}_* \left[\frac{1}{R_t} F(S_t) \middle| S_t = x \right]$$

where S_t satisfies

$$f(T_t, x) = x\Phi\left(\frac{\log\left(\frac{x}{k}\right) + \left(r + \frac{\sigma^2}{2}\right) t}{\sigma\sqrt{t}} \right) - e^{-rt}\Phi\left(\frac{\log\left(\frac{x}{k}\right) + \left(r - \frac{\sigma^2}{2}\right) t}{\sigma\sqrt{t}} \right)$$

which we can just plug in values. Note that we can get back σ using this formula.

Now, if the system is

$$dS_t = S_t[m(t,S_t)dt + \sigma(t,S_t)dB_t]; \quad dR_T = H(t,S_t)dB_t$$

and what to do is to change measure. Let

$$M_t = \exp\left\{ \int_0^t \frac{r(s,S_s) - m(s,S_s)}{\sigma(s,S_s)} dB_s - \frac{1}{2}\int_0^t \left(\frac{r(s,S_s) - m(s,S_s)}{\sigma(s,S_s)} \right)^2 ds \right\}$$

and hence

$$dM_t = \frac{r(s,S_s) - m(s,S_s)}{\sigma(s,S_s)} M_t dW_t$$

and so we need a Martingale for all these change of measure to make sense, and if it does then just plugging in we get $\mathbb{E}_*[|F(S_T)|] < \infty$ which guarantees everything. Now we use the following lemma to show the Martingale step:

Lemma 16.1. *(Martingale representation theorem)*

$$S_t = \int_0^t A_s dW_s$$

for some W, A.

16.2. **Conformal Invariance of two dimensional B.M..** Let $B_t = B_t^1 + iB_t^2$ and have domain $D \subset \mathbb{C}$ with $f : D \to f(D)$ is holomorphic if at each z, f' exists. Then $f(z) = u(z) + iv(z)$. Ito gives

$$du(B_t) = \nabla u \cdot dB_t + \frac{1}{2}\Delta u dt$$

and

$$dv(B_t) = \nabla v \cdot dB_t + \frac{1}{2}\Delta v dt$$

and by harmoniticity the dB_t term goes away and we further write out

$$du(B_t) = u_x(B_t)dB_t^1 + u_y(B_t)dB_t^2$$

and by Cauchy Riemann

$$dv(B_t) = v_x(B_t)dB_t^1 + v_y(B_t)dB_t^2 = -u_y(B_t)dB_t^1 + u_x(B_t)dB_t^2.$$

Further, we can compute

$$d\langle u(B_t)\rangle = (u_x^2 + u_y^2)dt = |f'(B_r)|^2 dt$$

similarly $d\langle v(B_t)\rangle = |f'(B_t)|^2 dt$ and

$$d\langle u(B_t), v(B_t)\rangle = u_x u_y - u_y u_x = 0.$$

and we can define $T(t)$ by

$$\int 0^T |f'(B_s)|^2 ds = t$$

for f entire and if $Y_t = f(B_{T(t)})$ is a complex B.M. we note that Brownian motion ignores rotation, and for the dialation at each time we use T to constraint.

17. 11/28: LEVY PROCESSES I

Similar to definition of Brownian motion, we define

Def 17.1. *A Levy process (starting at origin) is a process X_t for $t \geq 0$ that sastisfying*

1. $X_0 = 0$;
2. *Independent increment;*
3. *Stationary increment;*
4. *as $t \to 0$, $X_t \to 0$ in probability.*

Note that we don't require continuity here.

Example 17.2.

A Poisson process is just after some amount of time a people will get in the line. A compound Poisson process is such that, let N_t be a Poisson process with rate $\lambda > 0$ and $Y_1, Y_2, \ldots$ be iid variables independent with N_t with push-forward distribution $\mu^{\#}$. Then a compound Poisson process is

$$X_t := Y_1 + \cdots + Y_{N_t}$$

and $X_t = 0$ if $N_t = 0$.

Def 17.3. *A random variable X has an infinitely divisible distribution if $\forall n$, we can find $Y_1, \ldots, Y_n$ iid such that X has the same distribution*

$$X \overset{d}{\sim} Y_1 + \cdots + Y_n.$$

An observation is that if X_t is a Levy process, then for each t X_t has infinitely divisibility distribution:

$$X_t = \left(X_{\frac{t}{n}} - X_0\right) + \cdots + \left(X_{\frac{n}{n}t} - X_{\frac{n-1}{n}t}\right).$$

Def 17.4. *We define the character function to be*

$$\phi_{X_1}(s) := \mathbb{E}\left[e^{isX_1}\right].$$

Observe that because $\phi_{X_1}(s) = \phi_{X_t}(s)\phi_{X_1 - X_t}(s)$ so $\phi_{X_t}(s) = \left[\phi_{X_1}(s)\right]^t$ by independent increment.

Def 17.5. *The characterstic exponent $\Psi(s)$: is such that*

$$e^{\Psi(s)} := \phi_{X_1}(s).$$

where $\Psi(0) = 0$ and Ψ is continuous in s for along any branch cut.

For a Brownian motion with drift m and variance σ^2, we have

$$\Psi(s) = ims - \frac{\sigma^2}{2}s^2.$$

Now we move on to examine the compound Poisson process with notations same as in definition above. Let ϕ be the character function of Y_j, i.e.

$$\phi(s) = \int_{-\infty}^{\infty} e^{isx}\mu^{\#}(dx)$$

then we have

$$\begin{aligned}\phi_{X_1}(s) = \mathbb{E}\left[e^{isX_1}\right] &= \sum_{n=0}^{\infty}\mathbb{P}\{N=n\}\mathbb{E}\left[e^{isX_{N_1}}|N_1=n\right] = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}e^{-\lambda}e^{[}\phi(s)]^n = e^{-\lambda}e^{\lambda\phi(s)}\\ &= e^{\lambda[\phi(s)-1]} = \exp\left\{\lambda\int_{-\infty}^{\infty}[e^{isx}-1]\mu^{\#}(dx)\right\} := \exp\left\{\int_{-\infty}^{\infty}[e^{isx}-1]\mu(dx)\right\}\end{aligned}$$

where we've used the fact that $\mu^{\#}$ is a probability measure, and defined $\mu := \lambda\mu^{\#}$. We call μ the Levy measure for this particular process.

Facts: as $t \downarrow 0$,

$$\mathbb{P}\{N_t = 0\} = 1 - \lambda t + o(t); \quad \mathbb{P}\{N_t \geq 2\} = o(t); \quad \mathbb{P}\{X_t \in (a,b)\} = \mu(a,b)t + o(t)$$

and a convention is to take $\mu(\{0\}) = 0$, since we don't want to deal with the Brownian part here. So if $0 \notin (a,b)$ we have the last estimate above. Moreover, as is obvious from above,

$$\phi_{X_t}(s) = e^{t\Psi(s)}.$$

Now we consider the generator of X_t.

$$Lf(x) = \lim_{t\downarrow 0}\frac{\mathbb{E}^x[f(X_t)] - f(x)}{t}$$

and

$$\mathbb{E}^x[f(X_t)] = f(x)[1-\lambda t + o(t)] + t\int_{-\infty}^{\infty} f(x+y)\mu(dy) + o(t)$$

thus plugging in we have

$$Lf(x) = \int_{-\infty}^{\infty}[f(x+y) - f(x)]\mu(dy).$$

For Brownain motion part, we have already that

$$Lf(x) = mf'(x) + \frac{\sigma^2}{2}f''(x).$$

Thus, we have the following thoerem.

Proposition 17.6. *If X_t, Y_t are independent Levy processes with characteristic exponents Ψ_x, Ψ_y and generators L_x, L_y, then $X_t + Y_t$ is a Levy process with characterstic exponent $\Psi_x + \Psi_y$ and generator $L_x + L_y$.*

For X_t a Levy process that we know, define

$$m = \int_{-\infty}^{\infty} x\mu(dx), \sigma^2 = \int_{-\infty}^{\infty} x^2\mu(dx)$$

where we note that we have actually defined the variance to be $\mathbb{E}[X_t^2]$, and that's for a reason. In general $\mathbb{E}[X_t] = mt, \mathrm{Var}(X_t) = \sigma^2 t$.

Def 17.7. *A function $f : [0, \infty) \to \mathbb{R}$ is called* <u>*Cadlag*</u> *if the paths are right continuous and left limit exists.*

A compound Poisson process has cadlag paths. There is no problem in defining the stochastic integral

$$\int_0^t A_s dX_s = \sum_{\text{jumps}} A_s[X_s - X_{s-}].$$

Now, let $\mu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$ then X_t is a Martingale and we can define the Natural filtration $\mathcal{F}_t = \sigma\{X_t\}$. So we might want to require that $A_t \in \mathcal{F}_t$ so that we can let the integral be a Martingale(local or not), but that's actually not the case. Rather we take $A_t \in \mathcal{F}_{t-}$ then we are good.

Now we move on and define generalized compound Poisson process: Let

$$X_t := \sum_{s\le t} \left[X_s - X_{s-}\right]$$

then we allow infinitely many jumps in each interval. If $\sum_{s\le t} \left|X_s - X_{s-}\right| < \infty$ we'd be ok. In other words, we have bounded variation. We now can rewrite

$$X_t = \sum_{s\le t} \mathbb{1}_{|X_s - X_{s-}|\ge 1} \left(X_s - X_{s-}\right) + \sum_{n=0, s\le t}^{\infty} [X_s - X_{s-}]\mathbb{1}\left\{2^{-n-1} \le |X_s - X_{s-}| < 2^{-n}\right\}$$

and taking expectation

$$\mathbb{E}\left[\sum_{s\le t} |X_s - X_{s-}|\right] = \sum_{s\le t} |X_s - X_{s-}|\mathbb{P}(|X_s - X_{s-}| \ge 1) + \sum_{s\le t} |X_s - X_{s-}|\mathbb{P}(2^{-n-1} \le |X_s - X_{s-}| < 2^{-n}).$$

Now we take measure μ on $\mathbb{R}\backslash\{0\}$ that satisfies

$$\int_{-\infty}^{\infty} |x|\mu(dx) < \infty$$

where we note immediately we have

$$\int_{|x|>\varepsilon} \mu(dx) < \infty, \forall \varepsilon > 0$$

but $\mu(\mathbb{R}) = \infty$ is still possible. Then we know there exists a Levy process with Levy measure μ, for all kind of Levy processes we know of. Let $\mu^{(n)} = \mu|_{\{|x|\geq\frac{1}{n}\}}$ which we know is a finite measure, then let $X_t^{(n)}$ be a compound Poisson process with measure $\mu^{(n)}$. Then define

$$X_t = \lim_{n\to\infty} X_t^{(n)}$$

and correspondingly

$$\Psi(s) = \int_{-\infty}^{\infty} [e^{ixs} - 1]\mu(dx)$$

and to make sense of it note $e^{ixs} - 1 = ixs + O(x^2)$ so away from 0 we are good and close to 0 we know it is $\sim |x|\mu(dx)$ so finite. The generator of this is

$$Lf(x) = \int_{-\infty}^{\infty} [f(x+y) - f(x)]\mu(dy)$$

and so we need $f(x+y) - f(x) = yf'(x) + o(y)$ for the integral to exist, and need f to be bounded for large x.

18. 11/30: Levy Processes II

For conveniences, let's throw away all large jumps > 1. Now with slight difference to last time we define

$$\mu^{(n)} = \mu\Big|_{\frac{1}{2^n} \le |x| \le 1}$$

and

$$V_t^{(n)} := \sum_{s \le t} \left| X_s^{(n) - X_{s-}^{(n)}} \right|$$

we define $\tilde{\mu}(a, b) = \mu(a, b) + \mu(-b, -a)$ then we see that $V_t^n \uparrow$ and hence by MCT

$$V_t = \lim_{\infty} V_t^{(n)}$$

is well-defined. We can compute that

$$\mathbb{E}[V_t] = t \int |x| \mu(dx) =: tm$$

and V_{t-}, V_{t+} exists by monotone. Cadlag paths are checked and all the jumps are jumps at $X_t^{(n)}$.

Example 18.1. *Possitive stable processes.*

Choose $0 < \beta < 1$ and let $\mu(dx)$ be $\frac{c}{x^{1+\beta}}$ for $0 < x < \infty$, then it's easy to check that $\int_0^\infty |x| \mu(dx) < \infty$. To compute it's character we have

$$\Psi(s) = c \int_0^\infty [e^{isx} - 1] x^{-(1+\beta)} dx$$

and thus for $r > 0$ we have

$$\Psi(s) = c \int_0^\infty [e^{irsx} - 1] x^{-(1+\beta)} dx = c r^\beta \int_0^\infty [e^{isy} - 1] y^{1+\beta} dy = r^\beta \Psi(s)$$

by a change of variable. Thus, the character function of rX_1 is the same as X_{r^β}. Since we can write

$$X_1 = \frac{Z_1 + \cdots + Z_n}{n^{1/\beta}}$$

for

$$Z_j = n^{1/\beta} \left[X_{\frac{j}{n}} - X_{\frac{j-1}{n}} \right] \sim X_1$$

hence X is the natural limit in distribution for iid random variable $Y_1, Y_2, \dots$ with $\mathbb{P}\left\{ Y_j \ge x \right\} = cx^{-(1+\beta)}$ for $\dfrac{Y_1 + \cdots + Y_n}{n^{1/\beta}}$ and note that if $\beta > 1$ the distribution does not exist.

Example 18.2. *Gambler's process.*

Let $\lambda > 0$ be the rate and $\mu(dx) = \frac{e^{-\lambda x}}{x} dx$ and we can also check that it is GCPP I. Now we compute

$$\Psi(s) = \int_0^\infty [e^{isx} - 1] \frac{e^{-\lambda x}}{x} dx = \log \frac{\lambda}{\lambda - is}$$

and

$$\phi_{X_t}(s) = \left(\frac{\lambda}{\lambda - is} \right)^t$$

which we ususally call the Γ density and when $t = 1$ this is exponential with λ.

Now consider Levy process X_t within the types that we have seen, we know

$$\mathbb{E}[X_t] = mt, \quad m = \int |x| \mu(dx)$$

and hence to make it a Martingale we can compensate the process by defining $M_t = X_t - mt$.

Now we go on to generalize to GCCPP or GCPP II. We want to get rid of the large jumps, so we just let $\mu\{|x| > 1\} = 0$ and we want compensated so we would like it to be martingale. We require

$$\int_{-1}^1 x^2 \mu(dx) < \infty$$

and we define this via

$$\mu^{(n)} = \mu \bigg|_{\{|x| > \frac{1}{2^{-n}}\}}$$

where X_t^n is a compound poisson process with Levy measure $\mu^{(n)}$ since we don't consider near 0. Let

$$Y_t^n = X_t^{(n)} - tm^{(n)}$$

be the compensated process then $\mathbb{E}[Y_t^n] = 0$ and $\text{Var}(Y_t^{(n)}) = \mathbb{E}[(Y_t^n)^2]$. Then we can define

$$Y_t = \lim_{n \to \infty} Y_t^{(n)}$$

first in L^2, since Cauchy limit exists and eventually this can be passed to a.s. by Diatics.

When doing this limit we have that with probability 1 $\forall t \in [0, 1]$ the limits exist

$$Y_{t-} = \lim_{s \in D, s \uparrow t} Y_s; \quad Y_{t+} = \lim_{s \in D, s \downarrow t} Y_s$$

then we define $Y_t = Y_{t+}$ since we want cadlag. And this is easily a Levy process.

Example 18.3. *Cauchy Process*

Let B_t be a 2D Brownian motion with $\forall a > 0$, let $\tau_a = \inf\{t : B_t^2 = a\}$ then $(B_{\tau_a}^1, a)$ is the place of stop. Let $X_a = B_{tau_a}^1$ then X_a is a Levy process with Levy measure $\mu(dx) = \frac{1}{\pi x^2} dx$ and we note that X_a has jumps.

Now this is GCPP II and $Y_t^{(n)} = X_t^{(n)}$ since $X_t^{(n)}$ has symmetric jumps so $m^{(n)} = 0$. We compute

$$\Psi(s) = \int_{-1}^{1} [e^{isx} - 1 - isx]\mu(dx)$$

by just expanding one more terms in the Taylor, and the rest vanishes. The generator function is then

$$Lf(x) = \int_{-1}^{1} \left[f(x+y) - f(x) - yf'(x) \right] \mu(dx)$$

and hence we need $f \in C^2$.

Now we introduce the big theorem that categorizes all Levy process.

Theorem 18.4. *A random variable X has infinitely divisible distribution iff it's characteristic function is of the form*

$$\phi(s) = \exp\left\{ ims - \frac{\sigma^2 s^2}{2} + \int_{-\infty}^{\infty} \left[e^{isx - 1 - isx\mathbb{1}_{|x|\le 1}} \right] \mu(dx) \right\}$$

where $m \in \mathbb{R}, \sigma^2 \ge 0$ and μ is a measure on $\mathbb{R}\backslash\{0\}$ that satisfies

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \mu(dx) < \infty.$$

If we denote μ_Y to be μ on $|x| \le 1$ and μ_c otherwise, we can see that this theorem separates X into three parts: the Gaussian, the GCPP I with μ_c and GCPP II with μ_Y.

Here we only sketch a proof:

For X_t, the probability of jump size in (a, b) is roughly $\mu(a, b)t + o(t)$ as $t \to 0$. For $X_{\frac{1}{n}}$, which is one of the iid guys, we want to find the jump part then

$$\mu^{(n)}\{x \ge b\} = n\mathbb{P}\{X_{\frac{1}{n}} \ge b\}$$

and we can get $\mu^{(n)}(a, b) \to \mu(a, b)$ for points a, b with measure 0. By subsequence + Ascoli Arzela argument we know the measure satisfies the condition required.

Proposition 18.5. *Suppose X_t is a Levy process. Let*

$$M_t^{(n)} := \max\left\{ \left|X_{\frac{1}{n}}\right|, \left|X_{\frac{2}{n}} - X_{\frac{1}{n}}\right|, \dots, \left|X_{\frac{n}{n}} - X_{\frac{n-1}{n}}\right| \right\}$$

and $M_t^{(n)} \xrightarrow{p} 0$, then X_1 has a normal distribution.

This says that what collects at 0 is the normal part.

APPENDIX A. A

APPENDIX B. B

APPENDIX C. C

Acknowledgements.