MEASURE THEORETICAL PROBABILITY I HOMEWORK 2

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Discussed with classmates.

Exercise 1. *Prob 1.*

Proof.

Let $\{F_n\}_{n\in\mathbb{N}} \subset \mathcal{F} \cup X$ such that $X = \cup_{n\in\mathbb{N}} F_n$ and $\mu_1(F_n) < \infty$ for all $n \in \mathbb{N}$. Define

$$C_n = \{A \in \mathcal{A} : \mu_1(A \cap F_n) = \mu_2(A \cap F_n)\} \text{ for } n \in \mathbb{N}.$$

Claim: C_n is a $\lambda-$system.

(1) $\mu_1(X \cap F_n) = \mu_1(F_n) = \mu_2(F_n) = \mu_2(F_n \cap X) \Longrightarrow X \in C_n$
(2) Assume $A_1 \subset A_2 \in C_n$. Then,

$$\begin{aligned}\mu_1((A_2 \setminus A_1) \cap F_n) &= \mu_1(A_2 \cap F_n) - \mu_1(A_1 \cap F_n)\\ &= \mu_2(A_2 \cap F_n) - \mu_2(A_1 \cap F_n)\\ &= \mu_2((A_2 \setminus A_1) \cap F_n).\end{aligned}$$

Therefore, $A_2 \setminus A_1 \in C_n$
(3) Assume $\{A_i\}_{i\in\mathbb{N}} \subset C_n$ are pairwise disjoint.

$$\begin{aligned}\mu_1((\cup_{i\in\mathbb{N}} A_i) \cap F_n) &= \mu_1(\cup_{i\in\mathbb{N}}(A_i \cap F_n))\\ &= \sum_{i=1}^{\infty} \mu_1(A_i \cap F_n)\\ &= \sum_{i=1}^{\infty} \mu_2(A_i \cap F_n)\\ &= \mu_2(\cup_{i\in\mathbb{N}}(A_i \cap F_n))\\ &= \mu_2((\cup_{i\in\mathbb{N}} A_i) \cap F_n).\end{aligned}$$

Hence, $\cup_{i=1}^{\infty} A_i \in C_n$.

Now we have shown that C_n is a $\lambda-$system.

Claim: $\mathcal{F} \subset C_n$ for all n.

Let $F \in \mathcal{F}$. As $\sigma(\mathcal{F}) = \mathcal{A}$, $F \in \mathcal{A}$. Moreover, $\mathcal{F}$ is a $\pi-$system and $F \cap F_n \in \mathcal{F}$. By the definition of μ_1 and μ_2, $\mu_1(F \cap F_n) = \mu_2(F \cap F_n)$ and $F \in C_n$. Since the choice of F is arbitrary, $\mathcal{F} \subset C_n$.

By the HW1 Q6 (b) result, if a $\pi-$system $\mathcal{F}$ lies in a λ-system C_n, then $C_n \supset \sigma(\mathcal{F}) = \mathcal{A}$. Therefore, for every $A \in \mathcal{A}$,

$$\mu_1(A \cap F_n) = \mu_2(A \cap F_n) \text{ for all } n.$$

$\square$

Exercise 2. *Prob 2.*

Proof. We first prove that for all boxes B, $\mu_1(\partial B) = \mu_2(\partial B) = 0$, thus we can WLOG suppose that all boxes are closed. Then we prove the result for all boxes with rational vertices, then prove for all boxes.

For all boxes B, $\mu_1(\partial B) = \mu_2(\partial B) = 0$:

Assume, for contradiction, that exists a non-degenerate box $B = \prod_{i=1}^{d}[a_i, b_i]$ (with $a_i < b_i$) such that $\infty > \mu_1(\partial B) > 0$. Then we know that for one in 2^d sides (S) of the boundary of the box B, $\infty > \mu_1(S) > 0$ since there's only finitely many sides. Since the side is degenerate in one direction of the standard basis, we find that basis, say e_k, and we can translate the side any number of times to get that

$$\infty > \mu(B) > \mu\left(\sum_{\varepsilon\in[a_k,b_k]} \varepsilon e_k + S\right) = \infty \cdot \mu_1(S) = \infty$$

Contradiction. Thus $\mu_1(\partial B) = 0$.

If the box is itself degenerate in the direction k, we can simply expand it by letting $b_k' = a_k + 1$. Then the result follows.

For μ_2 it's the same argument.

μ_1 and μ_2 agrees on boxes with rational vertices:

Now we can WLOG suppose that all box we consider is closed.

For $B = \prod_{i=1}^{d}[a_i, b_i]$ where $a_i, b_i \in \mathbb{Q}$, we know that we can write each $a_i = \dfrac{p_i}{q_i}$ and $b_i = \dfrac{l_i}{m_i}$.
Then we can use the grid of cubes of side length $\prod_{i=1}^{d} \dfrac{1}{q_i m_i}$ to cut the whole $\mathbb{R}^d$ into small cubes. Then we see that

$$\mu_1\left(\left[0, \prod_{i=1}^{d}\frac{1}{q_i m_i}\right]^d\right) \cdot \prod_{i=1}^{d} q_i m_i = \mu_1([0,1]^d) = \mu_2([0,1]^d) = \mu_2\left(\left[0, \prod_{i=1}^{d}\frac{1}{q_i m_i}\right]^d\right) \cdot \prod_{i=1}^{d} q_i m_i$$

since we can view $[0,1]^d$ as the disjoint (up to boundary) union of small grid cubes translated to different places (since boundaries doesn't matter with respect to measures so we are good). The above equality implies

$$\mu_1\left(\left[0, \prod_{i=1}^{d}\frac{1}{q_i m_i}\right]^d\right) = \mu_2\left(\left[0, \prod_{i=1}^{d}\frac{1}{q_i m_i}\right]^d\right)$$

and since we can express B as the disjoint (up to boundary) union of grid cubes, and thus

$$\mu_1(B) = \mu_1\left(\left[0, \prod_{i=1}^{d} \frac{1}{q_i m_i}\right]^d\right) \cdot \prod_{i=1}^{d} p_i l_i = \mu_2\left(\left[0, \prod_{i=1}^{d} \frac{1}{q_i m_i}\right]^d\right) \cdot \prod_{i=1}^{d} p_i l_i = \mu_2(B).$$

<u>The result holds for all boxes:</u>

Since we can express any box as

$$B = \bigcap_{i=1}^{n} B_i$$

for B_i a decreasing sequence of boxes with rational vertices and $\mu_1(B_1) < \infty$. By downward monotone convergence we get that

$$\mu_1(B) = \lim_{n\to\infty} \mu_1(B_n) = \lim_{n\to\infty} \mu_2(B_n) = \mu_2(B)$$

hence we are done.

$\square$

Exercise 3. *Prob 3.*

Proof. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space with a measure μ. Show that if f is measurable, $f \geq 0$ and $\int f d\mu = 0$, then $f = 0$ μ-almost everywhere.

Define $E_n := \{x \in \Omega | f(x) > \frac{1}{n}\}$ for $n \in \mathbb{N}$. Since f is measurable, $E_n = f^{-1}(\frac{1}{n}, \infty)$ is measurable and $\mu(E_n)$ is well-defined. For all $n \in \mathbb{N}$, since f is non-negative,

$$\frac{1}{n}\mu(E_n) = \int_X \frac{1}{n}\mathcal{X}_{E_n} \leq \int_\Omega f d\mu = 0.$$

That is, $\mu(E_n) = 0$ for all $n \in \mathbb{N}$.

Let $\cup_{n\in\mathbb{N}} E_n := E = \{x \in f(x) : f(x) > 0\}$. Since $\{E_n\}$ is an increasing sequence and μ is continuous from below,

$$\mu(E) = \mu(\cup_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} 0 = 0.$$

Hence, $f(x) = 0$ $\mu-$almost everywhere.

□

Exercise 4. *Prob 4.*

Proof.

As is proven in prob 6 or in the notes, for every measurable function f there exists a sequence of simple functions s_n such that $s_n \to f$. We will also just use Monotone convergence theorem since the proof is in the notes.

We first show the following claim:

$\underline{\int \sum_{n=1}^{N} f_n d\mu = \sum_{n=1}^{N} \int f_n d\mu \text{ for measurable } f_n\text{:}}$

Since the sum is finite it can be think of as repeatedly adding two things, so we only need to prove that for f, g measurable

$$\int f + g d\mu = \int f d\mu + \int g d\mu.$$

Now by problem 6 below (whose prove does not depend on this problem) we know that there exists increasing sequences of simple functions

$$s_1 \leq s_2 \leq \dots, t_1 \leq t_2 \leq \dots$$

such that

$$s_i \to f,\ t_i \to g.$$

Let $u_i = s_i + t_i$ we have that u_i is increasing and by Monotone convergence theorem (u is simple thus measurable) the following:

$$\int f + g d\mu = \int \lim_{n\to\infty} u_n d\mu = \lim_{n\to\infty} \int s_n + t_n d\mu = \lim_{n\to\infty} \int s_n d\mu + \lim_{n\to\infty} \int t_n d\mu$$

where we can use monotone convergence theorem again to show that

$$\lim_{n\to\infty} \int s_n d\mu = \int \lim_{n\to\infty} s_n d\mu = \int f d\mu$$

and similar for g. Thus

$$\int f + g d\mu = \int f d\mu + \int g d\mu.$$

$\underline{\int \sum_{m=0}^{\infty} g_m d\mu = \sum_{m=0}^{\infty} \int g_m d\mu\text{:}}$

The first thing we are going to check is whether the above expression is valid. For each $N \in \mathbb{N}$,

$$\sum_{m=1}^{N} g_m$$

is a sum of measurable functions and is measurable (we proved it in Q6). Moreover, as g_m is non-negative for all $m \in \mathbb{N}$, for each $x \in \Omega$, $\sum_{i=1}^{\infty} g_m(x)$ is a non-decreasing sequence. The limit is finite or ∞. In either case, the limit is well-defined in $\mathbb{R}^*$. Hence $\sum_{m=1}^{n} g_m$ converges to $\sum_{m=1}^{\infty} g_m$ pointwise as $n \to \infty$. As shown in the class, the pointwise limit of a sequence of measurable functions is measurable, we have $\sum_{m=1}^{\infty} g_m$ is measurable and $\int \sum_{m=1}^{\infty} g_m d\mu$ is well-defined.

On the other hand, $\{\sum_{m=1}^{n} \int g_m d\mu\}_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathbb{R}^*$ as g_m is non-negative for all $m \in \mathbb{N}$. $\sum_{m=1}^{\infty} \int g_m d\mu$ converges in $\mathbb{R}^*$ and the expression is valid.

Let

$$f_n := \sum_{m=0}^{n} g_m$$

then since $g_m \geq 0$, f_n is an increasing sequence of functions with limit $f := \sum_{m=0}^{\infty} g_m$.

Now apply the monotone convergence theorem again we get

$$\begin{aligned}
\int \sum_{m=0}^{\infty} g_m d\mu &= \int f d\mu \\
\text{(Monotone convergence)} \quad &= \lim_{n\to\infty} \int f_n d\mu = \lim_{n\to\infty} \int \sum_{m=0}^{n} g_m d\mu \\
\text{(claim above)} \quad &= \lim_{n\to\infty} \sum_{m=0}^{n} \int g_m d\mu
\end{aligned}$$

and we are done.

□

Exercise 5. *Prob 5.*

Proof.

Claim: $\sigma(\mathcal{A}) = \mathcal{E} := \{A \subset \mathbb{R} | A \text{ is countable or } A^c \text{ is countable}\}$

Subclaim: $\mathcal{E}$ is a $\sigma-$algebra on $\mathbb{R}$.

(1) $\emptyset$ is countable $\implies \emptyset \in \mathcal{E}$. Since $\mathbb{R}^c = \emptyset \in \mathcal{E}$, $\mathbb{R} \in \mathcal{E}$;
(2) Let $A \in \mathcal{E}$. Then, A is countable or A^c is countable. If A is countable, then $(A^c)^c$ is countable and $A^c \in \mathcal{E}$. If A^c countable, we have $A^c \in \mathcal{E}$ by definition of $\mathcal{E}$.
(3) Let $\{A_i\}_{i\in\mathbb{N}} \subset \mathcal{E}$ and $A := \cup_{i\in\mathbb{N}} A_i$. If all A_i are countable, then A is a countable union of countable set, which is countable. If there exists $j \in \mathbb{N}$ such that A_j is uncountable, A_j^c will be countable. Then,
$$A^c = \cap_{i\in\mathbb{N}} A_i^c \subset A_j^c,$$
which is countable. Therefore, $A \in \mathcal{E}$.

Now we can conclude that $\mathcal{E}$ is a $\sigma-$algebra.

For every $x \in \mathbb{R}$, we have $\{x\} \in \sigma(\mathcal{A})$. Since $\sigma(\mathcal{A})$ is the smallest $\sigma-$algebra that contains all singleton sets and E contains all countable subsets of $\mathbb{R}$, we have $\sigma(\mathcal{A}) \subset \mathcal{E}$.

Let $A \in \mathcal{E}$.

If A is countable, then it can be written as a countable union of singleton sets. Therefore, $A \in \sigma(\mathcal{A})$.

If A^c is countable, then $A^c \in \sigma(\mathcal{A})$. Since $\sigma-$algebra is closed under taking compliment, $A \in \sigma(\mathcal{A})$. Therefore, $\mathcal{E} \subset \sigma(\mathcal{A})$.

Claim: f is measurable iff there exists $a \in \mathbb{R}$ such that $\{x : f(x) \neq a \text{ is countable}\}$.

Assume $f : (\mathbb{R}, \sigma(\mathcal{A})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable. Denote

$$A = \{x \in \mathbb{R} : f^{-1}(-\infty, x) \text{ is countable}\} \text{ and } B = \{x \in \mathbb{R} : f^{-1}(x, \infty) \text{ is countable}\}.$$

If A is empty, then $f^{-1}((-\infty, x))$ is uncountable for all $x \in \mathbb{R}$. Since f is measurable and $f^{-1}((-\infty, x)) \in \sigma(\mathcal{A})$, we then have $f^{-1}([x, \infty))$ is countable for all $x \in \mathbb{R}$. On the other hand,
$$\mathbb{R} = f^{-1}(\mathbb{R}) = \cup_{i\in\mathbb{Z}} f^{-1}([i, \infty)),$$
which is a countable union of countable sets. This contradicts the fact that $\mathbb{R}$ is uncountable. Therefore, A is not empty.

Similarly, if $B = \emptyset$, then $f^{-1}((x, \infty))$ is uncountable for all $x \in \mathbb{R}$. Since f is measurable and $f^{-1}((x, \infty)) \in \sigma(\mathcal{A})$, we then have $f^{-1}((-\infty, x])$ is countable for all $x \in \mathbb{R}$. On the other hand,

$$\mathbb{R} = f^{-1}(\mathbb{R}) = \cup_{i \in \mathbb{Z}} f^{-1}((-\infty, i])),$$

which is a countable union of countable sets. This contradicts the fact that $\mathbb{R}$ is uncountable. Therefore, B is not empty.

Claim: A is bounded above.

Assume A is not bounded above. Then, $\mathbb{R} = f^{-1}(\mathbb{R}) = \cup_{i=1}^{\infty} f^{-1}((-\infty, i))$, which is countable (contradicts the cardinality of $\mathbb{R}$).

Clam: B is bounded below.

Assume B is not bounded below, then $\mathbb{R} = f^{-1}(\mathbb{R}) = \cup_{i=1}^{\infty} f^{-1}((-i, \infty))$, which is countable (contradicts the cardinality o $\mathbb{R}$).

Now we can define

$$a = \sup_{A} \text{ and } b = \inf_{B}.$$

Claim: $a = b$.

1. Assume $a < b$. We can find $c \in (a, b)$ such that,

$$f^{-1}((-\infty, c)) \text{ and } f^{-1}([c, \infty)) \text{ uncountable.}$$

Since $f^{-1}((-\infty, c)) \in \sigma(\mathcal{A})$, either $f^{-1}((-\infty, c))$ or $[f^{-1}((-\infty, c))]^c = f^{-1}([c, \infty))$ is countable (contradiction).

2. Assume $a > b$. Then, for $\varepsilon, \delta > 0$, there exists $x_\varepsilon \in A$, $y_\delta \in B$ such that $x_\varepsilon > a - \varepsilon$ and $y_\delta < b + \delta$. Therefore, by controlling the size of ε and δ, we can choose

$$x \in A \text{ and } y \in B \text{ such that } x > y.$$

$$\mathbb{R} = f^{-1}((-\infty, x)) \cup f^{-1}((y, \infty)),$$

which is countable (contradicts the cardinality of $\mathbb{R}$).

Therefore, $a = b$.

$$\begin{aligned} f^{-1}(\{a\}) &= (f^{-1}((-\infty, a)) \cup f^{-1}((a, \infty)))^c \\ &= \underbrace{[\cup_{i \in \mathbb{N}} f^{-1}((-\infty, a - \frac{1}{n}))] \bigcup [\cup_{i \in \mathbb{N}} f^{-1}((a + \frac{1}{n}, \infty))]^c}_{\text{countable union of countable sets}}. \end{aligned}$$

Hence, $\{x \in \mathbb{R} | f(x) \neq a\}$ is countable.

Conversely, assume $f : \mathbb{R} \to \mathbb{R}$ and there exists $a \in \mathbb{R}$ such that $\{x \in \mathbb{R}, f(x) \neq a\}$ is countable. Assume $x \in \mathbb{R}$.

1. If $x \geq a$,

$$[f^{-1}((-\infty, x))]^c \subset [f^{-1}((-\infty, a])]^c$$

is countable and $f^{-1}((-\infty, x)) \in \mathcal{E}$.

2. If $x < a$,

$$f^{-1}((-\infty, x)) \subset f^{-1}((-\infty, a)),$$

is countable and $f^{-1}(-\infty, x) \in \mathcal{E}$.

Since the choice of $x \in \mathbb{R}$ is arbitrary, $f : (\mathbb{R}, \mathcal{E}) \to (\mathbb{R}, \mathcal{B})$ is measurable.

(As for why checking for sets like $(-\infty, t)$ is enough, just look at Lemma 1 in Prob 6, a similar argument to that will show this.)

$\square$

Exercise 6. *Prob 6*

Proof.

Lemma 1. *If a function $f : (X, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is such that $\forall t \in \mathbb{R}$*

$$f^{-1}\big([t, \infty)\big) \in \mathcal{F}$$

where f^{-1} means the pre-image, we know that f is measurable.

Proof. (Of Lemma):

We first show that $\mathfrak{M} := \sigma\left(\left\{[t,\infty)\middle| t \in \mathbb{R}\right\}\right) = \mathcal{B}(\mathbb{R})$, i.e. the smallest σ-algebra containing all sets of form $[t, \infty)$ is $\mathcal{B}(\mathbb{R})$, the Borel algebra on $\mathbb{R}$.

For any $O \in \mathcal{O}(\mathbb{R})$, i.e. O is open in $\mathbb{R}$, $O \in \mathfrak{M}$. But it suffices to show that a single open interval is in $\mathfrak{M}$, since O is the union of open intervals. Let $\varepsilon_n = 2^{-n}$, we have

$$(a, b) = \left(\bigcap [a - \varepsilon_n, \infty)\right) \backslash [b, \infty) = \left(\bigcap [a - \varepsilon_n, \infty)\right) \cap [b, \infty)^c$$

it is a combination of intersections and complements of sets of the form $[t, \infty)$, thus $(a, b) \in \mathfrak{M}$. By above argument, $O \in \mathfrak{M}$.

But then any element in $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{O}(\mathbb{R}))$, i.e. it is the smallest σ-algebra containing $\mathcal{O}(\mathbb{R})$. Yet $\mathcal{O}(\mathbb{R}) \subset \mathfrak{M}$, so $\mathcal{B}(\mathbb{R}) \subset \mathfrak{M}$. But then because every $[t, \infty)$ is in $\mathcal{B}(\mathbb{R})$, $\mathfrak{M} \subset \mathcal{B}(\mathbb{R})$. Hence $\mathfrak{M} = \mathcal{B}(\mathbb{R})$.

Now, any $M \in \mathcal{B}(\mathbb{R})$, we know $M \in \mathfrak{M}$, and thus M is a combination of unions and complements. So we still need to show that pre-images preserve these operations. Let's assume that $f^{-1}(T_i) \in \mathcal{F}$, for unions we have:

$$f^{-1}\left(\bigcup_{i=1}^{\infty} T_i\right) = \left\{x \middle| f(x) \in \bigcup_{i=1}^{\infty} T_i\right\} = \bigcup_{i=1}^{\infty} \left\{x \middle| f(x) \in T_i\right\} \in \mathcal{F}$$

and for complement:

$$f^{-1}\left(T_1^c\right) = \left\{x \middle| f(x) \in T_1^c\right\} = X \backslash \left\{x \middle| f(x) \in T_1\right\} \in \mathcal{F}.$$

Hence $f^{-1}(M) \in \mathcal{F}$, which means that f is measurable. □

Now we start prove the question:

Lemma 2. *Assume $f, g : (X, \Omega) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable, then $f + g$ is measurable.*

Let f, g be two measurable functions and $t \in \mathbb{R}$. Then, $f(x)+g(x) < t$ iff $f(x) < t-g(x)$ iff there exists $q \in \mathbb{Q}$ such that $f(x) < q < t - g(x)$. Therefore,

$$\begin{aligned}\{x \in X : f(x) + g(x) < t\} &= \{x \in X : f(x) < t - g(x)\} \\ &= \cup_{q\in\mathbb{Q}}[\underbrace{\{x \in X : f(x) < q\}}_{measurable} \cap \underbrace{\{x \in X : t - g(x) > q\}}_{measurable}].\end{aligned}$$

Since countable union of measurable sets is measurable, $(f+g)^{-1}(-\infty, t)$ is measurable and $f+g$ is measurable.

Claim: $\phi \in \mathcal{M}$ for every simple function ϕ.

Assume $\phi = \sum_{i=1}^{n} c_i \mathcal{X}_{A_i}$, where $c_m \in \mathbb{R}$ and $A_i \in \mathcal{F}$. Since the sum of measurable functions is measurable, we only need to show that $c_1\mathcal{X}_{A_1}$ is measurable. Let $c \in \mathbb{R}$, then,

$$\{w \in \Omega : c_1\mathcal{X}_{A_1}(w) < c\} = \begin{cases} \emptyset & c_1 < 0, c < c_1 \\ A_1 & c_1 < 0, c_1 \leq c < 0 \\ \Omega & c_1 < 0, c \geq 0 \\ \emptyset & c_1 \geq 0, c < 0 \\ A_1^c & c_1 \geq 0, 0 \leq c < c_1 \\ \Omega & c_1 \geq 0, c_1 \leq c \end{cases},$$

which is measurable in either scenario. Hence, $c_1\mathcal{X}_{A_1}$ is measurable and ϕ is measurable.

$\mathcal{M}$ is closed under point wise limits:

Assume $f_n \in \mathcal{M}$ and $\lim\limits_{n\to\infty} f_n = f$ point wise. Then we need to show $f \in \mathcal{M}$, which by lemma 1 we only need to show that $f^{-1}([t,\infty)) \in \mathcal{F}$ for any t. But this is because

$$f^{-1}([t,\infty)) = \left\{x \middle| f(x) \geq t\right\} = \left\{x \middle| \lim_{n\to\infty} f_n(x) \geq t\right\}$$

$$= \left\{x \middle| \limsup_{n\to\infty} f_n(x) \geq t\right\} = \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} f_m^{-1}([t,\infty)) \in \mathcal{F}.$$

Thus, $\mathcal{M}$ is closed under point wise limits.

(Yet another simpler way is that, since the point-wise limit is the limsup, and we've proven in class that limsup and liminf preserves measurability, so is the limit.)

Lemma 3. *Let $(\Omega, \mathcal{F})$ be a measure space. Assume $f, g : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable. Then,* $\max\{f, g\}$ *and* $\min\{f, g\}$ *is measurable.*

Let $t \in \mathbb{R}$,

$$\{x \in \Omega, \max\{f,g\} < t\} = \{x \in \Omega, f(x) < t, g(x) < t\} = \{x \in \Omega : f(x) < t\} \cap \{x \in \Omega, g(x) < t\},$$

which is an intersection of measurable sets. Therefore, $\max\{f,g\}$ is measurable. Similarly,

$$\{x \in \Omega, \min\{f,g\} > t\} = \{x \in \Omega, f(x) > t, g(x) > t\} = \{x \in \Omega : f(x) > t\} \cap \{x \in \Omega, g(x) > t\},$$

which is an intersection of measurable sets. Therefore, $\min\{f,g\}$ is measurable.

Lemma 4. *Let $(\Omega, \mathcal{F})$ be a measure space. Assume $f : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable and $c \in \mathbb{R}$, then cf is measurable.*

Let $t \in \mathbb{R}$. WLOG, we may assume $t \neq 0$. Otherwise, it can be viewed as a simple function with zero coefficient, which is measurable.

$$\{x \in \Omega, cf(x) < t\} = \begin{cases} \{x \in \Omega, f(x) < \frac{t}{c}\} & c > 0 \\ \{x \in \Omega, f(x) > \frac{t}{c}\} & c < 0 \end{cases},$$

which is measurable in either scenario. Hence, cf is measurable.

<u>Claim: $\mathcal{M} \subset \tilde{\mathcal{M}}$.</u>

Let $f \in M$. Then we can write $f = f^+ - f^-$, where $f^+ = \max\{f, 0\}$, $f^- = -\min\{f, 0\}$. Since f and 0 are measurable functions, by the previous lemmas, f^+ and f^- are measurable. Moreover, they are non-negative. As shown in the lecture note, there exist two sequences of simple functions $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ such that

$$a_n \to f^+ \text{ and } b_n \to f^- \text{ pointwise as } n \to \infty .$$

$$f(x) = f^+(x) - f^-(x) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n(x) - b_n(x) \text{ for all } x \in \Omega.$$

Lemma 5. *Assume f_1 and f_2 are two simple functions. Then, $f \pm g$ is a simple function.*

Let $\{r_1, \dots, r_M\}$ be the range of $f \pm g$ for some $M \in \mathbb{N}$. By the previous lemmas, $f \pm g$ is measurable. Moreover, $(f \pm g)^{-1}\{r_i\}$ is measurable for $i = 1, \dots, M$ as $\{r_i\} \in \mathcal{B}(\mathbb{R})$. Also,

$$f \pm g = \sum_{i=1}^{M} r_i \mathcal{X}_{(f\pm g)^{-1}\{r_i\}},$$

which is a simple function.

Define $z_n := a_n - b_n$ for $n \in \mathbb{N}$. Then,

$$f(x) = \lim_{n\to\infty} z_n(x) \text{ for all } x \in \Omega.$$

Since $\tilde{M}$ consists of all simple functions and is closed under the pointwise limit. We then have $f \in \tilde{\mathcal{M}}$ and $\mathcal{M} \subset \tilde{\mathcal{M}}$.

$\square$

Exercise 7. *Prob 7.*

Proof.

$\dfrac{\sin x}{x}$ is not integrable:

Let $f := \dfrac{\sin x}{x}$ on $[1, \infty)$. Then we can rewrite it's positive and negative part as

$$f^+(x) = \frac{\sin x}{x} \cdot \left(\mathbb{1}_{[1,\pi]} + \sum_{n=1}^{\infty} \mathbb{1}_{[2n\pi,(2n+1)\pi]} \right)$$

and

$$f^-(x) = -\frac{\sin x}{x} \cdot \left(\sum_{n=1}^{\infty} \mathbb{1}_{[(2n-1)\pi,2n\pi]} \right).$$

And hence

$$\begin{aligned}\int f^+ d\mu &= \int_1^{\pi} \frac{\sin x}{x} d\mu + \sum_{n=1}^{\infty} \int_{2n\pi}^{(2n+1)\pi} \frac{\sin x}{x} d\mu \geq \sum_{n=1}^{\infty} \int_{2n\pi}^{(2n+1)\pi} \frac{\sin x}{x} d\mu \\ &\geq \sum_{n=1}^{\infty} \frac{1}{(2n+1)\pi} \int_{2n\pi}^{(2n+1)\pi} \sin x d\mu = \sum_{n=1}^{\infty} \frac{1}{(2n+1)\pi} \cdot 1 = \infty\end{aligned}$$

and

$$\begin{aligned}\int f^+ d\mu &= \sum_{n=1}^{\infty} \int_{(2n-1)\pi}^{2n\pi} -\frac{\sin x}{x} d\mu \\ &\geq \sum_{n=1}^{\infty} \frac{1}{2n\pi} \int_{(2n-1)\pi}^{2n\pi} - \sin x d\mu = \sum_{n=1}^{\infty} \frac{1}{2n\pi} \cdot 1 = \infty\end{aligned}$$

which means that f is not integrable.

What can you say about the limit $\lim_{N\to\infty} \int_1^N \dfrac{\sin x}{x} dx$:

The limit exists since we can separate the integral into segments of length 2π (we can ignore the first segment since it's finite) i.e.

$$\int_1^N \frac{\sin x}{x} = \int_1^{2\pi} \frac{\sin x}{x} + \int_{2\pi}^{4\pi} \frac{\sin x}{x} + \cdots + \int_{2k\pi}^{N} \frac{\sin x}{x}$$

and for each small segment we have

$$\int_{2m\pi}^{2(m+1)\pi} \frac{\sin x}{x} \leq \int_{2m\pi}^{(2m+1)\pi} \frac{\sin x}{2m\pi} + \int_{2m\pi+\pi}^{2(m+1)\pi} \frac{\sin x}{2(m+1)\pi} = \frac{1}{m(m+1)\pi}$$

where since $\{a_n\} = \dfrac{1}{n(n+1)\pi}$ is absolutely summable, the integral is bounded by a absolutely convergent series, hence the limit exists, i.e. (WLOG assume $N > 2k\pi$ for convenience)

$$\lim_{N\to\infty}\int_1^N \frac{\sin x}{x}dx \leq \int_1^{2\pi}\frac{\sin x}{x} + \int_{2k\pi}^N \frac{\sin x}{x} + \sum_{n=1}^{k-1}\frac{1}{n(n+1)\pi} < \infty.$$

We can also tell this since $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$ and the function is positive on $[0,1]$ (the integral sign here is a limit of proper integrals).

Moreover, with any calculator we can get

$$\lim_{N\to\infty}\int_1^N \frac{\sin x}{x}dx \approx 0.6247132564277136.$$

Is the function $f(x) = x^2\sin\left(\frac{1}{x^2}\right)$ and it's derivative Lebesgue integral on $[0,1]$:

f is integrable since we can rewrite the integral in the following way:

$$\begin{aligned}\int_0^1 |x^2\sin\left(\frac{1}{x^2}\right)|dx &= \int_0^1 f^+dx + \int_0^1 f^-dx\\ &= \left(\int_{1/\pi}^1 f\,dx + \sum_{n=1}^\infty \int_{1/((2n+1)\pi)}^{1/(2n\pi)} f\,dx\right) + \left(\sum_{n=1}^\infty \int_{1/(2n\pi)}^{1/((2n-1)\pi)} f\,dx\right)\end{aligned}$$

but for each segment except the first we can change of variable to get

$$\int_{1/(k+1)\pi}^{1/k\pi} x^2\sin\left(\frac{1}{x^2}\right)dx = -\int_{1/(k+1)\pi}^{1/k\pi} x^4\sin\left(\frac{1}{x^2}\right)\left(-\frac{dx}{x^2}\right) = \int_{k\pi}^{(k+1)\pi}\frac{\sin(y^2)}{y^4}dy$$

and applying that to each segment we get

$$\int_{1/(k+1)\pi}^{1/k\pi} x^2\sin\left(\frac{1}{x^2}\right)dx = \int_1^\infty \frac{\sin(y^2)}{y^4}dy$$

where since $g(y) = \frac{1}{y^4}$ is L^1, we get that $\left(\frac{\sin(y^2)}{y^4}\right)^+ \leq g$, $\left(\frac{\sin(y^2)}{y^4}\right)^- \leq g$ and hence both are finite, which means that f is integrable.

Now as for $f'(x) = 2\sin\left(\frac{1}{x^2}\right)\frac{1}{x} - 2x\cos\left(\frac{1}{x^2}\right)$, after a similar change of variable the sin term becomes $2\sin(y^2)/y$, which we can use the very same method in the first part of this question to prove to be not integrable; Whereas the cos term goes into $2\cos(y^2)/y^3$ is integrable.

Thus f' is the sum of an integrable function and a non-integrable function, hence not integrable.

$\square$