APPROXIMATION THEORY HOMEWORK 4

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STAT 31220
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Discussed with classmates.

Exercise 1.

Proof.

(a):

$||f||_1 \leq c||f||_2$:

We use the Cauchy Schwartz inequality in the usual L^2 norm:

$$||f||_1^2 = \left(\int_a^b |f|w dx\right)^2 = \left(\int_a^b [|f|\sqrt{w}] \cdot \sqrt{w} dx\right)^2$$
$$\overset{C.S.}{\leq} \int_a^b (\sqrt{w})^2 dx \int_a^b [|f|\sqrt{w}]^2 dx = (c||f||_2)^2$$

and taking off the square on both sides we are done.

$||f||_2 \leq c||f||$:

This is direct:

$$||f||_2^2 = \int_a^b f^2 w dx \leq ||f||^2 \int_a^b w dx$$

and taking square root on both sides gives the result. Note that we can do this because everything is positive.

(b):

Because polynomials separate points and nowhere vanishes in $[a, b]$, so by Stone Wierstrass we know that they are dense in $C[a, b]$ in the uniform norm.

But this means for all $f \in C[a, b]$, $\exists p$ such that $||f - p||_1 \leq c||f - p||$ and $||f - p||_1 \leq c||f - p||$. So by choosing $||f - p|| \leq \varepsilon/c$ we've shown that all points in $C[a, b]$ can be approximated by polynomials in $|| \cdot ||_1$ and $|| \cdot ||_2$. Hence dense in those norms.

For not-completeness of $||\cdot||_1$, we pick point $c \in (a,b)$ such that $w \neq 0$. Then we construct the sequence of approximate identity on $\mathbb{R}$

$$g_n(x) = \begin{cases} \frac{n}{2} & |x-c| \leq \frac{1}{n} \\ 0 & \text{else} \end{cases}$$

and define the truncation on $[a,b]$ by

$$f_n = g_n \cdot \mathbb{1}_{[a,b]}$$

then we make it continuous by linear interpolation around the endpoints. Then f_n can be defined in that manner such that f_n has only ε length. Note that when n is large this indicator function does not affect our result since $c \in (a,b)$.

By continuity of w we know that for n large and ε small

$$\int_a^b f_n w dx \xrightarrow{\varepsilon\to 0} \fint_{c-1/n}^{c+1/n} w dx \to w(c) = \int_a^b \delta_c w dx$$

is the average of w on the ball around point c. Since there is a limit the sequence f_n is Cauchy in the $||\cdot||_1$ norm defined here, which has expression exactly as above,i.e.

$$||f_n - f_{m>n}||_1 = \int_a^b f_n w dx - \int_a^b f_m w dx \to 0$$

But then $f_n \to \delta_c$ and limit is unique in space of temperate distribution, which is not in $C[a,b]$, so $C[a,b]$ is not complete under $||\cdot||_1$.

Let's just take

$$h_n := \sqrt{\frac{f_n}{w(c)}}$$

then h_n is a sequence of squeezed approximated identity that has the good property of normalizing $w(c)$, that is

$$||h_n||_2 = \left(\int_a^b h_n^2 w\right)^{\frac{1}{2}} \to \left(\frac{w(c)}{w(c)}\right)^{\frac{1}{2}} = 1 = \int_a^b \frac{1}{w(c)}\delta_c w dx$$

and thus

$$||h_n - h_{m>n}||_1 = \left(\int_a^b h_n^2 w dx - \int_a^b h_m^2 w dx\right)^{\frac{1}{2}} \to 0$$

Cauchy by above convergence. Since the limit is $\dfrac{\delta_c}{w(c)} \notin C[a,b]$ we've shown that $C[a,b]$ is also not complete under $||\cdot||_2$.

(c):

It's basically a recap of the second part of Hilbert space projection theorem.

$\underline{\langle f - p_*, p\rangle_w = 0 \text{ for all } p \in P_{n-1}:}$

First, note that the $||\cdot||_2$ norm is really induced by $\langle \cdot, \cdot \rangle_w$ since

$$||f||_2^2 = \int_a^b f^2 w dx = \langle f, f \rangle_w$$

and so our deduction below are justified.

Since p_* is optimal we have that for any $\lambda \neq 0$ and $\forall p \in P_{n-1}$, also note that our inner product is symmetric for real functions (which is what we're considering) so

$$||f - p_*||_2^2 \leq ||f - p_* + \lambda p||_2^2 = ||f - p_*||_2^2 + ||\lambda p||_2^2 + 2\lambda \langle f - p_*, p \rangle$$

thus

$$-2\lambda \langle f - p_*, p \rangle \leq ||\lambda p||_2^2 = |\lambda|^2 ||p||_2^2$$

where this also work for $\lambda = -\lambda$ so we can apply absolute value on both sides and get

$$2|\lambda| \langle f - p_*, p \rangle \leq |\lambda|^2 ||p||_2^2$$

$$\Rightarrow 2\langle f - p_*, p \rangle \leq |\lambda| ||p||_2^2$$

so we just send $\lambda \to 0$ and we get

$$\langle f - p_*, p \rangle = 0.$$

Best approximation is unique:

Let's say that p and q are both best approximations, then

$$\langle f - p, q - p \rangle = 0 = \langle f - q, q - p \rangle$$

and thus

$$\langle q - p, q - p \rangle = \langle f - p, q - p \rangle - \langle f - q, q - p \rangle = 0$$

so $||p - q||_2^2 = 0$. This does imply $p = q$ since $(p - q)^2 \geq 0$ and $w > 0$ is strictly positive.

(d):

If it does not oscillate n times, then $f - p_*$ has at most $n - 1$ zeroes, call those points $y_1, \dots, y_m$ where $m \leq n - 1$. Then define (with multiplicity)

$$q = \pm\delta \prod_{i=1}^{m} (x - y_i)$$

where the sign is chosen so that $q \cdot (f - p_*) \leq 0$ everywhere (guaranteed by continuity). Then we just choose small enough δ such that for some x_* (hence a small ball around it)

$$|f - p_* + q|(x_*) < |f - p_*|(x_*)$$

and for all x

$$|f - p_* + q|(x) \leq |f - p_*|(x)$$

thus we have

$$||f - p_* + q||_2^2 = \int_a^b |f - p_* + q|^2 w dx < \int_a^b |f - p_*|^2 w dx = ||f - p_*||_2^2$$

contradiction to the fact that p_* is the best approximation.

(e):

Just by definition of what is the least-square best approximation we know

$$||f - p_{*,2}^{(n)}||_2 \leq ||f - p_{*,\infty}^{(n)}||_2$$

since the polynomial in the norm in RHS can be any $p \in P_{n-1}$.

But we know by (b) that

$$||f - p_{*,\infty}^{(n)}||_\infty \to 0$$

and hence by (b)

$$||f - p_{*,\infty}^{(n)}||_2 \leq c \cdot ||f - p_{*,\infty}^{(n)}||_\infty \to 0$$

thus

$$||f - p_{*,2}^{(n)}||_2 \leq ||f - p_{*,\infty}^{(n)}||_2 \to 0$$

and we are done.

(f): skip.

□

Exercise 2.

Proof.

(a) Of course the integral formula still makes sense because the contour we've chosen is everywhere fine on the contour integral. So let's just see what our l_j are.

Now, remember that we get the formula by noting that (for non-repeated roots)

$$\frac{1}{2\pi i}\int_{\Gamma_i}\frac{f(t)}{l(t)(x-t)} = \operatorname{Res}\left(\frac{f(t)}{l(t)(x-t)};x_i\right) = f(x_j)l_j(x)\frac{1}{l(x)}$$

and by timing $l(x)$ on both sides we get the result. So the only difference in generalizing is we compute the second or higher order residue (since denominator is polynomial it must be poles), and thus for x_i repeated k times that

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\Gamma_i}\frac{l(x)f(t)}{(x-t)\prod_{j=1}^n(t-x_j)} &= \operatorname{Res}\left(\frac{l(x)f(t)}{l(t)(x-t)};x_i\right)\\
&= \frac{1}{(k-1)!}\left((t-x_i)^k\cdot\frac{l(x)f(t)}{(x-t)\prod_{j=1}^n(t-x_j)}\right)^{(k-1)}\\
&= \sum_{i=0}^{k-1}F(t,x,i)\cdot f^{(i)}(t)
\end{aligned}$$

where $F(t,x,i)$ is a polynomial coefficient of $f^{(i)}(t)$.

Then we will get that our Hermite formula works if we do the same for all repeated roots.

So now we focus on the case where the given $x=[x_0,x_0,x_2,\dots,x_n]$, thus the $l_{k\geq 2}$ has the same formula as before, but for l_1 and l_0 we need to run the residue theorem to get

$$\begin{aligned}
&l_0(x_0)\cdot f(x_0)+l_1(x_0)\cdot f'(x_0) = \operatorname{Res}\left(\frac{l(x)f(t)}{l(t)(x-t)};x_0\right)\\
&= \frac{1}{1!}\left[\frac{l(x)(t-x_0)^2}{l(t)(x-t)}\cdot\left(\frac{1}{x-t}-\sum_{i=2}^n\frac{1}{t-x_i}\right)\cdot f(t)+\frac{l(x)(t-x_0)^2}{l(t)(x-t)}\cdot f'(t)\right]_{t=x_0}\\
&= \frac{l(x)}{(x-x_0)\prod_{i=2}^n(x_0-x_i)}\left(\frac{1}{x-x_0}-\sum_{i=2}^n\frac{1}{x_0-x_i}\right)f(x_0)+\frac{l(x)}{(x-x_0)\prod_{i=2}^n(x_0-x_i)}f'(x_0)
\end{aligned}$$

thus

$$l_0(x) = \frac{l(x)}{(x-x_0)\prod_{i=2}^n(x_0-x_i)}\left(\frac{1}{x-x_0}-\sum_{i=2}^n\frac{1}{x_0-x_i}\right)$$

and

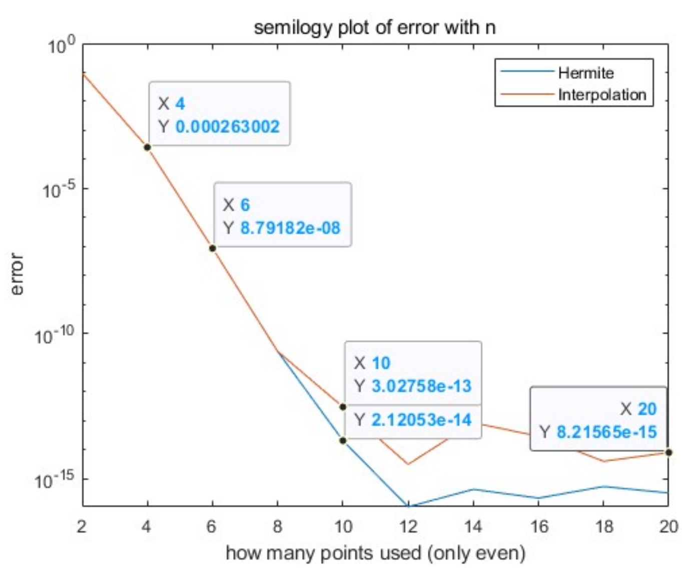
$$l_1(x) = \frac{l(x)}{(x-x_0)\prod_{i=2}^n(x_0-x_i)}.$$

(b):

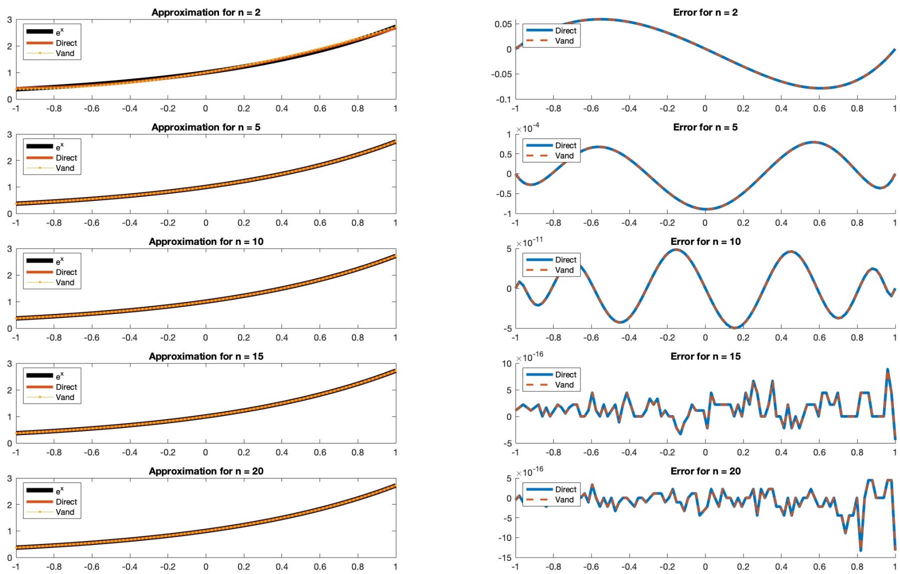
So I implemented the Hermite formula as well as the formula for new interpolations and summing things up, I got the following results for the points $y_1 = [-1, -1, 1, 1]$ and $y_1 = [-1, -1, 0, 0, 1, 1]$ (All generated by the file TommenixYu_q2b, except 1 pic later from last hw):

```
>> TommenixYu_q2b
Using points [-1,-1, 1, 1]

T =

  3×3 table

      Method               Value                    Error
    ___________     _________________     ____________________

    {'Hermite'}     0.904574415822641     0.000263002213318875
    {'Interp' }      0.90457441582264     0.000263002213319208
    {'f(1/3)' }      0.90483741803596                        0

Using points [-1, -1, 0, 0, 1, 1]

T =

  3×3 table

      Method               Value                    Error
    ___________     _________________     ____________________

    {'Hermite'}     0.904837330118218     8.79177420864963e-08
    {'Interp' }     0.904837330117742       8.791821770604e-08
    {'f(1/3)' }      0.90483741803596                        0
```

and for more points, say there's 2m points in total (so doubled first 10 Chebyshev points), then the decay of error is (generated by the file TommenixYu_q2b):



and recall from last time the chebyshev error is (Directly from pic in last hw):



and we see that the decay is a faster.

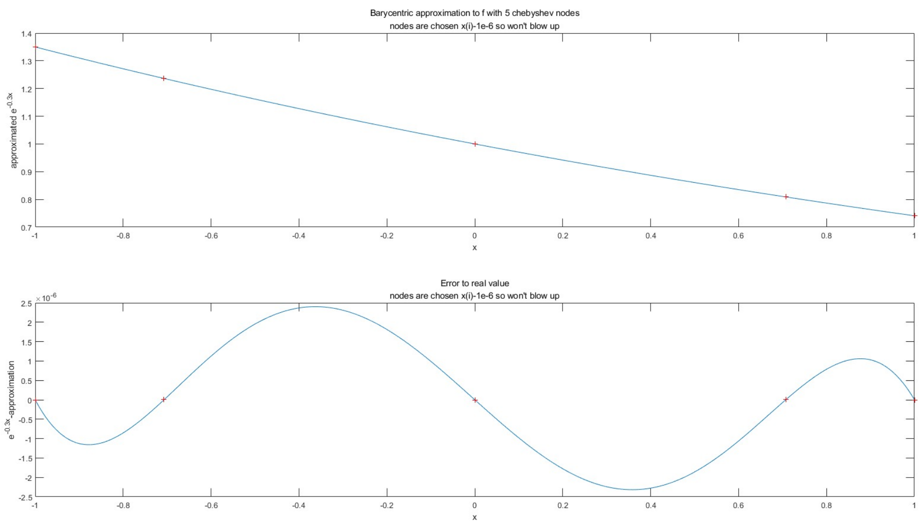
(c):

This is just by direct computation: We denote $C := d([-1,1],\Gamma)$, the we can bound

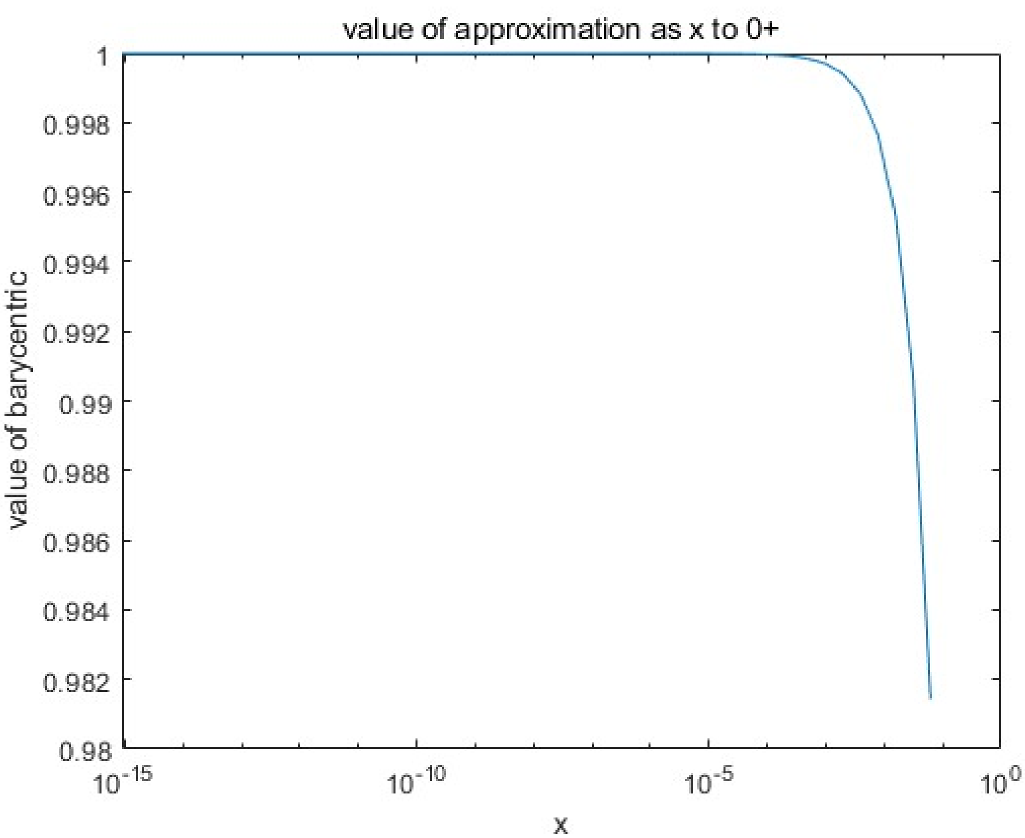
$$\begin{aligned}||f-p||_\infty &\le \frac{1}{2\pi}\int_\Gamma \left|\frac{l(x)f(t)}{l(t)(x-t)}\right| dt \le \frac{1}{2\pi C}||\Gamma|| \cdot ||f||_\infty \cdot \max_{x\in[-1,1],t\in\Gamma}\frac{|l(x)|}{|l(t)|}\\ &= \frac{1}{2\pi C}||\Gamma|| \cdot ||f||_\infty \cdot \frac{1}{\alpha_n^n}.\end{aligned}$$

(d):

I did it and the result of barycentric interpolation is (Generated by TommenixYu_q2d):



and since the barycentric formula has a removable singularity (denominator = nominator =0), so matlab will give NaN at the interpolation points, but we see that as the value goes to one of the points ($x_3 = 0$) from right side the curve is continuous (Generated by TommenixYu_q2d):



Note that this is just a semilogx scaled and zoomed in version of the first graph above.

So the function does converge to the exact value.

□

Exercise 3.

Proof.

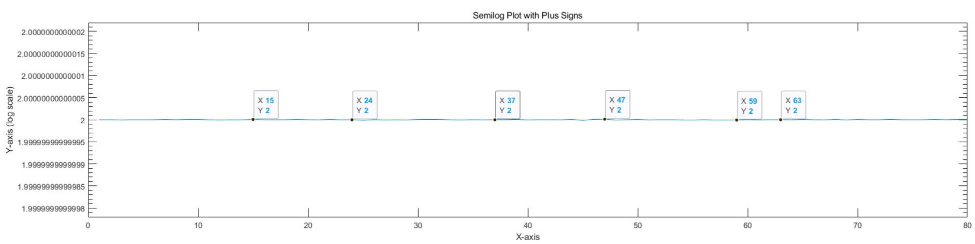
(a):

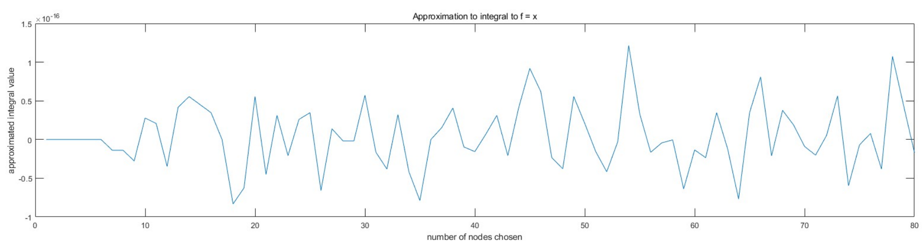
Note that $f \in L^2$ on a bounded space means $f \in L^1$. Thus using the fact that $e^{-(x-y)^2}$ is uniformly continuous on a compact set we can find suitable δ for each ε such that

$$T[f](x) - T[f](x+\delta) = \int_{-1}^{1} \left(e^{-(x-y)^2} - e^{-(x+\varepsilon-y)^2} \right) f(y)dy \leq \varepsilon ||f||_{L^1[-1,1]} \to 0$$

so $T[f]$ is continuous.

(b): The error is the following (generated by TommenixYu_q3b):





and we see that the first is just a line (even with log scale it does not change). In particular for $n = 40$ or 80 the results are very very good.

(c):

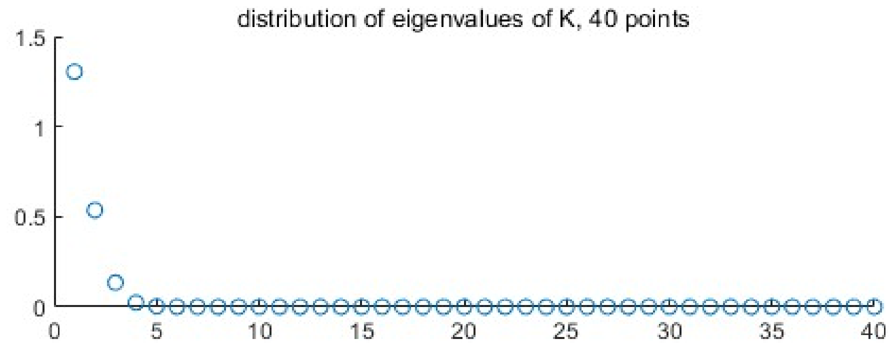
We compute that for each set of points, there is a corresponding w already known, and we want, say

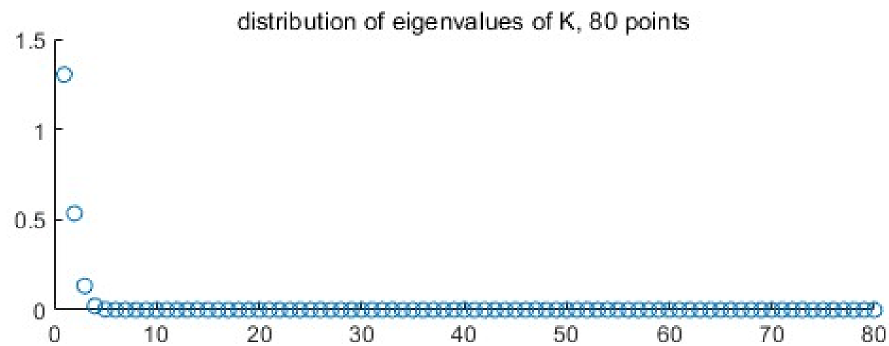
$$T[f](x_i) = \sum_{j=1}^{m} e^{-(x_i-x_j)^2} f(x_j) w_j$$

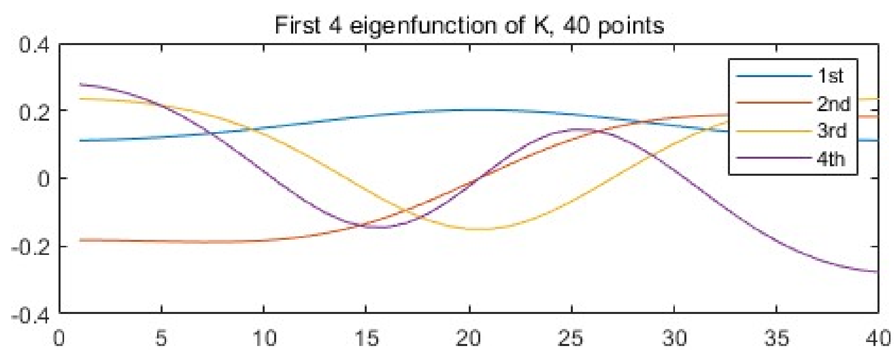
thus we know the matrix should be represented as

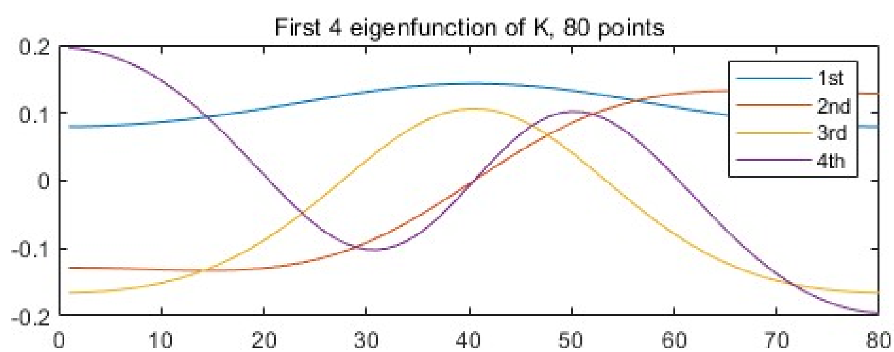
$$T_{i,j} = e^{-(x_i-x_j)^2} w_j.$$

And plotting it's eigenvalues and first 4 eigen functions we have (generated by TommenixYu_q3c):


distribution of eigenvalues of K, 40 points


distribution of eigenvalues of K, 80 points


First 4 eigenfunction of K, 40 points
1st
2nd
3rd
4th


First 4 eigenfunction of K, 80 points
1st
2nd
3rd
4th

which makes very much sense because the operator is a Hilbert-Schimdt operator, and thus its only accumulation point of eigenvalues is 0. $\square$