APPLIED DYNAMICAL SYSTEM HOMEWORK 4

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STAT 31410
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General ideas were discussed with many classmates in casual talks.

Exercise 1.

Proof. I solve the equation first.

For the system
$$\begin{cases} \dot{x} = x(x-1) \\ \dot{y} = -y \end{cases}$$
since x and y are not related and both equation is separable, we can just solve the ODEs by separation.

More specifically, if $x = 0$ then the solution in x is 0. And for $x \neq 0$,
$$\frac{dx}{dt} = x^2 - x \Rightarrow \frac{dx}{x^2 - x} = dt \Rightarrow x = \frac{1}{1 - ce^t}$$
and for $x(0) = x_0$,
$$x_0 = \frac{1}{1-c} \Rightarrow c = 1 - \frac{1}{x_0}$$
and for y the solution is simply
$$y = e^{-t} y_0.$$
Since $y_0 = 1$ on $\mathcal{B}$, the solution is
$$\begin{cases} x = \frac{1}{1 - \left(1 - \frac{1}{x_0}\right)e^t} \\ y = e^{-t} \end{cases}$$
for $(x_0, y_0) \in \mathcal{B} \backslash \{(0,1)\}$ and
$$\begin{cases} x = 0 \\ y = e^{-t} \end{cases}$$
for $(x_0, y_0) = (0, 1)$.

So we only need to construct a sequence that goes to $Y := \left(\frac{1}{2}, 0\right)$.

We do it by fixing t_n first and then fix $X_n := (x_n, 1)$ since whichever point we start in $\mathcal{B}$, the y coordinate after time t is the same.

Since we are not using start point $(0, 1)$ anyway, we define $\phi_t : \mathcal{B}\backslash\{(0, 1)\} \times [0, \infty) \to \mathbb{R}^2$ by:

$$\phi_t(x_0, 1) = \begin{pmatrix} \frac{1}{1-\left(1-\frac{1}{x_0}\right)e^t} \\ e^{-t} \end{pmatrix}.$$

So, by letting $e^{-t_n} = \dfrac{1}{2^n}$ we get the sequence

$$t_n = -\log\left(\frac{1}{2^n}\right) = n\log(2)$$

and we just let $\phi_{t_n}(x_n, 1) = \frac{1}{2}$ to get

$$\frac{1}{1-\left(1-\frac{1}{x_n}\right)e^{t_n}} = \frac{1}{2} \quad \Rightarrow \quad x_n = \frac{2^n - 1}{2^n}$$

which, as we'd expected, goes to 1.

So to sum up, let $(t_n)_{n=1,2,\ldots}$ be such that $t_n = n\log 2$, $(X_n)_{n=1,2,\ldots}$ such that $X_n = \left(\frac{2^n-1}{2^n}, 1\right)$, then we have

$$\phi_{t_n}(X_n) = \begin{pmatrix} 1/2 \\ 1/2^n \end{pmatrix} =: Y_n$$

then

$$Y_n \to \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

as $n \to \infty$.

So

$$Y := \left(\frac{1}{2}, 0\right) \in \omega(\mathcal{B}).$$

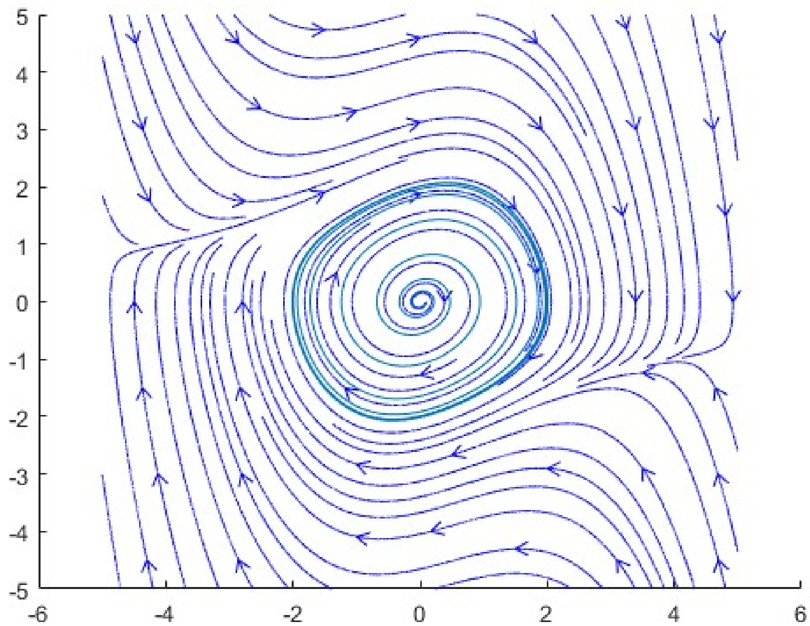
□

Exercise 2.

The ODE system is

$$\begin{cases} \dot{x} = 2y \\ \dot{y} = -2x + \frac{1}{2}(1 - x^2)y \end{cases}$$

For a start, I plotted the streamslice plot to get an idea of what's going on:



the blue flowline in middle is the flow starting from $(0.1, 0)$ (not in Σ).

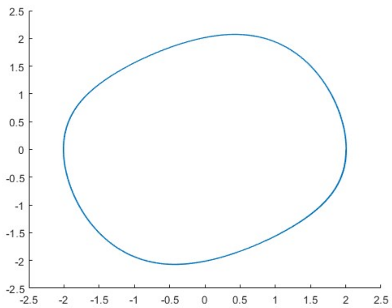
So we want to compute $Df(\gamma(t))$, the Monodromy matrix. For that, let's compute Df first:

$$Df = \begin{pmatrix} 0 & 2 \\ -2 - xy & \frac{1}{2}(1 - x^2) \end{pmatrix}$$

However, there was no way of actually finding out $\gamma(t)$ (except through some complicated Fourier series maybe), but there's a way to get around that by using a simulation on the cycle:

$$\begin{cases} \dot{y} = Df(x)y \\ \dot{x} = f(x) \end{cases} \tag{0.1}$$

This is a 4d ODE system. If we choose the initial value of x on the limit cycle, then the above first two lines were really a simulation of $\dot{y} = Df(\gamma(t))y$. So we first use ODE45 by starting from $(2, 0)$, which gives us:

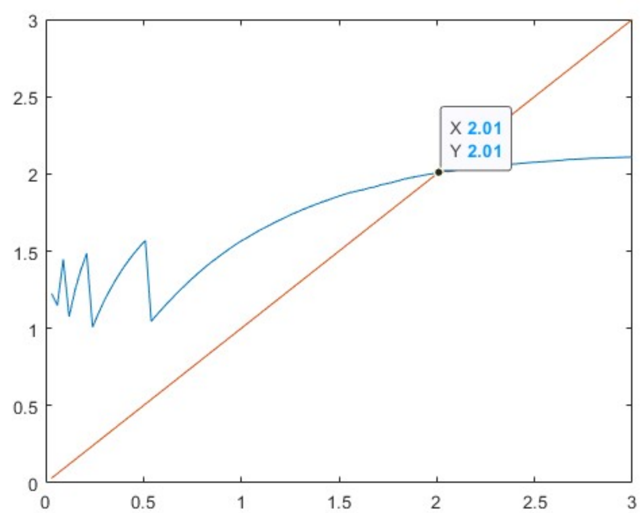


1.83109677297189	0.903043868557364
1.8583011848431	0.828746803726317
1.88317053077214	0.754432293964443
1.9057071632691	0.680324335964926
1.92592076895389	0.60663298021199
1.94382836855616	0.533554330981006

which is pretty close to our solution, and the result shown is the evolution after 10π. So this last data $(y_1, y_2) = (1.94382836855616, 0.533554330981006)$ should be a good approximation of a point on the cycle.

To get an approximate period T of γ, I wrote up a Poincare return function on the section $[1, 3] \times \{0\}$. The idea is that, since ODE45 returns a list of positions of x and the time t, I record the next time when the y-coordinate of x has absolute value below $tol = 0.01$. (I tried smaller tolerance, but even for 0.001 the step size required is smaller than $\pi/20000$). So when I use $(2, 0)$ as a start point, the Poincare map returns $t = 3.15$. (But the map should be larger than π since the tolerance actually make t smaller).

I also plotted the Poincare function and checked the fixed point:



And we have good reason to try find the derivative at that point $(2, 0)$. I used the center quotient $\dfrac{P(2+h) - P(2-h)}{2h}$ with $h = 0.1$ to get a result around 0.185.

Now we solve (0.1) using the initial condition $(1, 0, y_1, y_2)$ and $(0, 1, y_1, y_2)$ to the time T. The Monodromy matrix is the first two items of the above two initial conditions glued together, and the result is:

```
>> test2

ans =

        0.205239524057063         2.00919487330053        0.197582956256441       -0.0119493237179951
        0.992805734206353                    0.001        0.509541565670606          1.00046230200697
```

where the first column is the eigenvalue of M, the second is just a point on the cycle that is close to the x-axis (I used this to make sure to use $(2, 0)$ in some of the above tests), and the last for term is just M.

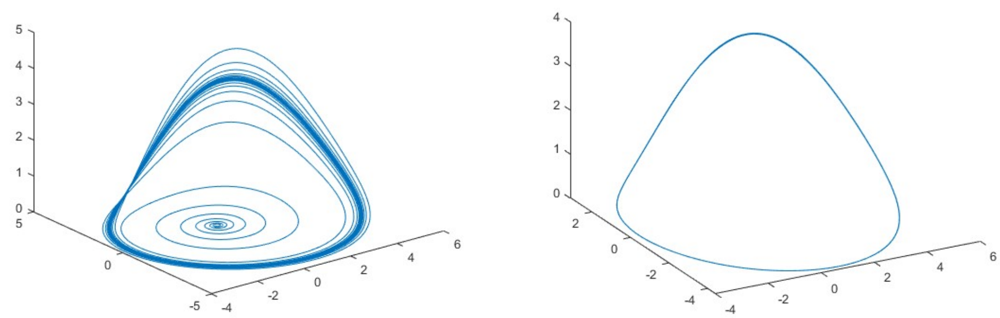
So indeed, one of the Floquet multipliers is around 1, and the other is 0.205, which is not that far from $P'(2, 0) \approx 0.185$. The error should come from the tolerance in my Poincare function.

Exercise 3.

The ODE system is

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = b + z(x - c) \end{cases}$$

As is discussed in class, the system is globally stable, so I just pick the origin to try to get to the limit cycle. And indeed within $t \leq 100\pi$ I've already arrived the limit cycle:



where the left is when I start at $(0,0,0)$, and the right is when I start from the endpoint in the first trial. I use the last data on the second trial for later use, which is

$$x(0) = (1.58266833818079, -4.37254296204156, 0.0879986516501191).$$

Now we do the same process as in question 2. We first find the period

$$T = 5.74581588378305$$

by finding the time corresponding to the first return to a ball of radius 0.01 to the point $x(0)$.

Then again we solve this equation

$$\begin{cases} \dot{y} = Df(x)y \\ \dot{x} = f(x) \end{cases} \tag{0.2}$$

with

$$Df = \begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x - c \end{pmatrix}$$

and starting point $(1, 0, 0, x(0))$, $(0, 1, 0, x(0))$, and $(0, 0, 1, x(0))$.

The resulting Floquet Multipliers are:

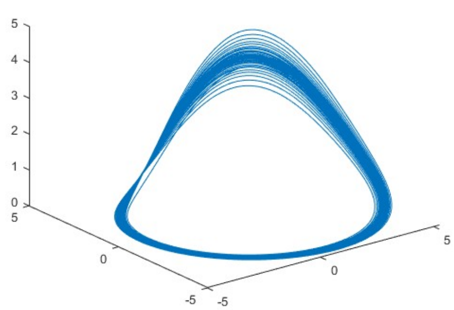
```
h =

       1.00381924914183
   0.000206875928545445
     -0.767604860547826

fx >>
```

So indeed it is stable, as it should be.

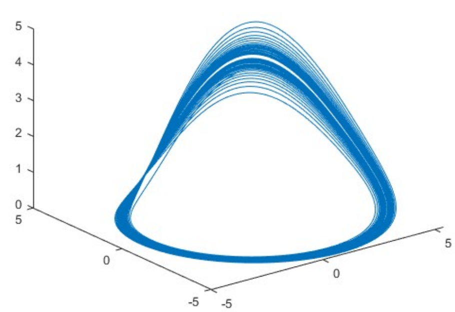
If we increase c, my result is that at $c \approx 2.765$, the small eigenvalue goes to around -0.95, then suddenly at $c = 2.766$, all three eigenvalues are positive. This I think is because of the tolerance of my period, i.e., the distance between the start point and the end point. And I suppose there's some error with it since there's occationally normal behavior of the eigenvalues for larger c. For instance, when $c = 2.773$, suddenly the smallest eigenvalue is negative and is -0.96.

After discussion with Su I realized that it might be the case that when we're near the bifurcation point, only the closeness of norm is not enough, so I tried to test for 20 consequent points, in this way even if there is bifurcation it should not change the result. However the result is the same. I did find that when $c \approx 2.81$ the eigenvalues starts to be complex. So I guess that there is the critical point.

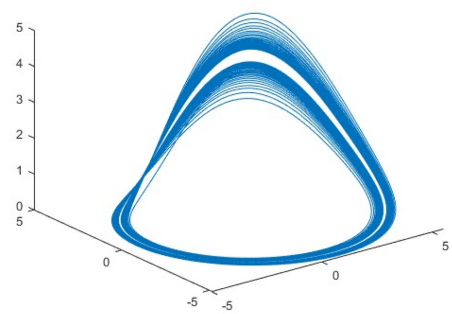
And indeed a bifurcation happens around that value. I have plotted a sequence for $c = 2.7, 2.75, 2.8, 2.85, 2.9$ during which period we can see a beautiful illustration of how this one limit cycle bifurcates into two:



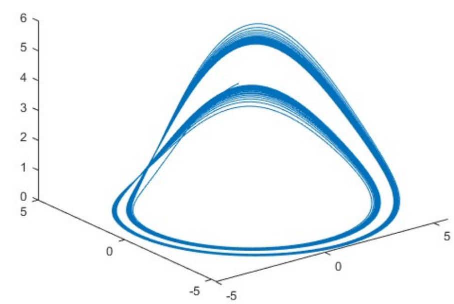
$c = 2.7$



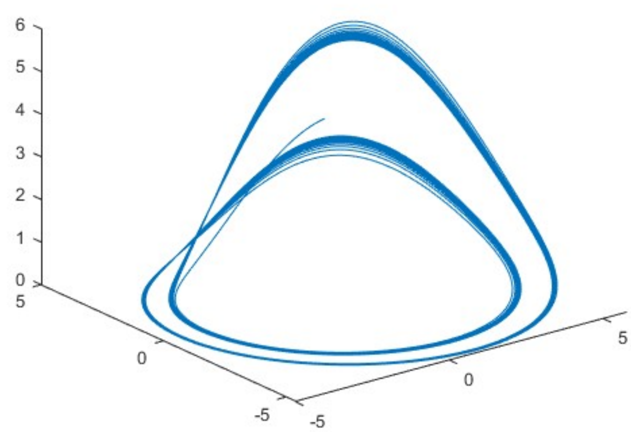
$c = 2.75$



$c = 2.8$



$c = 2.85$



$c = 2.9$