APPLIED FUNCTIONAL ANALYSIS HOMEWORK 2

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Discussed with classmates.

Exercise 1. *(2.3) in book*

Proof.

We first show that such an extension exists for uniform continuous f, then that it's unique, then give a counterexample to it for non-uniform continuous f.

Existence of extension:

Given $f : G \to \mathbb{R}$ uniformly continuous on open G, for every point $x \in G$, $\bar{f}(x) = f(x)$ by requirement that this is an extension. For $x \in \bar{G} \backslash G$, by definition of closeness we know that there exists some sequence $x_n \to x$.

Now we prove that the sequence in $\mathbb{R}$ $f(x_n) \to L$ for some $L \in \mathbb{R}$. We prove this by proving that the sequence $(f(x_n))$ is Cauchy, then use completeness of $\mathbb{R}$ we can conclude the existence of L.

For all $\varepsilon > 0$, $\exists \delta$ such that $\forall x_i, x_j \in X$ with $d(x_i, x_j) < \delta$, $|f(x_i) - f(x_j)| < \varepsilon$. But since $x_n \to x$, there exists N such that $\forall n, m > N$, $d(x_n, x_m) < \delta$ (convergent implies Cauchy). Hence, for that particular N, we have $\forall n, m > N$

$$|f(x_n) - f(x_m)| < \varepsilon$$

and since ε is arbitrary $f(x_n)$ is Cauchy, hence L exists. Let $\bar{f}(x) = L$.

We now show that it is well-defined, i.e. for any two sequences $(x_n) \to x$ and $(y_n) \to x$, the limit L is the same.

For the purpose of contradiction, assume $L_x \neq L_y$, then $|L_x - L_y| = c > 0$. Now let $\varepsilon = c/3$, then we can find corresponding δ for which $\forall a, b \in X$ with $d(a, b) < \delta$, $|f(a) - f(b)| < c/3$. Yet since $(x_n) \to x$ and $(y_n) \to x$, we can find N_x, N_y such that for all $n_x > N_x, n_y > N_y$, $d(x, x_{n_x}) < \delta/2$ and $d(x, y_{n_y}) < \delta/2$. Let $N = \max N_x, N_y$ we have for any $n, m > N$ we can get by triangle inequality

$$d(x_n, y_m) \leq d(x_n, x) + d(y_m, x) < \delta/2 + \delta/2 = \delta.$$

Now $\forall n, m > N$, we have

$$c = |L_x - L_y| \le |L_x - f(x_n)| + |f(x_n) - f(y_m)| + |f(y_m) - L_y| < 3 \cdot \frac{c}{3} = c$$

where the second inequality is because $d(x, x_n) < \delta/2 < \delta$, $d(x, y_m) < \delta/2 < \delta$ and $d(y_m, x_n) < \delta$. Contradiction! So the function $\bar{f}(x)$ is well-defined.

But is it continuous? We check that it is. Fix $\varepsilon > 0$, let δ be such that if $d(x, y) < \delta$, $|f(x) - f(y)| < \varepsilon/3$ for any $x, y \in G$. Since G is dense in $\bar{G}$, then $\forall x \in \bar{G}$ we can find $x_n \to x$ and $y_n \to y$. By convergence we know that there exist a M such that any $n_1 > M$ we have $d(x_{n_1}, x) < \delta/3$, $d(y_{n_1}, y) < \delta/3$.

Also, by definition of $\bar{f}$ there exists M' such that any $n_2 > M'$ we have

$$|\bar{f}(x) - \bar{f}(x_{n_2})| < \varepsilon/3 \text{ and } |\bar{f}(y) - \bar{f}(y_{n_2})| < \varepsilon/3.$$

Now choose $n = \max\{n_1, n_2\}$ we have

$$d(x_n, y_n) \le d(x_n, x) + d(x, y) + d(y, y_n) < \delta$$

so

$$|f(x_n) - f(y_n)| < \varepsilon/3.$$

Now we can just say that

$$|\bar{f}(x) - \bar{f}(y)| \le |\bar{f}(x) - \bar{f}(x_n)| + |\bar{f}(x_n) - \bar{f}(y_n)| + |\bar{f}(y_n) - \bar{f}(y)| \le 3 \cdot \frac{\varepsilon}{3} = \varepsilon$$

since f agrees with $\bar{f}$ on G. Now since x, y are arbitrary chosen we've proven that $\bar{f}$ is continuous.

$\bar{f}$ is unique:

This is almost obvious after proving that $\bar{f}$ is well-defined. If g, h both extends f, then they cannot disagree on G since they are extensions of f. If they disagree on $\bar{G}\backslash G$, then since they are both continuous, they are sequentially continuous, so for $x \in \bar{G}\backslash G$ with $x_n \to x$, by sequentially continuity

$$g(x) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} h(x_n) = h(x).$$

And so they must agree everywhere, i.e. $g - h \equiv 0$. So the extension is unique.

Counterexample:

Let $X = \mathbb{R}^2$, $G = \mathbb{R}^2\backslash\{0\}$ so that it's open. Then let

$$f(x, y) := \frac{y^2}{x^2 + y^2}$$

we will get that

$$\lim_{x\to 0+} f(x, 0) = 0 \neq 1 = \lim_{y\to 0+} f(0, y).$$

Also, f is continuous because it's a combination of continuous functions. It is not uniformly continuous around the origin. Anyway there's no extension, and we are done.

$\square$

Exercise 2. *(2.6) in book.*

Proof.

$\underline{C([a,b], ||\cdot||_1)\text{ is not complete:}}$

First we can WLOG assume $[a,b] = [0,1]$ since the function $f(x) = \dfrac{x-a}{b-a}$ is a diffeomorphism. (This is proven in class so I assume this).

Now we construct the sequence of functions $f_n \in C[0,1]$ be $f_n(x) := x^n$. Then f_n is Cauchy under $||\cdot||_1$ norm since $\forall \varepsilon > 0$, $\exists N = \left\lceil \dfrac{1}{\varepsilon} \right\rceil$ with $\forall n \geq N$

$$\int_0^1 |f_n(x)|dx = \frac{1}{n+1} < \varepsilon$$

i.e. $||f_n||_1 < \varepsilon$. Note that $f_n > 0$ is decreasing with n, so for any $N \leq n \leq m$

$$||f_n - f_m||_1 = \int_0^1 |f_n(x) - f_m(x)|dx \leq \int_0^1 |f_n(x)|dx < \varepsilon$$

so it's Cauchy. But the limit is the indicator function of $\{1\}$, which is not in $C[0,1]$, so the space is not complete.

Convergence in sup-norm means convergence in 1-norm:

If $f_n \to f$ in sup-norm we have that for any $\varepsilon > 0$, $|f_n(x) - f(x)| < \varepsilon$ for all $n > N$ and any $x \in [a,b]$, where N is fixed. But then we have

$$\int_a^b |f_n - f|dx \leq \varepsilon \cdot |b-a| = c \cdot \varepsilon$$

which means that by choosing $\varepsilon' = \dfrac{\varepsilon}{|b-a|}$ we can find the N with all $n > N$ satisfying $||f_n - f||_1 < \varepsilon$. So $f_n \to f$ in the 1-norm.

Convergence in 1-norm does not mean convergence in sup-norm:

For the same reason as above we assume $[a,b] = [0,1]$ WLOG and define a sequence of "spike function"

$$f_n(x) = \begin{cases} 0 & x \in \left[0, \frac{1}{n+1}\right] \\ \frac{2n^2+2n}{2n+1}\left(x - \frac{1}{n+1}\right) & x \in \left(\frac{1}{n+1}, \frac{2n+1}{2n^2+2n}\right] \\ -\frac{2n^2+2n}{2n+1}\left(x - \frac{1}{n}\right) & x \in \left(\frac{2n+1}{2n^2+2n}, \frac{1}{n}\right] \\ 0 & x \in \left(\frac{1}{n}, 1\right] \end{cases}$$

which is nothing but a spike on $\left(\frac{1}{n+1}, \frac{1}{n}\right)$. Since it's supported only on $\left(\frac{1}{n+1}, \frac{1}{n}\right)$ and $f_n(x) \leq 1$, $||f_n||_1 \leq \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n^2+n}$.

Now we show that $f_n \to f \equiv 0$ since $\forall \varepsilon > 0$, let N be such that $\frac{1}{N^2+N} \leq \varepsilon$, we have $\forall n > N$, $||f_n||_1 \leq \varepsilon/2$, which is just

$$||f_n - f||_1 < \varepsilon$$

so $f_n \to f$ in the 1-norm.

But every two f_n share no common support and each attains 1 at some point, so $||f_n - f_m||_\infty = 1$ for any $n \neq m$, which means that the sequence is not Cauchy, thus doesn't converge. $\square$

Exercise 3. *(2.9) in book.*

Proof.

$||\cdot||_w$ is a norm for $w > 0$ on $(0, 1)$:

We show property by property:

(1) $||f||_w \geq 0$ and $||f||_w = 0 \iff f = 0$.
Reason:

$$||f||_w = \sup_x \{w(x)|f(x)|\} \geq 0$$

since $w(x)|f(x)| \geq 0$ for all x.
($||f||_w \Rightarrow f = 0$):
Assume, for contradiction that $f \neq 0$. Then $f(x) \neq 0$ at some $x \in [0, 1]$. But then $w(x)|f(x)| > 0$ and thus $||f||_w$, the sup, is larger than 0. Contradiction! so $f = 0$.
($||f||_w \Leftarrow f = 0$):
Since $f = 0$ we have $|f(x)| = 0$ for all x and thus $||f||_w = 0$.

(2) $||\lambda f||_w = |\lambda| \cdot ||f||_w$ for $\lambda \in \mathbb{R}$.
Reason:

$$||\lambda f||_w = \sup_x \{w(x)|\lambda f(x)|\} = \lambda \sup_x \{w(x)|f(x)|\} = |\lambda| \cdot ||f||_w.$$

(3) $||f + g||_w \leq ||f||_w + ||g||_w$:
Reason:

$$\begin{aligned} ||f+g||_w &= \sup_x \{w(x)|f(x)+g(x)|\} \leq \sup_x \{w(x)|f(x)| + w(x)|g(x)|\} \\ &\leq \sup_x \{w(x)|f(x)|\} + \sup_x \{w(x)|g(x)|\} = ||f||_w + ||g||_w \end{aligned}$$

So it's a norm.

$||\cdot||_w$ is equivalent to the sup-norm for $w > 0$ on $[0, 1]$:

Since w is continuous and $[0, 1]$ is compact, its image is compact and hence bounded above and below by $0 < c \leq w(x) \leq C$. Now we have that

$$c||f||_\infty = \sup_x \{c|\lambda f(x)|\} \leq \sup_x \{w(x)|\lambda f(x)|\} \leq \sup_x \{C|\lambda f(x)|\} = C||f||_\infty$$

which is the same thing as

$$c||f||_\infty \leq ||f||_w \leq C||f||_\infty$$

so they are equivalent.

$||\cdot||_x$ is not equivalent to the sup-norm:

We construct the "truncated" $\dfrac{1}{x}$ function at $\dfrac{1}{n}$:

$$f_n = \begin{cases} n & x \le \frac{1}{n} \\ \frac{1}{x} & \frac{1}{n} < x \le 1 \end{cases}$$

and thus $||f_n||_x = 1$ where as $||f_n||_\infty = n$. So assume that they are equal, then $\exists c, C$ with

$$c||f||_\infty \le ||f||_x \le C||f||_\infty$$

but for any c we can find $n > \frac{1}{c}$ such that $nc > 1$, which means that they are not equal.

$\underline{C([0,1], ||\cdot||_x) \text{ is not Banach:}}$

Define

$$f_n(x) = (1-x)^n$$

which is nothing but the flipped x^n on the interval. Now we know that $f_n > f_m > 0$ for $n < m$ so

$$||f_n - f_m||_x = \sup_x \{x[(1-x)^n - (1-x)^m]\} \le \sup_x \{x(1-x)^n\}.$$

Let $g_n(x) = x(1-x)^n$, then taking the derivative we get

$$g_n'(x) = (1-x)^{n-1}(1-(n+1)x)$$

so it start decreasing at $x = \dfrac{1}{n+1}$.

But then

$$g_n\left(\frac{1}{n+1}\right) = \frac{1}{n+1} \cdot \left(\frac{n}{n+1}\right)^n$$

taking the limit we get

$$\lim_{n\to\infty} g_n\left(\frac{1}{n+1}\right) = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{1}{e} = 0$$

and thus $||f_n - f_m||_x \to 0$ as $N \to \infty$, so f_n is Cauchy. Yet the limit of f_n is the indicator function of $\{0\}$, hence not in the space, so the space is not Banach.

$\square$

Exercise 4. *(2.13) in book.*

Proof. We use theorem 2.26 in book to prove this.

For $\alpha \geq 1$, since u is continuous there exists some T such that for $|t| \leq T$, $u(t) \leq 1$.

Let the corresponding rectangle in 2.26 be

$$R = \{(t, u) | |t| \leq T, |u| \leq 1\}$$

then we have $|f| = |u|^\alpha \leq 1$ in R since $\alpha \geq 1$.

Then, let $\delta = \min\{T, 1\}$ we know that the solution is unique on $t \in [-\delta, \delta]$ since $f' \leq \alpha < \infty$, i.e. it's Lipschitz on the box.

But note that $u = 0$ is a solution to the equation, so $u = 0$ is the only solution in the box.

Now that we know $u(\delta) = 0$ we can shift the box by δ to the right and using the same method prove for $t \in [0, 2\delta]$. To the left the procedure is the same. In this manner we can prove for any $t \in \mathbb{R}$, t can be reached by a finite time of shifting, thus $u(t) = 0$ uniquely. So $u = 0$ is the unique solution.

Now for $0 < \alpha < 1$, we the solution is not unique because both

$$u = 0$$

and

$$u(t) = t^{\frac{1}{1-\alpha}}$$

are solutions.

But for $\alpha = 0$ the solution is uniquely $u(t) = t$ simply by integration.

$\square$

Exercise 5. *(3.5) in book.*

Proof.

I don't think the first statement is correct since for the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

we have

$$L = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix}; U = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}; D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and thus

$$||L||_\infty + ||U||_\infty = 2 + 2 > 3 = ||D||_\infty$$

since $||\cdot||_\infty$ is the maximal of the sum of the absolute value of elements in rows.

However, after discussing with classmates I realized that it's a mistake updated in the new version of textbook, so I can prove that $||L+U||_\infty < ||D||_\infty$. This is just because by definition

$$||L+U||_\infty = \max_j \sum_{i \neq j}^{n} (L+U)_{ji} < \max_j D_{jj} = ||D||_\infty$$

where the inequality is the definition of diagonally domination of the matrix.

<u>A is invertible and the scheme converge for diagonally dominant A:</u>

First, A is invertible because for $Ax = 0$, assume $x \neq 0$ then there exists $|x_i| \geq |x_j|$ for all $1 \leq j \leq n$, and that $|x_i| > 0$. Then, the i-th row of Ax becomes

$$\sum_{j \geq 1} a_{i,j} x_j = 0$$

which means

$$a_{i,i} x_i = -\sum_{j \neq i} a_{i,j} x_j$$

but we also know that (strict inequality due to $|x_i| > 0$)

$$|a_{i,i} x_i| = |a_{i,i}||x_i| > \sum_{j \neq i} |a_{i,j}||x_i| \geq \sum_{j \neq i} |a_{i,j}||x_j| \geq \left| -\sum_{j \neq i} a_{i,j} x_j \right|$$

contradiction! So $|x| = 0$, and thus A is invertible.

Now, for convenience let

$$f(x) = D^{-1}(L+U)x + D^{-1}b$$

and

$$g(x) = (D-L)^{-1}Ux + (D-L)^{-1}b$$

be the iterative method scheme for Jacobi and Gauss-Seidel method.

First, if these methods converge they converge to the solution of $Ax = b$. This is because

$$f(x) = x \Rightarrow (D - (L + U))x = b \Rightarrow Ax = b$$

since D is invertible. The same applies to

$$g(x) = x \Rightarrow ((D - L) - U)x = b \Rightarrow Ax = b$$

We now prove that the schemes converge. To prove that it converges we use the Banach contraction theorem, for which we still need to prove that the functions f and g are self maps and that they are contraction mappings.

The domain of this function is nothing but $\mathbb{R}^n$, and the image is in $\mathbb{R}^n$. So it is a self-map.

It is a contraction for both f and g because

$$|f(x) - f(y)| = |D^{-1}(L + U)x + D^{-1}b - D^{-1}(L + U)y - D^{-1}b| = |D^{-1}(L + U)(x - y)|$$

Since each row of A's non-diagonal term is divided by the diagonal term, by definition $||D^{-1}(L + U)||_\infty < 1$, which using the fact that $\rho(A) \leq ||A||$ for any norm, we know that $\rho(D^{-1}(L + U)) < 1$. Yet this implies that $(D^{-1}(L + U))^n x \to 0$ for any $x \in \mathbb{R}^n$. Therefore there exists some n for which

$$|(D^{-1}(L + U))^n x| \leq c|x|$$

for $0 < c < 1$.

Let $f_n = f(f(\dots f(x)\dots))$ be the function of f applied n times. Then

$$\begin{aligned} |f_n(x) - f_n(y)| =& |(D^{-1}(L + U)^n x + D^{-1}(L + U)^{n-1} D^{-1} b + \cdots + D^{-1} b) \\ & - (D^{-1}(L + U)^n y + D^{-1}(L + U)^{n-1} D^{-1} b + \cdots + D^{-1} b)| \\ =& |D^{-1}(L + U)^n (x - y)| < |x - y| \end{aligned}$$

Then f_n is a contraction by above reasoning and hence the scheme converges since f_n is a self map.

Now for the Gauss-Seidel method. First note that $D - L$ is a lower triangular matrix with non-zero diagonal entries, thus invertible, so the method is well-defined.

Again we only need to prove $\rho((D - L)^{-1}U) < 1$ since the above n time iterative method applies exactly the same for g and similarly constructed g_n.

Note that

$$(D - L) = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = D \begin{pmatrix} 1 & & & \\ \frac{a_{21}}{a_{22}} & 1 & & \\ \vdots & \vdots & \ddots & \\ \frac{a_{n1}}{a_{nn}} & \frac{a_{n2}}{a_{nn}} & \dots & 1 \end{pmatrix} := DQ$$

where

$$Q = I + N = I + \begin{pmatrix} 0 & & & \\ \frac{a_{21}}{a_{22}} & 0 & & \\ \vdots & \vdots & \ddots & \\ \frac{a_{n1}}{a_{nn}} & \frac{a_{n2}}{a_{nn}} & \dots & 0 \end{pmatrix}$$

and hence

$$Q^{-1} = (I + N)^{-1} = I - N + N^2 - N^3 + \dots$$

We know that $\sum_{n=1}^{n}(-1)^{n-1}N^n$ is a finite sum since it's nilpotent. Since each term in N has absolute value less than 1, the result in the finite is term-wise smaller than that of N (by explicit computation each term is N's corresponding term times a less than 1 number), which means that $||Q^{-1}||_\infty < 1$. $||D^{-1}U||_\infty < 1$ by diagonal dominance.

Thus

$$\rho((D - L)^{-1}U) \leq ||(D - L)^{-1}U||_\infty \leq ||Q^{-1}||_\infty ||D^{-1}U||_\infty < 1$$

and by the same argument as for f we are done.

As for the convergence rate, let x be that $Ax = b$ and $\varepsilon_k = x_k - x$. We know that for Jacobi method

$$\varepsilon_k = x_k - x = D^{-1}(L + U)x_{k-1} + D^{-1}b - (D^{-1}(L + U)x + D^{-1}b) = D^{-1}(L + U)\varepsilon_{k-1}$$

and so the rate of convergence is $\rho(D^{-1}(L + U))$. Similarly the rate of convergence for g is $\rho((D - L)^{-1}U)$.

$\square$

Exercise 6. *(3.6) in book.*

Proof.

Existence and uniqueness of solution for $a < \infty$:

Let

$$\Phi(f)(x) := 1 + \frac{1}{\pi}\int_{-a}^{a} \frac{1}{1+(x-y)^2} f(y)dy$$

be a map that takes in $f \in C[-a, a]$. Then Φ is a self-map if the output is also continuous.

Since f is bounded on the compact domain $[-a, a]$, $f(y) \le B$. Thus

$$|\Phi(f)(x) - \Phi(f)(z)| \le \left| \frac{1}{\pi}\int_{-a}^{a} B\left(\frac{1}{1+(x-y)^2} - \frac{1}{1+(z-y)^2}\right) dy \right|$$

which is small enough when $|x - z|$ is small due to continuity of

$$\frac{1}{1+(x-y)^2}$$

on $[-a, a]$ (the integral can be reduced by multiplying $2a$).

Hence $\Phi(f)(x) : C[-a, a] \to C[-a, a]$. Now we show that it's a contraction.

Let $f, g \in C[-a, a]$ then

$$\begin{aligned}
|\Phi(f)(x) - \Phi(g)(x)| &= \left| \frac{1}{\pi}\int_{-a}^{a} \frac{1}{1+(x-y)^2}(f(y) - g(y))dy \right| \\
&\le \left| \frac{1}{\pi} ||f - g||_\infty \int_{-a}^{a} \frac{1}{1+(x-y)^2} dy \right| \\
&= \frac{1}{\pi} ||f - g||_\infty \arctan(y - x)\Big|_{-a}^{a} \\
&\le c||f - g||_\infty
\end{aligned}$$

for some $0 \le c < 1$ since $\arctan(y-x)\Big|_{-a}^{a} \frac{1}{\pi} < 1$ for any a fixed. Thus by Banach contraction mapping theorem we know that there exists a unique solution in $C[-a, a]$ to the function $\Phi(f) = f$. Since it's continuous on a compact set it is bounded.

Non-negativity of solution:

Since if $f_n(x) \ge 0$ for all x, then $f_n(x)$ is 1 plus something positive, thus it's everywhere larger than 1. So we can just start from $f_0 \equiv 1$ and since f is the limit of this iteration it is larger than 1.

For $a = \infty$:

If $a = \infty$ the Lipschitz constant is 1 and we cannot use the contraction theorem to prove this. But (after discussing with Zihao) note that for any solution to the function f, $f_c(x) =$

$f(x+c)$ is also a solution because

$$f_c(x) = f(x+c) = 1 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+(x+c-y)^2} f(y)dy$$

$$= 1 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+(x-(y-c))^2} f(y)d(y-c) = 1 + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+(x-z)^2} f_c(z)d(z)$$

and hence any shift with respect to the x axis is a solution. So if the solution is unique it is a constant, assume it is k. Then we get the equation

$$k = f(x) = 1 + \frac{k}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2} d(y) = 1 + \frac{k}{\pi}\pi = 1 + k$$

which means that such k doesn't exist. So at least the solution is not unique.

□

Exercise 7. *(3.7) in book.*

Proof.

Integrating twice on u gives

$$u(x) = -\int_0^x \int_1^y [f(s) - \lambda \sin(u(s))]\, ds dy + C_1 x + C_2$$

where if we do the integration by part with

$$w = \int_1^y [f(s) - \lambda \sin(u(s))]\, ds, v = y, \quad \int w dv = wv - \int v dw$$

to get

$$\begin{aligned} u(x) &= -\left[y \int_1^y [f(s) - \lambda \sin(u(s))]\, ds \right]_0^x + \int_0^x yd \left[\int_1^y f(s) - \lambda \sin(u(s)) ds \right] + C_1 x + C_2 \\ &= -x \int_1^x [f(y) - \lambda \sin(u(y))]\, dy + \int_0^x y[f(y) - \lambda \sin(u(y))] dy + C_1 x + C_2 \end{aligned}$$

which by evaluating at 0 we get

$$0 = u(0) = C_2$$

and by evaluating at 1 we get

$$0 = u(1) = \int_0^1 y[f(y) - \lambda \sin(u(y))] dy + C_1$$

which means

$$C_1 = -\int_0^1 y[f(y) - \lambda \sin(u(y))] dy.$$

And we get an expression of u in terms of itself:

$$\begin{aligned} u(x) = &- x \int_1^x [f(y) - \lambda \sin(u(y))]\, dy + \int_0^x y[f(y) - \lambda \sin(u(y))] dy \\ &- x \int_0^1 y[f(y) - \lambda \sin(u(y))] dy \\ = &\int_x^1 x(1-y)[f(y) - \lambda \sin(u(y))] dy + \int_0^x y(1-x)[f(y) - \lambda \sin(u(y))] dy \\ = &\int_0^1 g(x,y)[f(y) - \lambda \sin(u(y))] dy \end{aligned}$$

where

$$g(x,y) = \begin{cases} x(1-y) & 0 \le x \le y \le 1 \\ y(1-x) & 0 \le y \le x \le 1. \end{cases}$$

Thus, we can define

$$\Phi(u)(x) = \int_0^1 g(x,y)[f(y) - \lambda \sin(u(y))]dy$$

then apply Banach contraction mapping theorem on it.

It is a self map because for u continuous, $\Phi(u)$ is nothing but a combination of continuous function, thus in $C[0,1]$.

It is a contraction since for u, v we have

$$\begin{aligned}|\Phi(u)(x) - \Phi(v)(x)| &= \left|\int_0^1 g(x,y)[f(y) - \lambda \sin(u(y))] - g(x,y)[f(y) - \lambda \sin(v(y))]dy\right| \\ &= |\lambda| \left|\int_0^1 g(x,y)[\sin(v(y)) - \sin(u(y))]dy\right| \\ &\le |\lambda| \int_0^1 |g(x,y)| \cdot |\sin(v(y)) - \sin(u(y))|dy \\ &\le |\lambda| \cdot |1-0| \cdot ||g(x,y)||_\infty \cdot \mathrm{Lip}(\sin) \cdot ||u-v||_\infty \\ &\le |\lambda| \cdot ||u-v||_\infty\end{aligned}$$

where the last step is because $||g(x,y)||_\infty \le 0$ by definition, and $\mathrm{Lip}(\sin) \le 1$ by derivative.

Hence, for $|\lambda| < 1$ we have by Banach contraction mapping theorem that there is a unique solution to the equation.

The beginning few terms in the sequence:

$$u_0 = 0;$$

$$u_1 = \Phi(u_0) = \int_0^1 g(x,y)[f(y) - \lambda \sin(0))]dy = \int_0^1 g(x,y) f(y)dy$$

which is notably the solution for $-u'' = f$ with the same boundary value.

$$\begin{aligned}u_2 = \Phi(u_1) &= \int_0^1 g(x,y)\left[f(y) - \lambda \sin\left(\int_0^1 g(x,y)f(y)dy\right)\right]dy \\ &= u_1 - \lambda \int_0^1 g(x,y)\sin\left(\int_0^1 g(x,y)f(y)dy\right)dy. \\ &\vdots\end{aligned}$$

And as we can see since it's a little bit troublesome to handle the integral within a sin function, I leave it there. But we can see the pattern that it's gradually going from the solution of $-u'' = f$ to $-u'' = f - \lambda \sin(u)$.

$\square$