MEASURE THEORETICAL PROBABILITY I HOMEWORK 6

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Discussed with classmates.

Exercise 0.

Proof.

(1): $\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} \overset{as}{\to} \infty$:

First, we note that it doesn't matter if we throw away the first few random variables in the summation as $n \to \infty$:

$$\begin{aligned}\limsup_{n\to\infty} \frac{X_1 + \cdots + X_n}{\sqrt{n}} &= \limsup_{n\to\infty} \frac{X_1 + \cdots + X_l}{\sqrt{n}} + \limsup_{n\to\infty} \frac{X_{l+1} + \cdots + X_n}{\sqrt{n}} \\ &= \limsup_{n\to\infty} \frac{X_{l+1} + \cdots + X_n}{\sqrt{n}}\end{aligned}$$

But as $n \to \infty$ we can construct sequence $\phi(n) \to \infty$ with $\phi(n) \ll n$ such that

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = \limsup_{n\to\infty} \frac{X_1 + \cdots + X_n}{\sqrt{n}} = \limsup_{n\to\infty} \frac{X_{\phi(n)+1} + \cdots + X_n}{\sqrt{n}}$$

where the right hand side is measurable with respect to the tail σ-algebra as $\phi(n) + 1 \to \infty$ and by definition of $\mathcal{T}$. But then for any c, we know that

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} > c\right) \in \{0, 1\}$$

by Kolmogorov's 0-1 law.

Yet on the other hand by Fatou's lemma we have

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} > c\right) \geq \limsup_{n\to\infty} \mathbb{P}\left(\frac{S_n}{\sqrt{n}} > c\right) = 1 - F(c)$$

where F is the cdf of $N(0, \sigma^2)$ by CLT. In particular the expression is strictly larger than 0, hence it's 1.

But this means for any $c \in \mathbb{R}$ we have

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} > c\right) = 1$$

which means

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = \infty\right) = 1$$

i.e.

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} \xrightarrow{as} \infty.$$

(2) $\frac{S_n}{\sqrt{n}}$ does not converge in probability:

Assume it converges in probability, that is, $\exists Y$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{\sqrt{n}} - Y\right| \geq \varepsilon\right) = 0$$

for all ε. In other words, for any subsequence $\phi(n)$, we know that for all $\delta > 0$, $\exists N$ such that for all $n > N$ we have

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_{\phi(n)}}{\sqrt{\phi(n)}} - Y\right| \geq \varepsilon\right) < \delta$$

where we let $\phi(n) = n!$ and by triangular inequality that

$$\left|\frac{S_{n!}}{\sqrt{n!}} - \frac{S_{(n+1)!}}{\sqrt{(n+1)!}}\right| \leq \left|\frac{S_{n!}}{\sqrt{n!}} - Y\right| + \left|\frac{S_{(n+1)!}}{\sqrt{(n+1)!}} - Y\right| \leq 2\varepsilon$$

with probability $1 - 2\delta$.

But consider

$$\begin{aligned}
&\mathbb{P}\left(\left|\frac{S_{n!}}{\sqrt{n!}} - \frac{S_{(n+1)!}}{\sqrt{(n+1)!}}\right| > 2\varepsilon\right) \geq \mathbb{P}\left(\frac{S_{n!}}{\sqrt{n!}} - \frac{S_{(n+1)!}}{\sqrt{(n+1)!}} > 2\varepsilon\right) \\
=&\mathbb{P}\left(\frac{X_1 + \cdots + X_{n!}}{\sqrt{n!}}\left(1 - \frac{1}{\sqrt{n+1}}\right) - \frac{X_{n!+1} + \cdots + X_{(n+1)!}}{\sqrt{(n+1)!}} > 2\varepsilon\right) \\
\geq&\mathbb{P}\left(\frac{X_1 + \cdots + X_{n!}}{\sqrt{n!}}\left(1 - \frac{1}{\sqrt{n+1}}\right) > \varepsilon, \frac{S_{(n+1)!} - S_{n!}}{\sqrt{(n+1)!}} < -\varepsilon\right) \\
\stackrel{ind}{=}&\mathbb{P}\left(\frac{X_1 + \cdots + X_{n!}}{\sqrt{n!}}\left(1 - \frac{1}{\sqrt{n+1}}\right) > \varepsilon\right) \cdot \mathbb{P}\left(\frac{S_{(n+1)!} - S_{n!}}{\sqrt{(n+1)!}} < -\varepsilon\right)
\end{aligned}$$

But we know that

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_{(n+1)!} - S_{n!}}{\sqrt{(n+1)!}} < -\varepsilon\right) \stackrel{CLT}{=} \mathbb{P}(N(0,1) < -\varepsilon) \geq \frac{1}{3}$$

and for $n > 3$

$$\mathbb{P}\left(\frac{X_1 + \cdots + X_{n!}}{\sqrt{n!}}\left(1 - \frac{1}{\sqrt{n+1}}\right) > \varepsilon\right) \geq \mathbb{P}\left(\frac{S_{n!}}{\sqrt{n!}} > 2\varepsilon\right) \stackrel{CLT}{\to} \mathbb{P}(N(0,1) > 2\varepsilon) \geq \frac{1}{3}$$

and hence

$$\mathbb{P}\left(\left|\frac{S_{n!}}{\sqrt{n!}} - \frac{S_{(n+1)!}}{\sqrt{(n+1)!}}\right| > 2\varepsilon\right) \geq \frac{1}{9} > 2\varepsilon$$

for small enough ε, contradiction! Thus there doesn't exist such Y.

$\square$

Exercise 1. *Prob 1.*

Proof.

Discussed with classmates.

Assume $\mathbb{E}[X_i^2] = \infty$.

Hint: Let $X_1', X_2', \dots$ be an independent copy of the original sequence. Let $Y_i = X_i - X_i'$, $U_i = Y_i \cdot \mathbb{1}_{|Y_i| \le A}$, $V_i = Y_i - U_i$. Then, for any $K > 0$ we have

$$\mathbb{P}\left(\sum_{m=1}^{n} Y_m \ge K\sqrt{n}\right) \ge \mathbb{P}\left(\sum_{m=1}^{n} U_m \ge K\sqrt{n}, \sum_{m=1}^{n} V_m \ge 0\right) \ge \frac{1}{2}\mathbb{P}\left(\sum_{m=1}^{n} U_m \ge K\sqrt{n}\right) \ge \frac{1}{5}.$$

We first see that assume the hint, how we can prove the question. We have

$$\begin{aligned}
\mathbb{P}\left(\sum_{m=1}^{n} Y_m \ge K\sqrt{n}\right) =& \mathbb{P}\left(\frac{S_n}{\sqrt{n}} - \frac{S_n'}{\sqrt{n}} > K\right) \le \mathbb{P}\left(\left|\frac{S_n}{\sqrt{n}} - \frac{S_n'}{\sqrt{n}}\right| > K\right) \\
\le& \mathbb{P}\left(\left|\frac{S_n}{\sqrt{n}}\right| + \left|\frac{S_n'}{\sqrt{n}}\right| > K\right) \le \mathbb{P}\left(\left|\frac{S_n}{\sqrt{n}}\right| > \frac{K}{2}\right) + \mathbb{P}\left(\left|\frac{S_n'}{\sqrt{n}}\right| > \frac{K}{2}\right) \\
=& 2F\left(\frac{K}{2}\right) + 2\left(1 - F\left(\frac{K}{2}\right)\right) < \frac{1}{10}
\end{aligned}$$

for large enough K. Thus we have a contradiction.

Now we prove the hint: The first inequality is direct since we have $Y_i = U_i + V_i$ and thus when the two events $\sum_{m=1}^{n} U_m \ge K\sqrt{n}, \sum_{m=1}^{n} V_m \ge 0$ happens at the same time, we must have $\sum_{m=1}^{n} Y_m \ge K\sqrt{n}$. This gives us

$$\mathbb{P}\left(\sum_{m=1}^{n} Y_m \ge K\sqrt{n}\right) \ge \mathbb{P}\left(\sum_{m=1}^{n} U_m \ge K\sqrt{n}, \sum_{m=1}^{n} V_m \ge 0\right).$$

The second inequality is because that, we note that either $U_m = 0$ or $V_m = 0$, that is, we can separate Y_is at any point ω with respect to whether the value of $Y_i(\omega)$ has absolute value larger than A or not. There's in total 2^n possibilities of this partition, and for each one of then we have (WLOG reorder such that the first k U_m is non-zero) (Call this state B)

$$\mathbb{P}\left(\sum_{m=1}^{n} U_m \ge K\sqrt{n}, \sum_{m=1}^{n} V_m \ge 0 \middle| B\right) = \mathbb{P}\left(\sum_{m=1}^{k} U_m \ge K\sqrt{n}, \sum_{m=k}^{n} V_m \ge 0 \middle| B\right).$$

But now we've really separated the variables such that the two events are independent. So we have

$$\mathbb{P}\left(\sum_{m=1}^{k} U_m \geq K\sqrt{n}, \sum_{m=k}^{n} V_m \geq 0 \middle| B\right) = \mathbb{P}\left(\sum_{m=1}^{k} U_m \geq K\sqrt{n} \middle| B\right) \cdot \mathbb{P}\left(\sum_{m=k}^{n} V_m \geq 0 \middle| B\right)$$

where we further note

$$\mathbb{P}\left(\sum_{m=k}^{n} V_m \geq 0 \middle| B\right) = \mathbb{P}\left(\sum_{m=k}^{n} V_m \leq 0 \middle| B\right)$$

since Y_i and $-Y_i$ are identical random variables. Thus, we know by doing for ≤ 0 that

$$\mathbb{P}\left(\sum_{m=1}^{k} U_m \geq K\sqrt{n}, \sum_{m=k}^{n} V_m \geq 0 \middle| B\right) = \mathbb{P}\left(\sum_{m=1}^{k} U_m \geq K\sqrt{n}, \sum_{m=k}^{n} V_m \leq 0 \middle| B\right)$$

which considered the possibility that $\sum_{m=k}^{n} V_m = 0$ might be double counted, we get

$$\mathbb{P}\left(\sum_{m=1}^{k} U_m \geq K\sqrt{n}, \sum_{m=k}^{n} V_m \geq 0 \middle| B\right) \geq \frac{1}{2}\mathbb{P}\left(\sum_{m=1}^{k} U_m \geq K\sqrt{n} \middle| B\right)$$

and now combining all such Bs we get

$$\mathbb{P}\left(\sum_{m=1}^{k} U_m \geq K\sqrt{n}, \sum_{m=k}^{n} V_m \geq 0\right) \geq \frac{1}{2}\mathbb{P}\left(\sum_{m=1}^{k} U_m \geq K\sqrt{n}\right)$$

by summation, as Bs are disjoint.

For the third inequality, we first note that $\mathbb{E}[U_i] = 0$ by it's definition, since $\mathbb{E}[X_i]$ and $\mathbb{E}[X_i']$ are the same. Now since $\mathbb{E}[X^2] = \infty$ we can WLOG suppose that the contribution from the part where $X_i \geq 0$ is infinity (since either that or the negative part).

Now we know $\mathbb{P}(X_i > 0) = c > 0$ as otherwise the expectation cannot be infinity. We also claim that $c < 1$ since if $c = 1$ then the mean is strictly positive, which means that $\frac{S_n}{\sqrt{n}} \xrightarrow{d} \infty$ by CLT (it "converges to" $N(n \cdot \mathbb{E}[X_i], \sigma^2) = \infty$ as $n \to \infty$), contradiction to assumption in problem. So $0 < c < 1$.

Thus, we can find L large such that $\mathbb{P}(X_i \leq L) \geq c$ as $c < 1$. Now we have

$$\begin{aligned}
\mathbb{E}[Y_i^2] = \mathbb{E}[(X_i - X_i')^2] &\geq \mathbb{P}(X_i' \leq L)\mathbb{E}[(X_i - X_i')^2 | X_i' \leq L] \\
&\geq \mathbb{P}(X_i' \leq L)\mathbb{E}[\mathbb{1}_{X_i \geq L}(X_i - L)^2 | X_i' \leq L] \\
&= \mathbb{P}\left(X_i' \leq L\right) \cdot \mathbb{E}\left[\mathbb{1}_{X_i \geq L}(X_i - L)^2\right]
\end{aligned}$$

where we can further bound

$$\mathbb{1}_{X_i \geq L}(X_i - L)^2 \geq \mathbb{1}_{X_i \geq 2L}\left(\frac{X_i}{2}\right)^2$$

where the right hand side is ∞ after taking expectation (since under expectation, the finite part does not matter). This means really that $\mathbb{E}[Y_i^2] \to \infty$ (which makes sense as the perturbation is extremely large).

Thus $\mathbb{E}[U_i^2] > C'$ for any large C' if A is large enough, since $U_i \uparrow Y_i$. So we can pick any unstated large C to be chosen later with $\mathbb{E}[U_i^2] = C$ such that

$$\mathrm{Var}(U_i) = \mathbb{E}[U_i^2] - \mathbb{E}[U_i]^2 = C.$$

Thus by CLT we have

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} U_i = N(0, C)$$

and hence

$$\mathbb{P}\left(\sum_{m=1}^{k} U_m \geq K\sqrt{n} \right) \geq \mathbb{P}(N(0,C) \geq K)$$

where as for large enough A the Gaussian is spread widely that

$$\mathbb{P}(N(0,C) \geq K) \geq \frac{2}{5} < \frac{1}{2}$$

where the last bound is the sup we can get out of this way of bounding. Hence

$$\frac{1}{2}\mathbb{P}\left(\sum_{m=1}^{k} U_m \geq K\sqrt{n} \right) \geq \frac{1}{5}$$

and we are done.

$\square$

Exercise 2. *Prob 2.*

Proof.

(1) Character is uniformly continuous:

We know that the character function is continuous since e^{itx} is continuous and bounded. Moreover, note that $\phi_X(t) = \mathbb{E}[e^{itx}]$ is 2π periodic, and hence it is uniformly continuous if it is uniformly continuous on $[0, 2\pi]$, but the latter is just because any continuous function on a compact set is uniformly continuous.

(2)The point wise limit of a sequence of character functions is a character:

As in the proof of Levy's therorem backward direction, we can show that X_n is a tight family. Assume $\phi_{X_n}(t) \to f$ point wise, then for fixed $\varepsilon > 0$ we can choose a small such that

$$\frac{1}{a}\int_{-a}^{a}(1-f)ds \leq \varepsilon$$

as long as $|f-1| \leq \frac{\varepsilon}{2}$. But the latter holds because $f(0) = 1$ and at a small neighborhood of 0 the convergence is uniform and hence continuity is also passed.

Now, by DCT (since $(1-\phi_{X_n}(s)) \leq 2$ and the measure is finite)

$$\lim_{n\to\infty}\frac{1}{a}\int_{-a}^{a}(1-\phi_{X_n}(s))ds = \frac{1}{a}\int_{-a}^{a}(1-f)ds \leq \varepsilon.$$

Then we can pick $t := \frac{2}{a}$ and Lemma in class says

$$\mathbb{P}(|X| \geq t) \leq \frac{t}{2}\int_{-t/2}^{t/2}(1-\phi_X(s))ds$$

we have

$$\limsup \mathbb{P}(|X_n| \geq t) \leq \varepsilon$$

for large t (really for small a). And thus getting rid of the lim sup we get

$$\mathbb{P}(|X_n| \geq t) \leq 2\varepsilon$$

and thus the tail of X_n is a tight family. So we can extract subsequence of $X_{k_n} \xrightarrow{d} X$ using Helly's selection theorem for some random variable X. Then, if we assume $X_n \xrightarrow{d} X$ don't hold, then we can find continuous bounded g for which we have

$$|\mathbb{E}[f(X_{j_n})] - \mathbb{E}[g(X)]| \geq \varepsilon$$

but we can extract another subsequence of j_n that converges to a random variable Y that has the same character as X (by another direction of Levy + tightness), contradiction since then $\mathbb{E}[g(Y)] = \mathbb{E}[g(X)]$.

Thus $X_n \xrightarrow{d} X$. Yet then by the forward direction of Levy we get $f = \phi_X(t)$, then we are done.

(3): Prove that on any bounded interval, the convergence of ϕ_n is uniform:

If the functions ϕ_n is equicontinuous, then we have the solution because for any $x \in K$ where K is the compact domain we care about, we have

$$|\phi_n(x) - \phi(x)| \le |\phi_n(x) - \phi_n(x+\delta)| + |\phi_n(x+\delta) - \phi(x+\delta)| + |\phi(x+\delta) - \phi(x)|$$

where we know that the difference between

$$|\phi_n(x) - \phi(x)| \quad and \quad |\phi_n(x+\delta) - \phi(x+\delta)|$$

cannot exceed 2ε if we can bound uniformly the other 2 terms both in n and in x. But by (a) we have ϕ_n for all n is uniformly continuous, and by our assumption we have ϕ_n is equicontinuous. This implies that ϕ_n is uniformly equicontinuity (proof of Ascoli Arzela uses this). Thus, we are justfied to use δ to bound the above inquality, i.e.

$$|\phi_n(x) - \phi(x)| \le 2\varepsilon + |\phi_n(x+\delta) - \phi(x+\delta)|.$$

Now we know

$$\left(\sup_{|x-x_0|<\delta} - \inf_{|x-x_0|<\delta}\right) |\phi_n(x) - \phi(x)| \le 2\varepsilon \to 0$$

as $\varepsilon \to 0$. So $\phi_n \to \phi$ locally uniformly. But since K compact we can extend the locally uniformly convergence to uniformly convergence (for all δ ball, a point inside converges pointwise, and hence every point converges uniformly; now δ balls is a finite cover of K).

So it suffices us to prove equicontinuity. This we need tightness of X_n. Note that in considering the integral we only need to consider the integral for $|x| \le t$ since the tail is always less than ε by tightness, i.e.

$$|\phi_n(t+\delta) - \phi_n(t)| \le \int_{|X_n|<t} e^{itx} - e^{i(t+\delta)x} d\mu_{X_n} + \varepsilon$$

but for x bounded by continuity of e^{itx} in t we can find the required δ that is uniform in n now. So ϕ_n is equicontinuous, and by above argument the result follows. □

Exercise 3. *Prob 3.*

Proof.

(1) Given $\phi'(0) = ia$, conclude $S_n/n \xrightarrow{p} a$:

The character of S_n/n is

$$\phi_n = \left[\phi_{X_1}\left(\frac{t}{n}\right)\right]^n = \left[\frac{t}{n} \cdot \frac{\phi_{X_1}(t/n) - 1}{t/n} + 1\right]^n$$

where note that

$$\lim_{n\to\infty} t \cdot \frac{\phi_{X_1}(t/n) - 1}{t/n} = t\phi'_{X_i}(0) = tia$$

and Theorem 3.4.2 claims that

$$\lim_{n\to\infty} \left[\frac{t}{n} \cdot \frac{\phi_{X_1}(t/n) - 1}{t/n} + 1\right]^n = \lim_{n\to\infty} \left[\frac{c_n}{n} + 1\right]^n = e^{iat} = \phi_a(t)$$

and thus by Levy we know $\dfrac{S_n}{n} \xrightarrow{d} a$ and since a is a constant we have

$$\frac{S_n}{n} \xrightarrow{p} a.$$

(2): If $S_n/n \xrightarrow{p} a$, then $\phi(t/n)^n \to e^{iat}$ as $n \to \infty$:

Since $\dfrac{S_n}{n} \xrightarrow{p} a$, we have $\dfrac{S_n}{n} \xrightarrow{d} a$, which gives us pointwise convergence of character functions, i.e. $\phi_n \to \phi_a$.

Yet writing out we have

$$\left[\phi_{X_1}\left(\frac{t}{n}\right)\right]^n = \phi_n \to \phi_a = e^{iat}.$$

(3): skip

We can conclude the result since positive direction by (1) and the reverse by (2), (3).

□

Exercise 4. *Prob 4*

Proof.

Suppose that $(X_1, ..., X_d)$ has the normal distribution with mean vector θ and covariance matrix Γ. Then we have that, for any $a = \begin{bmatrix} a_1 \\ \vdots \\ a_d \end{bmatrix} \in \mathbb{R}^d$

$$E\left[e^{ia^t \cdot x}\right] = \exp\left(ia^t\theta - \frac{1}{2}a^t\Gamma a\right)$$

Let $c = \begin{bmatrix} c_1 \\ \vdots \\ c_d \end{bmatrix}$ be a vector in $\mathbb{R}^d$. Then $c \cdot X$ is a random variable and setting $a = tc$ above we have that

$$E\left[e^{it(c^t \cdot x)}\right] = \exp\left(t(c^t\theta) - \frac{t^2}{2}c^t\Gamma c\right)$$

We see that the last equation is the characteristic function of a normal random variable with mean $c^t\theta$ and variance $c^t\Gamma c$, and since characteristic functions specify the law uniquely, then we can conclude that $c_1X_1 + ... + c_dX_d$ follows a normal distribution with mean $c^t\theta$ and variance $c^t\Gamma c$.

For the other direction, suppose that $Y = c_1X_1 + ... + c_dX_d$ has the normal distribution with mean $c^t\theta$ and variance $c^t\Gamma c$. Then we can conclude that

$$E\left[e^{ic^t \cdot x}\right] = \Phi_Y(1) = \exp\left(ic^t\theta - \frac{1}{2}c^t\Gamma c\right)$$

Since the above holds for all $c \in \mathbb{R}^d$, we have that X has characteristic function

$$\exp\left(ic^t\theta - \frac{1}{2}c^t\Gamma c\right)$$

Since the characteristic function uniquely determine the law of X, the random variable is normally distributed with mean c and covariance matrix Γ.

$\square$

Exercise 5. *Prob 5*

Proof.

Note that the $\mathrm{Cov}(X_i, X_j) = \mathbb{E}\left[(X_i - \mathbb{E}[X_i])(X_j - \mathbb{E}[X_j])\right]$, which means we can get by linear algebra that

$$v^T \Sigma v = v^T \mathbb{E}\left[(X - \mathbb{E}[X])(X - \mathbb{E}[X])^T\right] v = \mathbb{E}\left[(v^T(X - \mathbb{E}[X]))^2\right] \geq 0$$

since v is constant vector so we can pass it in. So it's positive semi definite.

□

Exercise 6. *Prob 6*

Proof.

<u>Check:</u>

Prob 6.

Firstly, want to show X_i is stationary

m-dependence.

For stationary. $\forall n$, $\forall k$

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_{n+m} \end{pmatrix} \overset{d}{\sim} \begin{pmatrix} Y_{1+k} \\ \vdots \\ Y_{n+m+k} \end{pmatrix}$$

Therefore

$$g\begin{pmatrix} Y_1 \\ \vdots \\ Y_{n+m} \end{pmatrix} \overset{d}{=} \begin{pmatrix} f(Y_1, \dots Y_{1+m}) \\ \vdots \\ f(Y_n, \dots Y_{n+m}) \end{pmatrix} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$

$$\overset{d}{\sim} g\begin{pmatrix} Y_{1+k} \\ \vdots \\ Y_{n+m+k} \end{pmatrix} = \begin{pmatrix} X_{1+k} \\ \vdots \\ X_{n+k} \end{pmatrix}$$

For m-dependence.

$\forall i, j \in N$ $j - i \geq m+1$.

$X_i = f(Y_i, \dots Y_{i+m})$

$X_j = f(Y_{i+j-i}, \dots Y_{i+j-i+m})$

$\{i, \dots, i+m\} \cap \{i+j-i, \dots, i+j-i+m\} = \phi$

Therefore

$\{Y_i, \cdots Y_{i+m}\} \perp\!\!\!\perp \{Y_{i+j-i}, \cdots Y_{i+j-i+m}\}$

$\Rightarrow X_i \perp\!\!\!\perp X_j$.

To prove $\dfrac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\,\sigma}$.

~~Divide~~ For given n, divide the sequence $X_1, \cdots X_n$ into two types of blocks.

$\underbrace{X_1, X_2, \cdots X_r}_{G}, \underbrace{X_{r+1} \cdots X_{r+m}}_{g}, \underbrace{X_{r+m+1}, \cdots X_{2r+m}}_{G}.$

$\underbrace{X_{2r+m+1} \cdots X_{2r+2m}}_{g}, \cdots\cdots X_n$

$S_n = \sum_{j=1}^{n} X_j, \quad T_n = \sum_{j\in G} X_j$

$W_n = \sum_{j\in g} X_j$

By m-dependence. $\forall X_i, X_j \in G$. but in different blocks, $X_i \perp\!\!\!\perp X_j$

$\forall X_i, X_j \in g$ but in different blocks $X_i \perp\!\!\!\perp X_j$

$\Rightarrow Y_i \perp\!\!\!\perp Y_j, \quad Z_i \perp\!\!\!\perp Z_j$

Consider Y_i iid, Y_i has mean $r\mu$, variance $Var(X_1 + \cdots X_{r})< \infty$.

By CLT.

$$\frac{\sqrt{kn}}{\sqrt{n}} \frac{\sum_{i=1}^{kn}(Y_i - r\mu)}{\sqrt{kn}} \xrightarrow{d} \frac{1}{\sqrt{r+m}} N(0, Var(Y_1))$$

Consider $\frac{Var(Y_1)}{r+m}$

$$\underset{\text{Since } m\text{-dependence}}{=} \frac{r\, Var(X_1) + 2r \sum_{i=2}^{m+1} cov(X_1, X_i)}{r+m}$$

$$\xrightarrow{cov(X_1, X_{1+m})=0} \sigma^2 \quad \text{as } r \to \infty.$$

Therefore $\lim_{r\to\infty} = \lim_{n\to\infty} \frac{1}{\sqrt{n}\sigma} \sum (Y_i - r\mu) \xrightarrow{d} N(0, 1)$

for $\frac{1}{\sqrt{n}} \sum_{i=1}^{kn} (Z_i - m r)$

$Z_i - mr \overset{iid}{\sim}$ w/ mean 0. $Var(Z_i)$.

therefore

$P\left(\cancel{\left|\frac{\sum_{1}^{kn}(Z_i - m\mu)}{\sqrt{n}}\right| \sum() \le \varepsilon}\right.$

By CLT.

$$\frac{\sum Z_i - \mu m}{\sqrt{n}} = \frac{\sqrt{kn}}{\sqrt{n}} \frac{\sum (Z_i - \mu m)}{\sqrt{kn}}$$

$$\xrightarrow[d]{n \to \infty} \frac{1}{\sqrt{r+m}} N(0, Var(Z_1)).$$

$$\frac{Var(Z_1)}{r+m} \le \frac{m^2 Var(X_1)}{m+r} \to 0 \quad r \to \infty$$

Therefore $\frac{\sum Z_i - \mu m}{\sqrt{n}} \to N(0,0) = 0$ $\quad$ $\left(\begin{array}{l} k \to \infty \\ n \xrightarrow{2} \infty \\ r << n. \end{array}\right.$

For Q_n. $Q_n = \sum_{i \in (G \cup g)^c} X_i$

Since $|G \cup g|^c \le r$

$$\left(\frac{\sum_{i \in (G \cup g)^c} (X_i - \mu)}{\sqrt{n}}\right) \xrightarrow{P} 0 \text{ by Chebyche}$$

Since $E\left(\frac{\sum (X_i - \mu)}{\sqrt{n}}\right)^2$

$$\leq \frac{r^2 \operatorname{var}(X_1)}{n} \to 0 \qquad \begin{array}{l} n \to \infty \\ r \to \infty \\ r^2 << n. \end{array}$$

Therefore

$$\lim_{\substack{n\to\infty \\ r\to\infty \\ r^2 << n}} \frac{\sum_{i=1}^{n} X_i - \mu}{\sqrt{n}\,\sigma} \to N(0,1)$$

By diagonal argument it's easy to show

Lemma: $F_{n,r}$, F_r, F, cdf.

$F_{n,r} \to F_r$ $n \to \infty$ for all continuity points

$F_r \to F$ $r \to \infty$ for all continuity points

$\exists$ subsequence of r, r_k s.t.

$F_{n, r_k} \to F$ $k \to \infty$, $n \to \infty$ for all continuity points

$\square$