Tommenix Yu HW2. Set Theory.

To Show:

Assume ~~MA~~ $MA(\kappa)$ and $A \subseteq \mathcal{P}(\mathbb{N})$ A B a.d. with $|A| \leq \kappa$. and for all finite $S \subseteq A$ we have ϕ

$$|\mathbb{N} \setminus \cup S| = \aleph_0$$

Then we can find an infinite $B \subseteq \mathbb{N}$ that B almost disjoint ~~fro~~ from A. In particular, $A \cup \{B\}$ is an almost disjoint family.

Pf:

Define partial order on $(s, \underline{S})$ ~~a~~ where $s \subseteq \mathbb{N}$, $\underline{S} \subseteq A$, $|s| < \omega$, $|\underline{S}| < \omega$.

that $(t, T) \leq (s, \underline{S})$ if $s \subseteq t$, $\underline{S} \subseteq T$. $(t \setminus s) \cap (\cup S) = \phi$.

Check it's a p.o.:

- reflexive: ✓
- Anti-symmetry: if $s \subseteq t$, $t \subseteq s \Rightarrow s = t$, same for $\underline{S} = T$ ✓
- Transitivity: $(t, T) \leq (s, \underline{S}) \leq (r, R)$.

  then ~~r ⊆ t~~ $r \subseteq t$, $R \subseteq T$ and $(t \setminus r) = (s \setminus r) \cup (t \setminus s)$

  where $(s \setminus r) \cap (\cup R) = \phi$, $(t \setminus s) \cap (\cup S) = \phi$

  and $\cup R \subseteq \cup S \Rightarrow (t \setminus r) \cap (\cup R) = \phi$ ✓

Check it's ccc:

$\forall S, T$ finite subsets of A. $\forall s \subseteq \mathbb{N}$ finite.

(s, S) and (s, T) are comparable since

$(s, S \cup T)$ ~~a~~ extends both.

So there's at most $|\{s : s \subseteq \mathbb{N}, s \text{ finite}\}|$ many pairwise incompatible sets.

since ← is ω, get ccc.

Find Dense sets for MA:

- $D_\alpha := \{(s, \underset{\sim}{S}) : A_\alpha \in S\}$
  - Dense because $\forall (t, T)$, $(t, T \cup \{A_\alpha\}) \leq (t, T)$.
  - Only $|A| \leq \kappa$ many such D_α
  - meeting it means have an element $(s, \underset{\sim}{S})$ s.t. $A_\alpha \in \underset{\sim}{S}$.
- $D^n := \{(s, \underset{\sim}{S}) : |s| \geq n\}$
  - Dense because by assumption $\mathbb{N} \setminus \bigcup S$ contains any finite number of elements, so choose n from there, call those s_n, then
    $(s \cup s_n, \underset{\sim}{S}) \leq (s, \underset{\sim}{S})$
  - only $\aleph_0$ many such D_α.
  - Meeting it means $\bigcup\{s : (s, S) \in G\}$ is infinite.
    defined later

Use ~~MA~~ $MA(\kappa)$ to construct G filter that meets all D_α and D^n for all α, n.

Now, declare $B := Y_G := \bigcup\{s : (s, S) \in G\}$ to be our ~~set~~ goal.

Check infinity: if finite, size $\leq n$ $\exists$ $|s| > n$ in D^{n+1} so $|Y_G| \geq n+1$, contradiction ✓

check almost disjoint: $\forall A_\alpha \in A$, G meets D_α means $\exists (s, \underset{\sim}{S})$ with $A_\alpha \in \underset{\sim}{S}$.

$\Rightarrow B \cap A_\alpha \subseteq S < \omega$ by definition of our order.

$\Rightarrow$ We've found B that satisfies

$\Rightarrow$ A is not maximal a.d. set.