Tommenix Yu Set Theory HW 1.

Q1: Prove ~~Δ-lemma~~ Δ-system Lemma.

A: • Let $\mathcal{F}$ be the uncountable set of finite sets, with $|\mathcal{F}| = \kappa$, ~~[illegible]~~

κ-uncountable. then. $\exists\, \mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| = \aleph_1$.

~~Let~~ Discard all things in $\mathcal{F}$ that is not in $\mathcal{F}'$, ~~then~~

• We can now sort $\mathcal{F}$ by each element's order, and since there's only finite sets in $\mathcal{F}$, $\mathcal{F} = \underbrace{F_1 \,\dot\cup\, F_2 \,\dot\cup\, \cdots}_{\text{countably, idex is order}}$

• By Regularity of $\aleph_1$, $\exists n$ s.t. $|F_n| = \aleph_1$. Now discard all thats not in F_n, then $\mathcal{F} = \mathcal{F}_n$.

• Now do induction to prove the rest. By Well-Ordering thm, we order $\bigcup \mathcal{F}_0$ with "$\leq_1$"

▷ $n = 1$, then either there is some element contained in unctbly many sets, or there's not.

① if $a \in \bigcup \mathcal{F}_0$ where $a \in \mathcal{F}_\alpha$ for unctbly many α, then choose all containg a, we're done with a unctb Δ-system.

② if DNE such a, then we partition $\mathcal{F}$ by the element it contains. There must ~~be~~ be unctb partitions (by regularity) then use choice function to pick out one such set in each partition, we've find a Δ-system.

↓ next page

$\triangleright$ for $n = m+1$, say result holds for $n = 1$ to m:

① if $\exists$ element a contained in unctb sets, pick those sets, then take away a from each set, then we're left with $n=m$ case, which we get by IH. Putting a back won't break the "Δ-system property".

② if DNE such element, then the idea is to find an uncountable disjoint family within.

Now use the order "$\leq_1$" on $\bigcup \mathcal{F}$. pick one (exist by def of $\bigcup\mathcal{F}$) a_1 containing least element under "$\leq_1$", we throw away all that intersects a_1 in $\mathcal{F}$. Under our condition, we've thrown away ctb×ctb sets, so still left with an unctb set. $\mathcal{F}^2$, find "$\leq_2$" on "$\mathcal{F}^2$"

In a similar way we choose a_2 s.t. a_2 contains the least element of $\mathcal{F}^2$ under "$\leq_2$". we discard all that inter-sects and left with $\mathcal{F}^3$.

Continue this we can map $k \xrightarrow{f} \mathcal{F}$ where k is index.

Then $f(k)$ is the Δ-system by construction.

↑ as a set

$\square$.

Tommenix Yu ~~Set #1~~

Q_2: If κ regular, there's no maximal a.d. family of κ.

Proof: Suppose we have a family $f := \{A_\alpha : \alpha < \kappa\}$ as a maximal a.d. family of κ. then, we construct

$$B_\alpha := A_\alpha - \bigcup_{\beta<\alpha} A_\beta$$

and $|B_\alpha| = \kappa$ since $|A_\alpha| = \kappa$; $|A_\alpha \cap A_\beta| < \kappa$. $\alpha < \kappa$. then regularity says $\left|\bigcup_{\beta<\alpha} |A_\alpha \cap A_\beta|\right| < \kappa$.

$$\& \ |B_\alpha| = \left| A_\alpha \setminus \left(\bigcup_{\beta<\alpha} |A_\alpha \cap A_\beta\right)\right| = \kappa.$$

So B_α is a size κ disjoint family. By AC we pick one element from each B_α. ~~then~~, call this set C. then

$$|C \cap A_\alpha| \leq \alpha < \kappa; \ |C| = \kappa$$

So ~~A_α~~ $f \cup \{C\}$ is an a.d. family strictly larger than f.

$\Rightarrow$ f is not maximal, contradiction.

$\square$.