CONVEX OPTIMIZATION HOMEWORK 1

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Exercise 1.

Proof.

Since S^n_{++} is a group, the inverse is also in the group. That is, $P^{-1} \in S^n_{++}$. Now, to prove

$$\mathcal{E} = \left\{ x | (x - x_c)^T P^{-1} (x - x_c) \right\}$$

is convex, we use the factorization $P^{-1} = QQ = Q^T Q$ for $Q \in S^n_{++}$.

We only need to prove that for $x \in \mathcal{E}$, $y \in \mathcal{E}$, $\alpha x + (1 - \alpha) y \in \mathcal{E}$. One more thing is that we'll just let $(x - x_c) = a$, $(y - x_c) = b$ for convenience.

So we check

$$\begin{aligned} &(\alpha x + (1-\alpha) y - x_c)^T P^{-1} (\alpha x + (1-\alpha) y - x_c) \\ =&(\alpha a + (1-\alpha) b)^T Q^T Q (\alpha a + (1-\alpha) b) \\ =&\alpha^2 \langle Qa, Qa \rangle + (1-\alpha)^2 \langle Qb, Qb \rangle + 2\alpha(1-\alpha)\langle Qa, Qb \rangle \end{aligned}$$

□

But note that $||Qa||^2 = a^T Q^T Q a \leq 1 \Rightarrow ||Qa|| \leq 1$, and for the same reason $||Qb|| \leq 1$. Hence $\langle Qa, Qb \rangle \leq ||Qa|| \cdot ||Qb|| \leq 1$ by Cauchy.

Since $\langle Qa, Qa \rangle \leq 1$, $\langle Qb, Qb \rangle \leq 1$, and $\langle Qa, Qb \rangle \leq 1$, we get that

$$\alpha^2 \langle Qa, Qa \rangle + (1-\alpha)^2 \langle Qb, Qb \rangle + 2\alpha(1-\alpha)\langle Qa, Qb \rangle \leq \alpha^2 + (1+\alpha)^2 + 2\alpha(1-\alpha) = 1$$

which implies from our above deduction that

$$(\alpha x + (1-\alpha) y - x_c)^T P^{-1} (\alpha x + (1-\alpha) y - x_c) \leq 1$$

and hence $\alpha x + (1-\alpha) y \in \mathcal{E}$.

Exercise 2.

Proof. Let

$$S := \left\{ a \in \mathbb{R}^k \middle| p(0) = 2, |p(t)| \leq 2, \alpha \leq t \leq \beta \right\}$$

where $p(t) = a_1 + a_2 t + \cdots + a_k t^{k-1}$.

If $a \in S$, $b \in S$, we want to show that $c = \theta a + (1-\theta) b \in S$ for $\theta \in [0,1]$.

To show this, we check the 2 conditions. Let's denote the polynomial with index a, b, c to clarify.

Since $p_a(0) = p_b(0) = 2$, we get $a_1 = b_1 = 2$, which implies $c_1 = \theta a_1 + (1-\theta) b_1 = 2$, hence $p_c(0) = 2$.

As for $|p_c(t)| \leq 2$, we use triangle inequality:

$$\begin{aligned} |p_c(t)| &= \left| (\theta a_1 + (1-\theta) b_1) + (\theta a_2 + (1-\theta) b_2) t + \cdots + (\theta a_k + (1-\theta) b_k) t^{k-1} \right| \\ &\leq \theta \left| a_1 + a_2 t + \cdots + a_k t^{k-1} \right| + (1-\theta) \left| b_1 + b_2 t + \cdots + b_k t^{k-1} \right| \\ &\leq 2(\theta + 1 - \theta) = 2 \end{aligned}$$

Thus S is convex.

As for the set $T := \left\{ a \in \mathbb{R}^k \middle| p(0) = 2, |p(t)| \geq 2, \alpha \leq t \leq \beta \right\}$ for the same p, it is not convex since we can pick $\alpha = \beta = 4$, $k = 2$, $a = (2, 0)$, $b = (2, -1)$ to get

$$T' := \left\{ a \in \mathbb{R}^2 \middle| p(0) = 2, |p(4)| \geq 2 \right\}$$

check: $p_a(0) = p_b(0) = 2$ by direct computation and

$$|p_a(4)| = |2| \geq 2, |p_b(4) = |-2| \geq 2|$$

so they both are in T'. But let $\theta = \frac{1}{2}$ and we get $c = 2, -1/2$, and $|p_c(4)| = 0 \ngeq 2$, so T' is not convex.

$\square$

Exercise 3.

$$C^0 = \left\{ y \in \mathbb{R}^n \middle| y^T x \leq 1, \forall x \in C \right\}$$

(a) show C^0 is convex;
(b) what is the polar of a cone;
(c) what is the polar of the unit ball in the Euclidean norm?

Proof.

(a):

For $y_1, y_2 \in C^0$, we want to show that $z_\theta = \theta y_1 + (1-\theta) y_2 \in C^0$.

This is because

$$z_\theta^T x = \theta y_1^T x + (1-\theta) y_2^T x \leq \theta + (1-\theta) \leq 1$$

so $z \in C^0$.

(b):

A cone for a set S is the conic combination of all elements in S. Call this cone T, then T^0 is all points such that the inner product of it with any point in T is less or equal to 1.

Note that $\forall x \in T$, $Lx \in T$ for any $L \geq 0$, so if $0 < yx \leq 1$ for $x \in T$, we know that $\exists x' \in T$ such that $yx' > 1$. Hence

$$T^0 = \left\{ y \in \mathbb{R}^n \middle| y^T x \leq 0, \forall x \in T \right\}.$$

To further simplify the condition we show

$$y^T x \leq 0, \forall x \in T \iff y^T x \leq 0, \forall x \in S$$

since $S \subset T$, $\Rightarrow$ direction is obvious. For $\Leftarrow$, we note that any element in T can be written as $\sum\limits_{i \in S} a_i s_i$, for non-negative a_i. Now that each summand of $\sum\limits_{i \in S} y^T a_i s_i$ is ≤ 0 so is the sum, which is nothing but $y^T x$ where x is arbitrary in T. Hence we have

$$T^0 = \left\{ y \in \mathbb{R}^n \middle| y^T x \leq 0, \forall x \in S \right\}.$$

Since we are dealing with $\mathbb{R}^n$, for each point $x \in S$ the solution to $y^T x \leq 0$ is a halfplane, and thus T^0 is the intersection of halfplanes, which is a cone pointing towards the other direction in the whole space whose angle depends on the original cone.

(c): It's the unit ball itself.

Reason: Let D denote the unit ball, then $\forall x, y \in D$, the unit ball, $y^T x \leq ||y|| \cdot ||x|| = 1$, so $D \subset D^0$.

Yet $\forall y \notin D$, $\frac{y}{||y||} \in D$ and $y^T \frac{y}{||y||} = ||y|| > 1$ since y is not in the unit ball. So $D^0 \subset D$.

Therefore $D = D^0$.

$\square$