Technical University of Munich

Finsler-Laplace-Beltrami Operators with Application to Shape Analysis

TECHNION
The Henry and Marilyn Taub Faculty of Computer Science

Simon Weber[1*], Thomas Dagès[2*], Maolin Gao[1], Daniel Cremers[1]

CVPR JUNE 17-21, 2024 SEATTLE, WA

[1]Technical University of Munich, [2]Technion - Israel Institute of Technology

*Equal contribution

Differential Geometry

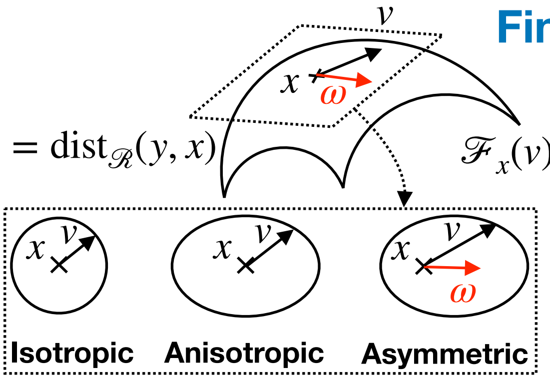
Riemann Geometry

- **Symmetric**

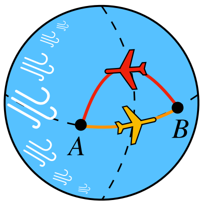
$$\mathcal{R}_x(v) = \mathcal{R}_x(-v) \implies \mathrm{dist}_{\mathcal{R}}(x,y) = \mathrm{dist}_{\mathcal{R}}(y,x)$$

- **Explicit** $\mathcal{R}_x(v) = \sqrt{v^\top M(x) v}$
- **Laplace-Beltrami (LBO)**

$$\Delta_X f = -\mathrm{div}_X(\nabla_X f)$$



Finsler Geometry



- **Asymmetric**

$$\mathcal{F}_x(v) \neq \mathcal{F}_x(-v) \implies \mathrm{dist}_{\mathcal{F}}(x,y) \neq \mathrm{dist}_{\mathcal{F}}(y,x)$$

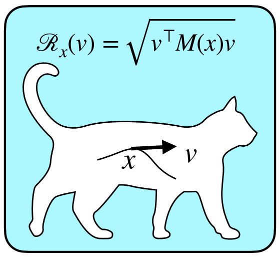
- **Randers** $\mathcal{F}_x(v) = \sqrt{v^\top M(x) v} + \omega(x)^\top v$
- **Laplace-Finsler**

$$\Delta_X f = -\mathrm{div}_X\left(\mathcal{F}^*_X(\nabla_X f)\nabla_X \mathcal{F}^*_x(\nabla_X f)\right)$$

Journey Through Finsler Geometry

Finding Riemann anisotropy via Randers asymmetry

Riemann metric

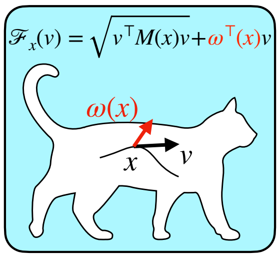


Heat equation → Laplace-Beltrami operator: $\Delta_X f = -\mathrm{div}_X(\nabla_X f)$ → Skew → Anisotropic diffusivity (heuristic): $D_{\alpha\theta}$ → Heuristic ALBO

$$\Delta_D f = -\mathrm{div}_X\left(D\nabla_X f\right)$$

if $\omega = 0$

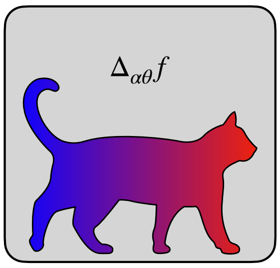
Finsler metric



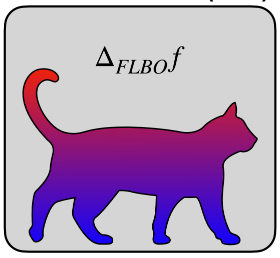
(optional: further skew $D_{\alpha\theta}$)

Heat equation → Laplace-Finsler operator: $\Delta_F f = -\mathrm{div}_X\left(\mathcal{F}^*_x(\nabla_X f)\nabla \mathcal{F}^*_x(\nabla_X f)\right)$ → Simplified special case → Anisotropic diffusivity (Finsler-based): $D_{\mathcal{F}^*_x} = M^* - \omega^*\omega^{*\top}$ → Finsler-based LBO (Ours)

Heuristic ALBO



Finsler-based LBO (Ours)



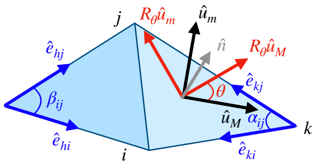
Finsler-Laplace-Beltrami Operator (FLBO)

Theoretically Motivated Anisotropic LBO

$$\Delta_{FLBO} f = -\mathrm{div}_X\left(\left(M^* - \omega^*\omega^{*\top}\right)\nabla_X f\right) \qquad \phi_k^{FLBO} = \lambda_k \Delta_{FLBO}\phi_k$$

Discretisation

- [1] **Heuristic Diffusivity** (Optional) $D_{\alpha\theta} = R_\theta U_{ijk} \overbrace{\begin{pmatrix} \frac{1}{1+\alpha} & \\ & 1 \end{pmatrix}}^{[2]} U_{ijk}^\top R_\theta^\top$
- **Metric construction** $M_{\alpha\theta} = D_{\alpha\theta}^{-1} \qquad \omega_\theta = \tau R_\theta \hat{u}_M$



- **FLBO** $\Delta_{FLBO}^{\alpha\theta} = -\underset{\text{Vertex area}}{S_{FLBO}^{-1}} \underset{\text{"Cotangent" weight scheme [1]}}{W_{FLBO}^{\alpha\theta}}$

$$w_{(i,j)\in E} = \frac{1}{2}\left(\frac{\langle \hat{e}_{kj}, \hat{e}_{ki}\rangle_{D^{\alpha\theta}_{\mathcal{F}^*_x}}}{\sin \alpha_{ij}} + \frac{\langle \hat{e}_{hj}, \hat{e}_{hi}\rangle_{D^{\alpha\theta}_{\mathcal{F}^*_x}}}{\sin \beta_{ij}}\right)$$

Convolution in the Spectral Domain

[2]

$$(f * g)_{\alpha\theta} = \sum \hat{f}(\lambda_{\alpha\theta,k})\hat{g}(\lambda_{\alpha\theta,k})\phi_{\alpha\theta,k} \qquad (f*g)_\alpha(x) = \int_\theta (f*g)_{\alpha\theta}(x)d\theta$$

Chebychev polynomial convolution filter $\hat{g}(\lambda_{\alpha\theta,k}) = \sum c_{\alpha\theta,s} T_s(\lambda_{\alpha\theta,k})$

Discretisation $(f*g)_{\alpha\theta} = \sum c_{\alpha\theta,s} T_s(\Delta_{FLBO}^{\alpha\theta}) f$

Shape Matching

Supervised CNN

[1] Boscaini, D, et al. "Learning shape correspondence with anisotropic convolutional neural networks." *NeurIPS* (2016).
[2] Li, Q, et al. "Shape correspondence using anisotropic Chebyshev spectral CNNs." *CVPR* (2020).