The Art of Mathematical Proof

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1 Introduction

What is a math proof? That is after all why you are reading this article.
Well I'm not going to answer that, at least not yet. First, I want you to
think about the following statements:

- The sum of two even integers is even.

- The sum of two odd integers is even.

- The sum of an even and an odd integer is odd.

You know these statements to be true, but how do you know they are
true? Maybe you quickly added some numbers in your head to confirm that
each statement is true for every example you ran through. That may be good
enough to have satiated any doubt you had, but it doesn't answer the more
important question - why are these statements true? All of mathematical
theory is focused around answering the question of why something is true;
and the tool used is the mathematical proof.
You should now have some understanding of what a mathematical proof
is - it's the tool used to answer the question of why something is true; put
bluntly, a mathematical proof is the proof (pun intended) that a statement
is true. Okay great, so what exactly does a math proof look like? From a
high-level, a math proof is a well-structured, logical argument showing that a
statement is true. How does it do this? A proof starts with the facts at hand
"I have two even integers, therefore I can represent the first number as $2 * j$
and the second as $2 * k$, where j and k are integers because even numbers are
defined to be 2 times some integer.", then deduces what must be true given
those facts,"Therefor, I can represent their sum as $(2 * j + 2 * k)$". With the
new information, more facts can then be uncovered "We can then factor out
the 2 from this representation: $(2 * j + 2 * k) = 2 * (j + k)$. The number

$(j + k)$ must be an integer because the sum of two integers is an integer."
This process continues on until the truth of the statement is uncovered, "I've
shown that the sum of the two even numbers is equal to 2 times the integer
$(j + k)$, which means that this number is even by definition.".

Don't worry if you don't see why the argument above does in fact prove
that the sum of two integers is even, we'll come back to it soon. Instead, focus
on the nature of the argument itself. You may have noticed that there's no
references to specific examples; that each 'step' in the argument was coupled
with a justification; or the reference to a specific definition of what it means
to be even - these are the hallmarks of a proof.

Okay, that's great Will, can you now tell me exactly what a proof is, and
how I can go about writing one? Well I can't... there's no easy way to define
what a proof is, nor a simple set of rules I can give for writing one - it'd
be like defining the color red, sure I could tell you its wavelength, but that
doesn't come close to explaining it. What I can do however is teach you how
to recognize what is a proof and what's not, as well as the tools to begin to
start proving things on your own.

2   The Fundamentals

In mathematics, definitions are the building blocks of all theory. A defini-
tion unambiguously and efficiently describes an 'object'. From the definition
springs all math theory - first, the properties that are inherent to the defini-
tion of the object are proved. From there, more complex facts are uncovered,
and then more, and more, and more (you get where I'm going). Each new
fact uses the previous facts and definitions to show that it too must also be
true. This process of building up facts endows mathematics with enormous
accountability. By the very nature of our exploration, we can always be con-
fident that the facts we discover are in fact true, no matter how far-fetched
or complicated they appear.

Let's start with one of the most simple definitions - the set of integers.
We'll define the set of integers $\mathbb{Z}$ to be the counting numbers $1, 2, 3, ...$ along
with 0, and the negative numbers $-1, -2, ...$ We can represent the integers
in set notation:

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$.

The attentive reader may have noticed that in the introduction proof, I justi-
fied that the sum of j and k was an integer because both j and k are integers.
We'll accept this statement as fact without proving it to be true. Although

simply accepting something as fact goes against the very foundation that theoretical mathematics is built on, there are times when we cannot escape having to do so. When we accept a statement as fact without proving it, we call it an axiom. An axiom does not need to be proven; instead, the truth of an axiom is self-evident (think of an axiom like existence - you know you exist, but it's very difficult to quantify a justification for that belief).