Chaos and periodicity in pendular systems

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Abstract

This report explores the chaotic dynamics of a driven, damped pendulum and a double pendulum system using numerical simulations in particular the Runge-Kutta fourth-order method. Methodologies such as Lyapunov exponents and bifurcation diagrams were employed to explore the role of initial conditions and system parameters on the systems behaviour. Results showed distinct regimes of chaos and periodicity. The study illustrates the complex nature of these dynamical systems and their potential applications in various fields where understanding and manipulating chaotic systems is crucial.

1 Introduction

1.1 What is Chaos?

The term chaos is often used to mean something that is broadly disordered. In physics and mathematics however, the definition of a chaotic system is slightly more specific. The key characteristic that defines a dynamical system as chaotic is its sensitivity to initial conditions. Small perturbations in the initial values of the system's state variables will lead to drastic differences in its behaviour as it evolves with time. This is popularly referred to as the 'butterfly effect', after a 1972 talk given by Edward Lorenz in which he speculated how the flapping of a butterflies wings in Brazil (a small perturbation of initial conditions) may result in a tornado in Texas [1]. Another important criteria for chaos is non-periodicity. Periodic motion is highly predictable and stable, characterized by repeated behavior over time that follows a consistent cycle. In contrast, chaotic systems do not settle into these stable, repetitive patterns over the long term. Although chaotic systems may exhibit brief episodes of periodic behavior, these are typically transient and the system's trajectory will eventually diverge due to its inherent sensitivity to initial conditions.

An important distinction to note is that of determinism as opposed to predictability. Chaotic motion is deterministic: its behaviour is determined by the equations of motion that govern the system, meaning its state can theoretically be determined at any time. In this sense, Chaotic motion is not Random, a common misconception. However, due to sensitivity to initial conditions and complex trajectories, the long term behaviour of a chaotic system is very difficult to predict.

You might imagine that the unpredictable and complex nature of chaotic systems would make them impossible to study in a meaningful way, but as this report will explore, there are in fact many ways to analyze and understand these systems that can yield valuable insights.

1.2 The driven, damped pendulum

The main focus of this report is on the chaotic system of a damped and driven pendulum. This consists of a pendulum that is subject to both gravitational forces and an external periodic driving force. It is damped, meaning the energy added by the driving force is dissipated via friction characterised by a damping factor. These elements combine to produce dynamics that can exhibit both chaotic and periodic behaviors depending on the values of the system's parameters. This is a good example of a nonlinear dynamical system that can transition from regular to chaotic motion as its parameters are varied.

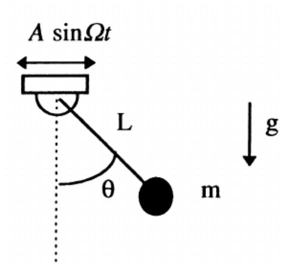

Figure 1: Diagram of set-up of the driven pendulum system

Figure 1 shows a potential experimental setup for this system with a pendulum length L and mass m driven by a horizontal oscillation. The angleof the pendulum θ is measured from the horizontal.

1.3 The double pendulum

Another chaotic system that was considered for this report was that of the double pendulum. This system consists of two massless rods and two bobs. The first rod is attached at one end to a fixed point, with a bob at the other end. The second rod is connected to the first bob, allowing the second bob to swing freely from it. In this case, at large angles, the chaos of the system comes the coupled nature of the two pendulums: they affect each others motion and as a result the system is highly sensitive to initial conditions and moves in complex and chaotic trajectories. The set-up

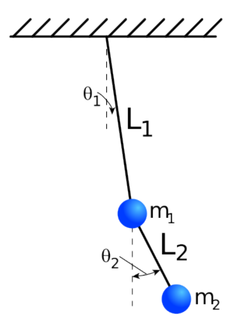

Figure 2: Diagram of set-up of the double pendulum system

is illustrated by Figure 2 with $L_1, L_2, m_1, m_2, \theta_1$ and θ_2 representing the length, mass and angle of the respective pendula.

1.4 Aims

This report aims to explore the nature of chaos within these dynamical systems, and to investigate methodologies for distinguishing between chaotic and periodic behaviors. The primary focus will be on the damped driven pendulum. Specifically, the report will delve into techniques such as Lyapunov exponents to quantify the sensitivity to initial conditions, phase space diagrams to visualise the nature of the trajectories, phase space attractors, and bifurcation diagrams to understand how the system's behavior changes with varying parameters. Through these methods, we seek to provide a deeper understanding of how and why chaos occurs in the driven, damped pendulum and what this implies about the predictability and control of chaotic systems more broadly. With a wide range of applications from biology to astronomy and beyond, the insights gained from studying such chaotic systems are invaluable.

2 Method

2.1 Equations of motion

The first step in modeling our system is to derive the equations of motion that govern the system. For the driven, damped pendulum, this can be done by simply summing the torques acting on the pendulum bob from gravity, damping, and the external driving force, and using Newtons second law for rotational motion. Note the damping force is proportional to the angular velocity of the pendulum, and the driving force is oscillatory in nature and thus takes the form of a sinusoidal function. The result is the following second order non-linear differential equation: [2]

$$I\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + \omega_0^2 I \sin(\theta) = A\sin(\omega_f t)$$

where:

- θ is the angle the pendulum makes with the vertical, measured anticlockwise.
- I is the moment of inertia of the pendulum.
- b is the damping coefficient.
- ω_0 is the natural frequency of the pendulum.
- A and ω_f are the amplitude and frequency of the external driving torque, respectively.

The following substitutions were then made in order to express this as a dimensionless equation: [2]

$$t' = \omega_0 t, \quad q = \frac{\omega_0 I}{b}, \quad f = \frac{A}{\omega_0^2 I}, \quad \omega_R = \frac{\omega_f}{\omega_0}$$

Using these substitutions, the dimensionless form of the equation becomes:

$$\frac{d^2\theta}{dt'^2} + \frac{1}{q}\frac{d\theta}{dt'} + \sin(\theta) = f\cos(\omega_R t')$$

This dimensionless equation simplifies the problem by reducing the number of parameters and allows for a more general analysis applicable to various physical situations with similar dimensionless parameters.

Regarding the double pendulum system, the following coupled equations of motion were derived using the Lagrangian method. This approach involves calculating the system's Lagrangian, which represents the difference between kinetic and potential energies, and then applying Lagrange's equations to each mass.

$$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g \sin(\theta_1) = 0$$

$$l_2\ddot{\theta}_2 + l_1\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - l_1\dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + g\sin(\theta_2) = 0$$

where:

- θ_1 and θ_2 are the angles that the top and bottom pendula respectively make with the vertical.
- m_1 and m_2 are the masses of the pendula.
- l_1 and l_2 are the lengths of the pendula.
- g is the acceleration due to gravity.

2.2 The Runge-Kutta 4th order method

Both of these sets of governing equations for our two systems are non-linear and this non linearity makes them impossible to solve analytically, as is the case with all chaotic systems. Instead they must be solved numerically. A popular method, and the one we employed to obtain solutions for our systems is the Runge-Kutta 4th order (RK4) method. This is a coding method for solving differential equations that is acclaimed for its good balance between accuracy and computational cost, and boasts a very small total error of the order of the fourth power of the step size[3], hence its name.

RK4 works by iteratively calculating four weighted estimates of changes in the system's state at different points within each time step and then combines these to produce a highly accurate approximation of the state at the next time step. Specifically, RK4 begins by calculating an initial rate of change of the state based on the current state variables using a gradient function derived from our equation of motion. It then uses this to estimate the state halfway through the time step, which is again used to calculate another estimate at the midpoint. A fourth and final estimate is calculated at the end of the time step using the third. These four values (k1, k2, k3, k4) are then combined in a weighted sum that prioritizes the mid-point estimates:

$$\text{new state} = \text{current state} + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \cdot \text{step size} \quad [3]$$

By repeating this process for each time step, RK4 provides a reliable and accurate way to trace the pendulum's motion over time, even in the presence of the extreme sensitivity to changes in state variables and system parameters typical of chaotic systems.

2.3 Validating the code

An important next step was to validate our code. Before we can trust the accuracy and validity of the results obtained by simulating the system with RK4, it is imperative to check that it behaves as we would expect for certain parameters that produce known results. By setting the amplitude parameter f in our equation of motion equal to zero, we have obtained the equation for an undriven, damped pendulum. This system can be solved analytically and its solutions are well understood. It behaves very predictably, namely it oscillates sinusoidally with constant period and with exponentially decaying amplitude (dependant on the damping factor). Figure 3 below shows the results obtained.

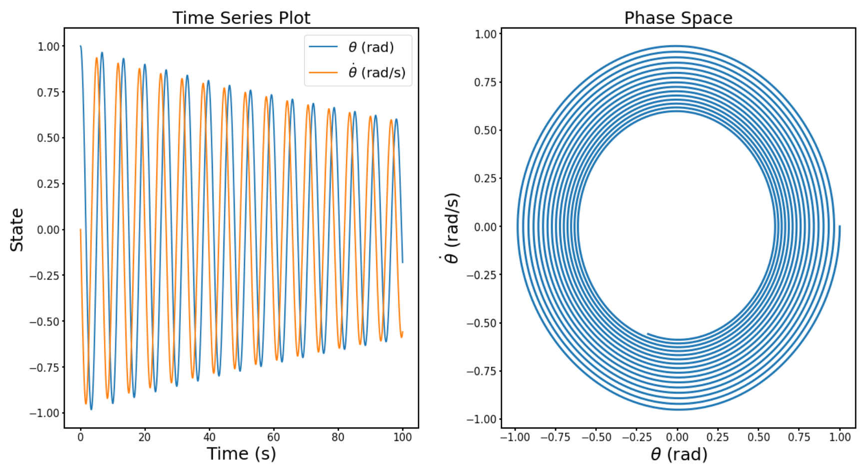

Figure 3: Exponential decay of oscillation amplitude in an undriven damped pendulum. Parameters are f=0, and b=100 to ensure light damping.

Clearly the expected outcome has been achieved for a system in light damping. On the left of Figure 3 you can see time series plot in which indicates the pendulum oscillating with constant period and decaying amplitude. on the right is a phase space diagram in which the trajectory of the pendulum is a circular spiral of decaying radius. The circular shape indicates the uniform oscillation, and the inward spiral due to its amplitude decreasing from damping .

Our code works as expected. This is significant evidence that we can proceed with our analysis of our system reassured of the validity of our results.

2.4 Methods of analysing chaotic systems

With the capability to generate accurate solutions of the system now established, the task becomes to interpret these trajectories in ways that yield meaningful insights. Effective analysis of these solutions is essential for uncovering the underlying properties and potential applications of the system. Several analytical methods have been employed, each contributing uniquely to our understanding of the chaotic behavior exhibited by the system and its responses under various conditions. This section details each of these approaches.

2.4.1 Time series and phase space plots

Time series and phase space plots are fundamental tools in dynamical system analysis, offering a straightforward method for visualizing the evolution of a system's states over time.

A time series plot displays how one or more variables (such as position or velocity) change over time, allowing for the detection of patterns such as periodic cycles or irregular fluctuations indicative of chaotic behavior. A phase space plot maps the trajectory of a system in a multidimensional space where each axis corresponds to one of the system's variables, in our case θ and $\dot{\theta}$. By plotting the trajectory in phase space, one can visually assess the system's dynamics. Regular, closed trajectories indicate periodic motion whereas more complex, scattered trajectories can indicate chaos. Such plots reveal whether the trajectories settle into stable cycles, converge to a point or exhibit the sensitivity to initial conditions typical of chaotic systems. For example, in Figure 3 we can clearly discern periodicity from the regular periodic nature of the time plot as well as the regular circular shape of the phase space plot. Whilst the phase space trajectory is not closed, this is simply due to the fact that the systems energy is decaying from damping, and if it were to be run indefinitely you can see that the trajectory would tend to the origin, representing the pendulum coming to rest. Equally, if the damping force were removed, the trajectory would be a closed circle of constant radius. This was successfully produced by our simulation.

2.4.2 Lyapunov exponents

Lyapunov exponents are a powerful tool for quantifying a systems sensitivity to initial conditions, and can clearly determine whether a system is displaying chaotic motion or not. Lyapunov exponents are the average exponential rates of convergence or divergence of nearby orbits in phase space.[4]At least one positive exponent indicates a chaotic regime, signifying that even infinitesimally close trajectories diverge exponentially over time.

Theoretically, the calculation of Lyapunov exponents is very involved. Namely it requires extensive Jacobian matrix calculations and eigenvalue analysis, and they are calculated in an infinite limit which is computationally impossible. Instead, a more computationally feasible approximation can be employed:

$$\lambda \approx \frac{1}{n\Delta t} \sum_{i=1}^{n} \ln\left(\frac{\|\delta x_i(t)\|}{\|\delta x_0\|}\right)$$ [4]

- λ: The maximal Lyapunov exponent.
- n: The number of measurements over which the divergence is calculated.
- Δt: The time step between each measurement.
- $\delta x_i(t)$: The distance in the phase space between the reference trajectory and the perturbed trajectory at time t_i, where $t_i = i\Delta t$.
- δx_0: The initial separation or perturbation between the reference trajectory and the perturbed trajectory at the beginning of the measurement period.

This approximation computes only the maximal Lyapunov exponent, the critical value for determining whether the system exhibits chaos (based on its sign). In practice, this approximation was implemented in the code as follows.

Begin by simulating two systems with initial states $\vec{x}_0$ and $\vec{x}_0+\vec{\delta x}_0$ where $\vec{x}_0 = [\theta_0, \dot{\theta}_0]$ and $\|\vec{\delta x}_0\|$ is small (to the order of 10^{-6}). These are the reference and perturbed trajectories, respectively. Both states are then evolved using RK4, and the Euclidean distance between the two trajectories in the phase space $\|\vec{\delta x}\| = \sqrt{(\delta\theta)^2 + (\delta\dot{\theta})^2}$ is calculated at regular intervals Δt. After each 'measurement' the perturbed trajectory is rescaled back to the initial small perturbation $\vec{\delta x}_0$ but kept along the direction it has evolved into. This step is critical because it prevents the perturbation from growing too large, which could lead to nonlinear effects that the Lyapunov exponent is not meant to measure. As per the given equation, the the natural logarithm of the ratio of the current distance between the trajectories to their initial separation is computed, quantifying the exponential rate of separation, and these values are averaged over the full run time of the code. The final value of the maximal Lyapunov exponent indicates the system's sensitivity to initial conditions. A positive value suggests chaos, as nearby trajectories diverge exponentially, whilst a negative value indicates that the trajectories converge meaning they are non-chaotic and stable.

2.4.3 Bifurcation plots

While phase space plots, time series and Lyapunov exponents provide information about the systems dynamics for specific parameters (f, q, ω_R), a Bifurcation Diagram provides information on how the system responds over a range of parameter values[2]. For the driven pendulum system which can display both periodic and chaotic motion dependent on system parameters, this is a hugely powerful tool and essentially acts as a map for locating and understanding 'regions' of chaos and stability. The term bifurcation refers to the change in the number of solutions of a differential equation as a result in a change to state parameters, and thus a bifurcation marks the change from stable to chaotic behaviour and vice versa.

Regarding how this method is specifically employed, there is a degree of freedom in the choices one can make. Particularly in the choice of how the system is measured at a given parameter value. In our case, the decision was to identify local maxima and minima (or turning points) of the pendulums angular velocity. This is a good choice, as it offers a clear distinction between chaos and periodicity. Within a periodic regime, the local extremes will continually occur at fixed values, whereas in a chaotic regime they will be scattered, taking a range of values. The code in question simulated the system via RK4, computed the first derivative of angular velocity and identified points where this derivative crossed zero with a change in sign: corresponding to local maxima and minima. The corresponding velocities were plotted for a range of a varying parameter. This process was repeated to obtain bifurcation plots for varying amplitude parameter f, damping parameter q, and relative driving frequency ω_R.

Filtering out transient behavior is crucial in bifurcation analysis as it allows us to focus solely on the long-term behavior of the system. Transient behavior refers to the initial transient responses of the system, which often arise due to the system's initial conditions or short-term disturbances. While transient behavior may exhibit interesting dynamics, it typically does not represent the system's true long-term behavior, which is of primary interest in bifurcation analysis. This was done by allowing the system to evolve for a sufficient duration and disregarding data points in this initial transient period.

2.4.4 Attractors

Attractors are specific sets in the phase space toward which a dynamical system evolves after a sufficient period of time, regardless of its initial conditions. In chaotic systems, these attractors play a crucial role in elucidating the complex behaviors that emerge over time. An attractor can manifest as a point, a curve, or more complex formations known as strange attractors, which exhibit fractal structures[5]. Strange attractors are particularly significant in chaotic systems because they characterize the state of the system in a manner that highlights both its unpredictability and the underlying patterns of its dynamics. They indicate the conditions under which the system stabilizes or remains bounded within a specific region of the phase space, despite the inherent sensitivity to initial conditions.

The identification and analysis of attractors involve numerical simulations in which the system's trajectory is plotted in the phase space over time. These trajectories, observed after the decay of transients, reveal the attractors, providing a powerful tool for studying and understanding the fundamental characteristics of chaotic systems.

3 Results

This section presents the outcomes of the numerical simulations conducted to explore the dynamics of the driven, damped pendulum system. Utilising the Runge-Kutta fourth-order (RK4) method, we investigated the system under various parameter settings, including changes in the driving force amplitude f, damping factor q, and the relative driving frequency ω_R. The results reveal how these parameters influence the chaotic behavior of the pendulum, as evidenced by time series plots, phase space trajectories, Lyapunov exponents, and bifurcation diagrams. Additionally, preliminary results have been produced for the double pendulum system. However, a comprehensive exploration of this more complex system was limited by time constraints, highlighting areas for future study.

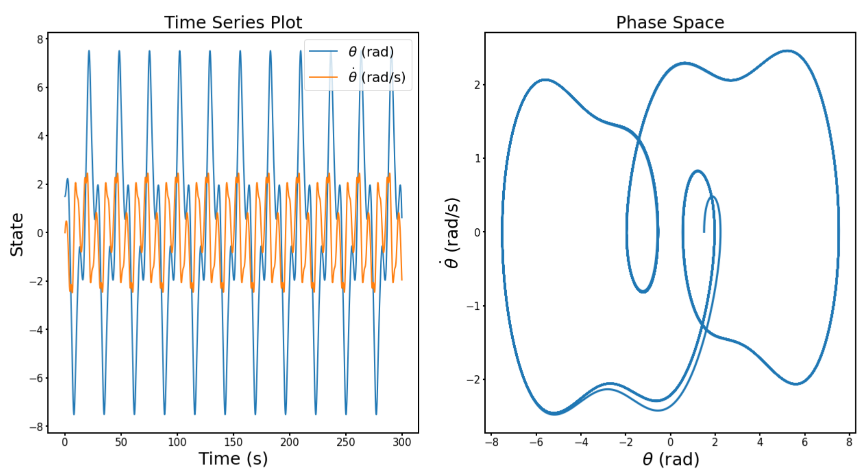

Figure 4: Time series and phase space plots of the driven damped pendulum under parameters $f = 1.75$, $q = 2$, $\omega_R = 0.7$. The system was given initial conditions $[\theta_0, \dot{\theta}_0] = [\frac{\pi}{2}, 0]$. The Lyapunov exponent was calculated at -0.0443.

Figure 4 shows the time series for angle and angular velocity (left) and the phase space plot for the driven, damped pendulum under specific parameters. The phase space trajectory forms a closed path with a small deviation visible in the lower section of the orbit. The time series is periodic in both variables. Note this simulation yielded a negative Lyapunov exponent.

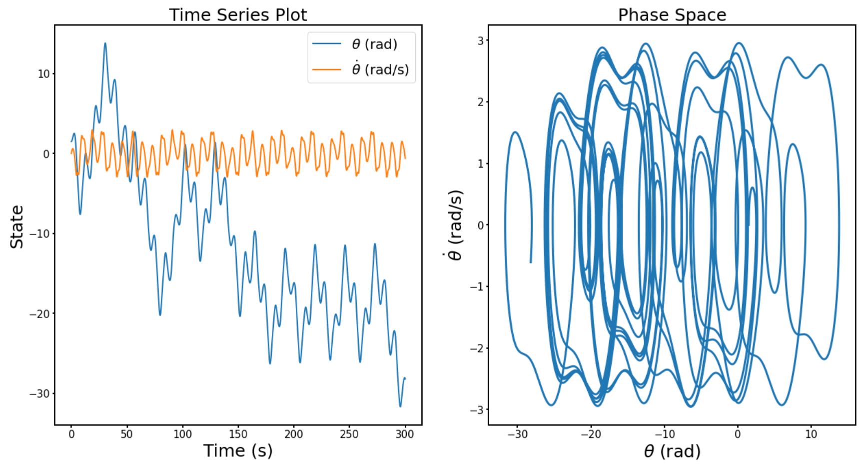

Figure 5: Time series and phase space plots of the driven-damped pendulum under parameters $f = 1.9$, $q = 2$, $\omega_R = 0.7$. The system was given initial conditions $[\theta_0, \dot{\theta}_0] = [\frac{\pi}{2}, 0]$. The Lyapunov exponent was calculated at 0.1273.

In Figure 5, The time series while oscillatory displays unpredictable and non-periodic behavior with transient repeating patterns visible from approximately 100-150s and 175-280s.The phase space plot shows a trajectory that is intricate and densely woven with some discernible repeated patterns/structures.

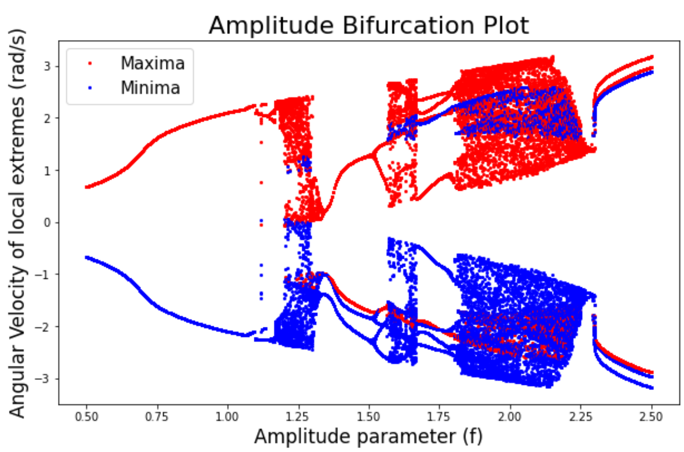

Figure 6: Bifurcation plot of angular velocity turning points against amplitude parameter f. The other system parameters were held constant at $q = 2$ and $\omega_R = 0.7$.

Figure 6 shows the distribution of local extremes in angular velocity as the amplitude parameter varies. The plot displays regions of clearly defined lines or strands, notably from approximately $f = 1.7 - 1.8$, and densely clouded regions ,notably from $f = 1.8 - 2.25$.

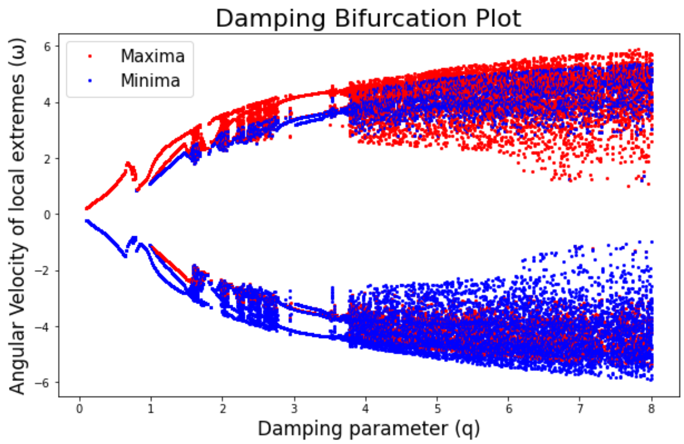

Figure 7: Bifurcation plot of angular velocity turning points against damping parameter q. The other system parameters were held constant at $f = 2.1$ and $\omega_R = 0.5$.

In Figure 7, a bifurcation plot of varying damping parameter, we similarly see regions with discreet defined maxima as well as scattered areas. Note that both bifurcation plots show symmetry in their maxima and minima about 0.

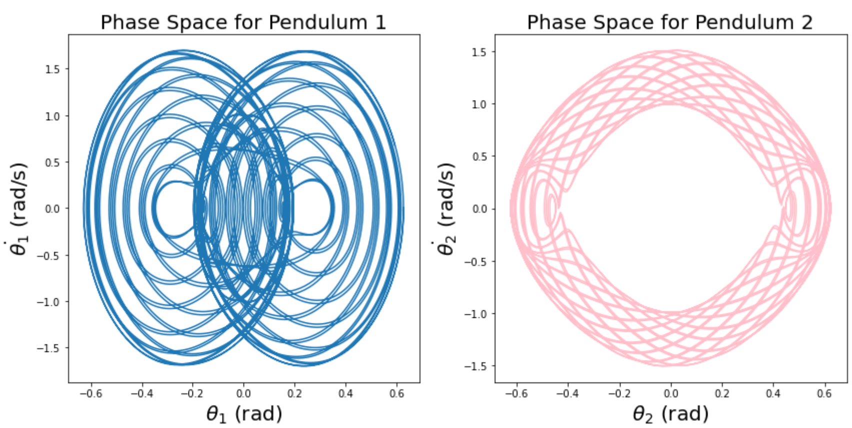

Figure 8: Phase space trajectories of the simulated double pendulum system with transient behaviour removed. The parameters and initial conditions were as follows: $l_1 = 1, l_2 = 3, m_1 = 1, m_2 = 1, [\theta_1(0), \dot{\theta}_1(0), \theta_2(0), \dot{\theta}_2(0)] = [\frac{\pi}{5}, 0.2, \frac{\pi}{7}, 0]$

Figure 8 displays phase space solutions to the double pendulum system. Transient behaviour was removed to reveal complex fractal structures.

4 Discussion

The results of our simulations, as illustrated in Figures 4 through 7, provide insightful contrasts between periodic and chaotic dynamics within the driven, damped pendulum system under varying parameters.

Figure 4 showcases a clear example of periodic motion. The phase space trajectory forms a closed loop, characteristic of periodic systems. The slight deviation observed from this loop is attributable to transient behavior stemming from the initial conditions. This indicates the system's orbit is stable trajectories converge toward it. Correspondingly, the time series plot confirms the periodic nature of the motion, with consistent oscillatory behavior over time.

In stark contrast, Figure 5 presents a state of chaos. The phase space trajectory does not settle into a closed loop but instead covers a broader area, suggesting erratic and unpredictable behavior. The time series backs this observation, displaying oscillations that appear to lack any consistent periodicity. However, despite the apparent randomness, there are subtle indications of underlying patterns and regimes of quasi-periodicity.

As expected, a negative Lyapunov exponent was computed for the solution represented in Figure 4. This confirms periodicity and convergence of neighbouring trajectories. A positive exponent was calculated for the solution shown in Figure 5 indicating chaotic motion characterised by sensitivity to initial conditions and diverging adjacent trajectories. This method of obtaining Lyapunov exponents has successfully and consistently identified chaotic and periodic solutions.

The Bifurcation analysis provided by Figures 6 is powerful as it maps out the regions of periodicity and chaos with varying amplitude of the driving force. Note that the highlighted ranges in Figure 6 contain the solutions shown in Figures 4 and 5, regions of periodicity and chaos respectively. This validates the bifurcation method. Figure 7 extends the analysis to variations in the damping factor, illustrating that the pendulum's response is not only sensitive to the driving force amplitude but to the systems other parameters as well. Similar bifurcation plots were produced exploring responses to varying driving frequency but these have been left out of the report in sake of brevity. That being said, variations in f yielded a greater number of distinct regimes of chaos and periodicity, and with greater clarity than changes in other parameters. It would appear that the amplitude of the driving force is the pivotal factor in eliciting the most diverse responses from the system, making it a key lever in exploring and controlling the transition between chaos and periodicity. Such insights are invaluable in applications where precise manipulation of dynamical systems is required. Interestingly, these bifurcation plots display consistent symmetry, another example of chaotic systems displaying underlying patterns

The key takeaway from these observations is the dual nature of the driven, damped pendulum system, and how it can display strikingly different responses over specific ranges of the system parameters, particularly the driving amplitude. These insights enhance our understanding of non linear dynamics with potential practical application across various fields. For example in engineering there are applications like vibration absorbers and stabilizers. The chaotic properties of dynamic systems are harnessed to develop devices that can efficiently adapt to varying frequencies without manual adjustments. By understanding and controlling the chaotic behavior through parameter manipulation, these devices can maintain optimal performance in diverse operational conditions. [6]In neural biology, chaotic systems play a crucial role in understanding the complex dynamics of neural networks. For instance, chaotic dynamics are evident at various levels of neural systems, from individual neurons to larger neural assemblies. This chaos is not just a random activity but has functional implications, such as in the synchronization of neuronal activity which is fundamental to cognitive functions like memory and perception.Specifically, chaotic dynamics in neural systems have been shown to enhance the brain's responsiveness to weak signals through a phenomenon known as chaotic resonance. This implies that chaotic neural networks can synchronize better with external stimuli.[7]

Due to time constraints, an in depth analysis of the double pendulum system was not achieved, leaving a more superficial understanding of the results. The complex fractal structures in the phase space shown in Figure 8 may represent Attractors, since transient behaviour was filtered out suggesting they are regions to which trajectories converge. Equally, they could represent complex periodic solutions as the pendulum enters small angle oscillations in which its solutions become analytic.

distinguishing between chaos and periodicity through numerical simulations. By altering system parameters, we have demonstrated the system's sensitivity to initial conditions and parameter values. These findings deepen our understanding of chaotic systems and underscore potential practical applications in engineering and neural biology where control and prediction of complex systems are crucial.

5 Conclusion

The numerical simulations conducted on the driven, damped pendulum and the double pendulum systems have been analysed with a range of methods providing valuable insights into the nature of chaotic dynamics. By varying system parameters, we demonstrated transition between chaos and periodicity. Bifurcation diagrams proved particularly effective in mapping these transitions, underscoring the amplitude of the driving force as a critical factor in producing varied responses in the driven, damped pendulum. Analysis via Lyapunov exponents proved a powerful method of confidently and rigorously identifying chaos and periodicity and highlighted the sensitivity to initial conditions. While comprehensive analysis of the double pendulum was constrained by time, preliminary findings indicate complex behavior potentially indicative of chaotic attractors or complex periodic orbits. Overall, this study enhances our understanding of chaotic systems, offering pathways for future research and practical applications such as in technology and biology, where precise control of dynamics is essential.

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