Mathematics For Me

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Chapter 1

Linear Algebra

1.1 The Beginning, Systems of Linear Equations

When we first started learning about variables in middle school, one of the earliest forms of an equation we were introduced to may have looked something like this:

$$mx + b = c$$

Your first instinct is probably, "Oh look! It's the slope-intercept formula!" and you'd be right! The slope-intercept formula is an equation representing a line in 2D space (because you can move in the x and y direction).

As we move along and learn about more complex equations, we most likely encountered equations of higher dimensions (i.e., containing more variables). For example:

$$ax + +cz = d \text{ Plane in 3D space.} \tag{1.1}$$

$$ax_1 + by + cz + dx_2 = e \text{ Something in 4D space.} \tag{1.2}$$

$$a_1x_1 + a_2x_2 + ... + a_nx_n = d \; n\text{D space.} \tag{1.3}$$

As you might have guessed from the equations above, we can easily scale our **linear equations** to higher dimensions just by adding another **constant** multiplied by a new **variable**. You might have been thrown off by the number of terms introduced, so let's give them a proper definition!

- **Constant**: Any term that does not change. For example, in the equation $2x + 3$, 2 and 3 are constants because we know their value. We also know that 2 and 3 are definite values, meaning they do not change at all, regardless of what x is. In the equations provided above, a, b, c, a_1, a_2, etc are constants.

- **Variable**: Any value that will change. Typically, these are represented by an alphabetic letter (x, y, z, etc). We are usually trying to solve the equation to find the value for these variables. For example, we might be provided a question that asks us to solve for x in: 10 = 2x + 5. In this case, x is the variable we are solving for.
- **Linear Equation**: A simple equation that *does not involve any exponents or square roots of a variable.* Basically, if you see equations like: $x^2+x=0$ or $x*y+x=2$, then they are *NOT* linear equations. Equations like: $x+2=10$ or $x+y+z=10$ would be considered linear equations (because they do not involve any powers or square roots, which would alter the **linearity** of the equation, basically it will no longer be a straight line).

In any linear equation, the *constants cannot all be 0's.* "Why not?" well, because if they were, then the equation would just be 0 = some number (remember, 0 times anything is going to be 0)! That's not going to be a straight line, or even a point! However, the d and e (shown in equation 1.3) *CAN* be 0. Any linear equation where the other side of the = is 0 is called a **homogeneous linear equation**.

Generally, a **linear equation in n variables** (where n is the number of variables for the linear equation) can be expressed via the equation shown in 1.3 (the nD one)!

1.1.1 There are more of them now!

Just like how we can add more variables, we can also add more equations.

A finite set of linear equations is called a **systems of linear equations** or **linear system** (we will be calling them linear systems from now on for ease of typing). In a linear system, the variables may also be called **unknowns**.

The equation below is an example of a simple linear system in 2 variables (x_1 and y_1):

$$\begin{aligned} 4x_1 + 10y_1 &= 6 \\ 6x_1 - y_1 &= 10 \end{aligned} \tag{1.4}$$

Again, like the number of variables...we can scale the number of equations as much as we want!

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_3x_3 + ... + a_{1n}x_n &= b_1 \\ &. \\ &. \\ &. \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m \end{aligned} \tag{1.5}$$

For a general linear system, we can say that a **solution** in n unknowns (x_1, x_2, ..., x_n) is a sequence of n numbers that, when substituted into unknowns x_1, x_2, ..., x_n, will make all linear equations in the linear system to be true.

Let's walk through an example. Given the linear system below, solve for x and y:

$$\begin{aligned} 2x - y &= 7 \\ 3x - 2y &= 10 \end{aligned} \tag{1.6}$$

Let's use the good old elimination method (if you need review, here is a great link: Khan Academy is great!). What variables can we easily eliminate? $-2y$ seems like a good candidate! Let's multiple $2x - y = 7$ by -2 and add the results to $3x - 2y = 10$.

$$\begin{aligned} 2x - y &= 7 \\ 3x - 4x - 2y + 2y &= 10 - 14 \\ \texttt{=>} \quad -x &= -4 \\ \texttt{=>} \quad x &= 4 \end{aligned} \tag{1.7}$$

Let's plug $x = 4$ back into our first equation to find y!

$$\begin{aligned} 2 * (4) - y &= 7 \\ \texttt{=>} \quad 8 - y &= 7 \\ \texttt{=>} \quad y &= 1 \end{aligned} \tag{1.8}$$

Now we know equation 1.6 has a solution: $x = 4$ and $y = 1$. We can also write the solution as coordinates too: (4, 1). More often times than not, we'll be writing the solution as coordinates because it is a lot easier to write out and visualize. Furthermore, we can think of the solutions to our linear systems as coordinates where all of the equations in the linear system intersect (cross). For example:

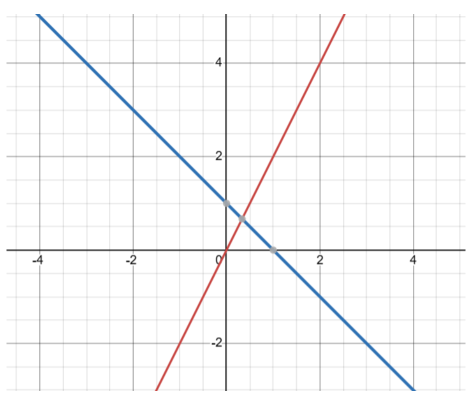

Figure 1.1: The two lines shown intersect only at point (0.33333, 0.66667).

Figure 1.1 shows two lines intersecting at one point. That is, it would only have **1 solution**. Okay, that is easy to understand...but what about other scenarios? What if there are so many solutions, we can't really pick one and be done? In that scenario, you have **infinitely many solutions**! You can think of it like *two lines that are on top of each other* or *two planes stacked on top of one another*.

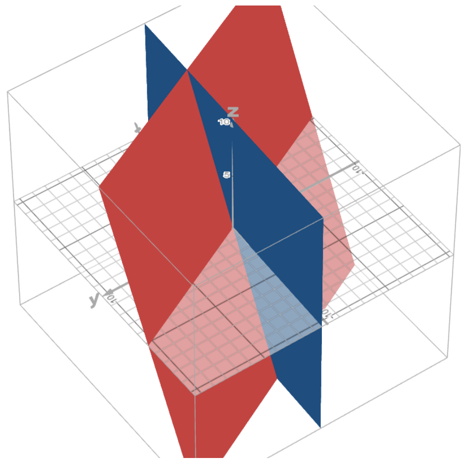

Figure 1.2: The red plane and the blue plane share an entire line where they both intersect each other. If you think back to calculus (specifically limits), you'll remember that there are infinitely many steps you can take along a line. In turn, the two planes share an infinite number of points of intersection. Likewise, if you stacked the plane on top of each other, they will also have infinitely many points of intersection, because they are basically crossing at every point!

Figure 1.2 showcases an example of infinitely many solutions. The two planes cross along an entire line, this indicates that the two planes share **MANY** points of intersections. You can think of it like this: If we stretched the planes out into infinity, the points of intersection will ALSO stretch to infinity.

In situations where we are expected to give an answer, but there are just too many answers to give, mathematicians created a neat way to provide a solution without explicitly mentioning a point! We call this way of writing our solution, **parametric equations**. Long story short, we rewrite one of the linear equations in our linear system. For example, the equation below has infinitely many solutions:

$$\begin{aligned} 4x + 2y &= 1 \\ 16x + 8y &= 4 \end{aligned} \tag{1.9}$$

If you tried solving for the linear system above, you would ultimately get $0 = 0$. This means that any solution that solves for $4x + 2y = 1$, will also solve for $16x + 8y = 4$ and vise versa! If you looked closely, you'll also notice that the *two equations are just scalar multiples of each other*!

Okay, now that we know for sure the linear system has infinitely many solutions, let's write a parametric equation! The steps are easy, really:

1. Rewrite the equation so that one variable is isolated to one side.
2. Substitute all other variables (except for the isolated variables) with a different letter.

For the first linear equation in 1.9, one of the parametric equation can be:

$$\begin{aligned} 4x + 2y &= 1 \\ \texttt{=>}\ 4x &= 1 - 2y \\ \texttt{=>}\ x &= \frac{1}{4} - \frac{y}{4} \\ \texttt{=>}\ x = \frac{1}{4} - \frac{t}{4}&, y = t \end{aligned} \tag{1.10}$$

If you had more unknowns, it's basically the same thing:

$$\begin{aligned} 2x - 3y + z - 9d &= 10 \\ \texttt{=>}\ 2x &= 10 + 3y - z + 9d \\ \texttt{=>}\ x &= \frac{10}{2} + \frac{3y}{2} - \frac{z}{2} + \frac{9d}{2} \\ \texttt{=>}\ x = \frac{10}{2} + \frac{3t}{2} - \frac{r}{2} + \frac{9s}{2}&, y = t, z = r, d = s \end{aligned} \tag{1.11}$$

Basically what we're doing in the last step is assigning a **parameter** to every unknown variable (except for the isolated variable).

There are some scenarios where you might also get **no solutions** too!

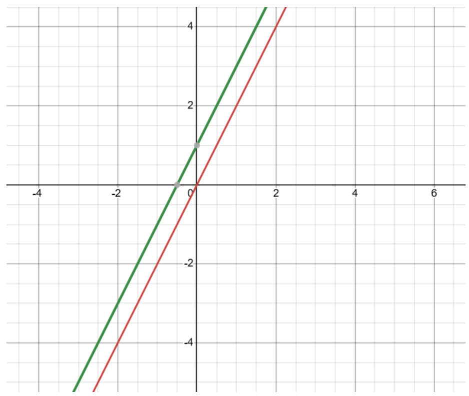

Figure 1.3: The lines are parallel, they will never cross.

Based on our investigation of one solution and infinitely many solutions, can you guess why Figure 1.3 has no solutions? If you said it was because they never cross, then you would be correct!

The two lines in Figure 1.3 never cross, they never intersect. This means that there are no solutions for x and y that satisfies both linear equations.

Okay...at this point you may be thinking, "Tommy, WHY are you saying all of this now? This seems like its coming out of left field!". Well, my curious and mathematically driven reader, the reason why I bring this up now is because of an interesting rule/property that comes with linear systems: **Every linear systems has zero, one or infinitely many solutions. There are no in betweens!** As such, a linear system can be **consistent** (if it has at least 1 solution) or **inconsistent** (if it has no solutions).

1.1.2 The Matrix: Not the 1999 Film

As we add more equations and more unknowns, it becomes increasingly more difficult to perform computations on them, both in terms of manual computation and on the computer.

For one, keeping track of all of the "+", "-", unknowns, etc will become increasingly more difficult the more equations and unknowns we have. To combat the visual complexity of a large linear system, we introduce **augmented matrices** (**augmented matrix** for 1 matrix).

Long story short, we can convert our long and tedious-to-write-out linear system into a matrix of numbers. So our long equation below:

$$
\begin{aligned}
2x_1 + 3x_2 + 4x_3 + ... + 10x_n &= 20 \\
&. \\
&. \\
&. \\
20x_1, +30x_2 + 12x_3 + ... + x_n &= 198
\end{aligned}
\tag{1.12}
$$

Can turn into the matrix:

$$
\begin{bmatrix}
2 & 3 & 4 & ... & 10 & 20 \\
. \\
. \\
. \\
20 & 30 & 12 & ... & 1 & 198
\end{bmatrix}
$$

As you may have noticed, the last column on the right represents the numbers on the right-hand side of the equation 1.12. Broadly, all numbers on the right-most column will be the c in $mx + b = c$, and all other columns will have the coefficients of the unknowns.

Here are a few more examples so you can see what an augmented matrix may look like.

$$
\begin{aligned}
2x - 3y &= 6 \\
x + 10y &= 7
\end{aligned}
\tag{1.13}
$$

$$
\begin{bmatrix}
2 & -3 & 6 \\
1 & 10 & 7
\end{bmatrix}
$$

$$
\begin{aligned}
3x - 1y - 20z &= 6 \\
10x + 5y + 8z &= 7 \\
10x + y + .5z &= 7
\end{aligned}
\tag{1.14}
$$

$$
\begin{bmatrix}
3 & -1 & -20 & 6 \\
10 & 5 & 8 & 7 \\
10 & 1 & .5 & 7
\end{bmatrix}
$$

Now that we know what an augmented matrix looks like, we can start describing how to perform operations on said matrix.

1.1.3 It's Elementary, My Dear Augmented Matrix!

When we perform operations on an augmented matrix to solve for unknowns, we are perform what are called **elementary row operations**. When it comes to row operations, there are 3 operations we can perform:

1. Multiple a row by a non-zero number.
2. Swap rows.
3. Add one row multiplied by a constant to another row.

TODOs:

- ~~one solution, infinitely many solutions, no solutions~~
- ~~consistent vs inconsistent~~
- ~~parametric equations~~
- elementary row operations
- ~~augmented matrix~~
- Examples