Ejercicio 1. Sea $\Sigma = \{a,b\}$ un alfabeto. Hallar:
$$\Sigma^0, \quad \Sigma^1, \quad \Sigma^2, \quad \Sigma^*, \quad \Sigma^+, \quad |\Sigma|, \quad |\Sigma^0|$$
($|A|$ indica la cantidad de elementos de A).

$\Sigma^0 = \{\lambda\}$ $\quad\Sigma^1 = \{a,b\}$ $\quad\Sigma^2 = \{aa, ab, ba, bb\}$

$\Sigma^* = \{\lambda, a, b, aa, ab, ba, bb, aaa...\}$ $\quad |\Sigma| = 2$

$\Sigma^+ = \{a, b, aa, ab, ba, bb, aaa...\}$ $\quad |\Sigma^0| = 1$

Ejercicio 2. Decidir si, dado $\Sigma = \{a,b\}$, vale:
$$\lambda \in \Sigma,\;✗ \quad \lambda \subseteq \Sigma,\;✗ \quad \lambda \in \Sigma^+,\;✗ \quad \lambda \in \Sigma^*,\;✓ \quad \Sigma^0 = \lambda,\;✗ \quad \Sigma^0 = \{\lambda\}\;✓$$

Ejercicio 3. Sea $\alpha = abb$ una cadena. Calcular:
$$\alpha^0, \quad \alpha^1, \quad \alpha^2, \quad \alpha^3, \quad \prod_{k=0,\ldots,3}\alpha^k = \alpha^0.\alpha^1.\alpha^2.\alpha^3, \quad \alpha^r$$

$\alpha^0 = \lambda$ $\quad \alpha^1 = abb$ $\quad \alpha^2 = abb\,abb$ $\quad \alpha^3 = abb\,abb\,abb$

$\prod_{k=0,..3}\alpha^k = \alpha^0.\alpha^1.\alpha^2.\alpha^3 = \therefore\; abb\,abb\,abb\,abb\,abb\,abb$

$\alpha^R = bba$

Ejercicio 4. Sean las cadenas $\alpha = abb$ y $\beta = acb$. Calcular:
$$\alpha\beta, \quad (\alpha\beta)^r, \quad \beta^r, \quad \beta^r\alpha^r, \quad \lambda\alpha, \quad \lambda\beta, \quad \alpha\lambda\beta, \quad \alpha^2\lambda^3\beta^2$$

$\alpha.\beta = abbacb$ $\quad(\alpha.\beta)^R = bcabba$ $\quad\beta^R = bca$

$\beta^R.\alpha^R = bcabba$ $\qquad \lambda\alpha = abb$ $\quad \lambda\beta = acb$ $\quad \alpha\lambda\beta = abbacb$ $\quad \alpha^2\lambda^3\beta^2 = abbabbacbacb$

Ejercicio 5. Dado un alfabeto Σ, sean $x,y \in \Sigma$ y $\alpha,\beta \in \Sigma^*$. Demostrar que:

a. $|x.(y.\alpha)| = 2 + |\alpha|$
b. $|\alpha^r| = |\alpha|$
c. $|\alpha x\beta| = |x\alpha\beta|$
d. $|\alpha.\alpha| = 2|\alpha|$
e. $(\alpha.\beta)^r = \beta^r.\alpha^r$
f. $(\alpha^r)^r = \alpha$
g. $(\alpha^r)^n = (\alpha^n)^r$

($|\alpha|$ indica la longitud de la cadena α).

a) $|x.(y.\alpha)| \overset{LON}{=} 1 + |y.\alpha| \overset{LON}{=} 2 + |\alpha|$

b) Inducción en α
Caso Base $\alpha = \lambda \to |\lambda^R| = |\lambda|$
$\overset{rev}{=} |\lambda|$ ✓

Caso inductivo $\alpha = (x.\beta) \to |(x.\beta)^R| = |x.\beta|$ Hi $|\beta^R| = |\beta|$
$|(x.\beta)^R| \overset{rev}{=} |\beta^R.x| \overset{L1}{=} 1 + |\beta^R| \overset{Hi}{=} 1 + |\beta| \overset{LON}{=} |x.\beta|$

Lema1 $|\alpha.x| = 1 + |\alpha|$
CB → $|\lambda.x| = 1 + |\lambda|$ Ci $|y.\beta.x| = 1 + |y.\beta|$ Hi: $|\beta.x| = 1 + |\beta|$
$|x.x| = |x| = 1 + 0$ $\overset{LON}{=} 1 + |\beta.x| \overset{LON}{=} 1 + 1 + |\beta|$
$= 1 + |\lambda|$ Hi $= 1 + 1 + |\beta|$ ✓

c) $|\alpha.x.\beta| = |x\alpha.\beta|$
$\overset{AST}{=} |\alpha.(x.\beta)|$ $\overset{Aso}{=} |x.(\alpha.\beta)|$
$\overset{L2}{=} |\alpha| + |x.\beta|$ $\overset{L2}{=} |x| + |\alpha.\beta|$
$\overset{L2}{=} |\alpha| + |x| + |\beta|$ $\overset{L2}{conmut} = |\alpha| + |x| + |\beta|$ ✓

Lema2 $\alpha, \beta \in \Sigma^*$ gvg $|\alpha.\beta| = |\alpha| + |\beta|$

Caso Base $\alpha = \lambda \to |\lambda.\beta| = |\lambda| + |\beta|$
$\overset{Neutro}{=} |\beta|$ $\overset{LON}{=} 0 + |\beta|$ ✓

Caso inductivo: $\alpha = x.\gamma \to |x.\gamma.\beta| = |x.\gamma| + |\beta|$ Hi $|\gamma.\beta| = |\gamma| + |\beta|$
$\overset{LON}{=} 1 + |\gamma.\beta|$ $\overset{LON}{=} 1 + |\gamma| + |\beta|$
$\overset{Hi}{=} 1 + |\gamma| + |\beta|$ ✓

d) $|\alpha.\alpha| = 2|\alpha|$
Caso Base $\alpha = \lambda \to |\lambda.\lambda| = 2|\lambda|$
$0 = 0$
Caso inductivo $\alpha = x.\beta$ $|x.\beta.x\beta| = 2|x.\beta|$ Hi $= |\beta.\beta| = 2|\beta|$
$\overset{L2}{=} |x.\beta| + |x.\beta|$
$= 2 \cdot |x.\beta|$ ✓

e) $(\alpha.\beta)^R = \beta^R.\alpha^R$
Caso Base $\alpha = \lambda$ $(\lambda.\beta)^R = \beta^R.\lambda^R$
$\qquad\qquad (\beta)^R = \beta^R.\lambda = \beta^R$ ✓
Caso inductivo $\alpha = x.\gamma$ $(x.\gamma.\beta)^R = \beta^R.(x.\gamma)^R$ Hi: $(\gamma.\beta)^R = \beta^R.\gamma^R$
$\overset{Rev}{=} (\gamma.\beta)^R.x$ $\overset{Rev}{=} \beta^R.\gamma^R.x$ ✓
Hi $= \beta^R.\gamma^R.x$

f) $(\alpha^R)^R = \alpha$
Caso Base $\alpha = \lambda \to (\lambda^R)^R = \lambda$
$\qquad\qquad = \lambda^R = \lambda$ ✓
Caso inductivo $\alpha = x.\beta \to ((x.\beta)^R)^R = x.\beta$ Hi: $(\beta^R)^R = \beta$
$\overset{Rev}{=} (\beta^R.x)^R \overset{L3}{=} x.\beta^{R\,R} \overset{Hi}{=} x.\beta$ ✓

Lema3 $(\alpha.x)^R = x.\alpha^R$ $x \in \Sigma$ $\alpha \in \Sigma^*$
Caso Base $\alpha = \lambda \to (\lambda.x)^R = x.\lambda^R$
$x.\lambda \overset{?}{=} x = x$ ✓
Caso inductivo $\alpha = y.\beta \to (y.\beta.x)^R = x.(y.\beta)^R$ Hi $(\beta.x)^R = x.\beta^R$
$\overset{Rev}{=} (\beta.x)^R.y$ $\overset{Rev}{=} x.\beta^R.y$ ✓
Hi $= x.\beta^R.y$

g) $(\alpha^R)^m = (\alpha^m)^R$
Caso Base $m = 0 \to (\alpha^R)^0 = (\alpha^0)^R$
$\qquad\qquad \lambda = \lambda^R = \lambda$ ✓
Caso inductivo $m = m+1 \to (\alpha^R)^{m+1} = (\alpha^{m+1})^R$ Hi: $(\alpha^R)^m = (\alpha^m)^R$
$\overset{Pot}{=} \alpha^R.(\alpha^R)^m$ $\overset{Pot}{=} (\alpha.\alpha^m)^R$
Hi $= \alpha^R.(\alpha^m)^R$ $\overset{e}{=} \alpha^R.(\alpha^m)^R$ ✓

Ejercicio 6. Dar ejemplos de cadenas que pertenezcan a los siguientes lenguajes:

a. $\mathcal{L} = \{a^n b^n \mid n \geq 0\}$
b. $\mathcal{L} = \{a^n b^n \mid n \geq 1\}$
c. $\mathcal{L} = \{a^n b^m \mid n \geq 1 \wedge m \geq 1\}$
d. $\mathcal{L} = \{a^n b^m \mid n \geq 1 \wedge m \geq 0\}$
e. $\mathcal{L} = \{a^n (ac)^p (bab)^q \mid n \geq 0 \wedge q = p + 2 \wedge p \geq 1\}$
f. $\mathcal{L} = \{a, b\}^3 \cap \Lambda$
g. $\mathcal{L} = \{\alpha \alpha^r \mid \alpha \in \{a, b\}^+\}$
h. $\mathcal{L} = \{\alpha \in \{a, b\}^+ \mid \alpha = \alpha^r\}$

a) $\lambda \in \mathcal{L}_a$, $(ab) \in \mathcal{L}_a$, $(aabb) \in \mathcal{L}_a$

b) Los mismos que $\mathcal{L}_a$ pero sin λ

c) $(aabb), (aaaab), (abb) \in \mathcal{L}_c$

d) $(a), (aab), (aaaa), (aabbb) \in \mathcal{L}_d$

e) $(ac\,bab\,bab\,bab), (aaa\,ac\,ac\,(bab)^4) \in \mathcal{L}_e$

f) $\mathcal{L}_F = \emptyset$ $\quad (\{a,b\}^3 = \{aaa, aab, abb, ...\}) \cap \{\lambda\} = \emptyset$

g) $(aa), (abbaa\,aabba), (bba\,abb) \in \mathcal{L}_g$

h) $(aaa), (aba), (aabab\,babaa) \in \mathcal{L}_h$.

Ejercicio 7. Definir por comprensión los siguientes lenguajes:

a. $\mathcal{L}_1 = \{ab, aabb, aaabbb, ...\}$
b. $\mathcal{L}_2 = \{aab, aaaabb, aaaaaabbb, ...\}$
c. $\mathcal{L}_3 = \{aaabccc, aaaabcccc, aaaaabccccc, ...\}$

(donde el «crecimiento» en la cantidad de cada símbolo es lineal en cada caso).

$\mathcal{L}_1 = \{a^m \cdot b^m \mid m \geq 1\}$ $\qquad \mathcal{L}_3 = \{a^m \cdot b \cdot c^m \mid m \geq 3\}$

$\mathcal{L}_2 = \{a^m \cdot b^m \mid m \geq 1 \wedge m = 2 \cdot m\}$

Ejercicio 8. Dados $\mathcal{L}_1 = \{a, bc\}$, $\mathcal{L}_2 = \{aaa, bc\}$, y siendo $\Lambda = \{\lambda\}$, calcular:

a. $\mathcal{L}_1 \cup \mathcal{L}_2$
b. $\mathcal{L}_1 \cap \mathcal{L}_2$
c. $\mathcal{L}_1 . \mathcal{L}_2$
d. $\mathcal{L}_1 . (\mathcal{L}_2)^0$
e. $\mathcal{L}_1 . (\mathcal{L}_2)^2$
f. $\mathcal{L}_1 . (\mathcal{L}_2)^+$
g. $(\mathcal{L}_1 . \mathcal{L}_2)^+$
h. $(\mathcal{L}_1 . \mathcal{L}_2)^*$
i. $\mathcal{L}_1 . \Lambda . \mathcal{L}_2$
j. $\mathcal{L}_1 \emptyset \mathcal{L}_2$
k. $(\mathcal{L}_1)^r$
l. $(\mathcal{L}_1 . \mathcal{L}_2)^r$

a) $\mathcal{L}_1 \cup \mathcal{L}_2 = \{a, bc, aaa\}$ b) $\mathcal{L}_1 \cap \mathcal{L}_2 = \{bc\}$

c) $\mathcal{L}_1 . \mathcal{L}_2 = \{a.aaa, a.bc, bc.aaa, bc.bc\}$

d) $\mathcal{L}_1 . (\mathcal{L}_2)^0 = \mathcal{L}_1 . \Lambda = \mathcal{L}_1$

e) $\mathcal{L}_1 (\mathcal{L}_2)^2 = \mathcal{L}_1 . \mathcal{L}_2 . \mathcal{L}_2 = \{aaaa.aaa, aaaa.bc, abc.aaa, abcbc, bcaaa.aaa, bcaaabc, bcbcaaa, bcbcbc\}$

f) $\mathcal{L}_1 . (\mathcal{L}_2)^+ = \mathcal{L}_1 . \{aaa, bc, aaaaaa, aaabc, bcaaa, bcbc....\}$

$= \{a.aaa, bc.aaa, abc, bcbc, aaaaaaa, bc\,aaaaaa, a.aaabc, bc.aaabc, a.bcaaa, bcbcaaa, abcbc, bcbcbc,...\}$

g) $(\mathcal{L}_1 . \mathcal{L}_2)^+ = \{\underset{A}{a.aaa}, \underset{B}{a.bc}, \underset{C}{bc.aaa}, \underset{D}{bc.bc}\}^+ = ...\{A,B,C,D, AA, AB, AC, AD, BA, BB, BC, BD, CA, CB, CC, CD..., AAA, AAB, AAC...\}$

h) $(\mathcal{L}_1 . \mathcal{L}_2)^* = \{\lambda, A, B, C, D, AA, AB, ...\}$

i) $\mathcal{L}_1 . \Lambda . \mathcal{L}_2 = (\mathcal{L}_1 . \Lambda) . \mathcal{L}_2 = (\mathcal{L}_1 . \{\lambda\}) . \mathcal{L}_2 = \{a.\lambda, bc\lambda\} . \mathcal{L}_2 = \{aaaa, abc, bcaaa, bcbc\}$

j) $\mathcal{L}_1 \emptyset \mathcal{L}_2$? Preguntar

k) $(\mathcal{L}_1)^R = \{a, cb\}$

l) $(\mathcal{L}_1 . \mathcal{L}_2)^R = \{aaaa, abc, bcaaa, bcbc\}^R = \{aaaa, cba, aaacb, cbcb\}$

Ejercicio 9. Determinar el complemento de los siguientes lenguajes, considerando los alfabetos indicados en cada caso.

a. $\mathcal{L} = \Lambda$ para $\Sigma = \{a, b\}$
b. $\mathcal{L} = \{\lambda, a\}$ para $\Sigma = \{a\}$ y $\Sigma = \{a, b\}$
c. $\mathcal{L} = \{b\alpha \mid \alpha \in \{a, b\}^*\}$ para $\Sigma = \{a, b\}$
d. $\mathcal{L} = \{a^{2n} \mid n \geq 0\}$ para $\Sigma = \{a\}$ y $\Sigma = \{a, b\}$
e. $\mathcal{L} = \{\alpha_1 b \alpha_2 \mid \alpha_1, \alpha_2 \in \{a, b\}^* \wedge |\alpha_1| > |\alpha_2|\}$ para $\Sigma = \{a, b\}$

a) $\mathcal{L}^c = \Sigma^+$

b) $\mathcal{L}^c = \Sigma^+ - \{a\}$ y $\mathcal{L}^c = \Sigma^+ - \{a\}$

c) $\mathcal{L}^c = \{\alpha \in \Sigma^* \mid \alpha = \lambda \vee \alpha = a.\beta \wedge \beta \in \Sigma^*\}$

d) $\mathcal{L}^c = \{a^m \mid m \bmod 2 \neq 0\}$ (impares) y $\mathcal{L}^c = \{a^m \mid m \bmod 2 \neq 0\} \cup \{\alpha \in \Sigma^* \mid |\alpha|_b \geq 1\}$

e) $\mathcal{L}^c = \{\alpha_1 b \alpha_2 \mid \alpha_1, \alpha_2 \in \{a,b\}^* \wedge |\alpha_1| \leq |\alpha_2|\} \cup \{\alpha_1 . a . \alpha_2 \mid \alpha_1, \alpha_2 \in \{a,b\}^* \wedge |\alpha_1| > |\alpha_2|+1\}$

Ejercicio 10. Sea $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$ lenguajes cualesquiera. Determinar si las siguientes afirmaciones son verdaderas o falsas. Si son verdaderas, demostrarlas. Si no, dar un contraejemplo.

a. $\mathcal{L}^+ \subseteq \mathcal{L}^+$.

b. $\mathcal{L}^+ \subsetneq \mathcal{L}^*$

c. $\mathcal{L}^n \mathcal{L}^m = \mathcal{L}^{n+m}$ para todo $n, m \geq 0$

d. $\mathcal{L}^n \subseteq \mathcal{L}^{n+1}$ para todo $n \geq 0$

e. $\mathcal{L}_1 \subseteq \mathcal{L}_2, n \geq 0 \Longrightarrow (\mathcal{L}_1)^n \subseteq (\mathcal{L}_2)^n$

f. $\mathcal{L}_1 \subseteq \mathcal{L}_2 \Longrightarrow (\mathcal{L}_1)^* \subseteq (\mathcal{L}_2)^*$

g. $(\mathcal{L}^*)^* = \mathcal{L}^*$

h. $(\mathcal{L}^+)^+ = \mathcal{L}^*$

i. $(\mathcal{L}^+)^* = \mathcal{L}^*$

j. $(\mathcal{L}_1 \cup \mathcal{L}_2)^* = (\mathcal{L}_1)^* \cup (\mathcal{L}_2)^*$

k. $(\mathcal{L}_1 \cap \mathcal{L}_2)^* = (\mathcal{L}_1)^* \cap (\mathcal{L}_2)^*$

l. $(\mathcal{L}^2)^* = \mathcal{L}^*$

m. $(\mathcal{L} \cup \mathcal{L}_2)^* = \mathcal{L}^*$

n. $(\mathcal{L}^n)^r = (\mathcal{L}^r)^n$ para todo $n \geq 0$

ñ. $(\mathcal{L}^*)^r = (\mathcal{L}^r)^*$

a) son iguales

b) Contraej: $\mathcal{L}^+ = \{\lambda, a\} = \mathcal{L}^* \mid$ son iguales si $\lambda \in \mathcal{L} \mid$

c) $\mathcal{L}^n \mathcal{L}^m = \mathcal{L}^{n+m} \quad \forall n, m \geq 0$

Caso Base $m = 0 \to \mathcal{L}^0 \mathcal{L}^m = \mathcal{L}^m$

$\Lambda \cdot \mathcal{L}^m = \mathcal{L}^m \checkmark$

Caso inductivo $m = \bar{m} + 1$ $\mathcal{L}^{\bar{m}+1} \mathcal{L}^m = \mathcal{L}^{\bar{m}+1+m}$, $H_i := \mathcal{L}^{\bar{m}} \cdot \mathcal{L}^m = \mathcal{L}^{\bar{m}+m}$

$\mathcal{L}^{\bar{m}+1} \cdot \mathcal{L}^m \overset{pot}{=} \mathcal{L} \cdot \mathcal{L}^{\bar{m}} \cdot \mathcal{L}^m \overset{H_i}{=} \mathcal{L} \cdot \mathcal{L}^{\bar{m}+m} \overset{pot}{=} \mathcal{L}^{\bar{m}+1+m} \checkmark$

d) $\mathcal{L}^m \subseteq \mathcal{L}^{m+1} \quad \forall m \geq 0$

Contraejemplo: $\mathcal{L} = \{a, b\}$ $m = 0$ $\mathcal{L}^0 = \{\lambda\} \not\subseteq \mathcal{L}^1 = \{0, b\}$, o $m = 2$ $\mathcal{L}^1 = \{0, b\} \not\subseteq \mathcal{L}^2 = \{aa, ab, ba, bb\}$

e) $\mathcal{L}_1 \subseteq \mathcal{L}_2$, $m \geq 0 \to (\mathcal{L}_1)^m \subseteq (\mathcal{L}_2)^m$

Caso Base $m = 0 \to \mathcal{L}_1^0 \subseteq \mathcal{L}_2^0 = \{\lambda\} \subseteq \{\lambda\} \checkmark$

Caso inductivo $m = m + 1 \to \mathcal{L}_1^{m+1} \subseteq \mathcal{L}_2^{m+1}$, $H_i : \mathcal{L}_1^m \subseteq \mathcal{L}_2^m$

$\overset{pot}{=} \mathcal{L}_1 \cdot \mathcal{L}_1^m \subseteq \mathcal{L}_2 \cdot \mathcal{L}_2^m$

Sea $\alpha = \mathcal{L}_1 \cdot \mathcal{L}_1^m$ y $\exists \beta \in \mathcal{L}_1, \exists \gamma \in \mathcal{L}_1^m / \alpha = \beta \cdot \gamma$.

$\left. \begin{array}{l} \beta \in \mathcal{L}_1, \ \mathcal{L}_1 \subseteq \mathcal{L}_1, \ \beta \in \mathcal{L}_2 \\ \gamma \in \mathcal{L}_1^m \ y \ \underset{H_i}{\underline{\mathcal{L}_1^m \subseteq \mathcal{L}_2^m}}, \gamma \in \mathcal{L}_2^m \end{array} \right\} \begin{array}{l} \text{so } \alpha = \beta \cdot \gamma \in \mathcal{L}_2 \cdot \mathcal{L}_2^m \\ \text{or } \mathcal{L}_1 \cdot \mathcal{L}_1^m \subseteq \mathcal{L}_2 \cdot \mathcal{L}_1^m \end{array}$

f) $\mathcal{L}_1 \subseteq \mathcal{L}_2 \to (\mathcal{L}_1)^* \subseteq (\mathcal{L}_2)^*$

$= \bigcup_{m \geq 0} \mathcal{L}_1^m \subseteq \bigcup_{m \geq 0} \mathcal{L}_2^m$, vers: $\mathcal{L}_1^m \subseteq \mathcal{L}_2^m$ con $m \geq 0$. e)

g) $(\mathcal{L}^*)^* = \mathcal{L}^*$

En clase $\subseteq \supseteq$