

## Ejercicio 5 \*

Mostrar en deducción natural que las siguientes fórmulas son teoremas **sin usar principios de razonamiento clásicos** salvo que se indique lo contrario. Recordemos que una fórmula  $\sigma$  es un teorema si y sólo si vale  $\vdash \sigma$ :

- I. Modus ponens relativizado:  
 $(\rho \Rightarrow \sigma \Rightarrow \tau) \Rightarrow (\rho \Rightarrow \sigma) \Rightarrow \rho \Rightarrow \tau$
- II. Reducción al absurdo:  $(\rho \Rightarrow \perp) \Rightarrow \neg \rho$
- III. Introducción de la doble negación:  $\rho \Rightarrow \neg \neg \rho$
- IV. Eliminación de la triple negación:  $\neg \neg \neg \rho \Rightarrow \neg \rho$
- V. Contraposición:  $(\rho \Rightarrow \sigma) \Rightarrow (\neg \sigma \Rightarrow \neg \rho)$
- VI. Adjunción:  $((\rho \wedge \sigma) \Rightarrow \tau) \Rightarrow (\rho \Rightarrow \sigma \Rightarrow \tau)$
- VII. de Morgan (I):  $\neg(\rho \vee \sigma) \Leftrightarrow (\neg \rho \wedge \neg \sigma)$
- VIII. de Morgan (II):  $\neg(\rho \wedge \sigma) \Leftrightarrow (\neg \rho \vee \neg \sigma)$ . Para la dirección  $\Rightarrow$  es necesario usar principios de razonamiento clásicos.
- IX. Conmutatividad ( $\wedge$ ):  $(\rho \wedge \sigma) \Rightarrow (\sigma \wedge \rho)$
- X. Asociatividad ( $\wedge$ ):  $((\rho \wedge \sigma) \wedge \tau) \Leftrightarrow (\rho \wedge (\sigma \wedge \tau))$
- XI. Conmutatividad ( $\vee$ ):  $(\rho \vee \sigma) \Rightarrow (\sigma \vee \rho)$
- XII. Asociatividad ( $\vee$ ):  $((\rho \vee \sigma) \vee \tau) \Leftrightarrow (\rho \vee (\sigma \vee \tau))$

¿Encuentra alguna relación entre teoremas de adjunción, asociatividad y conmutatividad con algunas de las propiedades demostradas en la práctica 2?

$$\begin{array}{c}
 \text{I)} \quad \frac{\frac{\frac{\frac{\Gamma \vdash \rho \Rightarrow \sigma \Rightarrow \tau}{\Gamma \vdash \rho \Rightarrow \sigma} \text{ax}}{\Gamma \vdash \rho \Rightarrow \tau} \text{ax}}{\Gamma \vdash \rho \Rightarrow \sigma} \text{ax}}{\Gamma \vdash \rho \Rightarrow \tau} \text{ax} \\
 \frac{\Gamma \vdash \rho \Rightarrow \sigma \Rightarrow \tau, \Gamma \vdash \rho \Rightarrow \sigma, \Gamma \vdash \rho}{\Gamma \vdash \rho \Rightarrow \tau} \Rightarrow e \\
 \frac{\Gamma \vdash \rho \Rightarrow \sigma \Rightarrow \tau, \Gamma \vdash \rho \Rightarrow \sigma \vdash \rho \Rightarrow \tau}{\Gamma \vdash \rho \Rightarrow \sigma \Rightarrow \tau \vdash \rho \Rightarrow \tau} \Rightarrow i \\
 \vdash (\rho \Rightarrow \sigma \Rightarrow \tau) \Rightarrow ((\rho \Rightarrow \sigma) \Rightarrow \rho \Rightarrow \tau) \Rightarrow
 \end{array}$$

$$\begin{array}{c}
 \text{II)} \quad \frac{\frac{\frac{\Gamma \vdash \rho \Rightarrow \perp, \Gamma \vdash \rho}{\Gamma \vdash \rho \Rightarrow \perp} \text{ax}}{\Gamma \vdash \rho \Rightarrow \perp} \text{ax}}{\Gamma \vdash \rho \Rightarrow \perp} \text{ax} \\
 \frac{\Gamma \vdash \rho \Rightarrow \perp, \Gamma \vdash \rho}{\Gamma \vdash \perp} \rightarrow e \\
 \frac{\Gamma \vdash \rho \Rightarrow \perp \vdash \neg \rho}{\vdash (\rho \Rightarrow \perp) \rightarrow \neg \rho} \rightarrow i
 \end{array}$$

$$\begin{array}{c}
 \text{III)} \quad \frac{\frac{\frac{\Gamma \vdash \rho}{\Gamma \vdash \neg \neg \rho} \text{ax}}{\Gamma \vdash \neg \neg \rho} \text{ax}}{\Gamma \vdash \neg \neg \rho} \text{ax} \\
 \frac{\Gamma \vdash \neg \neg \rho, \Gamma \vdash \neg \neg \neg \rho}{\Gamma \vdash \neg \neg \neg \rho} \neg e \\
 \frac{\Gamma \vdash \neg \neg \neg \rho}{\vdash \neg \neg \neg \neg \rho} \neg i \\
 \vdash \neg \neg \neg \neg \rho \Rightarrow \neg \neg \neg \rho
 \end{array}$$

$$\begin{array}{c}
 \text{IV)} \quad \frac{\frac{\frac{\Gamma \vdash \rho}{\Gamma \vdash \neg \neg \neg \neg \rho} \text{ax}}{\Gamma \vdash \neg \neg \neg \neg \rho} \text{ax}}{\Gamma \vdash \neg \neg \neg \neg \rho} \text{ax} \\
 \frac{\Gamma \vdash \neg \neg \neg \neg \rho, \Gamma \vdash \neg \neg \neg \neg \neg \rho}{\Gamma \vdash \neg \neg \neg \neg \neg \rho} \neg e \\
 \frac{\Gamma \vdash \neg \neg \neg \neg \neg \rho}{\vdash \neg \neg \neg \neg \neg \neg \rho} \neg i \\
 \vdash \neg \neg \neg \neg \neg \neg \rho \Rightarrow \neg \neg \neg \neg \neg \rho
 \end{array}$$

$$\begin{array}{c}
 \text{V)} \quad \frac{\frac{\frac{\frac{\Gamma \vdash \rho \Rightarrow \sigma}{\Gamma \vdash \rho \Rightarrow \sigma} \text{ax}}{\Gamma \vdash \rho \Rightarrow \sigma} \text{ax}}{\Gamma \vdash \rho \Rightarrow \sigma} \text{ax}}{\Gamma \vdash \rho \Rightarrow \sigma} \text{ax} \\
 \frac{\Gamma \vdash \rho \Rightarrow \sigma, \Gamma \vdash \neg \sigma}{\Gamma \vdash \neg \neg \sigma} \neg i \\
 \frac{\Gamma \vdash \neg \neg \sigma}{\vdash (\rho \Rightarrow \sigma) \rightarrow (\neg \neg \sigma)} \rightarrow i \\
 \vdash (\rho \Rightarrow \sigma) \rightarrow (\neg \neg \sigma)
 \end{array}$$

$$\begin{array}{c}
 \text{VI)} \quad \frac{\frac{\frac{\frac{\Gamma \vdash \rho \wedge \sigma \rightarrow \tau}{\Gamma \vdash \rho \wedge \sigma} \text{ax}}{\Gamma \vdash \rho \wedge \sigma} \text{ax}}{\Gamma \vdash \rho \wedge \sigma} \text{ax}}{\Gamma \vdash \rho \wedge \sigma} \text{ax} \\
 \frac{\Gamma \vdash \rho \wedge \sigma \rightarrow \tau, \Gamma \vdash \rho \wedge \sigma}{\Gamma \vdash \tau} \wedge e \\
 \frac{\Gamma \vdash \rho \wedge \sigma \rightarrow \tau, \Gamma \vdash \rho \wedge \sigma \vdash \tau}{\Gamma \vdash \rho \wedge \sigma \rightarrow \tau \vdash \tau} \wedge i \\
 \vdash ((\rho \wedge \sigma) \rightarrow \tau) \rightarrow (\rho \wedge \sigma \rightarrow \tau)
 \end{array}$$

$$\begin{array}{c}
 \frac{\frac{\frac{\frac{\Gamma \vdash \rho \rightarrow \sigma \rightarrow \tau}{\Gamma \vdash \rho \rightarrow \sigma} \text{ax}}{\Gamma \vdash \rho \rightarrow \sigma} \text{ax}}{\Gamma \vdash \rho \rightarrow \sigma} \text{ax}}{\Gamma \vdash \rho \rightarrow \sigma} \text{ax} \\
 \frac{\Gamma \vdash \rho \rightarrow \sigma, \Gamma \vdash \rho \wedge \sigma}{\Gamma \vdash \rho \wedge \sigma} \wedge i \\
 \frac{\Gamma \vdash \rho \wedge \sigma}{\vdash (\rho \rightarrow \sigma) \rightarrow ((\rho \wedge \sigma) \rightarrow \tau)} \rightarrow i \\
 \vdash (\rho \rightarrow \sigma) \rightarrow ((\rho \wedge \sigma) \rightarrow \tau)
 \end{array}$$

$$\begin{array}{c}
 \text{VII)} \quad \frac{\frac{\frac{\frac{\frac{\Gamma \vdash \rho \vee \sigma}{\Gamma \vdash \rho \vee \sigma} \text{ax}}{\Gamma \vdash \rho \vee \sigma} \text{ax}}{\Gamma \vdash \rho \vee \sigma} \text{ax}}{\Gamma \vdash \rho \vee \sigma} \text{ax}}{\Gamma \vdash \rho \vee \sigma} \text{ax} \\
 \frac{\Gamma \vdash \rho \vee \sigma, \Gamma \vdash \neg(\rho \vee \sigma)}{\Gamma \vdash \perp} \vee e \\
 \frac{\Gamma \vdash \perp}{\vdash \neg(\rho \vee \sigma)} \rightarrow e \\
 \vdash \neg(\rho \vee \sigma) \rightarrow \neg(\rho \vee \sigma)
 \end{array}$$

$$\begin{array}{c}
 \text{VIII)} \quad \frac{\frac{\frac{\frac{\frac{\Gamma \vdash \neg(\rho \wedge \sigma)}{\Gamma \vdash \neg(\rho \wedge \sigma)} \text{ax}}{\Gamma \vdash \neg(\rho \wedge \sigma)} \text{ax}}{\Gamma \vdash \neg(\rho \wedge \sigma)} \text{ax}}{\Gamma \vdash \neg(\rho \wedge \sigma)} \text{ax}}{\Gamma \vdash \neg(\rho \wedge \sigma)} \text{ax} \\
 \frac{\Gamma \vdash \neg(\rho \wedge \sigma), \Gamma \vdash \rho \wedge \sigma}{\Gamma \vdash \perp} \wedge e \\
 \frac{\Gamma \vdash \perp}{\vdash \neg(\rho \wedge \sigma)} \rightarrow e \\
 \vdash \neg(\rho \wedge \sigma) \rightarrow \neg(\rho \wedge \sigma)
 \end{array}$$

$$\begin{array}{c}
 \text{IX)} \quad \frac{\frac{\frac{\frac{\frac{\Gamma \vdash \rho \wedge \sigma}{\Gamma \vdash \rho \wedge \sigma} \text{ax}}{\Gamma \vdash \rho \wedge \sigma} \text{ax}}{\Gamma \vdash \rho \wedge \sigma} \text{ax}}{\Gamma \vdash \rho \wedge \sigma} \text{ax}}{\Gamma \vdash \rho \wedge \sigma} \text{ax} \\
 \frac{\Gamma \vdash \rho \wedge \sigma, \Gamma \vdash \neg(\rho \wedge \sigma)}{\Gamma \vdash \perp} \wedge e \\
 \frac{\Gamma \vdash \perp}{\vdash \neg(\rho \wedge \sigma)} \rightarrow e \\
 \vdash \neg(\rho \wedge \sigma) \rightarrow \neg(\rho \wedge \sigma)
 \end{array}$$

$$\begin{array}{c}
 \text{X)} \quad \frac{\frac{\frac{\frac{\frac{\Gamma \vdash \rho \vee \sigma}{\Gamma \vdash \rho \vee \sigma} \text{ax}}{\Gamma \vdash \rho \vee \sigma} \text{ax}}{\Gamma \vdash \rho \vee \sigma} \text{ax}}{\Gamma \vdash \rho \vee \sigma} \text{ax}}{\Gamma \vdash \rho \vee \sigma} \text{ax} \\
 \frac{\Gamma \vdash \rho \vee \sigma, \Gamma \vdash \neg(\rho \vee \sigma)}{\Gamma \vdash \perp} \vee e \\
 \frac{\Gamma \vdash \perp}{\vdash \neg(\rho \vee \sigma)} \rightarrow e \\
 \vdash \neg(\rho \vee \sigma) \rightarrow \neg(\rho \vee \sigma)
 \end{array}$$

$$\begin{array}{c}
 \text{XI)} \quad \frac{\frac{\frac{\frac{\frac{\Gamma \vdash \rho \wedge \sigma}{\Gamma \vdash \rho \wedge \sigma} \text{ax}}{\Gamma \vdash \rho \wedge \sigma} \text{ax}}{\Gamma \vdash \rho \wedge \sigma} \text{ax}}{\Gamma \vdash \rho \wedge \sigma} \text{ax}}{\Gamma \vdash \rho \wedge \sigma} \text{ax} \\
 \frac{\Gamma \vdash \rho \wedge \sigma, \Gamma \vdash \neg(\rho \wedge \sigma)}{\Gamma \vdash \perp} \wedge e \\
 \frac{\Gamma \vdash \perp}{\vdash \neg(\rho \wedge \sigma)} \rightarrow e \\
 \vdash \neg(\rho \wedge \sigma) \rightarrow \neg(\rho \wedge \sigma)
 \end{array}$$

$$\begin{array}{c}
 \text{XII)} \quad \frac{\frac{\frac{\frac{\frac{\Gamma \vdash \rho \vee \sigma}{\Gamma \vdash \rho \vee \sigma} \text{ax}}{\Gamma \vdash \rho \vee \sigma} \text{ax}}{\Gamma \vdash \rho \vee \sigma} \text{ax}}{\Gamma \vdash \rho \vee \sigma} \text{ax}}{\Gamma \vdash \rho \vee \sigma} \text{ax} \\
 \frac{\Gamma \vdash \rho \vee \sigma, \Gamma \vdash \neg(\rho \vee \sigma)}{\Gamma \vdash \perp} \vee e \\
 \frac{\Gamma \vdash \perp}{\vdash \neg(\rho \vee \sigma)} \rightarrow e \\
 \vdash \neg(\rho \vee \sigma) \rightarrow \neg(\rho \vee \sigma)
 \end{array}$$

[illegible]
$$\frac{\frac{\Gamma \vdash P \vee (\sigma \vee \tau) \text{ (a)} \quad \Gamma, P \vdash P \vee (\sigma \vee \tau) \text{ (b)} \quad \Gamma, (\sigma \vee \tau) \vdash P \vee (\sigma \vee \tau) \text{ (c)}}{\Gamma \vdash P \vee (\sigma \vee \tau) \vdash ((P \vee \sigma) \vee \tau) \text{ (d)}}}{P \vee (\sigma \vee \tau) \rightarrow ((P \vee \sigma) \vee \tau)} \rightarrow$$

v. Contraposición clásica:  $(\neg\rho \Rightarrow \neg\tau) \Rightarrow (\tau \Rightarrow \rho)$

vi. Análisis de casos:  $(\tau \Rightarrow \rho) \Rightarrow (\neg\tau \Rightarrow \rho) \Rightarrow \rho$

vii. Implicación vs. disyunción:  $(\tau \Rightarrow \rho) \Leftrightarrow (\neg\tau \vee \rho)$

$$\begin{array}{l} \text{V)} \\ \frac{\frac{\frac{}{\vdash t} \quad \frac{}{\vdash \neg t}}{\vdash t \wedge \neg t} \quad \frac{}{\vdash \neg t} \quad \frac{}{\vdash t}}{\vdash t \rightarrow \neg t} \quad \frac{}{\vdash \neg t} \quad \frac{}{\vdash t}}{\vdash t \leftrightarrow \neg t} \end{array}$$
[illegible]

2)

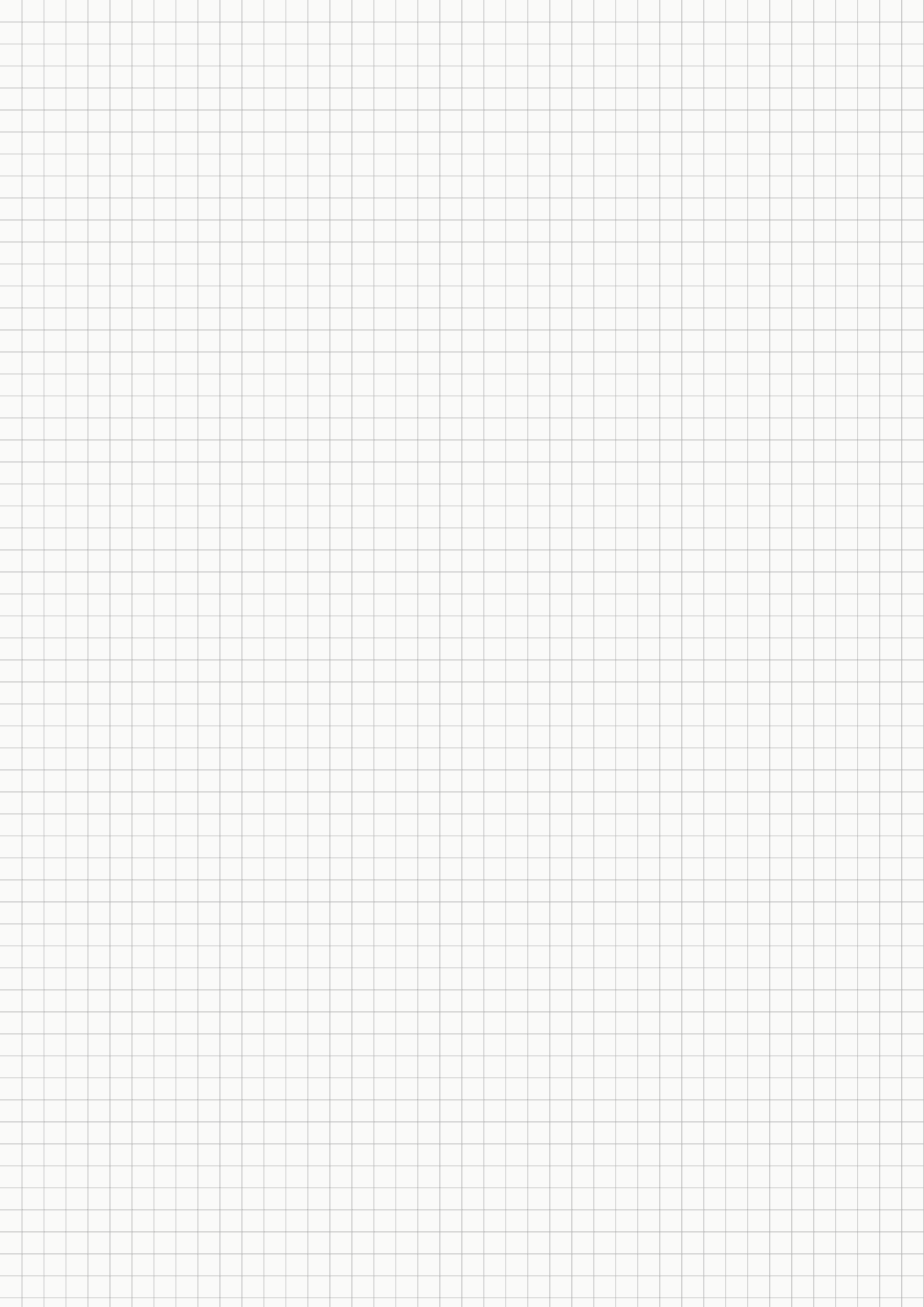
$$\begin{array}{c} \alpha \qquad \qquad \qquad \alpha \\ \hline \Gamma, \Gamma t, t \vdash t \quad \Gamma, \Gamma t, t \vdash \neg t \\ \hline \Gamma, \Gamma t, t \vdash \perp \\ \hline \alpha \quad \Gamma, \Gamma t, t \vdash P \qquad \vdash e \\ \vdash t \quad \Gamma, \Gamma t \vdash (t \rightarrow P) \qquad \rightarrow e \\ \hline \end{array}$$
$$\frac{\frac{\frac{\Gamma, \Gamma \vdash \Gamma}{\Gamma, \Gamma \vdash \perp} \text{ex} \quad \frac{\Gamma, \Gamma \vdash \perp}{\Gamma, \Gamma \vdash \perp} \text{Te}}{\Gamma, \Gamma \vdash \perp} \text{PBC} \quad \frac{\Gamma \vdash (\Gamma \rightarrow \perp) \rightarrow \perp}{(\Gamma \rightarrow \perp) \rightarrow \perp} \text{Te}$$

3)  $t \vee \neg t$  Lem

4)

[illegible]

$t \rightarrow p, \neg(t \vee p) \vdash \perp$	PBC
$t \rightarrow p \vdash \neg t \vee p$	
$(t \rightarrow p) \rightarrow (\neg t \vee p)$	



Ej Parcial

$$\begin{array}{c}
 \frac{\frac{\frac{}{\Gamma \vdash (P \rightarrow Q) \wedge (Q \rightarrow R)}{\text{ac2}}}{\Gamma \vdash Q \rightarrow R} \quad \frac{\frac{\frac{}{\Gamma \vdash (P \rightarrow Q) \wedge (Q \rightarrow R)}{\text{ac1}}}{\Gamma \vdash P \rightarrow Q} \quad \frac{}{\Gamma \vdash P} \rightarrow e}{\Gamma \vdash Q} \neg e \quad \frac{}{\Gamma \vdash \neg R} \neg e \\
 \frac{}{\Gamma \vdash R} \neg e \\
 \hline
 \Gamma \vdash ((P \rightarrow Q) \wedge (Q \rightarrow R)), \neg R, P \vdash \perp \\
 \hline
 \frac{}{((P \rightarrow Q) \wedge (Q \rightarrow R)), \neg R \vdash \neg P} \neg i \\
 \hline
 \frac{}{(P \rightarrow Q) \wedge (Q \rightarrow R) \vdash \neg R \rightarrow \neg P} \rightarrow i \\
 \hline
 \frac{}{(P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (\neg R \rightarrow \neg P)} \rightarrow c
 \end{array}$$