Understanding Analysis Solutions

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December 2016

Chapter 4

4.2

1. Use Definition 4.2.1 to supply a proof for the following statements

   (a) $\lim_{x \to 2} 2x + 4 = 8$

   (b) $\lim_{x \to 0} x^3 = 0$

   (c) $\lim_{x \to 2} x^3 = 8$

   (d) $\lim_{x \to \pi} [[x]] = 3$ where $[[x]]$ denotes the greatest integer less than or equal to x.

Proof. (a) Let $\epsilon > 0$ be given. Choose $\delta = \frac{\epsilon}{2}$. So $|x - 2| < \delta$ implies

$$|2x + 4 - 8| = |2x - 4| = 2|x - 2| < 2\frac{\epsilon}{2} = \epsilon$$

as desired.

(b) Let $\epsilon > 0$ be given. Choose $\delta = \sqrt[3]{\epsilon}$ so that $|x| < \delta$ implies

$$|x^3| = |x||x||x| < \delta^3 = \sqrt[3]{\epsilon}^3 = \epsilon$$

as desired.

(c) Let $\epsilon > 0$ be given. We note that that $x^3 - 8 = (x - 2)(x^2 + 2x + 4)$. Thus, if we have that $|x - 2| < 1$ then it follows that $7 < x^2 + 2x + 4 < 19$. So, by chosing $\delta = \min\{1, \frac{\epsilon}{19}\}$ we get

$$|x^3 - 8| = |x - 2||x^2 + 2x + 4| < \frac{\epsilon}{19} 19 = \epsilon$$

as desired.

(d) Let $\epsilon > 0$ be given. If we choose x so that $|x - \pi| < 0.01$ then $[[x]] = 3$. Thus

$$|[[x]] - 3| = 0 < \epsilon$$

as desired.

$\square$

2. Assume a particular $\delta > 0$ has been constructed as a suitable response to a particular ϵ challenge. Then, any *smaller* δ will also suffice.

3. Use Corollary 4.2.5 to show that each of the following limits does not exist.

   (a) $\lim_{x\to 0} \frac{|x|}{x}$

   (b) $\lim_{x\to 1} g(x)$ where $g(x)$ is Dirichlet's function from 4.1.

   Proof. (a) Let (x_n) be the sequence defined by $x_n = -\frac{1}{n}$ and (y_n) be the sequence defined by $y_n = \frac{1}{n}$. Then $\lim x_n = \lim y_n = 0$ while $x_n \neq 0$ and $y_n \neq 0$. However,

   $$\lim_{n\to\infty} \frac{|x_n|}{x_n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{-\frac{1}{n}} = \lim_{n\to\infty} -1 = -1$$

   while

   $$\lim_{n\to\infty} \frac{|y_n|}{y_n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} 1 = 1$$

   which, by Corollary 4.2.5, implies that $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

   (b) By density of $\mathbb{Q}$ in $\mathbb{R}$ we can select a sequence of rationals, (x_n), each different from 1, such that $x_n \to 1$. This will then imply that $g(x_n) \to 0$. On the other hand, if (y_n) is a sequence of irrationals, each different from 1, such that $y_n \to 1$, it follows that $g(y_n) = 1$. Therefore, by Corollary 4.2.5 we get that the limit does not exist.

   $\square$

4. Review the definition of Thomae's function $t(x)$ from Section 4.1.

   (a) Construct three different sequences (x_n), (y_n), and (z_n), each of which converges to 1 without using the number 1 as a term in the sequence.

   (b) Now, compute $\lim t(x_n)$, $\lim t(y_n)$, and $\lim t(z_n)$.

   (c) Make an educated conjecture for $\lim_{x\to 1} t(x)$, and use Definition 4.2.1B to verify the claim.

   Lemma. *Let $(x_n) \to x$ and $\forall_n x_n \in \mathbb{Q}$. If $x_n = \frac{a_n}{b_n}$ and $|b_n| \not\to \infty$ then there exists N such that $\forall_{n>N} x_n = x$.*

   Proof. Let $x_n = \frac{a_n}{b_n}$ with $|b_n| \not\to \infty$. Since $a_n, b_n \in \mathbb{Z}$ it follows there exists $\epsilon > 0$ such that

   $$|a_n - b_n| < \epsilon$$

   iff $a_n = b_n x$ iff $\frac{a_n}{b_n} = x$. Notice then that this is equivalent to saying that

   $$|a_n - b_n x| > \epsilon$$

unless $\frac{a_n}{b_n} = x$. Furthermore, since $|b_n| \not\to \infty$ it follows there exists $M > 0$ such that

$$|b_n| < M$$

for every $n \in \mathbb{N}$. Thus

$$|x_n - x| = |\frac{a_n}{b_n} - x| = \frac{|a_n - b_n x|}{|b_n|} \geq \frac{\epsilon}{|b_n|} \geq \frac{\epsilon}{M}$$

unless $\frac{a_n}{b_n} = x$. However, by assumption, we have that $x_n \to x$. Thus, there exists $N \in \mathbb{N}$ such that $\forall\, n \geq N$ we have

$$|x_n - x| < \frac{\epsilon}{M}$$

Therefore, $\forall n \geq N$ $x_n = x$. $\square$

Corollary. *Let $x_n = \frac{a_n}{b_n} \in \mathbb{Q}$ for every n and $x_n \to x$. If $x \in \mathbb{R} \setminus \mathbb{Q}$, then $|b_n| \to \infty$.*

Proof. Suppose, to the contrary, that $|b_n| \not\to \infty$. Then, by the Lemma, there exists $N \in \mathbb{N}$ such that $x_n = x$ for every $n \geq N$, contradicting with $x_n \in \mathbb{Q}$ and $x \notin \mathbb{Q}$. $\square$

Proof. (a)

$$\begin{aligned} x_n &= \frac{n+1}{n} \\ y_n &= \frac{n-1}{n} \\ z_n &= \frac{n+\sqrt{2}}{n} \end{aligned}$$

(b)

$$\lim t(x_n) = \lim t(\frac{n+1}{n})$$

and since n and $n+1$ are relatively prime, it follows that this fraction is in lowest common terms, thus

$$\begin{aligned} &= \lim \frac{1}{n} \\ &= 0 \end{aligned}$$

similarly, we get

$$\begin{aligned} \lim t(y_n) &= 0 \\ \lim t(z_n) &= \lim t\left(\frac{n+\sqrt{2}}{n}\right) \end{aligned}$$

and since the sum of a rational and an irrational is irrational, it follows that

$$= \lim 0$$
$$= 0$$

(c) Our conjecture will be that $\lim_{x\to 1} t(x) = 0$. So, following the hint, let $\epsilon > 0$ be given and let $S = \{x \in \mathbb{R} : t(x) \geq \epsilon\}$. Now, if x is a limit point of S, then there must be some $x_n \to x$ where $\forall_n x_n \in S$ and $x_n \neq x$. This immediately implies that for all n we have $x_n \in \mathbb{Q}$ for otherwise we would get that $t(x_n) = 0 < \epsilon$ implying that $x_n \notin S$ which is contrary to our assumption. By our Lemma and Corollary, we know that either there exists an $N \in \mathbb{N}$ such that $\forall_{n\geq N} x_n = x$ or $t(x_n) \to 0$. In the former, we would have a contradiction in that, by assumption, $x_n \neq x$ for all n. In the latter, we would have a contradiction in that for n big enough, there would be m such that we would have $t(x_n) = \frac{1}{m} < \epsilon$ contradicting with $x_n \in S$ for every n. Thus, it follows that S has no limit points.

So, we wish to show that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $x \neq 1$ and $x \in V_\delta(1)$ implies that $t(x) \in V_\epsilon(0)$. If $\epsilon \geq 1$ then we are done since $\forall_x t(x) \leq 1$. Otherwise we have that $1 \in S$ and, by our previous argument, it follows that 1 is a isolated point of S. Thus there exists $\delta > 0$ such that $V_\delta(1) \cap S = \{1\}$. Thus, for the same δ we have that $x \neq 1$ and $x \in V_\delta(1)$ implies that $x \notin S$ which, in turn, implies that $t(x) < \epsilon$ which is equivalent to $t(x) \in V_\epsilon(0)$. Thus, by Definition 4.2.1B, we get that $\lim_{x\to 1} t(x) = 0$.

$\square$

5. (a) Supply the details for how Corollary 4.2.4 part (ii) follows from the sequential criterion for functional limits in Theorem 4.2.3 and the Algebraic Limit Theorem for sequences proved in Chapter 2.

   (b) Now, write another proof of Corollary 4.2.4 part (ii) directly from Definition 4.2.1 without using the sequential criterion in Theorem 4.2.3.

   (c) Repeat (a) and (b) for Corollary 4.2.4 part (iii).

Proof. (a) From the sequential criterion for functional limits we know that $\lim_{x\to c}[f(x) + g(x)] = L + M$ if and only if for every sequence $x_n \to c$ such that $x_n \neq c$ we must have $\lim_{n\to\infty}[f(x_n) + g(x_n)] = L + M$. Now by assumption,

$$f(x) \underset{x\to c}{\to} L$$

$$g(x) \underset{x\to c}{\to} M$$

so, again, by the sequential criterion for functional limits, we get that for every $x_n \to c$ such that $x_n \neq c$ we must have $(f(x_n)) \underset{n\to\infty}{\to} L$ and $(g(x_n)) \underset{n\to\infty}{\to} M$. Thus from the Algebraic Limit Theorem from Chapter 2 we have that for all such sequences

$$\lim_{n\to\infty} f(x_n) + g(x_n) = L + M$$

Thus, it follows that

$$\lim_{x\to c} f(x) + g(x) = L + M$$

(b) Without utilizing the sequential criterion, we let $\epsilon > 0$ be given. By assumption, $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Thus, for some $\delta > 0$, we can obtain

$$|f(x) - L| < \frac{\epsilon}{2}$$

and

$$|g(x) - M| < \frac{\epsilon}{2}$$

whenever $|x - c| < \delta$. So $|x - c| < \delta$ implies

$$|f(x) + g(x) - L - M| \leq |f(x) - L| + |g(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, $\lim_{x\to c} f(x) + g(x) = L + M$.

(c) Using the sequential criterion is quite simple: whenever f and g converge to L and M respectively and $x_n \to c$ with $x_n \neq c$ we know, from Chapter 2, that

$$(f(x_n)g(x_n)) \underset{n\to\infty}{\to} LM$$

Thus, by the sequential criterion we have

$$(f(x)g(x)) \underset{x\to c}{\to} LM$$

as desired.

Now, proceeding without the sequential criterion, let $\epsilon > 0$ be given. By assumption, $f \underset{x\to c}{\to} L$ and $g \underset{x\to c}{\to} M$. So, there exists $\delta_1 > 0$ such that

$$|f - L| < \frac{\epsilon}{2|M|}$$

Furthermore, this is equivalent to saying

$$\left|L - \frac{\epsilon}{2|M|}\right| < |f| < \left|L + \frac{\epsilon}{2|M|}\right|$$

To make matters simpler then, we conclude that there exists $K > 0$ such that $|f(x)| < K$ whenever $|x - c| < \delta_1$. Similarly, there exist $\delta_2 > 0$ such that

$$|g(x) - M| < \frac{\epsilon}{2K}$$

whenever $|x - c| < \delta_2$. Thus we have then that if $\delta = \min\{\delta_1, \delta_2\}$ then

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) + fM - fM - LM| \\
&\leq |f(x)g(x) - f(x)M| + |f(x)M - LM| \\
&= |f(x)||g - M| + |f(x) - L||M| \\
&< K|g(x) - M| + |f(x) - L||M| \\
&< K\frac{\epsilon}{2K} + \frac{\epsilon}{2|M|}|M| \\
&= \epsilon
\end{aligned}$$

whenever $|x - c| < \delta$. Therefore, $\lim_{x\to c} f(x)g(x) = LM$.

□

6. Let $g : A \to \mathbb{R}$ and assume that f is a bounded function on $A \subseteq \mathbb{R}$ (i.e. there exists $M > 0$ satisfying $|f(x)| \leq M$ for all $x \in A$). Show that if $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} g(x)f(x) = 0$ as well.

   Proof. Given $\epsilon > 0$ there is $\delta > 0$ such that $|x - c| < \delta$ implies

   $$|f(x)g(x)| = |f(x)||g(x)| < M|g(x)| < M\frac{\epsilon}{M} = \epsilon$$

   Therefore, $\lim_{x\to c} f(x)g(x) = 0$ as desired. □

7. (a) The statement $\lim_{x\to 0} \frac{1}{x^2} = \infty$ certainly makes intuitive sense. Construct a rigorous definition in the "challenge-response" style of Definition 4.2.1 for a limit statement of the form $\lim_{x\to c} f(x) = \infty$ and use it to prove the previous statement.

   (b) Now, construct a definition of the statement $\lim_{x\to\infty} f(x) = L$. Show $\lim_{x\to\infty} \frac{1}{x} = 0$.

   (c) What would a rigorous definition for $\lim_{x\to\infty} f(x) = \infty$ look like? Give an example of such a limit.

   Proof. (a) Let $f : A \to \mathbb{R}$ and let c be a limit point of A. We say $\lim_{x\to c} f(x) = \infty$ provided that, for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|f(x)| > \epsilon$. Now, to use this to demonstrate that $\lim_{x\to 0} = \infty$ we let $\epsilon > 0$ be given. Choosing x such that $0 < |x| < \sqrt{\frac{1}{\epsilon}}$ gives us that

   $$\left|\frac{1}{x^2}\right| = \frac{1}{x^2} = \frac{1}{|x|}\frac{1}{|x|} > \frac{1}{\sqrt{\frac{1}{\epsilon}}}\frac{1}{\sqrt{\frac{1}{\epsilon}}} = \frac{1}{\frac{1}{\epsilon}} = \epsilon$$

which then implies that $\lim_{x\to 0} \frac{1}{x^2} = \infty$.

(b) Let $f : A \to \mathbb{R}$ and let there exist a sequence (x_n) such that $x_n \in A$ and for every $M > 0$ there exists $N \in \mathbb{N}$ such that $x_n > M$ whenever $n \geq N$. Then we say $\lim_{x\to\infty} f(x) = L$ when for every $\epsilon > 0$ there exists $M > 0$ such that whenever $x > M$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$. To demonstrate that $\lim_{x\to\infty} \frac{1}{x} = 0$ we let $\epsilon > 0$ be given. If $x > \frac{1}{\epsilon}$ then

$$\left|\frac{1}{x}\right| = \frac{1}{|x|} < \frac{1}{\frac{1}{\epsilon}} = \epsilon$$

as desired.

(c) Let $f : A \to \mathbb{R}$ and let there exist a sequence (x_n) such that $x_n \in A$ and for every $M > 0$ there exists $N \in \mathbb{N}$ such that $x_n > M$ whenever $n \geq N$. We say $\lim_{x\to c} f(x) = \infty$ provided that, for all $\epsilon > 0$, there exists $M > 0$ such that whenever $x > M$ (and $x \in A$) it follows that $|f(x)| > \epsilon$. For example we claim $\lim_{x\to\infty} x = \infty$. To demonstrate, let $\epsilon > 0$ be given. Then, selecting $x > \epsilon$ we get that

$$|x| = x > \epsilon$$

Therefore, $\lim_{x\to\infty} x = \infty$.

$\square$

8. Assume $f(x) \geq g(x)$ for all x in some set A on which f and g are defined. Show that for any limit point c of A we must have

$$\lim_{x\to c} f(x) \geq \lim_{x\to c} g(x)$$

Proof. Suppose, to the contrary, that

$$\lim_{x\to c} f(x) < \lim_{x\to c} g(x)$$

while $f(x) \geq g(x)$ for all $x \in A$. Then, by the sequential criterion, we must have for every sequence (x_n) in A with $x_n \neq c$ we get that

$$f(x_n) \geq g(x_n) \ \ \forall n$$

and

$$\lim_{n\to\infty} f(x_n) < \lim_{n\to\infty} g(x_n)$$

contradicting with Theorem 2.3.4. Therefore

$$\lim_{x\to c} f(x) \geq \lim_{x\to c} g(x)$$

$\square$

9. (Squeeze Theorem) Let f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. If $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$ at some limit point c of A, show $\lim_{x\to c} g(x) = L$.

Proof. By the previous problem, we know that

$$\lim_{x\to c} f(x) = L \leq \lim_{x\to c} g(x)$$

and

$$\lim_{x\to c} g(x) \leq \lim_{x\to c} h(x) = L$$

So

$$L \leq \lim_{x\to c} g(x) \leq L$$

implying then that

$$\lim_{x\to c} g(x) = L$$

as desired. ☐

4.3

1. Let $g(x) = \sqrt[3]{x}$.
   (a) Prove that g is continuous at $c = 0$.
   (b) Prove that g is continuous at a point $c \neq 0$. (The identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ will be helpful.)

Proof. (a) We first observe that

$$|\sqrt[3]{x}| = \sqrt[3]{|x|}$$

and that if $x_1 \leq x_2$ then $\sqrt[3]{x_1} \leq \sqrt[3]{x_2}$, that is say, $\sqrt[3]{x}$ is increasing. The first observation is obvious. To demonstrate the latter observation suppose, to the contrary, that $\sqrt[3]{x_2} < \sqrt[3]{x_1}$. Then we would have

$$x_2 = \sqrt[3]{x_2}^3 < \sqrt[3]{x_1}^3 = x_1$$

which contradicts with our assumption that $x_1 \leq x_2$. So, if $\epsilon > 0$ is given, then choosing $|x| < \epsilon^3$ it follows that

$$|\sqrt[3]{x}| = \sqrt[3]{|x|} < \sqrt[3]{\epsilon^3} = \epsilon$$

Therefore, $\sqrt[3]{x}$ is continuous at 0.

(b) Let $c \neq 0$ and let $\epsilon > 0$ be given. We observe that

$$x - c = (x^{\frac{1}{3}} - c^{\frac{1}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}}c^{\frac{1}{3}} + c^{\frac{2}{3}})$$

If we let $y = x^{\frac{1}{3}}$ then we get that

$$x^{\frac{2}{3}} + c^{\frac{1}{3}}x^{\frac{1}{3}} + c^{\frac{2}{3}} = y^2 + c^{\frac{1}{3}}y + c^{\frac{2}{3}}$$

which means that this expression is quadratic in y. Furthermore, one can easily verify, using the quadratic formula, that since $c \neq 0$ it follows that c is not a root of $y^2 + c^{\frac{1}{3}}y + c^{\frac{2}{3}}$. Thus, since $x^{\frac{1}{3}}$ is continuous at 0 it follows that there exists $\delta > 0$ and $M_2 > M_1 > 0$ such that whenever $|x - c| < \delta_1$ we will have that

$$M_1 < |x^{\frac{2}{3}} + c^{\frac{1}{3}}x^{\frac{1}{3}} + c^{\frac{2}{3}}| < M_2$$

Notice this implies that $\frac{M_1}{|x^{\frac{2}{3}} + c^{\frac{1}{3}}x^{\frac{1}{3}} + c^{\frac{2}{3}}|} < 1$ for x close enough to c. So, if $\delta = \min\{d_1, \epsilon M_1\}$ then it follows that whenever $|x - c| < \delta$ we get

$$|x^{\frac{1}{3}} - c^{\frac{1}{3}}| = \frac{|x - c|}{|x^{\frac{2}{3}} + c^{\frac{1}{3}}x^{\frac{1}{3}} + c^{\frac{2}{3}}|} < \frac{\epsilon M_1}{|x^{\frac{2}{3}} + c^{\frac{1}{3}}x^{\frac{1}{3}} + c^{\frac{2}{3}}|} < \epsilon$$

which was to be demonstrated.

□

2. (a) Supply a proof for Theorem 4.3.9 using the ϵ-δ characterization of continuity.

   (b) Give another proof of this theorem using the sequential characterization of continuity.

Proof. (a) Let $\epsilon > 0$ be given. By continuity of g at $f(c)$ we know that there exists $\delta_1 > 0$ such that $|f(x) - f(c)| < \delta_1$ implies $|g(f(x)) - g(f(c))| < \epsilon$. Furthermore, by continuity of f at c, we know that there exists $\delta_2 > 0$ such that $|x - c| < \delta_2$ implies $|f(x) - f(c)| < \delta_1$. Thus, for $|x - c| < \delta_2$ we have that

$$|g(f(x)) - g(f(c))| < \epsilon$$

as desired.

(b) Let $(x_n) \to c$ with $x_n \in A$. Since f is continuous at c, it follows $f(x_n) \to f(c)$. But then, by continuity of g at $f(c)$ it follows $g(f(x_n)) \to g(f(c))$.

□

3. Using the ϵ-δ characterization of continuity (and thus using no previous results about sequences), show that the linear function $f(x) = ax + b$ is continuous at every point of $\mathbb{R}$.

Proof. Let $\epsilon > 0$ be given and let $c \in \mathbb{R}$. If $|x - c| < \frac{\epsilon}{|a|}$ then it follows that

$$|f(x) - f(c)| = |ax + b - ac - b| = |ax - ac| = |a||x - c| < |a|\frac{\epsilon}{|a|} = \epsilon$$

Thus $ax + b$ is continuous on $\mathbb{R}$. □

4. (a) Show using Definition 4.3.1 that any function f with domain $\mathbb{Z}$ will necessarily be continuous at every point in its domain.
   (b) Show in general that if c is an isolated point of $A \subseteq \mathbb{R}$, then $f : A \to \mathbb{R}$ is continuous at c.

Proof. (a) Let $\epsilon > 0$ be given. Notice $|x - c| < \frac{1}{2}$ and $x \in \mathbb{Z}$ implies $x = c$. Thus for all such x

$$|f(x) - f(c)| = 0 < \epsilon.$$

(b) Let $\epsilon > 0$ be given. If c is isolated in $A \subseteq \mathbb{R}$ then there exists $\delta > 0$ s.t. $V_\delta(c) \cap A = \{c\}$. Thus $\forall x \in V_\delta(c) \cap A$ we have

$$|f(x) - f(c)| = 0$$

Therefore, for all such x we have

$$f(x) \in V_\epsilon(f(c))$$

□

5. In thoerem 4.3.4, statement (iv) syas that $f(x)/g(x)$ is continuous at c if both f and g are, provided that the quotient is defined. Show that if g is continuous at c and $g(c) \neq 0$, then there exists an open interval containing c on which $f(x)/g(x)$ is always defined.

Proof. We begin by noting that $f(x)/g(x)$ is defined where f and g are defined and g is non-zero. Since, by assumption in Theorem 4.3.4, f and g are both defined on the same domain A we need only then to show that there is an open interval containing c such that $g(x) \neq 0$ for every x in the interval. To demonstrate, suppose, to the contrary, that every open interval containing c contains an x such that $g(x) = 0$. Then by choosing the sequence of nested intervals given by

$$I_k = (c - \frac{1}{k}, c + \frac{1}{k})$$

and selecting a sequence of points

$$x_k \in I_k$$

such that $g(x_k) = 0$, we would get that $x_k \to c$ while $g(x_k) \to 0$ contradciting with $g(c) \neq 0$ and g continuous at c. thus there is an open interval, I, containing c such that $g(x) \neq 0$ for every $x \in I$. Therefore, $f(x)/g(x)$ is always defined on I. ☐

6. (a) Referring to the proper theorems, give a formal argument that Dirichlet's function from Section 4.1 is nowhere-continuous on $\mathbb{R}$.
   (b) Review the definition of Thomae's function in Section 4.1and demonstrate that it fails to be continuous at every rational point.
   (c) Use the characterization of continuity in theorem 4.3.2 (iii) to show that Thomae's function is continuous at every irrational point in $\mathbb{R}$.

Proof. (a) Let $c \in \mathbb{R}$. We recall that Direichlet's function is defined by

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

If $x_n \to c$ with $\forall_n x_n \in \mathbb{Q}$ we get $g(x) \to 1$. On the other hand, if $y_n \to c$ with $\forall_n y_n \notin \mathbb{Q}$ we get $g(y_n) \to 0$. So, by Theorem 4.3.2, it follows that g is not continuous at c. Therefore, g is nowhere-continuous on $\mathbb{R}$.

(b) Let $c \in \mathbb{Q}$. By our Lemma in Exercise 4 from 4.2 we know that if $x_n \to c$ with $\forall_n x_n \in \mathbb{Q}$ we will have $t(x_n) \to 0$. However, $t(c) = 1 \neq 0$. Thus, again, by Theorem 4.3.2, it follows t is nowhere-continuous on $\mathbb{Q}$.

(c) Let $\epsilon > 0$ be given and let $c \notin \mathbb{Q}$. So $t(c) = 0$. As was argued in Exercise 4(c) from 4.2 we know that $S = \{x \in \mathbb{R} : t(x) \geq \epsilon\}$ has no limit points. Thus c is not a limit point of S. Therefore there exists $\delta > 0$ such that $x \in V_\delta(c)$ implies $x \notin S$. Thus, $x \in V_\delta(c)$ implies $|t(x) - t(c)| = t(x) < \epsilon$ which is equivalent to

$$t(V_\delta(c)) \subset V_\epsilon(t(c))$$

Therefore, t is continuous at c.

☐

7. Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Proof. Let l be a limit point of K. So there exists $k_n \to l$ such that $\forall_n k_n \neq l$ and $k_n \in K$. By continuity of h on $\mathbb{R}$, we get that

$$h(k_n) \to h(l)$$

But $h(k_n) = 0$ for all n. Thus $h(k_n) \to 0$. Thus, by uniqueness of the limit, it follows that $h(l) = 0$. therefore $l \in K$. So K contains all its limit points. Therefore, K is closed. □

8. (a) Show that if a function is continuous on all of $\mathbb{R}$ and equal to 0 at every rational point, then it must be identically 0 on all of $\mathbb{R}$.

   (b) If f and g are continuous on all of $\mathbb{R}$ and $f(r) = g(r)$ at every rational point, must f and g be the same function?

Proof. (a) Suppose, to the contrary, that there exists $c \notin \mathbb{Q}$ such that

$$f(c) \neq 0$$

By density of $\mathbb{Q}$ in $\mathbb{R}$ there is

$$x_n \to c$$

where $\forall_n x_n \neq c$ and $x_n \in \mathbb{Q}$. By continuity of f at c we would then have

$$f(x_n) \to f(c)$$

But $f(x_n) \to 0$ contradicting with $f(c) \neq 0$. Thus $f(c) = 0$. Thus $f(x) = 0$ for all $x \in \mathbb{R}$.

(b) We claim that, indeed, f and g must be the same function. To demonstrate, suppose to the contrary, that there are f and g, both continuous on $\mathbb{R}$ and which agree on $\mathbb{Q}$, which are different. So there exists $x \notin \mathbb{Q}$ such that $f(x) - g(x) \neq 0$. Thus, if we define $h(x) = f(x) - g(x)$, by the Algebraic Continuity Theorem, we know that h is continuous on $\mathbb{R}$ and that $h(x) \neq 0$. By reasoning quite the same as in Exercise 5 above, we know then there is an open interval, I, about x such that $h(x) \neq 0$ for all $x \in I$. But by density of $\mathbb{Q}$ in $\mathbb{R}$, it would then follow that there exists $r \in \mathbb{Q} \cap I$ such that $h(r) \neq 0$. So $f(r) - g(r) \neq 0$. So $f(r) \neq g(r)$ contradicting with our assumption that $f(r) = g(r)$ for all $r \in \mathbb{Q}$.

□

9. (Contraction Mapping Theorem). Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbb{R}$.

(a) Show that f is continuous on $\mathbb{R}$.

(b) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \ldots)$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence. Hence we may let $y = \lim y_n$.

(c) Prove that y is a fixed point of f.

(d) Finally, prove that if x is any arbitrary point in $\mathbb{R}$ then the sequence $(x, f(x), f(f(x)), \ldots)$ converges to y in (b).

Proof. (a) Let $\epsilon > 0$ be given and let $z \in \mathbb{R}$. If $|x - z| < \frac{\epsilon}{c}$ then

$$|f(x) - f(z)| < c|x - z| < c\frac{\epsilon}{c} = \epsilon$$

as desired.

(b) We first wish to show that the sequence (y_n) is bounded. So given $n \in \mathbb{N}$ we have

$$\begin{aligned}|y_n| &= |y_1 + (y_2 - y_1) + (y_3 - y_2) + (y_4 - y_3) + \ldots + (y_{n-1} + y_{n-2}) + (y_n - y_{n-1})| \\ &\leq |y_1| + |y_2 - y_1| + |y_3 - y_2| + |y_4 - y_3| + \ldots + |y_n - y_{n-1}| \\ &< |y_1| + |y_2 - y_1| + c|y_2 - y_1| + c^2|y_2 - y_1| + \ldots + c^{n-2}|y_2 - y_1|\end{aligned}$$

recognizing that this a geometric series, we get

$$= |y_1| + \frac{|y_2 - y_1|(1 - c^{n-1})}{1 - c}$$

by recalling that $0 < c < 1$ we get that

$$< |y_1| + \frac{|y_2 - y_1|}{1 - c}$$

which implies then that (y_n) is bounded. It therefore follows that there exists $K > 0$ such that for all $n \in \mathbb{N}$ we have that $|y_n - y_1| < K$. So, choosing N so that $c^N < \frac{\epsilon}{K}$ we get that for all $n > m > N$ we have

$$\begin{aligned}|y_n - y_m| &< c|y_{n-1} - y_{m-1}| \\ &< c^2|y_{n-2} - y_{m-2}| \\ &\vdots \\ &< c^{m-1}|y_{n-(m-1)} - y_1| \\ &< c^{m-1}K \\ &< \frac{\epsilon}{K}K \\ &= \epsilon\end{aligned}$$

Therfore, (y_n) is a Cauchy sequence and we set $y = \lim y_n$.

(c) By (a) we know that f is continuous. Therefore

$$f(y) = f(\lim y_n) = \lim f(y_n) = \lim y_{n+1} = y$$

It follows, by a previous theorem, that y is therefore unique.

(d) Choose $x \in \mathbb{R}$ and let the sequence (x_n) be defined by $x_{n+1} = f(x_n)$. Then choosing $N \in \mathbb{N}$ so that $c^N < \frac{\epsilon}{|y-x|}$ implies that for all $n > N$ we have

$$\begin{aligned} |y_n - x_n| &< c|y_{n-1} - x_{n-1}| \\ &\vdots \\ &< c^{n-1}|y-x| \\ &< \frac{\epsilon}{|y-x|}|y-x| \\ &= \epsilon \end{aligned}$$

implying that $\lim x_n = \lim y_n = y$.

$\square$

10. Let f be a function defined on all of $\mathbb{R}$ that satisfies the additive condition $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

(b) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbb{R}$.

(c) Let $k = f(1)$ show that $f(n) = kn$ for all $n \in \mathbb{N}$, and then prove that $f(z) - kz$ for all $z \in \mathbb{Z}$. Now, prove that $f(r) = kr$ for all $r \in \mathbb{Q}$.

(d) Use (b) and (c) to conclude that $f(x) = kx$ for all $x \in \mathbb{R}$. Thus, any additive function that is continuous at $x = 0$ must necessarily be a linear function through the origin.

Proof. (a) We observe that

$$\begin{aligned} f(0) &= f(0+0) \\ &= f(0) + f(0) \end{aligned}$$

and therefore, by subtracting on both sides by $f(0)$, we get

$$\begin{aligned} f(0) - f(0) &= f(0) + f(0) - f(0) \\ 0 &= f(0) \end{aligned}$$

We then have that

$$\begin{aligned} 0 &= f(0) \\ &= f(x + (-x)) \\ &= f(x) + f(-x) \end{aligned}$$

and so, subtracting $f(x)$ from both sides yields

$$\begin{aligned} 0 - f(x) &= f(x) + f(-x) - f(x) \\ -f(x) &= f(-x) \end{aligned}$$

(b) Let f be continuous at 0, $c \in \mathbb{R}$, and $\epsilon > 0$ be given. Since f is continuous at 0 it follows that there exists $\delta > 0$ such that $|y| < \delta$ implies $|f(y)| < \epsilon$. Thus if $|x - c| < \delta$ then we have

$$|f(x) - f(c)| = |f(x - c)| < \epsilon$$

So, f is continuous at c and so on $\mathbb{R}$. It follows that f is continuous on $\mathbb{R}$.

(c) Let $n \in \mathbb{N}$ and $k = f(1)$. We observe that

$$f(n) = f\left(\sum_{i=1}^{n} 1\right) = \sum_{i=1}^{n} f(1) = kn$$

If $z \in \mathbb{Z}$ then either $z \geq 0$ or $z < 0$. If the former then clearly $f(z) = kz$. If the latter then $z = -n$ for some $n \in \mathbb{N}$.

$$f(z) = f(-n) = -f(n) = -kn = k(-n) = kz$$

Now, let $r \in \mathbb{Q}$. So r is of the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$. We first observe that if $q > 0$

$$\begin{aligned} f(1) &= f\left(\sum_{i=1}^{q} \frac{1}{q}\right) \\ &= \sum_{i=1}^{q} f\left(\frac{1}{q}\right) \\ &= f\left(\frac{1}{q}\right) \sum_{i=1}^{q} 1 \\ &= f\left(\frac{1}{q}\right) q \end{aligned}$$

and so

$$\begin{aligned} \frac{f(1)}{q} &= f\left(\frac{1}{q}\right) \\ \frac{k}{q} &= f\left(\frac{1}{q}\right) \end{aligned}$$

So then

$$\begin{aligned} -\frac{k}{q} &= -f\left(\frac{1}{q}\right) \\ &= f\left(-\frac{1}{q}\right) \end{aligned}$$

and so if $p = 0$ we are done. If $sgn(p) = sgn(q)$ then

$$\begin{aligned}
f\left(\frac{p}{q}\right) &= f\left(\sum_{i=1}^{|p|} \frac{1}{|q|}\right) \\
&= \sum_{i=1}^{|p|} f\left(\frac{1}{|q|}\right) \\
&= f\left(\frac{1}{|q|}\right) \sum_{i=1}^{|p|} 1 \\
&= f\left(\frac{1}{|q|}\right) |p| \\
&= \frac{k}{|q|}|p| \\
&= k\frac{p}{q}
\end{aligned}$$

On the other hand, if $sgn(p) \neq sgn(q)$ then

$$\begin{aligned}
f\left(\frac{p}{q}\right) &= f\left(-\frac{|p|}{|q|}\right) \\
&= -f\left(\frac{|p|}{|q|}\right) \\
&= -k\frac{|p|}{|q|} \\
&= k\frac{p}{q}
\end{aligned}$$

thus $f(r) = kr$ for all $r \in \mathbb{Q}$.

(d) From (b) we know that f is continuous on all of $\mathbb{R}$. From (c) we also know that f and $g(x) = kx$ must be identical on all of $\mathbb{Q}$. Therefore, from Exercise 8, we know that $f = g$ on all of $\mathbb{R}$. Therefore $f(x) = kx$ for all $x \in \mathbb{R}$.

$\square$

11. For each of the following choices of A, construct a function $f : \mathbb{R} \to \mathbb{R}$ that has discontinuities at every point x in A and is continuous on A^c.

(a) $A = \mathbb{Z}$

(b) $A = \{x : 0 < x < 1\}$

(c) $A = \{x : 0 \leq x \leq 1\}$

(d) $A = \{\frac{1}{n} : n \in \mathbb{N}\}$

Proof. (a)

$$f(x) = \begin{cases} 0 & , x \in \mathbb{Z} \\ 1 & , \text{elsewhere} \end{cases}$$

(b)

$$f(x) = \begin{cases} 0 & , x \leq 0 \\ x & , x \in \mathbb{Q} \cap (0, \frac{1}{2}] \\ 0 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap (0, \frac{1}{2}] \\ -x & , x \in \mathbb{Q} \cap (\frac{1}{2}, 1) \\ 0 & , x \in (\mathbb{R} \setminus \mathbb{Q}) \cap (\frac{1}{2}, 1) \\ 0 & , x \geq 1 \end{cases}$$

(c)

$$f(x) = \begin{cases} \frac{1}{2} & , x < 0 \\ 0 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1] \\ 1 & , x \in \mathbb{Q} \cap [0, 1] \\ \frac{1}{2} & , x > 1 \end{cases}$$

(d)

$$f(x) = \begin{cases} x & , x \in A \\ 2 & x \notin A \end{cases}$$

□

12. Let C be the Cantor set constructed in Section 3.1. Define $g : [0, 1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & x \in C \\ 0 & x \notin C \end{cases}$$

(a) Show that g fails to be continuous at any point $c \in C$.

(b) Prove that g is continuous at every point $c \notin C$.

Proof. (a) Let $x \in C$. Recall that C does not contain any intervals. So $\forall \delta > 0$ we have $V_\delta(x) \cap C^c \neq \emptyset$. It is easy then to show that $\exists \epsilon > 0 \, \forall \delta > 0 \, \exists z \in V_\delta(x)$ such that $f(z) \notin V_\epsilon(f(x))$.

(b) Let $x \notin C$.

□

4.4

1. (a) Show that $f(x) = x^3$ is continuous on all of $\mathbb{R}$.
   (b) Argue, using Theorem 4.4.6 that f is not uniformly continuous on $\mathbb{R}$.
   (c) Show that f is uniformly continuous on any bounded subset of $\mathbb{R}$.

Proof. (a) Let $c \in \mathbb{R}$ and let $\epsilon > 0$ be given. Now we observe that

$$|x^3 - c^3| = |x - c||x^2 + cx + c^2|$$

Since $x^2 + cx + c^2$ is clearly a parabola, it follows that it is bounded on $[c-1, c+1]$. That is $|x - c| < 1$ implies

$$|x^2 + cx + c^2| < K$$

Thus if we take $|x - c| < \min\{\frac{\epsilon}{K}, 1\}$ then

$$|x^3 - c^3| = |x - c||x^2 + cx + c^2| < \frac{\epsilon}{K}|x^2 + cx + c^2| < \frac{\epsilon}{K}K = \epsilon$$

as desired.

(b) Let $\epsilon = 1$ and let (x_n) and (y_n) be sequences defined by

$$\begin{aligned} x_n &= n \\ y_n &= n + \frac{1}{n} \end{aligned}$$

Then

$$|y_n - x_n| = |n + \frac{1}{n} - n| = |\frac{1}{n}| \to 0$$

Now we observe that, for all $n \in \mathbb{N}$, we have

$$\begin{aligned} |f(y_n) - f(x_n)| &= \left| \left(n + \frac{1}{n}\right)^3 - n^3 \right| \\ &= \left| n^3 + 3n + \frac{3}{n} + \frac{1}{n^3} - n^3 \right| \\ &= \left| 3n + \frac{3}{n} + \frac{1}{n^3} \right| \\ &= 3n + \frac{3}{n} + \frac{1}{n^3} \\ &> 3n \\ &> 1 \end{aligned}$$

Thus, by Theorem 4.4.6, it follows that $f(x) = x^3$ is not uniformly continuous on $\mathbb{R}$.

(c) Let $A \subset \mathbb{R}$ be bounded. Thus there exists $a > 0$ such that $A \subset [-a, a]$. From (a), we know that x^3 is continuous on $[-a, a]$. Thus, by compactness of $[-a, a]$ and Theorem 4.4.8, we get that x^3 is uniformly continuous on $[-a, a]$ and therefore on A.

□

2. Show that $f(x) = \frac{1}{x^2}$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

Proof. We first observe that, by Theorem 4.3.4 (iii), we have that $f(x) = \frac{1}{x^2}$ is continuous on $\mathbb{R} \setminus \{0\}$. To show that f is uniformly continuous on $[1, \infty)$ we begin by letting $\epsilon > 0$ be given. Then, by Archi, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \epsilon$$

and so $\forall x \geq \sqrt{n}$ we have

$$0 < \frac{1}{x^2} < \epsilon \tag{1}$$

Now, since $f(x)$ is continuous on $[1, \infty)$ it follows then, by Theorem 4.4.8, that is uniformly continuous on $[1, \sqrt{n}]$. So there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$ so long as $x, y \in [1, \sqrt{n}]$. On the other hand if $x, y \in [\sqrt{n}, \infty)$, by (1), we have then that

$$\left| \frac{1}{x^2} - \frac{1}{y^2} \right| < |\epsilon - 0| = \epsilon$$

thus for $x, y \in [\sqrt{n}, \infty)$ such that $|x - y| < \delta$ we have

$$|f(x) - f(y)| < \epsilon$$

Therefore, f is uniformly continuous on $[1, \infty)$. To demonstrate that f is *not* uniformly continuous on $(0, 1]$ we let $\epsilon = 1$ and

$$x_n = \frac{1}{2n}$$
$$y_n = \frac{1}{n}$$

It is not hard to show that $|x_n - y_n| \to 0$. However

$$|f(x_n) - f(y_n)| = \left| \frac{1}{\frac{1}{2n}^2} - \frac{1}{\frac{1}{n}^2} \right| = \left| 4n^2 - n^2 \right| = 3n^2 > 1 = \epsilon$$

Thus, f is not uniformly continuous on $(0, 1]$. □

3. Furnish the details (including an argument for Exercise 3.3.1 if is not already done) for the proof for the Extreme Value Theorem.

Proof. We begin by demonstrating Exercise 3.3.1. If K is compact then, by Theorem 3.3.8, it is closed and bounded. Therefore, by boundedness of K, both $\sup K$ and $\inf K$ exist. Furthermore, by closure of K, both $\sup K, \inf K \in K$. Now if $f : K \to \mathbb{R}$ is continuous, by theorem 4.4.2, we have that $f(K)$ is compact. By our previous argument we then know that $\sup f(K), \inf f(K) \in f(K)$. Thus there exists $x_0 \in K$ such that $f(x_0) = \inf f(K)$ and $x_1 \in K$ such that $f(x_1) = \sup f(K)$. This, in turn, implies that, for all $x \in K$ we have $f(x_0) = \inf f(K) \leq f(x) \leq \sup f(K) = f(x_1)$ as desired. □

4. Show that if f is continuous on $[a, b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $\frac{1}{f}$ is bounded on $[a, b]$.

Proof. We observe that $[a, b]$ is compact and, by continuity of f, we therefore have that $f([a, b])$ is compact as well. Compactness of $f([a, b])$ implies that $f([a, b])$ is closed and bounded. Thus, $\forall x \in [a, b]$ we have

$$l \leq f(x) \leq u$$

and thus

$$\frac{1}{l} \geq \frac{1}{f(x)} \geq \frac{1}{u}$$

Thus $\frac{1}{f}$ is bounded on $[a, b]$. □

5. Using the advice that follows Theorem 4.4.6, provide a complete proof for this criterion for nonuniform continuity.

Proof. $\Rightarrow$: Suppose f is not uniformly continuous on A. Then there exists $\epsilon_0 > 0$ such that for all $\delta > 0$ there exist $x, y \in A$ such that $|x - y| < \delta$ while $|f(x) - f(y)| \geq \epsilon_0$. In particular for all $n \in \mathbb{N}$ there exist $x_n, y_n \in A$ such that $|x_n - y_n| < \frac{1}{n}$ while $|f(x_n) - f(y_n)| \geq \epsilon_0$. Clearly we have then that $|x_n - y_n| \to 0$.

$\Leftarrow$: Suppose there exists $\epsilon_0 > 0$ and (x_n), (y_n) in A satisfying $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$. Then for ϵ_0 we have that for all $\delta > 0$ there exist $x_n, y_n \in A$ satisfying $|x_n - y_n| < \delta$. However, by assumption, we have that $|f(x_n) - f(y_n)| \geq \epsilon_0$. Therefore, f is not uniformly continuous on A. □

6. Give an example of each of the following, or state that such a request is impossible.

   (a) a continuous function $f : (0, 1) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence;

   (b) a continuous function $f : [0, 1] \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence;

(c) a continuous function $f : [0, \infty) \to \mathbb{R}$ and a Cauchy sequence (x_n) such that $f(x_n)$ is not a Cauchy sequence;

(d) a continuous bounded function f on $(0, 1)$ that attains a maximum value on this open interval but not a minimum value.

Proof. (a) Let $f(x) = \frac{1}{x}$ and $x_n = \frac{1}{n}$. Then $(x_n) \to 0$ and is therefore Cauchy. On the other hand, $f(x_n) = n$ which is clearly not Cauchy.

(b) The request is impossible to satisfy. If f is continuous on $[0, 1]$ then, by compactness of $[0, 1]$ we know that if $(x_n) \subset [0, 1]$ and $x_n \to L$, then, $L \in [0, 1]$. But then, by continuity of f we would get that $f(x_n) \to f(L)$ implying that $f(x_n)$ is Cauchy.

(c) The request is impossible to satisfy. Notice $[0, \infty)$ is a closed set. Thus again if $(x_n) \subset [0, \infty)$ and $x_n \to L$, then $L \in [0, \infty)$. Thus, by continuity we would have $f(x_n) \to f(L)$ implying that $f(x_n)$ is Cauchy.

(d) Let $f : (0, 1) \to \mathbb{R}$ be defined by

$$f(x) = -\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}$$

It is an easy matter to show that f is continuous on $(0, 1)$ and that it attains a maximum but no minimum.

□

7. Assume that g is defined on an open interval (a, c) and it is known to be uniformly continuous on $(a, b]$ and $[b, c)$, where $a < b < c$. Prove that g is uniformly continuous on (a, c).

Proof. Let $\epsilon > 0$ be given. So, by uniform continuity on $(a, b]$, there exists $\delta_1 > 0$ such that whenever $x, y \in (a, b]$ and $|x - y| < \delta_1$ we get $|f(x) - f(y)| < \frac{\epsilon}{2}$. Similarly, there is $\delta_2 > 0$ such that $x, y \in [b, c)$ and $|x - y| < \delta_2$ implies $|f(x) - f(y)| < \frac{\epsilon}{2}$. Thus choosing $\delta = \min\{\delta_1, \delta_2\}$ we get that whenever $x, y \in (a, b]$ and $|x - y| < \delta$ we have

$$|f(x) - f(y)| < \frac{\epsilon}{2} < \epsilon$$

we also have whenever $x, y \in [b, c)$ and $|x - y| < \delta$ it follows that

$$|f(x) - f(y)| < \frac{\epsilon}{2} < \epsilon$$

Lastly, wlog, if $x \in (a, b]$ and $y \in [b, c)$ and $|x - y| < \delta$ then

$$|f(x) - f(y)| \leq |f(x) - f(b)| + |f(b) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, for all $x, y \in (a, c)$ if $|x - y| < \delta$ it follows that

$$|f(x) - f(y)| < \epsilon$$

establishing that f is uniformly continuous on (a, c). □

8. (a) Assume that $f : [0, \infty) \to \mathbb{R}$ is continuous at every point in its domain. Show that if there exists $b > 0$ such that f is uniformly continuous on the set $[b, \infty)$, then f is uniformly continuous on $[0, \infty)$.

   (b) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Proof. (a) Notice that since f is continuous on $[0, b]$ and $[0, b]$ is compact, it follows that f is uniformly continuous on $[0, b]$. Applying similar reasoning as in 7, we will get the desired result.

(b) Let $\epsilon > 0$ be given. To demonstrate that $\sqrt{x}$ is uniformly continuous on $[\epsilon^2, \infty)$ we let $x, y \in [\epsilon^2, \infty)$ such that, wlog, $x < y$ and $y - x < \epsilon^2$. Then

$$\sqrt{y} - \sqrt{x} < \sqrt{x + \epsilon^2} - \sqrt{x} \leq \sqrt{x} + \sqrt{\epsilon^2} - \sqrt{x} = \epsilon$$

Thus $\sqrt{x}$ is uniformly continuous on $[\epsilon^2, \infty)$. Thus, by continuity of $\sqrt{x}$, which is a trivial matter to show, and part (a), we know that $\sqrt{x}$ is uniformly continuous on $[0, \infty)$.

□

9. A function $f : A \to \mathbb{R}$ is called *Lipschitz* if there exists a bound $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x, y \in A$. Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.

   (a) Show that if $f : A \to \mathbb{R}$ is Lipschitz, then it is uniformly continuous on A.

   (b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

Proof. (a) Let $f : A \to \mathbb{R}$ be Lipschitz and let $x, y \in A$ such that $|x - y| < \frac{\epsilon}{M}$. Then

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

which implies

$$|f(x) - f(y)| \leq M|x - y| < M\frac{\epsilon}{M} = \epsilon$$

Thus f is uniformly continuous on A.

(b) No the converse is not true. Take $f(x) = \sqrt{x}$. As we have shown, f is uniformly continuous on $[0, \infty)$. However, we observe that, given $M > 0$, for all $0 < x < \frac{1}{M^2}$ we have

$$\left| \frac{\sqrt{x} - \sqrt{0}}{x - 0} \right| = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} > \frac{1}{\sqrt{\frac{1}{M^2}}} = M$$

□

10. Do uniformly continuous functions preserve boundedness? If f is uniformly continuous on a bounded set A, is $f(A)$ necessarily bounded?

Proof. Yes, uniform continuity does, indeed, preserve boundedness. To demonstrate we proceed by way of contradiction. To this end, suppose that A is bounded, f is uniformly continuous on A, and that $f(A)$ is unbounded. Then there exists a sequence $(x_n) \subseteq A$ such that $f(x_n) \to \pm\infty$ as $n \to \infty$. Now, by boundedness of A and Bolzano-Weierstrass, it follows that there exists a convergent subsequence (x_{n_k}). Clearly $\lim_{k\to\infty} x_{n_k} = L \notin A$ since, otherwise, by continuity of f on A, we would have

$$f(L) = \pm\infty$$

which contradicts with $f(A) \subset \mathbb{R}$. So L must be a limit point of A. Now let $\epsilon > 0$ be given. Then there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. But there exists K such that $\forall k > K$ we have

$$|x_{n_k} - L| < \frac{\delta}{2}$$

and so, it follows that for all $s, k > K$ we have

$$|x_{n_k} - x_{n_s}| < \delta$$

which, in turn, implies that

$$|f(x_{n_k}) - f(x_{n_s})| < \epsilon$$

for all $s, k > K$ which, finally, is contrary to our assumption that $f(x_n) \to \pm\infty$. Therefore, $f(A)$ is, necessarily, bounded. □

11. (Topological Characterization of Continuity). Let g be defined on all of $\mathbb{R}$. If A is a subset of $\mathbb{R}$, define the set $g^{-1}(A)$ by

$$g^{-1}(A) = \{x \in \mathbb{R} : g(x) \in A\}$$

Show that g is continuous if and only if $g^{-1}(O)$ is open whenever $O \subset \mathbb{R}$ is an open set.

Proof. $\Rightarrow$: Let O be open and $x \in g^{-1}(O)$. So $g(x) \in O$. Since O is open, there exists $V_\epsilon(g(x)) \subset O$. By continuity, there then exists $V_\delta(x)$ such that $f(V_\delta(x)) \subset V_\epsilon(g(x))$. So $V_\delta(x) \subset g^1(O)$. Thus $g^{-1}(O)$ is an n-hood of x. Therefore, $g^{-1}(O)$ is open.
$\Leftarrow$: Let that $g^1(O)$ is open whenever O is open and let $c \in \mathbb{R}$. Given $\epsilon > 0$ we have that $g^1(V_\epsilon(g(c)))$ is open, and therefore that there is $V_\delta(c) \subset g^1(V_\epsilon(g(c)))$. So $f(V_\delta(x)) \subset V_\epsilon(g(x))$. Therefore, g is continuous. $\square$

12. Construct an alternate proof of Theorem 4.4.8 using the open cover characterization of compactness from 3.3.8 (iii).

Proof. The following proof is adapted from Rudin. Let $\epsilon > 0$ be given. Since f is continuous, for each point $p \in K$ there is a positive number $\phi(p)$ such that

$$\text{if } q \in K \text{ and } |q-p| < \phi(p) \text{ then } |f(q) - f(p)| < \frac{\epsilon}{2}$$

Now, let $J(p) = \{q : |q-p| < \frac{1}{2}\phi(p)\}$. Then $\{J(p) : p \in K\}$ forms an open cover of K. Thus, by compactness of K, we get that there exists a finite subcover $\{J(p_1), J(p_2), \ldots, J(p_n)\}$. Now if we let $\delta = \frac{1}{2}\min\{\phi(p_1), \phi(p_2), \ldots, \phi(p_n)\} > 0$ then we have that for $|q-p| < \delta$, there exists p_i such that $p \in J(p_i)$ and therefore

$$|p - p_i| < \frac{1}{2}\phi(p_i)$$

But then

$$|q - p_i| \leq |q-p| + |p - p_i| < \delta + \frac{1}{2}\phi(p_i) < \phi(p_i)$$

Therefore

$$|f(q) - f(p)| \leq |f(q) - f(p_i)| + |f(p_i) - f(p)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, f is continuous on K. $\square$

13. (a) Show that a uniformly continuous function preserves Cauchy sequences; that is, if $f : A \to \mathbb{R}$ is uniformly continuous and $(x_n) \subset A$ is a Cauchy sequence, then show $f(x_n)$ is a Cauchy sequence.

    (b) Let g be a continuous function on the open interval (a, b). Prove that g is uniformly continuous on (a, b) if and only if it is possible to define values $g(a)$ and $g(b)$ at the endpoints so that the extended function g is continuous on $[a, b]$.

Proof. (a) Let f be uniformly continuous on A, $(x_n) \subset A$ be Cauchy, and $\epsilon > 0$ be given. Since f is uniformly continuous, there exists $\delta > 0$

such that $|x-y|<\delta$ implies $|f(x)-f(y)|<\epsilon$. Since (x_n) is Cauchy, there exists N such that $m,n \geq N$ implies $|x_n - x_m| < \delta$ which further implies that for all $m,n \geq N$ we have $|f(x_n) - f(x_m)| < \epsilon$. Therefore, $f(x_n)$ is Cauchy.

(b) $\Leftarrow$: Trivial.

$\Rightarrow$: Let g be uniformly continuous on (a,b) and let $(x_n),(y_n) \subset (a,b)$ such that $x_n \to a$ and $y_n \to b$. So (x_n) and (y_n) are Cauchy. So, by (a), $f(x_n)$ and $f(y_n)$ are Cauchy. So there exist L and M such that $f(x_n) \to L$ and $f(y_n) \to M$. So if $g(a) = L$ and $g(b) = M$ we claim that g is continuous on $[a,b]$. To show g is continuous at a we let $\epsilon > 0$ be given. Since g is uniformly continuous on (a,b) there exists $\delta > 0$ such that $\forall x,y \in (a,b)$ such that $|x-y|<\delta$ we have $|f(x)-f(y)| < \frac{\epsilon}{2}$. Since $x_n \to a$ there exists N_1 such that $n \geq N_1$ implies $|x_n - a| < \frac{\delta}{2}$. Furthermore, since $f(x_n) \to L$ there exists N_2 such that $\forall n \geq N_2$ we have $|f(x_n) - f(a)| < \frac{\epsilon}{2}$. Thus for $N = \max N_1, N_2$ we have $\forall n \geq N$

$$|x_n - a| < \frac{\delta}{2}$$

and

$$|f(x_n) - f(a)| < \frac{\epsilon}{2}$$

So, if $|x-a| < \frac{\delta}{2}$ and $n \geq N$ we get

$$|x - x_n| \leq |x-a| + |a - x_n| < \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

which implies

$$|f(x) - f(x_n)| < \frac{\epsilon}{2}$$

Finally, we then get that, together, this implies that

$$|f(x) - f(a)| \leq |f(x) - f(x_n)| + |f(x_n) - f(a)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, g is continuous at a when $g(a) = \lim_{n\to\infty} g(x_n) = L$. A similar proof demonstrates that g is continuous at b when $g(b) = \lim_{n\to\infty} g(y_n) = M$.

$\square$

4.5

1. Show how the Intermediate Value Theorem follows as a corollary to Theorem 4.5.2.

Proof. Let $f : [a, b] \to \mathbb{R}$ be continuous and L a real number between $f(a)$ and $f(b)$. Since $[a, b]$ is connected and f is continuous, by Theorem 4.5.2 we have that $f([a, b])$ is connected. Since every connected set in $\mathbb{R}$ is an interval it follows that $f([a, b])$ is also an interval. Since $f(a), f(b) \in f([a, b])$ and $f([a, b])$ is an interval, it follows that every element between $f(a)$ and $f(b)$ is in $f([a, b])$. In particular $L \in f([a, b])$. Thus there $c \in (a, b)$ such that $f(c) = L$. □

2. Decide on the validity of the following conjectures.

   (a) Continuous functions take bounded open intervals to bounded open intervals.

   (b) Continuous functions take bounded open intervals to open sets.

   (c) Continuous functions take bounded closed intervals to bounded closed intervals.

   Proof. (a) This is false. $f(x) = x^2$ is continuous and takes $(-1, 1)$ to $[0, 1)$.

   (b) This is false. See the above.

   (c) This is true. Notice a bounded closed interval is compact. By Theorem 4.5.2 we know that that the image, by a continuous function, of a bounded closed interval must also be an interval. However, from Theorem 4.4.2 it must also be compact which implies that it must be bounded and closed. Therefore, it must be a bounded and closed interval.

   □

3. Is there a continuous function on all of $\mathbb{R}$ with range $f(\mathbb{R})$ equal to $\mathbb{Q}$?

   Proof. No. If f is continuous on $\mathbb{R}$ and $f(\mathbb{R}) = \mathbb{Q}$, then there exist $a, b \in \mathbb{R}$ such that $f(a) = 0$ and $f(b) = 1$. Suppose $a < b$. Then $f([a, b]) \subset \mathbb{Q}$ must be connected. However, we know that there exists $y \in \mathbb{R} \setminus \mathbb{Q}$ such that $0 < y < 1$. Thus, there must exist $c \in (a, b)$ such that $f(c) = y$ contradicting with $f(\mathbb{R}) = \mathbb{Q}$. A similar argument demonstrates the same holds when $b < a$. □

4. A function f is increasing on A if $f(x) \leq f(y)$ for all $x < y$ in A. Show that the Intermediate Value Theorem does have a converse if we assume f is increasing on $[a, b]$.

   Proof. Let f have the IVP and be increasing on $[a, b]$ and let $\epsilon > 0$ be given.

(a) Continuity at a: Let $y \in (a, b]$. If

$$f(y) < f(a) + \epsilon$$

then, by f increasing, $a < z < y$ implies

$$f(a) \leq f(z) \leq f(y) < f(a) + \epsilon$$

so that

$$|f(z) - f(a)| < \epsilon$$

On the other hand, if $f(y) \geq f(a) + \epsilon$ then, by IVP, there exists $c \in (a, y)$ such that $f(c) = f(a) + \frac{\epsilon}{2} < f(a) + \epsilon$. Then, by our previous argument, we will have that $a < z < c$ implies that $|f(z) - f(a)| < \epsilon$. Therefore, f is continuous at a.

(b) Continuity at b: Extremely similar to the above.

(c) Continuity on (a, b): Let $c \in (a, b)$. Notice f is then increasing and has the IVP on $[a, c]$. So following from (b), we will get that there exists $\delta_1 > 0$ such that for all x such that $x < c$ and $c - x < \delta_1$ we will have $|f(x) - f(c)| < \epsilon$. Similarly f is increasing and has the IVP on $[c, b]$. So following form (a), we will get that there exists $\delta_2 > 0$ such that for all x such that $c < x$ and $x - c < \delta_2$ we will have $|f(x) - f(c)| < \epsilon$. Thus if $\delta = \min\{\delta_1, \delta_2\}$ then $|x - c| < \delta$ implies that $|f(x) - f(c)| < \epsilon$. Thus f is continuous at c.

Therefore, altogether, we have that f is continuous on $[a, b]$. □

5. Finish the proof of the Intermediate Value Theorem using the Axiom of Completeness started previously.

Proof. Continuing the argument, we first show that $f(c) \leq 0$. We observe that either $c \in K$ or, by c being the supremum of K, there is $(x_n) \subset K$ such that $x_n \to c$. The former implies that $f(c) \leq 0$. On the other hand, in the latter case, by compactness of $[a, b]$, it follows then that $c \in [a, b]$ and so $f(c)$ is well-defined. By continuity of f, we get $f(x_n) \to f(c)$. Now, since $f(x_n) \leq 0$ for all $n \in \mathbb{N}$, it follows from the Order Limit Theorem that $f(c) \leq 0$. Thus, in all cases $f(c) \leq 0$. Now, since $f(b) > 0$, it follows that $c \neq b$. Furthermore for all $x \in (c, b]$ we have $f(x) > 0$. If $(y_n) \subset (c, b]$ such that $y_n \to c$, then, again, by continuity of f, we must have $f(y_n) \to f(c)$. This then implies that $f(c) = 0$, for otherwise we would have that $f(y_n) > 0$ for all n and $\lim_{n\to\infty} f(y_n) = f(c) < 0$ contradicting with the Order Limit Theorem. Therefore, $f(c) = 0$. Now then, if $g : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $L \in (g(a), g(b))$ then, by the previous argument, there exists c such that $g(c) - L = 0$. Thus there exists c such that $g(c) = L$. This also holds when $g(b) < g(a)$. □

6. Finish the proof of the Intermediate Value Theorem using the Nested Interval Property started previously.

Proof. Continuing the argument, we denote each of the intervals by $I_n = [x_n, y_n]$. By the NIP, we must have that there exists $c \in \cap_{n=0}^{\infty} I_n$. Furthermore, $x_n, y_n \to c$. By continuity of f we have then that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n) = f(c)$. Since $f(x_n) < 0$ for all n and $f(y_n) \geq 0$ for all n, it follows by the Order Limit Theorem, that $f(c) \geq 0$ and $f(c) \leq 0$. Therefore $f(c) = 0$. Thus, if $g : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $L \in (g(a), g(b))$ it follows from the previous argument that there exists c such that $g(c) - L = 0$. Therefore, there exists $c \in (a, b)$ such that $g(c) = L$. The same holds when $g(b) < g(a)$. □

7. Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show $f(x) = x$ for at least one value of $x \in [0, 1]$.

   Proof. Notice that $f(0) \geq 0$ and $f(1) \leq 1$. Define $g(x) = x - f(x)$. Clearly $g(0) = 0 - f(0) \leq 0$ and $g(1) = 1 - f(1) \geq 0$. Moreover, by the Algebraic Continuity Theorem, we know that g is continuous on $[0, 1]$. Thus by the Intermediate Value Theorem, there exists $c \in (0, 1)$ such that $g(c) = 0$. That is, there is $c \in (0, 1)$ such that $c - f(c) = 0$, which, in turn, implies that $c = f(c)$. Therefore, f has a fixed point in $[0, 1]$. □

Chapter 5

5.2

1. Supply the proofs for part (i) and (ii) of Theorem 5.2.4.

   Proof. (i)

$$\begin{aligned}
(f+g)'(c) &= \lim_{x\to c} \frac{(f+g)(x)-(f+g)(c)}{x-c} \\
&= \lim_{x\to c} \frac{f(x)+g(x)-f(c)-g(c)}{x-c} \\
&= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \frac{g(x)-g(c)}{x-c} \\
&= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\
&= f'(c)+g'(c)
\end{aligned}$$

   (ii)

$$\begin{aligned}
(kf)'(c) &= \lim_{x\to c} \frac{(kf)(x)-(kf)(c)}{x-c} \\
&= \lim_{x\to c} \frac{kf(x)-kf(c)}{x-c} \\
&= k\lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\
&= kf'(c)
\end{aligned}$$

   □

2. (a) Use Definition 5.2.1 to produce the proper formula for the derivative of $f(x) = \frac{1}{x}$.
   (b) Combine the result in (a) with the chain rule (Theorem 5.2.5) to supply a proof for part (iv) of Theorem 5.2.4.

(c) Supply a direct proof of Theorem 5.2.4 (iv) by algebraically manipulating the difference quotient for $\frac{f}{g}$ in a style similar to the proof of Theorem 5.2.4 (iii).

Proof. (a) Let $c \in \mathbb{R}$ such that $c \neq 0$. Then

$$\begin{aligned}\lim_{x\to c} \frac{\frac{1}{x} - \frac{1}{c}}{x - c} &= \lim_{x\to c} \frac{\frac{c}{cx} - \frac{x}{cx}}{x - c} \\ &= \lim_{x\to c} \frac{-(x-c)}{(cx)(x-c)} \\ &= \lim_{x\to c} -\frac{1}{cx} \\ &= -\frac{1}{c^2}\end{aligned}$$

Thus, in general, $f'(x) = -\frac{1}{x^2}$

(b) Notice we can rewrite $\frac{f}{g}$ as fg^{-1}. Thus, if $h(x) = \frac{1}{x}$ then $(h \circ g) = \frac{1}{g} = g^{-1}$ which, in turn implies $\frac{f}{g} = fg^{-1} = f(h \circ g)$. Thus, by (iii) of Theorem 5.2.4, we get that when c is chosen so that $g(c) \neq 0$ we have

$$\left(\frac{f}{g}\right)'(c) = (f(h\circ g))'(c) = f'(h\circ g)(c) + f(h \circ g)'(c)$$

By The Chain Rule, we get that $(h \circ g)'(c) = h'(g(c))g'(c)$. From (a), we know that

$$h'(g(c))g'(c) = -\frac{1}{(g(c))^2}g'(c)$$

Thus, we have

$$\begin{aligned}f'(h\circ g)(c) + f(h\circ g)'(c) &= \frac{f'(c)}{g(c)} + f(c)\left(-\frac{1}{(g(c))^2}g'(c)\right) \\ &= \frac{f'(c)}{g(c)} + \left(-\frac{f(c)g'(c)}{(g(c))^2}\right) \\ &= \frac{g(c)f'(c) - f(c)g'(c)}{(g(c))^2}\end{aligned}$$

So, altogether, we get that

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{(g(c))^2}$$

as desired.

(c) Let c be chosen such that $g(c) \neq 0$. Then

$$\begin{aligned}
\left(\frac{f}{g}\right)'(c) &= \lim_{x \to c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c} \\
&= \lim_{x \to c} \frac{\frac{g(c)f(x) - f(c)g(x)}{g(c)g(x)}}{x - c} \\
&= \lim_{x \to c} \frac{1}{x - c} \left(\frac{g(c)f(x) + f(x)g(x) - f(x)g(x) - f(c)g(x)}{g(x)g(c)} \right) \\
&= \lim_{x \to c} \frac{1}{x - c} \left(\frac{f(x)(g(c) - g(x)) + g(x)(f(x) - f(c))}{g(x)g(c)} \right) \\
&= \lim_{x \to c} \frac{1}{x - c} \left(\frac{g(x)(f(x) - f(c)) - f(x)(g(x) - g(c))}{g(x)g(c)} \right) \\
&= \lim_{x \to c} \frac{1}{g(c)} \left(\frac{f(x) - f(c)}{x - c} - \frac{f(x)}{g(x)} \frac{g(x) - g(c)}{x - c} \right)
\end{aligned}$$

Now, since f and g are differentiable at c and $g(c) \neq 0$, it follows that both f and g are continuous at c. So we have

$$\begin{aligned}
&= \frac{1}{g(c)} \lim_{x \to c} \frac{f(x) - f(c)}{x - c} - \left(\lim_{x \to c} \frac{f(x)}{g(x)} \right) \left(\lim_{x \to c} \frac{g(x) - g(c)}{x - c} \right) \\
&= \frac{1}{g(c)} \left(f'(c) - \frac{f(c)}{g(c)} g'(c) \right) \\
&= \frac{f'(c)}{g(c)} - \frac{f(c)g'(c)}{g(c)^2} \\
&= \frac{g(c)f'(c) - f(c)g'(c)}{g(c)^2}
\end{aligned}$$

as desired.

□

3. By imitating the Dirichlet constructions in Section 4.1, construct a function on $\mathbb{R}$ that is differentiable at a single point.

Proof. Let

$$f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

First, to demonstrate that f is differentiable at $x = 0$, we let $\epsilon > 0$ be given. If $|x| < \epsilon$, then, for all such x we have either

$$f(x) = 0$$

or

$$f(x) = x^2$$

If $f(x) = 0$ then

$$\left|\frac{f(x) - f(0)}{x - 0}\right| = \left|\frac{0 - 0}{x - 0}\right| = 0 < \epsilon$$

On the other hand, if $f(x) = x^2$ then

$$\left|\frac{f(x) - f(0)}{x - 0}\right| = \left|\frac{x^2 - 0}{x - 0}\right| = |x| < \epsilon$$

Thus

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = 0$$

Now, to show that f is not differentiable at any other point, let $c \neq 0$. Now, there exist sequences $(x_n) \subset \mathbb{Q}$ and $(y_n) \subset \mathbb{R}\backslash\mathbb{Q}$ such that $x_n, y_n \to c$ while, for every n, $x_n \neq c$ and $y_n \neq c$. If $f(c) = 0$ then

$$\begin{aligned}
\frac{f(x_n) - f(c)}{x_n - c} &= \frac{x_n^2}{x_n - c} \\
&= \frac{x_n^2 - c^2 + c^2}{x_n - c} \\
&= \frac{(x_n + c)(x_n - c) + c^2}{x_n - c} \\
&= x_n + c + \frac{c^2}{x_n - c}
\end{aligned}$$

and clearly

$$x_n + c + \frac{c^2}{x_n - c} \to \pm\infty$$

Thus $\lim_{x\to c} \frac{f(x)-f(c)}{x-0}$ does not exist. On the other hand, if $f(c) = c^2$ then

$$\begin{aligned}
\frac{f(y_n) - f(c)}{y_n - c} &= \frac{0 - c^2}{y_n - c} \\
&= -\frac{c^2}{y_n - c}
\end{aligned}$$

Again, clearly we have

$$-\frac{c^2}{y_n - c} \to \pm\infty$$

Thus, $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ does not exist when $f(c) = c^2$. Thus, in all cases, when $c \neq 0$ we have that $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ does not exist. Therefore, f is differentiable a solely at $c = 0$. $\square$

4. Let $f_a(x) = \begin{cases} x^a & x \geq 0 \\ 0 & x < 0 \end{cases}$

   (a) For which values of a is f continuous at zero?

   (b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?

   (c) For which values of a is f twice-differentiable?

   First a few Lemmas. Throughout, we assume some basic properties of $\ln(x)$ and e^x (such as monotonicity) without proof.

   Lemma. *For $b > 0$ and $b \neq 1$, b^x is injective.*

   Proof. Let $b > 0$ and $b \neq 1$. Now suppose

$$b^{x_1} = b^{x_2}$$

   Wlog, assume that $x_1 \geq x_2$, so we have

$$\begin{aligned} b^{x_2+k} &= b^{x_2} \\ b^{x_2}b^k &= b^{x_2} \\ b^k &= 1 \end{aligned}$$

   We claim this then implies that $k = 0$. For otherwise we would have that

$$b = \left(b^k\right)^{\frac{1}{k}} = 1^{\frac{1}{k}} = 1$$

   contradicting with our assumption that $b \neq 1$. So $k = 0$. Therefore, $x_1 = x_2$. $\square$

   Corollary. *$log_b(x)$ is injective.*

   Lemma. *$\ln(x)$ is continuous at $x = 1$.*

   Proof. Let $\epsilon > 0$ be given and choose $\delta = \min\{|e^{-\epsilon} - 1|, e^{\epsilon} - 1\}$. Then $|x - 1| < \delta$ implies that

$$\begin{array}{ccccc} e^{-\epsilon} - 1 & < & x - 1 & < & e^{\epsilon} - 1 \\ e^{-\epsilon} & < & x & < & e^{\epsilon} \\ -\epsilon & < & \ln(x) & < & \epsilon \end{array}$$

   This implies that $|\ln(x)| < \epsilon$ which, in turn, implies that

$$|\ln(x) - \ln(1)| = \left|\ln\left(\frac{x}{1}\right)\right| = |\ln(x)| < \epsilon$$

   Therefore, $\ln(x)$ is continuous at $x = 1$. $\square$

Corollary. $\ln(x)$ *is continuous on* $(0, \infty)$.

Proof. Let $\epsilon > 0$ be given and $c \in (0, \infty)$. So, we have

$$|\ln(x) - \ln(c)| = \left|\ln\left(\frac{x}{c}\right)\right|$$

and so if $z = \frac{x}{c}$ then we have

$$= |\lim(z)|$$

Now, by our previous Lemma, we know there exists $\delta > 0$ such that $|z - 1| < \delta$ implies $|\ln(z)| < \epsilon$. So then $\left|\frac{x}{c} - 1\right| < \delta$ implies that $\left|\ln\left(\frac{x}{c}\right)\right| < \epsilon$. Thus $|x - c| < c\delta$ implies

$$|\ln(x) - \ln(c)| = \left|\ln\left(\frac{x}{c}\right)\right| < \epsilon$$

Therefore, $\ln(x)$ is continuous on $(0, \infty)$. $\square$

Lemma. *If* $c \in (0, \infty)$ *then* $\lim_{x \to c} \ln(x^a) = \ln(c^a)$

Proof.

$$\begin{aligned} \lim_{x \to c} \ln(x^a) &= \lim_{x \to c} a \ln(x) \\ &= a \lim_{x \to c} \ln(x) \end{aligned}$$

and by continuity, we have

$$\begin{aligned} &= a \ln(c) \\ &= \ln(c^a) \end{aligned}$$

$\square$

Corollary. x^a *is continuous on* $(0, \infty)$.

Proof. Let $c \in (0, \infty)$. By injectivity and continuity of $\ln(x)$ and the above Lemma, we have

$$\ln(\lim_{x \to c} x^a) = \lim_{x \to c} \ln(x^a) = \ln(c^a) \iff \lim_{x \to c} x^a = c^a$$

$\square$

Lemma. $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$ *for* $x \in (0, \infty)$

Proof. Let $c \in (0, \infty)$. We simply compute

$$\begin{aligned}
\lim_{x \to c} \frac{\ln(x) - \ln(c)}{x - c} &= \lim_{x \to c} \frac{\ln\left(\frac{x}{c}\right)}{x - c} \\
&= \lim_{x \to c} \frac{\ln\left(1 + \frac{x-c}{c}\right)}{x - c} \\
&= \lim_{x \to c} \ln\left[\left(1 + \frac{x-c}{c}\right)^{\frac{1}{x-c}}\right] \\
&= \lim_{x \to c} \ln\left[\left(1 + \frac{\frac{1}{c}}{\frac{1}{x-c}}\right)^{\frac{1}{x-c}}\right]
\end{aligned}$$

and by continuity of $\ln(x)$, we have

$$= \ln\left[\lim_{x \to c}\left(1 + \frac{\frac{1}{c}}{\frac{1}{x-c}}\right)^{\frac{1}{x-c}}\right]$$

and by definition of e^x, we have

$$\begin{aligned}
&= \ln(e^{\frac{1}{c}}) \\
&= \frac{1}{c}
\end{aligned}$$

□

Corollary. $\frac{d}{dx}[x^a] = ax^{a-1}$ *for* $a \in \mathbb{R}$ *and* $x \in (0, \infty)$.

Proof. Let $a \in \mathbb{R}$ and $x \in (0, \infty)$. Then by the above Lemma, the Chain Rule, and linearity of the derivative, we have the following

$$\frac{1}{x^a}\frac{d}{dx}[x^a] = \frac{d}{dx}[\ln(x^a)] = \frac{d}{dx}[a\ln(x)] = a\frac{d}{dx}[\ln(x)] = a\frac{1}{x}$$

Thus,

$$\frac{1}{x^a}\frac{d}{dx}[x^a] = a\frac{1}{x}$$

which yields

$$\frac{d}{dx}[x^a] = ax^{a-1}$$

as desired. □

Now we continue with our proof

Proof. (a) We claim f_a is continuous at zero iff $a > 0$. To begin, we observe that $f_a(0) = 0^a$ is defined only when $a > 0$. Thus if f_a is continuous at zero then $a > 0$. Now if $a > 0$ and $\epsilon > 0$ is given, $|x| < \sqrt[a]{\epsilon}$ implies

$$|f_a(x) - f_a(0)| = |0 - 0| = 0 < \epsilon$$

whenever $-\sqrt[a]{\epsilon} < x < 0$ and

$$|f_a(x) - f_a(0)| = |x^a - 0| = |x^a| = |x|^a < \sqrt[a]{\epsilon}^a = \epsilon$$

whenever $0 < x < \sqrt[a]{\epsilon}$. Therefore if $a > 0$ then f_a is continuous at zero.

(b) Notice that f_a is differentiable at zero iff

$$\lim_{x\to 0^-} f_a(x) = 0 = \lim_{x\to 0^+} f_a(x)$$

We claim that $0 = \lim_{x\to 0^+} f_a(x)$ iff $a > 1$. So if $0 = \lim_{x\to 0^+} f_a(x)$, then for every $\epsilon > 0$ there exists $\delta > 0$ such that $0 < x < \delta$ implies

$$\left|\frac{f_a(x) - f_a(0)}{x}\right| = \left|\frac{x^a}{x}\right| = |x^{a-1}| = x^{a-1} < \epsilon$$

However, if $a = 1$ then $x^{a-1} = 1$ for all x such that $0 < x < \delta$. So for $\epsilon = \frac{1}{2}$ we would then have that $1 < \frac{1}{2}$ which is clearly absurd. If $a < 1$ then $0 < 1 - a$ and we can therefore choose x so that

$$0 < x^{1-a} < \min\{\delta, 1/\epsilon\}$$

yielding $x^{a-1} = \frac{1}{x^{1-a}} > \frac{1}{1/\epsilon} = \epsilon$ which contradicts with our assumption that $0 < z < \delta$ implies $z^{a-1} < \epsilon$. Therefore, $a > 1$.

Now, if $a > 1$ then $0 < x < \epsilon^{\frac{1}{a-1}}$ implies

$$\left|\frac{f_a(x) - f_a(0)}{x - 0}\right| = x^{a-1} < (\epsilon^{\frac{1}{a-1}})^{a-1} = \epsilon$$

and, therefore, $0 = \lim_{x\to 0^+} f_a(x)$. Therefore f_a is differentiable at zero iff $a > 1$.

Now we wish to show that, in this case, indeed, f_a' is continuous. It is obvious from the definition of f_a and our previous argument that f_a' is continuous on $(\infty, 0]$. By our Corollary we know that, for $x \in (0, \infty)$ $\frac{d}{dx}[x^a] = ax^{a-1}$. By our Lemma we know that, for $a \in \mathbb{R}$, x^a is continuous on $(0, \infty)$. In particular, for x^{a-1}. Thus, f_a' is continuous on $(0, \infty)$. Therefore, f_a' is continuous when $a > 1$.

(c) Notice, from previous considerations, we get that f_a is infinitely differentiable on $(-\infty, 0) \cup (0, \infty)$. However, for f_a to be twice-differentiable on $\mathbb{R}$ we must have

$$\lim_{x \to 0^-} \frac{f_a'(x) - f_a'(0)}{x - 0} = 0 = \lim_{x \to 0^+} \frac{f_a'(x) - f_a'(0)}{x - 0}$$

But observe that

$$\begin{aligned} \lim_{x \to 0^+} \frac{f_a'(x) - f_a'(0)}{x - 0} &= \lim_{x \to 0^+} \frac{f_a' - 0}{x - 0} \\ &= \lim_{x \to 0^+} \frac{ax^{a-1}}{x} \\ &= \lim_{x \to 0^+} ax^{a-2} \end{aligned}$$

and $\lim_{x \to 0^+} ax^{a-2} = 0$ if and only if $a > 2$.

□

5. Let

$$g_a(x) = \begin{cases} x^a \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Find a particular (potentially noninteger) value for a so that

(a) g_a is differentiable on $\mathbb{R}$ but such that g_a' is unbounded on $[0, 1]$.

(b) g_a is differentiable on $\mathbb{R}$ with g_a' continuous but not differentiable at zero.

(c) g_a is differentiable on $\mathbb{R}$ and g_a' is differentiable on $\mathbb{R}$, but such that g_a'' is not continuous at zero.

Proof. (a)

□

6. (a) Assume that g is differentiable on $[a, b]$ and satisfies $g'(a) < 0 < g'(b)$. Show that there exists a point $x \in (a, b)$ where $g(a) > g(x)$, and a point $y \in (a, b)$ where $g(y) < g(b)$

(b) Now complete the proof of Darboux's Theorem

Lemma. *Let f be differentiable on $[a, b]$. If $c \in [a, b]$ is a minimum for f on $[a, b]$, then c is either an endpoint of $[a, b]$ or $f'(c) = 0$*

Proof. Let the assumptions on f and c hold, and suppose that c is not an endpoint of $[a, b]$. By differentiability of f at c, we have

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

By our assumptions on c, we have $f(x) - f(c) \geq 0$ for all $x \in [a, b]$. Thus we have

$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \leq 0 \leq \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

Together, with the above equality, we get then that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = 0$$

□

We now proceed with the exercise.

Proof. (a) We have

$$\lim_{x \to a^+} \frac{g(x) - g(a)}{x - a} < 0$$

and

$$\lim_{y \to b^-} \frac{g(y) - g(b)}{y - b} > 0$$

Since $x > a$ and $y < b$, then the respective inequalities, along with the Sequential Criterion for Functional Limits and the Order Limit Theorem, imply that there exist $x, y \in (a, b)$ such that

$$g(x) < g(a)$$
$$g(y) < g(b)$$

Corollary. *The assumptions in (a) imply that the minimum of g on $[a, b]$ is not at the end points.*

Proof. From (a), we have

$$\min\{g(x), g(y)\} < g(a), g(b)$$

□

(b) To demonstrate Darboux's Theorem, we let f be differentiable on $[a, b]$ and let α be such that $f'(a) < \alpha < f'(b)$ or $f'(b) < \alpha < f'(a)$. If we define $g(x) = f(x) - \alpha x$ we get that $g'(x) = f'(x) - \alpha$. By differentiability of g on $[a, b]$, it follows that g is continuous on $[a, b]$ and so, by compactness, g achieves a minimum, c, on $[a, b]$. Now, from the above inequalities, we get that

$$g'(a) < 0 < g'(b)$$

or

$$g'(b) < 0 < g'(a)$$

and therefore, by (a), that c is not an endpoint of $[a, b]$. Thus $c \in (a, b)$. Furthermore, from the Lemma, we get that we must then have that $g'(c) = 0$. Thus $f'(c) - \alpha = 0$ which is equivalent to $f'(c) = \alpha$. Therefore, there exists $c \in (a, b)$ such that $f'(c) = \alpha$.

$\square$

5.3

1. Recall from Exercise 4.4.9 that a function $f : A \to \mathbb{R}$ is Lipschitz on A if there exists $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x, y \in A$. Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

Proof. Since f' is continuous on $[a, b]$ and $[a, b]$ is compact, it follows that f' realizes a maximum and minimum on $[a, b]$. That is, there exists $M > 0$ such that for all $x \in [a, b]$

$$|f'(x)| \leq M$$

Given $x, y \in [a, b]$, by the MVT, we have that there exists $c \in (x, y) \subset [a, b]$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(c)| \leq M$$

Thus, f is Lipschitz on $[a, b]$. $\square$

2. Recall from Exercise 4.3.9 that a function f is contractive on a set A if there exists a constant $0 < s < 1$ such that

$$|f(x) - f(y)| < s|x - y|$$

for all $x, y \in A$. Show that if f is differentiable and f' is continuous and satisfies $|f'(x)| < 1$ on a closed interval, then f is contractive on this set.

Proof. Let the assumptions on f hold. Since f' is continuous on $[a, b]$, which is compact, it follows that f' achieves a maximum and a minimum there. Since, by assumption, $|f'(x)| < 1$ for all $x \in [a, b]$, it follows there is $0 < s < 1$ such that for all $x \in [a, b]$

$$|f'(x)| \leq s$$

It is not to hard to show that, from the above proof, we have that this implies that for all $x, y \in [a, b]$

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq s$$

which in turn gives

$$|f(x) - f(y)| \leq s|x - y|$$

Therefore, f is a contraction mapping. $\square$

3. Let h be a differentiable function defined on the interval $[0, 3]$, and assume that $h(0) = 1, h(1) = 2$, and $h(3) = 2$.

   (a) Argue that there exists a point $d \in [0, 3]$ where $h(d) = d$.

   (b) Argue that at some point c we have $h'(c) = \frac{1}{3}$.

   (c) Argue that $h'(x) = \frac{1}{4}$ at some point in the domain.

Proof. (a) Since h is differentiable on $[0, 3]$ it follows that it is continuous there. So if

$$g(x) = h(x) - x$$

Then g is also continuous on $[0.3]$. We observe that

$$\begin{aligned} g(1) &= h(1) - 1 \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} g(3) &= h(3) - 3 \\ &= 2 - 3 \\ &= -1 \end{aligned}$$

So by the IVT, it follows that there exists $d \in (a, b)$ such that $g(d) = 0$. Thus there exists $d \in (0, 3)$ such that $h(d) - d = 0$. That is $h(d) = d$. Therefore, h has a fixed point in $[0, 3]$.

(b) By the MVT, we have that there exists $c \in [0, 3]$ such that

$$h'(c) = \frac{h(3) - h(0)}{3 - 0} = \frac{2 - 1}{3} = \frac{1}{3}$$

(c) By the MVT, there exists $c_1, c_2 \in (0, 3)$ such that

$$\begin{aligned} f'(c_1) &= \frac{h(1) - h(0)}{1 - 0} = 1 \\ f'(c_2) &= \frac{h(3) - h(1)}{3 - 1} = 0 \end{aligned}$$

Since $0 < \frac{1}{4} < 1$ we then have, by Darboux, that there exists $c_3 \in (0, 3)$ such that

$$f'(c_3) = \frac{1}{4}$$

$\square$

4. (a) Supply the details for the proof of Cauchy's Generalized Mean Value Theorem.

   (b) Give a graphical interpretation of the generalized Mean Value Theorem analogous to the one given for the Mean Value Theorem at the beginning of Section 5.3 (Consider f and g as parametric equations on a curve.)

Proof. (a) Following with the author, we simply apply the MVT to the function $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$. So there exists $c \in (a, b)$ such that

$$\begin{aligned} h'(c) &= \frac{h(b) - h(a)}{b - a} \\ &= \frac{[f(b) - f(a)]g(b) - [g(b) - g(a)]f(b) - ([f(b) - f(a)]g(a) - [g(b) - g(a)]f(a))}{b - a} \\ &= 0 \end{aligned}$$

On the other hand, we also have that

$$h'(c) = [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c)$$

which, together with the above, gives us

$$\begin{aligned} 0 &= [f(b) - f(a)]g'(c) - [g(b) - g(a)]f'(c) \\ [g(b) - g(a)]f'(c) &= [f(b) - f(a)]g'(c) \end{aligned}$$

as desired.

(b)

□

5. A *fixed point* of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Proof. Let f be differentiable on an interval, I, with $f'(x) \neq 1$ on I. Now, suppose to the contrary, that there exist two different fixed points of f. That is there is $x_1 < x_2$ such that $f(x_1) = x_1$ and $f(x_2) = x_2$. Then, by the MVT there exists a $c \in I$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{x_2 - x_1}{x_2 - x_1} = 1$$

which contradicts with our assumption that $f'(c) \neq 1$ on I. Therefore, there cannot be two distinct fixed points of f. □

6. Let $g : [0,1] \to \mathbb{R}$ be twice-differentiable (i.e. both g and g' are differentiable functions) with $g''(x) > 0$ for all $x \in [0,1]$. If $g(0) > 0$ and $g(1) = 1$, show that $g(d) = d$ for some point $d \in (0,1)$ if and only if $g'(1) > 1$. (This geometrically plausible fact is used in the introductory discussion to Chapter 6.)

Proof. We first demonstrate that $g'(x)$ is strictly increasing on $[0,1]$. So, suppose to the contrary, that there exists $x_1 < x_2 \in [0,1]$ such that $g'(x_1) \geq g'(x_2)$. So

$$\frac{g'(x_2) - g'(x_1)}{x_2 - x_1} \leq 0$$

But then, by the MVT, there exists $k \in (0,1)$ such that

$$g'(k) \leq 0$$

contradicting with our assumption that $g''(x) > 0$ for all $x \in [0,1]$. Thus if $x_1 < x_2$ then $g'(x_1) < g'(x_2)$. Now, supposing that there is $d \in (0,1)$ such that $g(d) = d$, then, by the MVT, there exists $c \in (d,1)$ such that

$$g'(c) = \frac{g(1) - g(d)}{1 - d} = \frac{1 - d}{1 - d} = 1$$

However, since $g'(x)$ is strictly increasing on $[0,1]$

$$c < 1 \text{ implies } g'(c) < g'(1)$$

So

$$1 < g'(1)$$

On the other hand, if $\forall\, x \in (0,1)$ we have that $g(x) \neq x$. Since g is differentiable on $(0,1)$ is also continuous there. Thus, as a consequence of the IVT, it follows that $g(x) \geq x$ for all $x \in [0,1]$. So for all $x \in [0,1]$ we have

$$\begin{aligned}
g(x) - x &\geq 0 \\
\frac{g(x) - x}{x - 1} &\leq 0 \\
\frac{g(x) - 1 + 1 - x}{x - 1} &\leq 0 \\
\frac{g(x) - 1}{x - 1} + \frac{1 - x}{x - 1} &\leq 0 \\
\frac{g(x) - 1}{x - 1} - 1 &\leq 0 \\
\frac{g(x) - 1}{x - 1} &\leq 1
\end{aligned}$$

and so, by Exercise 4.2.8,

$$\begin{aligned}\lim_{x\to 1} \frac{g(x)-1}{x-1} &\leq 1\\ g'(1) &\leq 1\end{aligned}$$

□

7. (a) Recall that a function $f : (a,b) \to \mathbb{R}$ is *increasing* on (a,b) if $f(x) \leq f(y)$ whenever $x < y$ in (a,b). Assume f is differentiable on (a,b). Show that f is increasing on (a,b) if and only if $f'(x) \geq 0$ for all $x \in (a,b)$.

   (b) Show that the function

$$g(x) = \begin{cases} \frac{x}{2} + x^2 \sin(\frac{1}{x}) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

   is differentiable on $\mathbb{R}$ and satisfies $g'(0) > 0$. Now prove that g is *not* increasing over any open interval containing 0.

Proof. (a) Let f be increasing on (a,b) and let $c \in (a,b)$. Since f is differentiable on (a,b) we know that $f'(c)$ exists. In particular, by exercise 4.2.8, we have

$$f'(c) = \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} \geq 0$$

On the other hand, if there are $x < y \in (a,b)$ such that $f(x) > f(y)$ then, by the MVT, we have that there exists $c \in (x,y)$ such that

$$f'(c) = \frac{f(y)-f(x)}{y-x} < 0$$

(b) First, we see that g is clearly differentiable on $\mathbb{R} \setminus \{0\}$. Indeed, for $x \in \mathbb{R} \setminus \{0\}$

$$g'(x) = \frac{1}{2} + 2x\sin(1/x) - \cos(1/x)$$

Now,

$$\begin{aligned}\frac{g(x)-g(0)}{x-0} &= \frac{g(x)-0}{x}\\ &= \frac{g(x)}{x}\end{aligned}$$

and for $x \neq 0$

$$= \frac{1}{2} + x\sin(1/x)$$

thus

$$\begin{aligned} g'(0) &= \lim_{x\to 0} \frac{1}{2} + x\sin(1/x) \\ &= \frac{1}{2} \\ &> 0 \end{aligned}$$

Now, let $(a,b) \subset \mathbb{R}$ be an interval be such that $0 \in (a,b)$. Then, for n large enough, we will have $\left[-\frac{1}{2\pi n}, \frac{1}{2\pi n}\right] \subset (a,b)$. So,

$$g'\left(\frac{1}{2\pi n}\right) = \frac{1}{2} + 2\frac{1}{2\pi n}\sin(2\pi n) - \cos(2\pi n) = \frac{1}{2} - 1 = -\frac{1}{2} < 0$$

Thus, by (a) , we know that g is not increasing on (a,b).

$\square$

8. Assume $g : (a,b) \to \mathbb{R}$ is differentiable at some point $c \in (a,b)$. If $g'(c) \neq 0$, show that there exists a δ-neighborhood $V_\delta(c) \subset (a,b)$ for which $g(x) \neq g(c)$ for all $x \in V_\delta(c)$.

   Proof. Suppose not. Then $\forall\, \delta > 0$ there exists $x \in V_\delta(c)$ such that $g(x) = g(c)$. So then there exists a sequence (x_n) such that, $x_n \to c$, and for all n we have $x_n \neq c$ and $g(x_n) = g(c)$. Since g is assumed to be differentiable at c, it follows then that

   $$g'(c) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} 0 = 0$$

   which contradicts with our assumption that $g'(c) \neq 0$. $\square$

9. Assume that $\lim_{x\to c} f(x) = L$, where $L \neq 0$, and assume $\lim_{x\to c} g(x) = 0$. Show that $\lim_{x\to c} \left|\frac{f(x)}{g(x)}\right| = \infty$.

   Proof. Let $M > 0$ be given. Notice, first, that $\lim_{x\to c} f(x) = L$ implies that, in some neighborhood of c, f is bounded below by a nonzero number. That is, there exists $\delta_1 >$ such that $|x - c| < \delta_1$ implies

   $$|f(x)| \geq K$$

   for some $K > 0$. Furthermore, there exists $\delta_2 > 0$ such that $|x - c| < \delta_2$ implies

   $$|g(x)| < \frac{K}{M}$$

   Notice then that $\frac{K}{M} > 0$. Thus if $\delta = \min\{\delta_1, \delta_2\}$, then $|x - c| < \delta$ implies

   $$\left|\frac{f(x)}{g(x)}\right| = \frac{|f(x)|}{|g(x)|} \geq \frac{K}{|g(x)|} > \frac{K}{\frac{K}{M}} = K\frac{M}{K} = M$$

   where the above is valid since both $K, \frac{K}{M} > 0$. Thus, $\lim_{x\to c} \left|\frac{f(x)}{g(x)}\right| = \infty$. $\square$

10. Let f be a bounded function and assume $\lim_{x\to c} g(x) = \infty$. Show that $\lim_{x\to c} \frac{f(x)}{g(x)} = 0$.

Proof. So there is $M > 0$ such that $|f(x)| \leq M$ for all x. Furthermore, given $\epsilon > 0$, there is $\delta > 0$ such that $|x - c| < \delta$ implies

$$|g(x)| \geq \frac{2M}{\epsilon}$$

Thus, for $|x - c| < \delta$ we have

$$\left|\frac{f(x)}{g(x)}\right| = \frac{|f(x)|}{|g(x)|} \leq \frac{M}{|g(x)|} \leq \frac{M}{\frac{2M}{\epsilon}} = \frac{\epsilon}{2} < \epsilon$$

Therefore, $\lim_{x\to c} \frac{f}{g} = 0$. $\square$

11. Use the Generalized Mean Value Theorem to furnish a proof of the 0/0 case of L'Hospital's rule.

Proof. Let the assumptions from Theorem 5.3.6 hold. Notice that $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$ implies that there exists an interval, I, containing a, on which $g'(x)$ is never zero. Now let $(x_n), (y_n)$ be sequences such that

$$y_n \to a^+$$
$$x_n \to a^-$$

where $\forall_n \; x_n, y_n \neq a$. We wish to show that

$$\lim_{n\to\infty} \frac{f(y_n)}{g(y_n)} = L$$

and

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = L$$

which will establish that

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^-} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)}{g(x)} = L$$

To this end, we observe that, for n large enough we have that $(x_n), (y_n) \subset I$. Thus, by the Generalized Mean Value Theorem, we get that there exist sequences $(k_n), (k_n^*) \subset I$ such that

(a) $k_n \to a^+$

(b) $k_n^* \to a^-$

(c) $\frac{f(y_n)-f(a)}{g(y_n)-g(a)} = \frac{f(y_n)}{g(y_n)} = \frac{f'(k_n)}{g'(k_n)}$

(d) $\frac{f(a)-f(x_n)}{g(a)-g(x_n)} = \frac{f(x_n)}{g(x_n)} = \frac{f'(k_n^*)}{g'(k_n^*)}$

Notice that our assumption that $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, together with (a), (b), (c), and (d) above, imply that

$$\lim_{n\to\infty} \frac{f(y_n)}{g(y_n)} = \lim_{n\to\infty} \frac{f'(k_n)}{g'(k_n)} = L$$

and

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{f'(k_n^*)}{g'(k_n^*)} = L$$

Therefore

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^-} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)}{g(x)} = L$$

□

12. Assume f and g are as described in Theorem 5.3.6, but now add the assumption that f and g are differentiable at a and f' and g' are continuous at a. Find a short proof for the 0/0 case of L'Hospital's rule under this stronger hypothesis.

Proof. Differentiability of f and g at a, together with the assumption that f' and g' are continuous at a, yield,

$$L = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

□

13. Review the hypothesis of Theorem 5.3.6. What happens if we do not assume that $f(a) = g(a) = 0$, but assume only that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$? Assuming we have a proof for Theorem 5.3.6 as it is written, explain how to construct a valid proof under this slightly weaker hypothesis.

Chapter 6

6.1

1. Let
$$f_n(x) = \frac{nx}{1+nx^2}$$

   (a) Find the pointwise limit of (f_n) for all $x \in (0,\infty)$.
   (b) Is the convergence uniform on $(0,\infty)$?
   (c) Is the convergence uniform on $(0,1)$?
   (d) Is the convergence uniform on $(1,\infty)$?

Proof. (a)
$$\lim_{n\to\infty} \frac{nx}{1+nx^2} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n}} \frac{nx}{1+nx^2} = \lim_{n\to\infty} \frac{x}{\frac{1}{n}+x^2} = \frac{1}{x}$$

(b) No, convergence is not uniform on $(0,\infty)$. Notice
$$\left|\frac{nx}{1+nx^2} - \frac{1}{x}\right| = \left|-\frac{1}{x+nx^3}\right|$$
and on $(0,\infty)$ we get
$$= \frac{1}{x+nx^3}$$
So, given $\epsilon > 0$
$$\left|\frac{nx}{1+nx^2} - \frac{1}{x}\right| < \epsilon \iff \frac{1-\epsilon x}{\epsilon x^3} < n$$
Clearly there is no n satisfying this for all $x \in (0,\infty)$

(c) Since
$$\lim_{x\to 0^+} \frac{1-\epsilon x}{\epsilon x^3} = \infty$$
then, together with the above observations, it becomes clear that convergence is still not uniform on $(0,1)$.

(d) Notice that, on $(1, \infty)$ we have

$$\frac{1-\epsilon x}{\epsilon x^3} \leq \frac{1-\epsilon x}{\epsilon} \leq \frac{1}{\epsilon}$$

and there is indeed and $N \in \mathbb{N}$ so that

$$\frac{1}{\epsilon} < N$$

Therefore, there is an N such that, for all $x \in (1, \infty)$ and for all $n \geq N$ we have

$$\left|\frac{nx}{1+nx^2} - \frac{1}{x}\right| < \epsilon$$

Therefore, convergence is uniform on $(1, \infty)$.

□

2. Let

$$g_n(x) = \frac{nx + \sin(nx)}{2n}$$

Find the pointwise limit of (g_n) on $\mathbb{R}$. Is the convergence uniform on $[-10, 10]$? Is the convergence uniform on all of $\mathbb{R}$?

Proof. To find the pointwise limit, we examine

$$\lim_{n\to\infty} \frac{nx + \sin(nx)}{2n} = \lim_{n\to\infty} \frac{x}{2} + \frac{1}{2}\frac{\sin(nx)}{n} = \frac{x}{2}$$

which is, indeed, well-defined on $\mathbb{R}$. To determine where convergence is uniform, we consider

$$\left|\frac{nx + \sin(nx)}{2n} - \frac{x}{2}\right| = \left|\frac{\sin(nx)}{2n}\right| \leq \frac{1}{2n}$$

Clearly, regardless of x, for n large enough, we will have that, for a given $\epsilon > 0$

$$\frac{1}{2n} < \epsilon$$

Thus convergence is uniform on all of $\mathbb{R}$ □

3. Consider the sequence of functions

$$h_n(x) = \frac{x}{1+x^n}$$

over the domain $[0, \infty)$.

(a) Find the pointwise limit of (h_n) on $[0, \infty)$.

(b) Explain how we know that the convergence *cannot* be uniform on $[0, \infty)$?

(c) Choose a smaller set over which the convergence is uniform and supply an argument to show that this is indeed the case.

Proof. (a) If $x \in (1, \infty)$ then

$$\lim_{n\to\infty} \frac{x}{1+x^n} = \lim_{n\to\infty} \frac{1}{\frac{1}{x} + x^{n-1}} = 0$$

when $x = 1$ we have

$$\lim_{n\to\infty} \frac{1}{1+1^n} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

If $x \in [0, 1)$ then

$$\lim_{n\to\infty} \frac{x}{1+x^n} = x$$

Thus, our pointwise limit is given by

$$h(x) = \begin{cases} x & 0 \le x < 1 \\ \frac{1}{2} & x = 1 \\ 0 & x > 1 \end{cases}$$

(b) Observe that $h(x)$ is not continuous at $x = 1$ while $h_n(x)$ is. Therefore, by negation of the consequent and Theorem 6.2.6, we get h_n does not converge to h uniformly on $[0, \infty)$.

(c) Notice when we take our set to simply be $\{1\}$ pointwise convergence of h_n implies uniform convergence of h_n.

□

4. For each $n \in \mathbb{N}$, find the points on $\mathbb{R}$ where the function $f_n(x) = \frac{x}{1+nx^2}$ attains its maximum and minimum values. Use this to prove (f_n) converges uniformly on $\mathbb{R}$. What is the limit function?

Proof. To find the extrema of f_n we utilize the Interior Extremum Theorem. We first observe that

$$f_n'(x) = \frac{1 + nx^2 - 2nx^2}{(1+nx)^2}$$

and consider when

$$\begin{aligned} 0 &= \frac{1 + nx^2 - 2nx^2}{(1+nx)^2} \\ 0 &= 1 - nx^2 \\ \pm\frac{1}{\sqrt{n}} &= x \end{aligned}$$

Thus, $\pm\frac{1}{\sqrt{n}} = x$ are the points on $\mathbb{R}$ where f_n attains its maximum and minimum values. Notice the maximum and the minimum are actually attained since $\left[-\frac{1}{\sqrt{n}} - 1, \frac{1}{\sqrt{n}} + 1\right]$ is compact and f_n is continuous and differentiable there. Plugging in the critical values back into f_n yields

$$|f_n(x)| \leq \frac{1}{2\sqrt{n}}$$

This then implies that $\lim_{n\to\infty} \sup\{|f_n(x)| : x \in \mathbb{R}\} = 0$. Thus, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f_m(x)| < \epsilon$$

whenever $n, m \geq N$. Thus, by the Cauchy Criterion for Uniform Convergence, it follows that (f_n) converges uniformly on $\mathbb{R}$. In particular,

$$\lim_{n\to\infty} f_n(x) = 0$$

Thus, the limit function is given by $f(x) = 0$. □

5. For each $n \in \mathbb{N}$, define f_n on $\mathbb{R}$ by

$$f_n = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{n} \\ n|x| & \text{if } |x| < \frac{1}{n} \end{cases}$$

(a) Find the pointwise limit of (f_n) on $\mathbb{R}$ and decide whether or not the convergence is uniform.

(b) Construct an example of a pointwise limit of a continuous function that converges everywhere on the compact set $[-5, 5]$ to a limit function that is unbounded on this set.

Proof. (a) At $x = 0$ we can see that

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} n|0| = 0$$

Now, if $x \neq 0$ then, given $\epsilon > 0$, we can choose N such that

$$\frac{1}{N} \leq |x|$$

so that $\forall\, n \geq N$ we have

$$|f_n(x) - 1| = |1 - 1| = 0 < \epsilon$$

Thus, for $x \neq 0$ we have that

$$\lim_{n\to\infty} f_n(x) = 1$$

Thus, $f_n \to f$ where

$$f = \begin{cases} 0 & x = 0 \\ 1 & \text{elsewhere} \end{cases}$$

Clearly then, convergence of (f_n) is not uniform since f is not continuous.

(b)

$$f = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{n} \\ n(n+1)|x| - n & \text{if } |x| < \frac{1}{n} \end{cases}$$

□

6. Using the Cauchy Criterion for convergent sequences of real numbers, supply a proof of Theorem 6.2.5.

Proof. $\Rightarrow$: Let $f_n \to f$ uniformly. This implies that for every $x \in A$, we have that $f_n(x)$ is a convergent sequence and therefore Cauchy. Moreover, by uniformity of convergence, we get that our choice of $n \in \mathbb{N}$ depends only on ϵ. Therefore, given ϵ there exists $N \in \mathbb{N}$ such that whenever $n, m \geq N$ we have

$$|f_n(x) - f_m(x)| < \epsilon$$

for every $x \in A$.

$\Leftarrow$: Notice then for each $x \in A$, $(f_n(x))$ forms a Cauchy sequence of real numbers and therefore converges to some real number. Let $f(x)$ be defined so that for every $x \in A$, $f_n(x) \to f(x)$. Now,e wish to demonstrate that $f_n \to f$ uniformly. So, let $\epsilon > 0$ be given. We can choose $N \in \mathbb{N}$ so that for $n, m \geq N$ we have

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$$

and

$$|f_m(x) - f(x)| < \frac{\epsilon}{2}$$

for all $x \in A$. So

$$|f_n(x) - f(x)| < |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $x \in A$. Therefore, $f_n \to f$ uniformly on A. □

7. Assume that (f_n) converges uniformly to f on A and that each f_n is uniformly continuous on A. Prove that f is uniformly continuous on A.

Proof. We can select $N \in \mathbb{N}$ so that $n \geq N$ implies

$$|f_n(z) - f(z)| < \frac{\epsilon}{3}$$

for all $z \in A$. Furthermore, there is also $\delta > 0$ so that $|x - y| < \delta$ implies, for a given $n \geq N$, that

$$|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$$

Therefore, for the same $n \geq N$ and for $|x - y| < \delta$ we have

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f(y)| \\
&\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

□

8. Decide which of the following conjectures are true and which are false. Supply a proof for those that are valid and a counterexample for each one that is not.

   (a) If $f_n \to f$ pointwise on a compact set K, then $f_n \to f$ uniformly on K.

   (b) If $f_n \to f$ uniformly on A and g is a bounded function on A, then $f_n g \to fg$ uniformly on A.

   (c) If $f_n \to f$ uniformly on A, and if each f_n is bounded on A, then f must also be bounded.

   (d) If $f_n \to f$ uniformly on a set A, and if $f_n \to f$ uniformly on a set B, then $f_n \to f$ on $A \cup B$.

   (e) If $f_n \to f$ uniformly on an interval, and if each f_n is increasing, then f is also increasing.

   (f) Repeat conjecture above, assuming only pointwise convergence.

   Proof. (a) This is not generally true. The example given on page 153 can be used as a counterexample. That is $g_n(x) = x^n$ converges pointwise to

   $$g(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

   but not uniformly on $[0, 1]$.

(b) This is valid. By boundedness of g on A, we have that $|g(x)| \leq M$ for all $x \in A$. Thus there exists $N \in \mathbb{N}$ such that

$$|f_n(x)g(x)-f(x)g(x)| \leq |f_n(x)-f(x)||g(x)| \leq |f_n(x)-f(x)|M < \frac{\epsilon}{M}M = \epsilon$$

for all $x \in A$.

(c) This is valid, for, suppose to the contrary, there is $f_n \to f$ uniformly with each f_n bounded and f unbounded on A. So, given $\epsilon > 0$, there exists N such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n \geq N$ and $x \in A$. This is equivalent to

$$f(x) - \epsilon < f_n(x) < f(x) + \epsilon$$

for all $n \geq N$ and $x \in A$. Fixing $n \geq N$, we have by boundedness of f_n, there is M such that $|f_n(x)| \leq M$ for all $x \in A$. This is equivalent to saying

$$-M < f_n(x) < M$$

for all $x \in A$. On the other hand, by unboundedness of f, there is x such that $|f(x)| > M + \epsilon$. So, either

$$M + \epsilon < f(x)$$

or

$$M + \epsilon < -f(x)$$

So, either

$$M < f(x) - \epsilon$$

or

$$-M > f(x) + \epsilon$$

Therefore, either

$$M < f(x) - \epsilon < f_n(x)$$

or

$$-M > f(x) + \epsilon > f_n(x)$$

both of which contradict with $-M < f_n(x) < M$. Therefore, f must be bounded on A.

(d) This is valid. Letting the assumptions hold and letting $\epsilon > 0$ be given, we get that there is N_1 such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n \geq N_1$ and for all $x \in A$. Furthermore, there is N_2 such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $n \geq N_2$ and for all $x \in B$. Thus for $N = max\{N_1, N_2\}$, we have that

$$|f_n(x) - f(x)|$$

for all $n \geq N$ and for all $x \in A \cup B$. Therefore, $f_n \to f$ uniformly on $A \cup B$.

(e) This is valid. To demonstrate, suppose to the contrary, that $f_n \to f$ uniformly, where f_n is increasing, for every n, and f is not. So there exist $x_1 < x_2$ such that $f(x_1) > f(x_2)$ while, for all n, we have $f_n(x_1) \leq f_n(x_2)$. By uniformity, we have that there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$|f_n(x) - f(x)| < \frac{f(x_1) - f(x_2)}{2}$$

for all x in the interval. In particular, this is true for x_1. Therefore, by the above inequality, along with the fact that f_n is increasing, we will have that

$$\frac{f(x_2) + f(x_1)}{2} < f_n(x_1) \leq f_n(x_2)$$

and therefore, that

$$\frac{f(x_2) + f(x_1)}{2} < f_n(x_2)$$

So it follows that

$$\frac{f(x_1) - f(x_2)}{2} < f_n(x_2) - f(x_2)$$

which implies that

$$|f_n(x_2) - f(x_2)| > \frac{f(x_1) - f(x_2)}{2}$$

contradicting with our assumption that $f_n \to f$ uniformly.

(f) This is valid. Again, suppose to the contrary, that $f_n \to f$ pointwise with f_n increasing and f not increasing. So there exist $x_1 < x_2$ such that $f(x_1) > f(x_2)$. Then, skipping some simple details, we can easily see that there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$|f_n(x_1) - f(x_1)| < \frac{f(x_1) - f(x_2)}{2} \text{ and } |f_n(x_2) - f(x_2)| < \frac{f(x_1) - f(x_2)}{2}$$

Thus

$$\frac{3f(x_2) - f(x_1)}{2} < f_n(x_2) < \frac{f(x_1) + f(x_2)}{2}$$

and
$$\frac{f(x_1)+f(x_2)}{2} < f_n(x_1) < \frac{3f(x_1)-f(x_2)}{2}$$
implying that
$$f_n(x_2) < \frac{f(x_1)+f(x_2)}{2} < f_n(x_1)$$
Contradicting with our assumption that f_n was increasing.

□

9. Assume (f_n) converges uniformly to f on a compact set K, and let g be a continuous function on K satisfying $g(x) \neq 0$. Show (f_n/g) converges uniformly on K to f/g.

Proof. Let the assumptions hold. Since g is continuous on K, which is compact, it follows that $g(K)$ is bounded. Therefore, since we also have that $g(x) \neq 0$ it follows that there exist $S \geq M > 0$ such that
$$M \leq |g(x)| \leq S$$
for all $x \in K$. Now let $\epsilon > 0$ be given. By uniform convergence of f_n it follows that there exists $N \in \mathbb{N}$ such that $n \geq N$ implies
$$|f_n(x) - f(x)| < \epsilon M$$
for all $x \in K$. Therefore, for $n \geq N$ we have
$$\left|\frac{f_n(x)-f(x)}{g(x)}\right| = \frac{|f_n(x)-f(x)|}{|g(x)|} \leq \frac{|f_n(x)-f(x)|}{M} < \frac{\epsilon M}{M} = \epsilon$$
for all $x \in K$. Therefore, $\frac{f_n}{g} \to \frac{f}{g}$ uniformly on K. □

10. Let f be uniformly continuous on all of $\mathbb{R}$, and define a sequence of functions by $f_n(x) = f\left(x + \frac{1}{n}\right)$. Show that $f_n \to f$ uniformly. Give an example to show that this proposition fails if f is only assumed to be continuous and not uniformly continuous on $\mathbb{R}$.

Proof. Since f is uniformly continuous on $\mathbb{R}$ it follows that, given $\epsilon > 0$, there exists $\delta > 0$ such that
$$|f(x) - f(y)| < \epsilon$$
whenever $|x - y| < \delta$. So, for all $n \geq N$, where $\frac{1}{N} < \delta$, we have that $|x + \frac{1}{n} - x| = |\frac{1}{n}| < \delta$ and therefore that
$$|f_n(x) - f(x)| = |f\left(x + \frac{1}{n}\right) - f(x)| < \epsilon$$

Therefore, $f_n \to f$ uniformly on $\mathbb{R}$. On the other hand we see this fails to be generally true if f is simply continuous on $\mathbb{R}$. Consider $f(x) = x^2$, which is continuous on $\mathbb{R}$ but not uniformly continuous on $\mathbb{R}$. So, if $\epsilon = 1$, then, if

$$|f_n(x) - f(x)| < 1$$

then we must have

$$\left| \left(x + \frac{1}{n}\right)^2 - x^2 \right| < 1$$
$$\left| \frac{2xn + 1}{n^2} \right| < 1$$

and so

$$-1 < \frac{2xn + 1}{n^2} < 1$$
$$-n(n + 2x) < 1 < n(n - 2x)$$

implying that the choice of n used to satisfy the above will depend on x. Therefore, f_n fails to converge uniformly to f. $\square$

11. Assume (f_n) and (g_n) are uniformly convergent sequences of functions.

    (a) Show that $(f_n + g_n)$ is a uniformly convergent sequence of functions.

    (b) Give an example to show that the product $(f_n g_n)$ may not converge uniformly.

    (c) Prove that if there exists an $M > 0$ such that $|f_n| \leq M$ and $|g_n| \leq M$ for all $n \in \mathbb{N}$, then $(f_n g_n)$ does converge uniformly.

Proof. (a) We demonstrate that $(f_n + g_n)$ is Cauchy. By the assumption that (f_n) and (g_n) are uniformly convergent, it follows that they are Cauchy. Therefore, given $\epsilon > 0$, there exists N such that for all $n \geq m \geq N$ we have

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$$

and

$$|g_n(x) - g_m(x)| < \frac{\epsilon}{2}$$

for all x. Thus $n \geq m \geq N$ implies that

$$|f_n(x)+g_n(x)-f_m(x)-g_m(x)| \leq |f_n(x)-f_m(x)|+|g_n(x)-g_m(x)| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$$

for all x. Therefore $(f_n + g_n)$ is Cauchy. Therefore, it is uniformly convergent.

(b) Let

$$f_n(x) = x + \frac{1}{n}$$
$$g_n(x) = x + 1 + \frac{1}{n}$$

It is clear then that if $f(x) = x$ and $g(x) = x + 1$ then

$$f_n(x) = f\left(x + \frac{1}{n}\right)$$
$$g_n(x) = g\left(x + \frac{1}{n}\right)$$

Since it is clear that f and g are uniformly continuous on $\mathbb{R}$, then it follows from the previous exercise that $f_n \to f$ and $g_n \to g$ uniformly on $\mathbb{R}$. Now, if $\epsilon = 1$ then

$$|(f_n g_n)(x) - (fg)(x)| < 1$$

for every $n \geq N$ for some N and for every $x \in \mathbb{R}$ will imply that

$$\left|\frac{2xn + n + 1}{n^2}\right| < 1$$

which, in turn, will imply that

$$(-n)(n + 2x + 1) < 1 < n(n - 2x - 1)$$

implying that N will depend on the value of x. Therefore, $f_n g_n$ will not converge to fg uniformly on $\mathbb{R}$.

(c) So we suppose that f_n and g_n are uniformly convergent sequences such that, there exists $M > 0$ such that $|f_n| < M$ and $|g_n| < M$ for all $n \in \mathbb{N}$. So there exists $N \in \mathbb{N}$ such that for all $n \geq m \geq N$ we will

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2M}$$

and

$$|g_n(x) - g_m(x)| < \frac{\epsilon}{2M}$$

for all x. Thus for all $n \geq m \geq N$ we will have

$$\begin{aligned}
|(f_n g_n)(x) - (f_m g_m)(x)| &= |(f_n g_n)(x) - (f_m g_n)(x) + (f_m g_n)(x) - (f_m g_m)(x)| \\
&\leq |(f_n g_n)(x) - (f_m g_n)(x)| + |(f_m g_n)(x) - (f_m g_m)(x)| \\
&= |f_n(x) - f_m(x)||g_n(x)| + |f_m(x)||g_n(x) - g_m(x)| \\
&< \frac{\epsilon}{2M} M + M \frac{\epsilon}{2M} \\
&= \epsilon
\end{aligned}$$

$\square$

6.3

1. (a) Let
$$h_n(x) = \frac{\sin(nx)}{n}$$
Show that $h_n \to 0$ uniformly on $\mathbb{R}$. At what points does the sequence of derivatives h'_n converge?

   (b) Modify this example to show that it is possible for a sequence (f_n) to converge uniformly but for (f'_n) to be unbounded.

Proof. (a) Let $\epsilon > 0$ be given. We observe then that if $\frac{1}{\epsilon} < N$ then $n \geq N$ implies that
$$-\epsilon < -\frac{1}{n} \leq \frac{\sin(nx)}{n} \leq \frac{1}{n} < \epsilon$$
and therefore, that
$$\left|\frac{\sin(nx)}{n}\right| < \epsilon$$
for all x.

(b) It is quite easy to see that if $g_n = \frac{\sin(n^2 x)}{n}$, then $g_n \to 0$ uniformly. On the other hand
$$g'_n = n\cos(n^2 x)$$
and therefore (g'_n) is unbounded.

$\square$

2. Consider the sequence of functions defined by
$$g_n(x) = \frac{x^n}{n}$$

   (a) Show (g_n) converges uniformly on $[0,1]$ and find $g = \lim g_n$. Show that g is differentiable and compute $g'(x)$ for all $x \in [0,1]$.

   (b) Now, show that (g'_n) converges on $[0,1]$. Is the convergence uniform? Set $h = \lim g'_n$ and compare h and g'. Are they the same?

Proof. (a) We claim $g_n \to 0$ uniformly on $[0,1]$. To demonstrate, let $\epsilon > 0$ be given. If $N > \frac{1}{\epsilon}$, then for all $n \geq N$ and for all $x \in [0,1]$, we have
$$\left|\frac{x^n}{n}\right| = \frac{x^n}{n} \leq \frac{1}{n} < \epsilon$$
therefore, $g_n \to 0$ on $[0,1]$. Clearly $g(x) = 0$ is differentiable and $g'(x) = 0$ for all $x \in [0,1]$.

(b) First, it is easy to see that

$$g_n'(x) = x^n - 1$$

So, for $x \in [0, 1)$, we have

$$\lim_{n\to\infty} x^{n-1} = 0$$

and for $x = 1$, we have

$$\lim_{n\to\infty} x^{n-1} = 1$$

Thus, if

$$h(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$$

the $g_n' \to h$ on $[0, 1]$. We see that (g_n') does not converge uniformly on $[0, 1]$ since each (g_n') is continuous at 1 while h is not. Furthermore, we see that $h \neq g'$ since $h(1) = 1$ and $g'(1) = 0$.

□

3. Consider the sequence of functions

$$f_n(x) = \frac{x}{1 + nx^2}$$

It can be shown from a previous exercise (6.2.4) that (f_n) converges uniformly on $\mathbb{R}$. Now, let $f = \lim f_n$. Compute $f_n'(x)$ and find all the values of x for which $f'(x) = \lim f_n'(x)$.

Proof. We know from Exercise 6.2.4 that $f_n \to 0$ uniformly on $\mathbb{R}$. Thus $f'(x) = 0$. On the other hand

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2}$$

and thus, it is a simple matter to show that, for $x \neq 0$ we get that $\lim f_n'(x) = 0$. On the other hand, if $x = 0$, then $\lim f_n'(0) = 1$. Therefore, $\lim f_n'(x) = f'(x)$ for all $x \neq 0$. □

4. Let

$$g_n(x) = \frac{nx + x^2}{2n}$$

and set $g(x) = \lim g_n(x)$. Show that g is differentiable in two ways:

(a) Compute $g(x)$ by algebraically taking the limit as $n \to \infty$ and then find $g'(x)$.

(b) Compute $g_n'(x)$ for each $n \in \mathbb{N}$ and show that the sequence of derivatives (g_n') converges uniformly on every interval $[-M, M]$. Use Theorem 6.3.3 to conclude $g'(x) = \lim g_n'(x)$.

Proof. (a)

$$\lim g_n(x) = \lim \frac{nx + x^2}{2n} = \lim \frac{x + \frac{x^2}{n}}{2} = \frac{x}{2} = g(x)$$

Therefore,

$$g'(x) = \frac{1}{2}$$

(b) We first compute

$$g_n'(x) = \frac{1}{2} + \frac{x}{n}$$

We claim that $g_n' \to \frac{1}{2}$ uniformly on $[-M, M]$ for all $M > 0$. To this end, we let $\epsilon > 0$ be given. If $\frac{M}{\epsilon} < N$ then we will have, for all $n \geq N$ and all $x \in [-M, M]$

$$\left| \frac{1}{2} + \frac{x}{2} - \frac{1}{2} \right| = \frac{|x|}{n} < \frac{M}{n} < \frac{M}{\frac{M}{\epsilon}} = \epsilon$$

Therefore $g_n' \to \frac{1}{2}$ uniformly on $[-M, M]$ for all $M > 0$. Notice then that, for $x = 0$ we have that $g_n(0) \to 0$. Since $0 \in [-M, M]$ for all $M > 0$, it follows that $g_n \to g$ uniformly on $[-M, M]$ for all $M > 0$. That is, $g_n \to g$ uniformly on $\mathbb{R}$.

□

5. Prove Theorem 6.3.2

Proof. Let $\epsilon > 0$ be given. So we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0))| + |f_m(x_0) - f_n(x_0)|$$

So, applying the Mean Value Theorem to $f_n - f_m$ yields

$$|f_n(x) - f_m(x)| \leq |f_n'(c) - f_m'(c)||x - x_0| + |f_m(x_0) - f_n(x_0)| \leq |f_n'(c) - f_m'(c)|(b-a) + |f_m(x_0) - f_n(x_0)|$$

for some c between x and x_0. Thus by convergence of $(f_n(x_0))$ and by uniform convergence of (f_n'), we get that there exists $N \in \mathbb{N}$ such that, for all $n \geq N$ and for all $x \in [a, b]$ we have

$$|f_n(x) - f_m(x)| \leq |f_n'(c) - f_m'(c)|(b-a) + |f_m(x_0) - f_n(x_0)| < \frac{\epsilon}{2(b-a)}(b-a) + \frac{\epsilon}{2} = \epsilon$$

Therefore, by the Cauchy Criterion, we have that (f_n) is uniformly convergent on $[a, b]$. □

6.4

1. Prove that if $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to zero.

   Proof. So, we have that (s_n) is Cauchy. That is, given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that, for all $n \geq m \geq N$ we have

   $$|s_n(x) - s_m(x)| < \epsilon$$

   for all x. In particular for $m = n - 1$. So

   $$|s_n(x) - s_{n-1}(x)| = |g_n(x)| < \epsilon$$

   for all x. Thus $g_n \to 0$ uniformly. □

2. Supply the details for the proof of the Weierstrass M-Test (Corollary 6.4.5)

   Proof. Since $\sum_1^{\infty} M_n$ converges, the sequence of partial sums, (t_n) must converge. So, let $\epsilon > 0$ be given and choose N so that for all $n \geq m \geq N$ we have

   $$|t_n - t_m| = |M_{m+1} + M_{m+2} + \ldots + M_n| = M_{m+1} + \ldots + M_n < \epsilon$$

   Then, if (s_n) is the sequence of the partial sums of (f_n), then, for all $x \in A$ and for all $n \geq m \geq N$, we will have

   $$|s_n(x) - s_m(x)| = |f_{m+1}(x) + \ldots + f_n(x)| \leq |f_{m+1}(x)| + \ldots + |f_n(x)| \leq M_{m+1} + \ldots + M_n < \epsilon$$

   Therefore, $\sum f_n$ converges uniformly on A. □

3. (a) Show that $g(x) = \sum_{n=1}^{\infty} \frac{\cos(2^n x)}{2^n}$ is continuous on all of $\mathbb{R}$.

   (b) Prove that $h(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is continuous on $[-1, 1]$.

   Proof. (a) We observe that

   $$|f_n(x)| = \left|\frac{\cos(2^n x)}{2^n}\right| \leq \frac{1}{2^n} = M_n$$

   for all $x \in \mathbb{R}$. Moreover, $\sum M_n = \sum (1/2)^n$ is convergent. Therefore, by the M-Test, we have that $\sum \frac{\cos(2^n x)}{2^n}$ is uniformly convergent on $\mathbb{R}$. Since f_n is continuous on $\mathbb{R}$ for every n, it follows, by Theorem 6.4.2, that $\sum f_n = \sum \frac{\cos(2^n x)}{2^n}$ is continuous on $\mathbb{R}$

(b) we observe that, on $[-1,1]$, we get that

$$\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2} = M_n$$

Furthermore, $\sum \frac{1}{n^2}$ is convergent, and therefore, by the M-Test, we get that $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ is uniformly convergent on $[-1,1]$. Furthermore, since

□

4. In Section 5.4, we postponed the argument that the nowhere differentiable function

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$$

is continuous on $\mathbb{R}$. Use the Weierstrass M-Test to supply the missing proof.

Proof. Observe that

$$\left|\frac{1}{2^n} h(2^n x)\right| \leq \frac{1}{2^n} = M_n$$

for all $x \in \mathbb{R}$. Thus, for similar reasons as in the previous exercise we will get that $\sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$ converges uniformly on $\mathbb{R}$. Furthermore, by the Algebraic Continuity Theorem, we get that $\frac{1}{2^n} h(2^n x)$ is continuous on $\mathbb{R}$ for all n. Therefore, again, by Theorem 6.4.2, we get that $g(x)$ is continuous on $\mathbb{R}$. □

5. Let

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}$$

(a) Show that $f(x)$ is differentiable and that the derivative $f'(x)$ is continuous.

(b) Can we determine if f is twice differentiable?

Proof. (a) Let

$$f_n(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}$$

Then, we see that if $[a,b]$ is an interval containing zero, then there is a point in a, b for which $\sum f_n$ is convergent. In particular if $x = 0$ we have

$$\sum_{k=1}^{\infty} f_n(0) \to 0$$

Furthermore,

$$f_n'(x) = \frac{\cos(kx)}{k^2}$$

Now, since

$$|f_n'(x)| \leq \frac{1}{k^2} = M_k$$

it follows by the convergence of $\sum \frac{1}{k^2}$ and by the M-Test that $\sum f_n'(x)$ is uniformly convergent on $\mathbb{R}$. Then, by Theorem 6.4.3, it follows $\sum \frac{\sin(kx)}{k^3} \to f(x)$ uniformly on $\mathbb{R}$ and that $f(x)$ is differentiable on $\mathbb{R}$. Furthermore,

$$f'(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$

By uniform convergence of $\sum \frac{cos(kx)}{k^2}$ on $\mathbb{R}$ and continuity of $\frac{\cos(kx)}{k^2}$, it follows by Theorem 6.4.2 that $f'(x)$ is continuous on $\mathbb{R}$.

(b) Not by the methods outlined in this section. We see that

$$|f_n''(x)| \leq \frac{1}{k} = M_k$$

but $\sum M_k$ does not converge. Thus, we cannot use the M-Test, as in the above, to establish uniform convergence of $\sum f_n''(x)$.

□

6. Observe that the series

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots$$

converges for every x in the half-open interval $[0, 1)$ but does not converge when $x = 1$. For a fixed $x_0 \in (0, 1)$, explain how we can still use the Weierstrass M-Test to prove that f is continuous at x_0.

Proof. Let $f_n = \frac{x^n}{n}$. So $f_n'(x) = x^{n-1}$. Then, clearly, on $(0, 1)$, the M-Test establishes that $\sum f_n'(x)$ is uniformly convergent. By convergence of $f(x_0)$, it follows that on some interval $[a, b] \subset (0, 1)$ such that $x_0 \in [a, b]$, we get that f is differentiable. But then f must be continuous on $[a, b]$. Therefore f is continuous at x_0. □

7. Let

$$h(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2}$$

(a) Show that h is a continuous function defined on all of $\mathbb{R}$.

(b) Is h differentiable? If so, is the derivative function h' continuous?

8. Let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the set of rational numbers. For each $r_n \in \mathbb{Q}$, define

$$u_n(x) = \begin{cases} \frac{1}{2^n} & \text{for } x > r_n \\ 0 & \text{and } x \leq r_n \end{cases}$$

Now, let $h(x) = \sum_{n=1}^{\infty} u_n(x)$. Prove that h is a monotone function defined on all of $\mathbb{R}$ that is continuous at every irrational point.

Chapter 7

7.2

1. Let f be a bounded function on $[a, b]$,and let P be an arbitrary partition on $[a, b]$. First explain why $U(f) \geq L(f, P)$. Now prove Lemma 7.2.6.

   Proof. Suppose, to the contrary, that there exists P such that $L(f, P) > U(f)$. But then there would be a partition Q such that
   $$U(f) \leq U(f, Q) < L(f, P)$$
   which contradicts with
   $$L(f, P) \leq U(f, Q)$$
   To prove Lemma 7.2.6 is now a simple matter, since we observe then, by the above, that $U(f)$ is an upper bound for $\{L(f, P) : P \in \mathcal{P}\}$. Thus $L(f) \leq U(f)$. $\square$

2.

3. Show directly (without appealing to Theorem 7.2) that the constant function $f(x) = k$ is integrable over any closed interval $[a, b]$. What is $\int_a^b f$?

   Proof. Let $[a, b] \subset \mathbb{R}$ and $f(x) = k$. We claim $U(f) = k(b - a) = L(f)$. We first notice that for all partitions $P \in \mathcal{P}$, we have $M_i = k = m_i$ for all $1 \leq i \leq n_P$. So, given a partition $P \in \mathcal{P}$, we have
   $$\begin{aligned} U(f, P) &= \sum_{i=1}^{n} M_i \Delta x_i \\ &= \sum_{i=1}^{n} k \Delta x_i \\ &= \sum_{i=1}^{n} m_i \Delta x_i \\ &= L(f, P) \end{aligned}$$

Thus, for every partition $P \in \mathcal{P}$ we have that $U(f, P) = k(b-a) = L(f, P)$. Therefore, $U(f) = k(b-a) = L(f)$. Therefore $f(x) = k$ is integrable over every interval $[a, b]$ and $\int_a^b f = k(b-a)$. $\square$

4. (a) Prove that a bounded function f is integrable on $[a, b]$ if and only if there exists a sequence of partitions $(P_n)_{n=1}^{\infty}$ satisfying

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0$$

(b) For each n, let P_n be a partition of $[0, 1]$ into n equal subintervals. Find formulas for $U(f, P_n)$ and $L(f, P_n)$ if $f(x) = x$.

(c) Use the sequential criterion for integrability from (a) to show directly that $f(x) = x$ is integrable on $[0, 1]$.

Proof. (a) Let f be bounded and integrable on $[a, b]$. So, for every n we have that there exists P_n such that

$$U(f, P_n) - L(f, P_n) < \frac{1}{n}$$

Thus

$$0 \leq \lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] \leq \lim_{n\to\infty} \frac{1}{n} = 0$$

implying, by the Squeeze Theorem, that

$$\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0$$

Now, let there exist a sequence (P_n) such that $\lim[U(f, P_n) - L(f, P_n)] = 0$ and let $\epsilon > 0$ be given. Then there exists an $n \in \mathbb{N}$ such that $U(f, P_n) - L(f, P_n) < \epsilon$. Thus, by Theorem 7.2.8, it follows that f is integrable on $[a, b]$.

(b)

$$\begin{aligned} U(f, P_n) &= \sum_{k=1}^{n} M_k \Delta x_k \\ &= \sum_{k=1}^{n} \frac{k}{n}\left(\frac{1}{n}\right) \\ &= \frac{1}{n^2}\left(\sum_{k=1}^{n} k\right) \\ &= \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right) \\ &= \frac{n+1}{2n} \end{aligned}$$

$$
\begin{aligned}
L(f, P_n) &= \sum_{k=1}^{n} m_k \Delta x_k \\
&= \sum_{k=1}^{n} \frac{k-1}{n} \left(\frac{1}{n}\right) \\
&= \frac{1}{n^2} \sum_{k=1}^{n} k - 1 \\
&= \frac{1}{n^2} \sum_{s=0}^{n-1} s \\
&= \frac{1}{n^2} \left(\frac{(n-1)n}{2}\right) \\
&= \frac{n-1}{2n}
\end{aligned}
$$

(c) We observe that, from (b), on $[0, 1]$ we have that $f(x) = x$ admits

$$U(f, P_n) - L(f, P_n) = \frac{n+1}{2n} - \frac{n-1}{2n} = \frac{1}{n}$$

and thus

$$\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = \lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore, $f(x) = x$ is integrable on $[0, 1]$

$\square$

5. Assume that, for each n, f_n is an integrable function on $[a, b]$. If $f_n \to f$ uniformly on $[a, b]$, prove that f is also integrable on this set.

Lemma. *Let $f_n \to f$ uniformly on $[a, b]$ and let $[c, d] \subseteq [a, b]$. Then, for all $\epsilon > 0$, there exists N such that $n \geq N$ implies for the suprema on $[c, d]$*

$$|M^f - M^{f_n}| \leq \epsilon$$

and for the infima on $[c, d]$

$$|m^f - m^{f_n}| \leq \epsilon$$

Proof. So, let $\epsilon > 0$ be given, let N be such that $n \geq N$ implies

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}$$

and let $[c, d] \subseteq [a, b]$. So if M^f is the sup of f on $[c, d]$ and m^f is the inf of f on $[c, d]$, then

$$m^f - \frac{\epsilon}{2} \leq f(x) - \frac{\epsilon}{2} \leq f_n(x) \leq f(x) + \frac{\epsilon}{2} \leq M^f + \frac{\epsilon}{2}$$

So $f_n(x)$ is bounded by $m^f - \frac{\epsilon}{2}$ and $M^f + \frac{\epsilon}{2}$ on $[c, d]$. So if M^{f_n} is the sup of f_n on $[c, d]$ and m^{f_n} is the inf of f on $[c, d]$, then

$$m^f - \frac{\epsilon}{2} \leq m^{f_n} \leq M^{f_n} \leq M^f + \frac{\epsilon}{2}$$

If $M^{f_n} \geq M^f$ then this implies that

$$|M^{f_n} - M^f| \leq \frac{\epsilon}{2} < \epsilon$$

On the other hand, when $M^{f_n} < M^f$ then $|M^{f_n} - M^f| > \epsilon$ implies $M^f - M^{f_n} > \epsilon$. But then there would exist x such that

$$|M^f - f(x)| \leq \frac{\epsilon}{2}$$

and

$$|f(x) - f_n(x)| < \frac{\epsilon}{2}$$

and therefore

$$|M^f - f_n(x)| < \epsilon$$

This in turn would imply that $f_n(x) > M^{f_n}$ contradicting with M^{f_n} being the supremum. Therefore, when $M^{f_n} < M^f$ we still have

$$|M^{f_n} - M^f| < \epsilon$$

A similar argument shows that

$$|m^f - m^{f_n}| < \epsilon$$

□

Proof. First, since each f_n is integrable on $[a, b]$, it follows that each f_n is bounded on $[a, b]$. By Exercise 6.7.8 (c) and uniform convergence of f_n on $[a, b]$, it follows that f is bounded on $[a, b]$. So, let $\epsilon > 0$ be given. From our Lemma, we can choose N so that $n \geq N$ will imply

$$|M^f - M^{f_n}| < \frac{\epsilon}{3(b-a)}$$

and

$$|m^{f_n} - m^f| < \frac{\epsilon}{3(b-a)}$$

on any interval in $[a, b]$. Furthermore, for any such n, by integrability of f_n we can choose a partition P so that

$$\sum_k^s (M_k^{f_n} - m_k^{f_n})\Delta x_k < \frac{\epsilon}{3}$$

Thus

$$\begin{aligned}
\sum_k^s (M_k^f - m_k^f)\Delta x_k &\le \sum_k^s |M_k^f - m_k^f|\Delta x_k \\
&\le \sum_k^s |M_k^f - M_k^{f_n}|\Delta x_k + \sum_k^s |M_k^{f_n} - m_k^{f_n}|\Delta x_k + \sum_k^s |m_k^{f_n} - m_k^f|\Delta x_k \\
&< \frac{\epsilon}{3(b-a)} \sum_k^s \Delta x_k + \frac{\epsilon}{3} + \frac{\epsilon}{3(b-a)} \sum_k^s \Delta x_k \\
&= \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

implying that f is integrable on $[a, b]$. □

6. Let $f : [a, b] \to \mathbb{R}$ be increasing on the set $[a, b]$. Show that f is integrable on $[a, b]$.

Proof. Let $\epsilon > 0$ be given. Then for a given $n \in \mathbb{N}$ we can choose a partition so that

$$\Delta x_i = \frac{b-a}{n}$$

Now, since f is increasing, it follows that

$$M_i = f(x_i) \text{ and } m_i = f(x_{i-1})$$

Thus

$$\begin{aligned}
\sum_{i=1}^n (M_i - m_i)\Delta x_i &= \sum_{i=1}^n (f(x_i) - f(x_{i-1}))\frac{b-a}{n} \\
&= \frac{b-a}{n} \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \\
&= \frac{b-a}{n}(f(b) - f(a)) \\
&< \epsilon
\end{aligned}$$

for n large enough. □

7.4

1. (a) Let f be a bounded function on a set A, and set

$$M = \sup\{f(x) : x \in A\}, \quad m = \inf\{f(x) : x \in A\},$$

$$M' = \sup\{|f(x)| : x \in A\}, \quad m' = \inf\{|f(x)| : x \in A\}$$

Show that $M - m \ge M' - m'$

(b) Show that if f is integrable on the interval $[a, b]$, then $|f|$ is also integrable on this interval.

(c) Provide the details for the argument that in this case we have $|\int_a^b f| \leq \int_a^b |f|$.

Proof. (a) If M and m are either both nonnegative or both nonpositive, then we equality

$$M - m = M' - m'$$

On the other hand, if $m < 0 < M$ then it follows

$$m' \leq M$$

since

$$M < m'$$

would imply that there exists $f(x)$ such that

$$0 < f(x) \leq M < m'$$

contradicting with $m' \leq |f(x)|$. So, we then have that

$$m < 0 < m' \leq M \leq M'$$

If $M = M'$ then clearly

$$M - m = M' - m > M' - m'$$

If $M < M'$ then it is not too hard to show that $M' = -m$. So

$$M - m = M' + M > M' - m'$$

Therefore,

$$M - m \geq M' - m'$$

(b) So for every $\epsilon > 0$ we have a partition so that

$$\sum_k (M_k - m_k)\Delta x_k < \epsilon$$

From (a), we know that $M_k - m_k \geq M'_k - m'_k$ on each $[x_{k-1}, x_k]$. Thus

$$\sum_k (M'_k - m'_k)\Delta x_k \leq \sum_k (M_k - m_k)\Delta x_k < \epsilon$$

Therefore, $|f|$ is integrable on $[a, b]$.

(c) We claim that $|M| \leq M'$. To demonstrate, suppose $M \geq 0$. So $f(x) \leq |f(x)| \leq M'$ and so M' is an upper bound. Since, in this case, $|M| = M$ we would have $|M| = M \leq M'$, since M is the least upper bound. If $M < 0$ then $|M| = -M$. Futhermore, there exists $f(x) \leq M$. So $|f(x)| = -f(x)$. Thus $|M| = -M \leq -f(x) = |f(x)| \leq M'$. Therefore, $|M| \leq M'$. Thus, for every partition we have

$$\left| \sum_k M_k \Delta x_k \right| \leq \sum_k |M_k| \Delta x_k \leq \sum_k M' \Delta x_k$$

Therefore, $| \int_a^b f| \leq \int_a^b |f|$.

□

2. Show that if $c \leq a \leq b$ and f is integrable on the interval $[c, b]$, then it is still the case that

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof. If $c \leq a \leq b$, then

$$\int_a^c f + \int_c^b f = -\int_c^a f + \int_c^b f = -\int_c^a f + \int_c^a f + \int_a^b f = \int_a^b f$$

□

3. Prove Theorem 7.4.4.

Proof. Let $\epsilon > 0$ be given. Then, by our assumptions on f and f_n and Theorem 7.4.2 (i), we have

$$\left| \int_a^b f - \int_a^b f_n \right| \leq \left| \int_a^b (f - f_n) \right|$$

and by Theorem 7.4.2 (v)

$$\leq \int_a^b |f - f_n|$$

and by Theorem 7.4.2 (iv) and uniform convergence

$$\leq \int_a^b \frac{\epsilon}{b-a}$$

and by Exercise 7.2.3

$$= \frac{\epsilon}{b-a}(b-a)$$
$$= \epsilon$$

□

4. Decide which of the following conjectures are true and supply a short proof. For those that are not true, give a counterexample.

   (a) If $|f|$ is integrable on $[a, b]$ then f is also integrable on this set.

   (b) Assume g is integrable and $g \geq 0$ on $[a, b]$. If $g(x) \geq 0$ for an infinite number of points $x \in [a, b]$, then $\int g > 0$.

   (c) If g is continuous on $[a, b]$ and $g \geq 0$ with $g(x_0) > 0$ for at least one point $x_0 \in [a, b]$, then $\int_a^b g > 0$.

   (d) If $\int_a^b f > 0$ there is an interval $[c, d] \subseteq [a, b]$ and a $\delta > 0$ such that $f(x) > \delta$ for all $x \in [c, d]$.

Proof. (a) False. Let

$$f = \begin{cases} 1 & x \text{ is rational} \\ -1 & x \text{ is irrational} \end{cases}$$

Notice on $[0, 1]$ we get that $|f| = 1$ and so $\int_a^b |f| = (b - a)$. On the other hand, much like Dirichlet's function, f is not integrable on $[0, 1]$.

(b) False. See Exercise 7.3.5

(c) This is true. By continuity of g on $[a, b]$ it follows that there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|g(x) - g(x_0)| < \frac{g(x_0)}{2}$$

Thus if we define

$$f(x) = \begin{cases} \frac{g(x_0)}{2} & x \in (x - \delta, x + \delta) \\ 0 & \text{elsewhere in } [a, b] \end{cases}$$

then clearly

$$\int_a^b f = g(x_0)\delta$$

But $f(x) \leq g(x)$ on $[a, b]$. Thus

$$0 < g(x_0)\delta = \int_a^b f \leq \int_a^b g$$

(d) This is true, and we demonstrate by contrapositive. So for all $[c, d] \subseteq [a, b]$ there exists $x_0 \in [c, d]$ such that

$$f(x_0) \leq 0$$

Thus for every partition $P \in \mathcal{P}$ there is $x_k \in I_k$ such that

$$f(x_k) \leq 0$$

and

$$m_k \leq f(x_k) \leq M_k$$

Thus

$$L(f,P) \leq \sum_k f(x_k)\Delta x_k \leq U(f,P)$$

However, since $f(x_k) \leq 0$ it follows $\sum_k f(x_k)\Delta x_k \leq 0$. Thus by integrability of f and the squeeze theorem, it follows that

$$\int_a^b f \leq 0$$

□

5. Let f and g be integrable functions on $[a,b]$.

   (a) Show that if P is any partition of $[a,b]$, then

   $$U(f+g,P) \leq U(f,P) + U(g,P)$$

   Provide a specific example where the inequality is strict. What does the corresponding inequality for lower sums look like?

   (b) Prove Theorem 7.4.2 (i)

Proof. (a) Without going through the details we simply observe that

$$U(f+g,P) \leq U(f,P) + U(g,P)$$

holds since $\sup f+g \leq \sup f + \sup g$ which is true since $f(x)+g(x) \leq \sup_x f(x) + \sup_x g(x)$. A simple example would be $[a,b] = [0,1]$, $f = \sin$, and $g = x^2$. Similar reasoning shows that

$$L(f,P) + L(g,P) \leq L(f+g,P)$$

(b) By the above we have

$$L(f,P) + L(g,P) \leq L(f+g,P) \leq U(f+g,P) \leq U(f,P) + U(g,P)$$

By integrability of f and g it follows that for every $\epsilon > 0$ there exists a $P \in \mathcal{P}$ such that

$$L(f,P) + L(g,P), U(f,P) + U(g,P) \in V_\epsilon\left(\int_a^b f + \int_a^b g\right)$$

which, by the above inequality, implies the same for $L(f+g,P)$ and $U(f+g,P)$. Therefore

$$\int_a^b f + \int_a^b g = \int_a^b f + g$$

□

6.

7.5

1. We have seen that not every derivative is continuous, but explain how we know that every continuous is a derivative.

   Proof. We know that if a function, f, is continuous then it is integrable. By FToC we know that if

   $$F(x) = \int_a^x f$$

   and f is continuous at x, then $F'(x) = f(x)$. Since we know f is continuous, it follows $f(x) = F'(x)$ and is therefore a derivative. □

2. (a) Let $f(x) = |x|$ and define $F(x) = \int_{-1}^x f$. Find a formula for $F(x)$ for all x. Where is F continuous? Where is F differentiable? Where does $F'(x) = f(x)$?

   (b) Repeat for the function

   $$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0 \end{cases}$$

   Proof. (a) We observe that if

   $$F(x) = \begin{cases} \frac{1}{2} - \frac{x^2}{2} & x < 0 \\ \frac{1}{2} + \frac{x^2}{2} & x \geq 0 \end{cases}$$

   Then $F(x) = \int_{-1}^x f$. It is obvious from that F is continuous on all of $\mathbb{R}$. It can easily be shown that for all $c \in \mathbb{R}$ we will have that $F'(c) = c$, even at $c = 0$, and is thus differentiable everywhere.

   (b)

   □