Multivariable Mathematics Solutions Manual

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1 VECTORS AND MATRICES

1.1 Vectors in $\mathbb{R}^n$

1.1.1

Given $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ calculate the following both algebraically and geometrically.

1. $\mathbf{x} + \mathbf{y}$
2. $\mathbf{x} - \mathbf{y}$
3. $\mathbf{x} + 2\mathbf{y}$
4. $\frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y}$
5. $\mathbf{y} - \mathbf{x}$
6. $2\mathbf{x} - \mathbf{y}$
7. $\|\mathbf{x}\|$
8. $\frac{\mathbf{x}}{\|\mathbf{x}\|}$

Proof. 1. $\mathbf{x} + \mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 + (-1) \\ 3 + 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

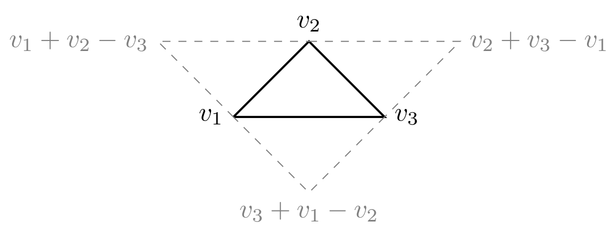
The idea is similar for the rest. □

1.1.2

Three vertices of a parallelogram are $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}$. What are all the possible positions of the fourth vertex? Give your reasoning.

Proof. The three vertices given already define a plane, which implies that any fourth vertex would have to belong to the plane as well. Thus the problem is reduced to finding the three points in the plane that will produce a parallelogram. Each solution is given by the sum of the three original vectors where one of them is negative. The three points then are given by:

$$\begin{bmatrix}1\\2\\1\end{bmatrix}+\begin{bmatrix}2\\4\\3\end{bmatrix}-\begin{bmatrix}3\\1\\5\end{bmatrix}=\begin{bmatrix}0\\5\\-1\end{bmatrix}$$
$$\begin{bmatrix}2\\4\\3\end{bmatrix}+\begin{bmatrix}3\\1\\5\end{bmatrix}-\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}4\\3\\7\end{bmatrix}$$
$$\begin{bmatrix}3\\1\\5\end{bmatrix}+\begin{bmatrix}1\\2\\1\end{bmatrix}-\begin{bmatrix}2\\4\\3\end{bmatrix}=\begin{bmatrix}2\\-1\\3\end{bmatrix}$$



□

1.1.3

The origin is at the center of a regular polygon.

1. What is the sum of the vectors to each of the vertices of the polygon? Give your reasoning.

2. What is the sum of the vectors from one fixed vertex to each of the remaining vertices? Give your reasoning

Proof. 1. Let one of the vertices of the polygon be at $(r, 0)$, where $r > 0$. By symmetry of the polygon, it follows that the sum of all the y-components of the vectors is 0. One can use the identity

$$\sum_{k=0}^{n-1} \cos(a + kd) = \cos\left(\frac{a + (n-1)d}{2}\right) \frac{\sin\left(n\frac{d}{2}\right)}{\sin\left(\frac{d}{2}\right)}$$

to show that the x-components of the vectors sum to 0. Thus the sum of the vectors is $\mathbf{0}$.

2. Let v_i denote the vector from the origin to the i^{th} vertex, where the vector from the center to the source vertex is v_1. Let w_i denote the vector from the source vertex to the i^{th} vertex. Then $w_i = v_i - v_1$. Thus

$$
\begin{aligned}
\sum_{i=2}^{n} w_i &= \sum_{i=2}^{n} (v_i - v_1) \\
&= \sum_{i=2}^{n} v_i - (n-1)v_1
\end{aligned}
$$

and by the previous problem

$$= -nv_1$$

$\square$

1.1.4

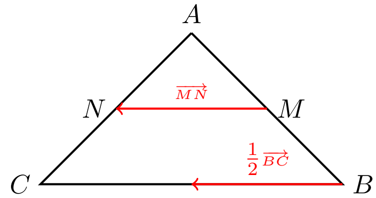
Given $\triangle ABC$, let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Prove that $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$.

Proof. Set $A = (0,0)$. Then

$$
\begin{aligned}
\vec{M} &= \frac{1}{2}\overrightarrow{AB} \\
\vec{N} &= \frac{1}{2}\overrightarrow{AC}
\end{aligned}
$$

So that

$$
\begin{aligned}
\overrightarrow{MN} &= \vec{M} - \vec{N} \\
&= \frac{1}{2}\overrightarrow{AB} - \frac{1}{2}\overrightarrow{AC} \\
&= \frac{1}{2}\left(\overrightarrow{AB} - \overrightarrow{AC}\right) \\
&= \frac{1}{2}\overrightarrow{BC}
\end{aligned}
$$



$\square$

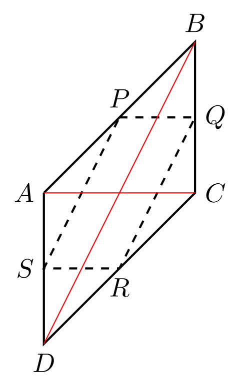
1.1.5

Let $ABCD$ be an arbitrary quadrilateral. Let P, Q, R, and S be the midpoints of $\overline{AB}, \overline{BC}, \overline{CD}$, and $\overrightarrow{DA}$, respectively. Use vector methods to prove that $PQRS$ is a parallelogram.

Proof. By the previous exercise, we know that

$$\overrightarrow{SP} = \frac{1}{2}\overrightarrow{DB} = \overrightarrow{RQ}$$
$$\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AC} = \overrightarrow{SR}$$

implying that $PQRS$ is a parallelgram.



□

1.2 Dot Product

1.2.1

For each of the following pairs of vectors x and y, calculate $x \cdot y$ and the angle θ between the vectors.

$$x = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, y = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

Proof.

$$x \cdot y = 2(-5) + 5(2) = 0$$

and

$$\theta = \cos^{-1} \frac{x \cdot y}{\|x\|\|y\|} = \frac{\pi}{2}$$

The rest of the problems are similar. □

1.2.2

For each pair of vectors in Exercise 1, calculate $\text{proj}_y x$ and $\text{proj}_x y$

Proof. Since we saw that $x \cdot y = 0$, we have

$$\text{proj}_x y = \frac{x \cdot y}{\|x\|^2} x = \text{proj}_y x = \frac{x \cdot y}{\|y\|^2} y = 0$$

The remaining are similar. □

1.2.3

Find the angle between the long diagonal of a cube and a face diagonal.

Proof. Wlog, take the unit cube. Denote the long diagonal $l = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and, wlog, the short diagonal $s = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Then

$$\theta = \cos^{-1} \frac{l \cdot s}{\|l\|\|s\|} = \frac{2}{\sqrt{3}\sqrt{2}} = \sqrt{\frac{2}{3}}$$

□

1.2.4

Find the angle that the long diagonal of a $3 \times 4 \times 5$ rectangular box makes with the longest edge.

Proof. □

1.2.5

Suppose $x, y \in \mathbb{R}^n$, $\|x\| = 2, \|y\| = 1$, and the angle θ between x and y is $\theta = \arccos \frac{1}{4}$. Prove that the vectors $x - 3y$ and $x + y$ are orthogonal.

Proof. Since $\theta = \arccos \frac{1}{4}$ it follows that

$$\begin{aligned} \frac{1}{4} &= \cos\theta = \frac{x \cdot y}{\|x\|\|y\|} \\ \frac{1}{2} &= x \cdot y \end{aligned}$$

Thus

$$\begin{aligned}(x-3y)\cdot(x+y) &= (x\cdot x)+(x\cdot y)-3(y\cdot x)+(y\cdot y)\\ &= 4-2(x\cdot y)-3\\ &= 0\end{aligned}$$

□

1.2.6

Suppose $x, y, z \in \mathbb{R}^2$ are unit vectors satisfying $x+y+z=0$. What can you say about the angles between each pair?

Proof. Note that, more generally, we have

$$\|x_1\| = \|x_2\| = \|x_3\| = 1$$

and

$$x_3 = -x_1 - x_2$$

Thus

$$\begin{aligned}0 &= (x_1+x_2+x_3)\cdot(x_1+x_2+x_3)\\ &= 3+2\left((x_1\cdot x_2)+(x_1\cdot x_3)+(x_2\cdot x_3)\right)\\ &= 3+2\left((x_1\cdot x_2)+(x_1\cdot(-x_1-x_2))+(x_2\cdot(-x_1-x_2))\right)\\ &= 3+2\left((x_1\cdot x_2)-2-2(x_1\cdot x_2)\right)\\ &= 3-4-2(x_1\cdot x_2)\\ &= -1-2(x_1\cdot x_2)\end{aligned}$$

thus

$$-\frac{1}{2} = x_1\cdot x_2$$

It therefore follows that the angle, θ, between any pair is given by

$$\theta = \arccos -\frac{1}{2} = \frac{2\pi}{3}$$

□

1.2.7

Let $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ be the so-called *standard basis vectors* of $\mathbb{R}^3$. Let $x \in \mathbb{R}^3$ be a nonzero vetor. For $i = 1, 2, 3$, let θ_i denote the angle between x and e_i. Compute $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3$.

Proof. Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Then

$$\begin{aligned}
\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 &= \frac{\sum_{i=1}^n (x_i \cdot e_i)^2}{\|x\|^2} \\
&= \frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^2} \\
&= 1
\end{aligned}$$

□

1.2.8

Let $x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 1 \\ 2 \\ 3 \\ \vdots \\ n \end{bmatrix} \in \mathbb{R}^n$. Let θ_n be the angle between x and y. Find $\lim\limits_{n\to\infty} \theta_n$.

Proof. Note that

$$\theta_n = \arccos \frac{x \cdot y}{\|x\|\|y\|}$$

We observe that

$$\begin{aligned}
\frac{x \cdot y}{\|x\|\|y\|} &= \frac{\sum_{i=1}^n i}{\sqrt{n}\sqrt{\sum_{i=1}^n i^2}} \\
&= \left(\frac{n(n+1)}{2}\right)\left(\frac{\sqrt{6}}{n\sqrt{n+1}\sqrt{2n+1}}\right) \\
&= \frac{1}{2}\left(\sqrt{\frac{6n+6}{2n+1}}\right)
\end{aligned}$$

and therefore

$$\begin{aligned}
\lim_{n\to\infty} \frac{x \cdot y}{\|x\|\|y\|} &= \lim_{n\to\infty} \frac{1}{2}\left(\sqrt{\frac{6n+6}{2n+1}}\right) \\
&= \frac{\sqrt{3}}{2}
\end{aligned}$$

Thus, by continuity of arccos, we have

$$\begin{aligned}
\lim_{n\to\infty} \theta_n &= \lim_{n\to\infty} \arccos \frac{1}{2}\left(\sqrt{\frac{6n+6}{2n+1}}\right) \\
&= \arccos \frac{\sqrt{3}}{2} \\
&= \frac{\pi}{6}
\end{aligned}$$

□

1.2.9

With regard to the proof of Proposition 2.3, how is $t_0 y$ related to $x^{||}$? What does this say about $\mathrm{proj}_y x$?

Proof. Note that

$$x^{||} = \frac{x \cdot y}{\|y\|^2} y = \mathrm{proj}_y x = -t_0 y$$

Intuitively, this simply means that the function g, presented in the proof of Proposition 2.3, is minimized when the only non-zero component of $x + ty$ is $x^{\perp}$. That is

$$\begin{aligned}
\|x + t_0 y\|^2 &= \|x^{||} + x^{\perp} + t_0 y\|^2 \\
&= \|x^{\perp}\|^2
\end{aligned}$$

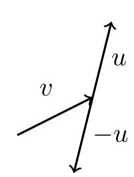
□

1.2.10

Use vector methods to prove that a parallelogram is a rectangle iff its diagonals have the same length.

Proof. Note that if $u, v \neq 0$, then

$$\begin{gathered}
\|v + u\|^2 = \|v - u\|^2 \\
(v + u) \cdot (v + u) = (v - u) \cdot (v - u) \\
v \cdot v + 2v \cdot u + u \cdot u = v \cdot v - 2v \cdot u + u \cdot u \\
\text{iff} \\
v \cdot u = 0
\end{gathered}$$



Thus, if ABCD is a parallelogram with diagonals $D_1, D_2 \neq 0$, then

$$\|D_1\|^2 = \|\overrightarrow{AD} + \overrightarrow{DC}\|^2 = \|\overrightarrow{BC} + \overrightarrow{CD}\|^2 = \|\overrightarrow{AD} - \overrightarrow{DC}\|^2 = \|D_2\|^2$$
$$\text{iff}$$
$$\overrightarrow{AD} \cdot \overrightarrow{DC} = 0$$

which implies the result. □

1.2.11

Use the fundamental properties of the dot product to prove that

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

Proof.

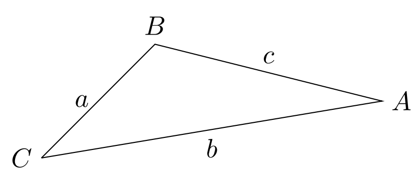
$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 &= (x+y)\cdot(x+y) + (x-y)\cdot(x-y)\\ &= (x\cdot x) + 2(x\cdot y) + (y\cdot y) + (x\cdot x) - 2(x\cdot y) + (y\cdot y)\\ &= 2(x\cdot x) + 2(y\cdot y)\\ &= 2\left(\|x\|^2 + \|y\|^2\right)\end{aligned}$$

□

1.2.12

Use the dot product to prove the law of cosines

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



Proof.

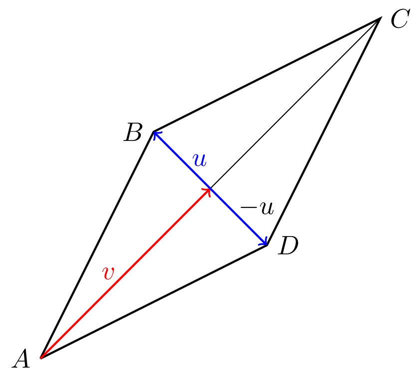
$$\begin{aligned}c^2 &= \|\overrightarrow{CA} - \overrightarrow{CB}\|^2\\ &= (\overrightarrow{CA} - \overrightarrow{CB})\cdot(\overrightarrow{CA} - \overrightarrow{CB})\\ &= \|\overrightarrow{CA}\|^2 + \|\overrightarrow{CB}\|^2 - 2\left(\overrightarrow{CA}\cdot\overrightarrow{CB}\right)\\ &= a^2 + b^2 - 2ab\cos\theta\end{aligned}$$

□

1.2.13

Use vector methods to prove that the diagonals of a parallelogram are orthogonal if and only if the parallelogram is a rhombus (i.e., has all sides of equal length).

Proof. Consider the parallelogram



From Exercise 1.2.10,

$$\|v + u\| = \|v - u\| \text{ iff } v \cdot u = 0$$

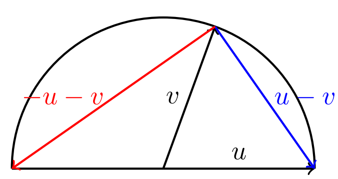
□

1.2.14

Use vector methods to prove that a triangle inscribed in a circle and having a diameter as one of the sides must be a right triangle.

Geometric Challenge: More generally, given two points A and B in the plane, what is the locus of points X so that $\angle AXB$ has a fixed measure?

Proof. Note that this is a simple consequence of the inscribed angle theorem. Using vector methods, given

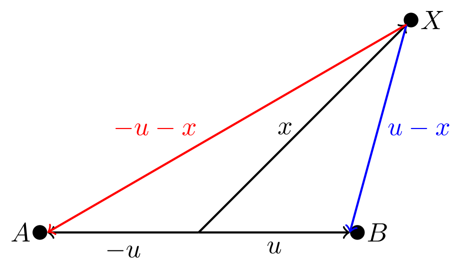


Note $\|v\| = \|u\|$. It follows

$$(u - v) \cdot (-u - v) = -\|u\|^2 + \|v\|^2 = 0$$

implying the result.

Geometric Challenge: Suppose X is chosen so that $\angle AXB = \theta$. Then



So that

$$\cos\theta = \frac{(u-x)\cdot(-u-x)}{\|u-x\|\|u+x\|} = \frac{\|x\|^2 - \|u\|^2}{\|u-x\|\|u+x\|}$$

which defines our locus. □

1.3 Subspaces of $\mathbb{R}^n$

1.3.1

trivial

1.3.2

trivial

1.3.3

Suppose $x, v_1, \ldots, v_k \in \mathbb{R}^n$ and x is orthogonal to each of the vectors $v_1, \ldots, v_k$. Prove that x is orthogonal to any linear combination $c_1v_1 + c_2v_2 + \ldots + c_kv_k$.

Proof.

$$x \cdot \sum_{i=1}^{k} c_i v_i = \sum_{i=1}^{k} c_i (x \cdot v_i) = 0$$

□

1.3.4

$V^\perp$ is a subspace.

Proof. Let V be a subspace, then, given $v \in V$, there is $w \in V^\perp$ such that $v \cdot w = 0$.

1. $0 \cdot v = 0$, $\forall v \in V$
2. $(w_1 + w_2) \cdot v = (w_1, v) + (w_2, v) = 0 + 0 = 0$
3. $cw \cdot v = c(w \cdot v) = 0$

□

1.3.5

Given vectors $v_1, \ldots, v_k \in \mathbb{R}^n$, prove that $V = \text{Span}(v_1, \ldots, v_k)$ is the *smallest* subspace containing them all. That is, prove that if $W \subset \mathbb{R}^n$ is a subspace and $v_1, \ldots, v_k \in W$, then $V \subset W$.

Proof. Note if $v_1, \ldots, v_k \in W$ then $\sum \alpha_i v_i \in W$. Thus $\forall_{v \in V} v \in W$. □

1.3.6

1. Let U and V be subspaces of $\mathbb{R}^n$. Define
$$U \cap V = \{x \in \mathbb{R}^n : x \in U \text{ and } x \in V\}$$
Prove that $U \cap V$ is a subspace of $\mathbb{R}^n$.
2. Is $U \cup V = \{x \in \mathbb{R}^n : x \in U \text{ or } x \in V\}$ a subspace of $\mathbb{R}^n$? Give a proof or counterexample.
3. Let U and V be subspaces of $\mathbb{R}^n$. Define
$$U + V = \{x \in \mathbb{R}^n : x = u + v \text{ for some } u \in U \text{ and } v \in V\}$$
Prove that $U + V$ is a subspace of $\mathbb{R}^n$.

Proof.

1. (a) $0 \in U \cap V$
   (b) If $y, x \in U \cap V$ then $x + y \in U \cap V$
   (c) $\forall_{x \in U \cap V} cx \in U \cap V$
2. No. Let $U = \text{Span} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $V = \text{Span} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin U \cup V$
3. (a) $0 = 0 + 0$
   (b) $x_1 + x_2 = u_1 + v_1 + u_2 + v_2 = (u_1 + u_2) + (v_1 + v_2)$
   (c) $cx = c(u_1 + v) = cu + cv$

□

1.3.7

Let $v_1, \ldots, v_k \in \mathbb{R}^n$ and let $v \in \mathbb{R}^n$. Prove that

$$\operatorname{Span}(v_1, \ldots, v_k) = \operatorname{Span}(v_1, \ldots, v_k, v) \iff v \in \operatorname{Span}(v_1, \ldots, v_k)$$

Proof. Let $v \in \operatorname{Span}(v_1, \ldots, v_k)$ then

$$v = \sum \alpha_i v_i$$

Clearly $\operatorname{Span}(v_1, \ldots, v_k) \subset \operatorname{Span}(v_1, \ldots, v_k, v)$. Now if $u \in \operatorname{Span}(v_1, \ldots, v_k, v)$ then

$$\begin{aligned} u &= \sum_{i=1}^{k} \beta_i v_i + \beta_{k+1} v \\ &= \sum \beta_i v_i + \beta_{k+1} (\sum \alpha_i v_i) \end{aligned}$$

Now let $\operatorname{Span}(v_1, \ldots, v_k) = \operatorname{Span}(v_1, \ldots, v_k, v)$. Then

$$\sum_{i=1}^{k} \alpha_i v_i + \alpha_{k+1} v = \sum \beta_i v_i$$

$$\alpha_{k+1} v = \sum_{i=1}^{k} (\beta_i - \alpha_i) v_i$$

□

1.3.8

Let $V \subset \mathbb{R}^n$ be a subspace. Prove that $V \cap V^\perp = \{0\}$.

Proof. Note that $x \cdot x = 0 \iff x = 0$, so that $\forall_{v \in V} x \cdot v = 0 \iff x = 0$. Thus since $V \cap V^\perp \subset V$. So $u \in V \cap V^\perp$ iff $u \cdot u = 0$ iff $u = 0$. Thus $V \cap V^\perp = \{0\}$. □

1.3.9

Suppose $U, V \subset \mathbb{R}^n$ are subspaces and $U \subset V$. Prove that $V^\perp \subset U^\perp$.

Proof. Let $U \subset V$ and let $v' \in V^\perp$. So $\forall_{v \in V} v' \cdot v = 0$. Thus $\forall_{u \in U} v' \cdot u = 0$. Thus $v' \in U^\perp$. □

1.3.10

Let $V \subset \mathbb{R}^n$ be a subspace. Prove that $V \subset \left(V^\perp\right)^\perp$. Do you think more is true?

Proof.

$$x \in \left(V^\perp\right)^\perp \text{ iff } \forall_{v' \in V^\perp} x \cdot v' = 0 \text{ iff } x \in V$$

□

1.4 Linear Transformations and Matrix Algebra

1.4.1

Trivial

1.4.2

1. If A is an $m \times n$ matrix and $Ax = 0$ for all $x \in \mathbb{R}^n$, prove that $A = 0$.
2. If A and B are $m \times n$ matrices and $Ax = Bx$ for all $x \in \mathbb{R}^n$, prove that $A = B$.

Proof. 1. Suppose $A \neq 0$. Then there is i, j such that $[A]_{ij} \neq 0$. Let $x = [A]_{i\cdot}^T$. Then

$$[Ax]_{i\cdot}^T = x \cdot x > 0$$

2. For all x we have

$$\begin{aligned} Ax &= Bx \\ (A - B)x &= 0 \end{aligned}$$

so by the previous, $A - B = 0$. So $A = B$.

□

1.4.3

Trivial

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