MATH 3MB3 Group Project
Fall 2019

Weather Structure, The Lorenz System, and Chaos Theory

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August 20, 2020

Abstract

The Lorenz System, a simplified model of the Earth's atmosphere, created by meteorologist Edward Norton Lorenz in 1963, is analyzed in the attempt to justify the structure of the weather and to provide insight on the art of weather prediction in general. Mathematical properties and methods are discussed including fixed point analysis using linearization and Lyapunov methods, bifurcations, phase space visualization, strange attractors, and chaos theory.

Contents

1 Introduction

The weather is a complex system that has a tremendous effect on our daily lives. Not only has it been an integral part of human history by shaping customs, cultures, and beliefs, but it also has the power to take lives which is best characterized by severe weather and its relation to natural disasters. In order to prevent such tragedies and to provide a smoother existence, humans have entertained the idea of weather forecasting and have synthesized it into a very powerful tool. Despite our progress, modern meteorologists can only accurately forecast for about one week [5]. In such an advanced society, why have we not found a way to accurately predict the weather for longer periods of time? Using mathematics, we can translate this question into, what property of dynamical systems makes the weather unique when compared to other dynamical systems? In 1963, mathematician and meteorologist Edward Norton Lorenz, was fascinated with these questions in the hopes of explaining some of the weather's unpredictable behaviour. Lorenz first modelled the Earth's atmosphere by deriving a large system of partial differential equations based on the theory of thermal fluids (thermodynamics, heat transfer, fluid mechanics, combustion), however, the details were difficult to analyze, and the truth was unclear [2]. In an attempt to provide more transparent results, he simplified this model into a nonlinear, autonomous, first order system of ordinary differential equations, which is now known as The Lorenz System. Lorenz explained that one day, he was running computer simulations to produce weather data; everything was going well, however, he wanted to reexamine the behaviour of some of the earlier data in greater detail. Thus, he halted the program and initiated the simulation at previous values based on a printout created that very morning. With the weather simulation reset and running, Lorenz left for about an hour to get a cup of coffee, during which time, two months of weather data had been produced. Upon his return, Lorenz was astonished to see that the results of this simulation were completely different than before. At first, he rationalized this drastic change as a computational error, however, Lorenz soon realized that the true culprit was the printout, as it had rounded some of the data. It was at this moment that Lorenz discovered chaos, or a system's sensitivity to initial conditions [3]. In this paper, our goal is to analyze Lorenz's simplified model of the atmosphere and attempt to answer the questions: what does it mean for the weather to be a chaotic system and what does that imply with respect to the structure of the weather and the art weather prediction in general? To be clear, this model is not being used for specific weather forecasting, (ex. it is going to be sunny on this day and rainy on this day) as The Lorenz System is far too simple to model the Earth's atmosphere in totality; however, we do note that some of the weathers inherent *structure* is preserved in this system, so it is still useful for inference with regards to our purposes here. We will also state most definitions and theorems for the benefit of the reader and will provide calculations where it is thought they might be meaningful, but proofs will be left out to avoid confusion.[1] Sample python code for phase space visualization is provided at the end. [2]

[1] All definitions, theorems, and proofs can be found in [4] *Differential Equations, Dynamical Systems, and an Introduction to Chaos.*

[2] Python code was modified from [1] *Introduction to the modeling and analysis of complex systems.*

2 Model Construction

To begin, we describe the Lorenz system in terms of state variables, parameters, and assumptions.

2.1 State Variables in Continuous Time

The three variables of interest for the simplified atmosphere are the following.

- $x(t)$ is proportion to the intensity of the convection motion on day t

- $y(t)$ is proportional to the horizontal temperature variation on day t

- $z(t)$ is proportional to the vertical temperature variation on day t

2.2 Parameters

Below are the three parameters of the system.

- σ is the Prandtl number of the fluid, which is the dimensionless ratio of the fluid viscosity to the thermal diffusivity (in other words, it is the ratio of thickness of the fluid to the speed at which heat goes from the hot end to the cold end of fluid).

- ρ is the temperature difference between the top and the bottom of the atmospheric layer

- b is the physical dimension of the atmospheric layer, which is the ratio of its width to height

2.3 Assumptions

We assume that the atmosphere is a two-dimensional fluid layer uniformly warmed from below and cooled from above; under convection, the fluid would respectively rise and fall. Further, we assume that the atmosphere is constant in all other variables and its state is entirely and uniquely determined by the variables x, y and z. We also restrict the parameters such that $\sigma, \rho, b > 0$ and $\sigma > b + 1$.

2.4 The Lorenz System

With respect to the previously mentioned state variables, parameters, and assumptions, we can construct a system of partial differential equations to model the atmosphere which can then be simplified into The Lorenz System. The derivation of the original system and the simplification is a rather lengthy exercise of fluid dynamics and complex analysis, which is out of the scope of this paper [2]. [3]

[3] if you are interested in the full derivation, please see [2] Page 2 and 3.

As such, we proceed with the system.

$$\begin{cases} \frac{dx}{dt} = \sigma(y-x) \\ \\ \frac{dy}{dt} = \rho x - y - xz \\ \\ \frac{dz}{dt} = xy - bz \end{cases} \tag{1}$$

3 Stability Analysis

The first step in understanding the behaviour of The Lorenz System is to perform nonlinear stability analysis, which we formalize next.

Definition 3.1 (Topologically Conjugate)**.** Let the *flow* $\phi_t(X_0)$ be the solution of a system with initial condition X_0. Then we say that two flows $\phi, \psi : M \to M$ are *topologically conjugate* if there exists a homeomorphism $h : M \to M$ such that $\phi \circ h = h \circ \psi$.

Definition 3.2 (Hyperbolic)**.** A matrix A is *hyperbolic* if none of its eigenvalues has real part 0. If this is the case, we call the system $X' = AX$ hyperbolic. Further, an equilibrium point X_0 of a system is called hyperbolic if the linearized system $V' = DF_{X_0}V$ is hyperbolic.

Using these definitions, we can classify the criterion such that the flows of a nonlinear system will share the same fate or long term behaviour as the flows of its linear counterpart.

Theorem 3.3 (The Linearization Theorem)**.** *Suppose the n-dimensional system $X' = F(X)$ has an equilibrium point at X_0 that is hyperbolic. Then the nonlinear flow is conjugate to the flow of the linearized system $V' = DF_{X_0}V$ in a neighbourhood of X_0.*

3.1 Equilibrium Points

Motivated by the Linearization Theorem, we find the equilibrium points; at equilibrium, the system satisfies,

$$\begin{cases} 0 = \sigma(y-x) \\ 0 = \rho x - y - xz \\ 0 = xy - bz \end{cases} \tag{2}$$

Solving the first and third equations for y and substituting in the second results in,

$$(y = x) \implies (z = \frac{y^2}{b}) \implies (\rho y - y - \frac{y^3}{b}) \implies y = 0, \pm\sqrt{b(\rho - 1)}$$

Using these y values to solve for x and z gives us three equilibrium points, namely

$$E_{1,2,3} = (0,0,0), (\pm\sqrt{b(\rho-1)}, \pm\sqrt{b(\rho-1)}, \rho - 1) \tag{3}$$

3.2 Bifurcations

Notice that the number of equilibrium points of the system depends on the parameter ρ; in fact, different values of this parameter will cause *smooth*, qualitative changes in the behaviour of the system.

Definition 3.4 (Bifurcation)**.** A *bifurcation* is a topological change of a system's phase space that occurs when some parameters are slightly varied across their critical thresholds.

It is clear that a *pitchfork* bifurcation occurs at $\rho = 1$ since two equilibrium points are born at this critical value. Thus, we must analyze the system on two different domains of ρ, namely, $\rho \in (0, 1)$ which has equilibrium point E_1, and $\rho > 1$, which has equilibrium points $E_{1,2,3}$.

3.3 Stability at the Origin

After calculating the Jacobian, the linearized system is,

$$V' = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -b \end{bmatrix} V \tag{4}$$

Now solving for the characteristic polynomial,

$$\det \begin{bmatrix} -\sigma - \lambda & \sigma & 0 \\ \rho - z & -1 - \lambda & -x \\ y & x & -b - \lambda \end{bmatrix} = 0 \iff 0 = x \det \begin{bmatrix} -\sigma - \lambda & \sigma \\ y & x \end{bmatrix} - (b+\lambda) \det \begin{bmatrix} -\sigma - \lambda & \sigma \\ \rho - z & -1 - \lambda \end{bmatrix} \iff$$

$$\lambda^3 + \lambda^2[1 + b + \sigma] + \lambda[x^2 + b(\sigma + 1) + \sigma(1 + z - \rho)] + \sigma[(x^2 + xy + b(1 + z - \rho)] = 0 \tag{5}$$

At the origin, the eigenvalues of the Jacobian are,

$$\lambda_1 = -b, \lambda\pm = \frac{1}{2}\Big(- (\sigma + 1) \pm \sqrt{(\sigma + 1)^2 - 4\sigma(1 - \rho)}\Big)$$

For $\rho > 1$, we will have $\lambda_1, \lambda_- < 0 < \lambda_+$, so the origin is a saddle point with a two dimensional stable surface and an unstable curve. For $\rho \in (0, 1)$, we introduce a powerful method using Lyapunov functions to provide global stability.

Theorem 3.5 (Lyapunov Stability)**.** *Let X^* be an equilibrium point for $X' = F(X)$ and let $L : U \to \mathbb{R}$ be a differentiable function defined on an open set U containing X^*. If L satisfies*

1. $L(X^*) = 0 \wedge L(X) > 0, X \neq X^*$
2. $\dot{L}(X) \leq 0, X \in U - X^*$
   then X^* is stable and L is called a Lyapunov function. Further, if L also satisfies
3. $\dot{L}(X) < 0, X \in U - X^*$

then X^* is asymptotically stable and L is called a Strict Lyapunov function.

Consider the function $L(x, y, z) = x^2 + \sigma y^2 + \sigma z^2$ which is positive everywhere except at the origin. Then, using implicit differentiation, we get the time derivative $\dot{L} = -2\sigma(x^2 + y^2 - (1-\rho)xy) - 2\sigma bz^2$ which is strictly negative for all $(x, y, z) \neq (0, 0, 0)$. Thus, L is a strict Lyapunov function and a solution through any point in $\mathbb{R}^3$ will asymptotically approach the origin for $\rho \in (0, 1)$.

3.4 Non-Origin Equilibrium Points

At the other equilibrium points, $E_{2,3}$, we have the characteristic polynomial

$$f(\lambda) = \lambda^3 + \lambda^2[1 + b + \sigma] + \lambda[b(\sigma + \rho)] + 2b\sigma(\rho - 1) = 0 \tag{6}$$

$E_{2,3}$ only exist for $\rho > 1$ so this is where we must analyze them. By using the assumption $\sigma > b+1$ and solving the characteristic polynomial at various parameter values, one can notice that every eigenvalue is negative until ρ crosses a critical threshold where the real part of two eigenvalues becomes zero. This is known as a *Hopf* bifurcation, or more specifically, where a fixed point loses stability when complex conjugate eigenvalues cross the imaginary axis. This implies that $E_{2,3}$ are sinks provided that $1 < \rho < \rho_c$. To find ρ_c, we note that at this value, there will be complex conjugate eigenvalues with zero real part, namely $\lambda_\pm = \pm i\beta, \beta \neq 0$. Substituting both values into the characteristic polynomial and setting real and imaginary parts equal gives us two expressions of β which we equate to solve for ρ_c.

$$f(0 \pm i\beta) \implies \beta^2 = \frac{2\sigma b(\rho - 1)}{\sigma + b + 1} \wedge \beta^2 = b(\sigma + \rho) \implies \rho_c = \sigma\Big(\frac{\sigma + b + 3}{\sigma - b - 1}\Big) \tag{7}$$

Remark 3.6. For a brief summary of the previous results, see Table 1; equilibrium points can be found explicitly in equation (3).

Table 1: Fixed Point Classification

ρ	Fixed Points	Stability
$\rho \in (0, 1)$	E_1	stable
$\rho \in (1, \rho_c)$	(E_1, E_2, E_3)	(unstable, stable, stable)
$\rho > \rho_c$	(E_1, E_2, E_3)	(unstable, unstable, unstable)

4 Chaos

One of the most interesting properties of The Lorenz System is its chaotic behaviour.

Definition 4.1 (Chaotic System)**.** A dynamical system is chaotic if

1. It is sensitive to initial conditions
2. It is topologically transitive
3. It has dense periodic orbits

The first condition implies no matter how arbitrarily close two solutions start together, their paths will always diverge in finite time. The second and third essentially say that every point is approached arbitrarily close by a periodic orbit and that any two open sets will eventually have non-empty intersection after enough iterations. Next we examine The Lorenz System to determine where it displays chaos. We will be graphing the phase space over 42 days to view the overall behaviour; parameter values are fixed at $\sigma = 10$ and $b = \frac{8}{3}$, as these are the original values that Lorenz believed best modelled the Earth's atmosphere [4], along with the initial condition $(1, -2, 1)$. Then, to test sensitivity to initial conditions, we examine each state variable vs time individually at initial condition $(1, -2, 1)$ against the slightly perturbed initial condition $(1.001, -2.001, 1.001)$. Since ρ is defined to be a temperature difference, the analysis is separated into three cases representing places one would find small, moderate, and large temperature variation; to provide context, we let case one to three represent the weather at the equator, a neighbourhood of the equator, and elsewhere, respectively.

4.1 Case 1: $\rho \in (0, 1)$

Recall that we used the Lyapunov method to show that all solutions converge asymptotically to the origin for $\rho \in (0, 1)$. As you can see, Figure 1 shows a very tame solution curve converging to $(0, 0, 0)$ and the difference between the red and black lines is barely distinguishable implying no sensitivity to initial conditions. Thus, there is no chaos for $\rho \in (0, 1)$ and we have around 15 days of accurate forecasting before they reach equilibrium.

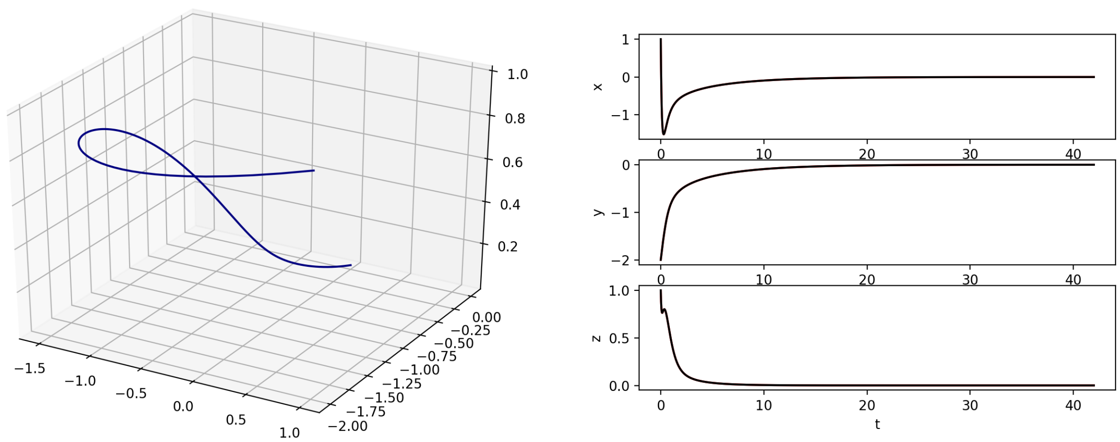

Figure 1: Phase space at $(\rho, \sigma, b) = (0.8, 10, 8/3)$, $X_0 = (1, -2, 1)$ and state variables vs time graphs for $X_0 = (1, -2, 1)$ in red and $X_0 = (1.001, -2.001, 1.001)$ in black

4.2 Case 2: $\rho \in (1, \rho_c = 24.7)$

In equation (7), we derived an upper bound for the stability of the equilibrium points $E_{2,3}$, namely, at the Hopf bifurcation value $\rho_c = \sigma\left(\frac{\sigma+b+3}{\sigma-b-1}\right)$. For $\sigma = 10, b = \frac{8}{3}$, we have $\rho_c \approx 24.7$. In Figure 2, one observes a well-behaved solution with an unstable node at the origin and a spiral sink at one of the other equilibrium points. Furthermore, the state variable vs time graphs are almost identical as they approach equilibrium; once again, no chaos is found and we have about 13 days of accurate forecasting before equilibrium.

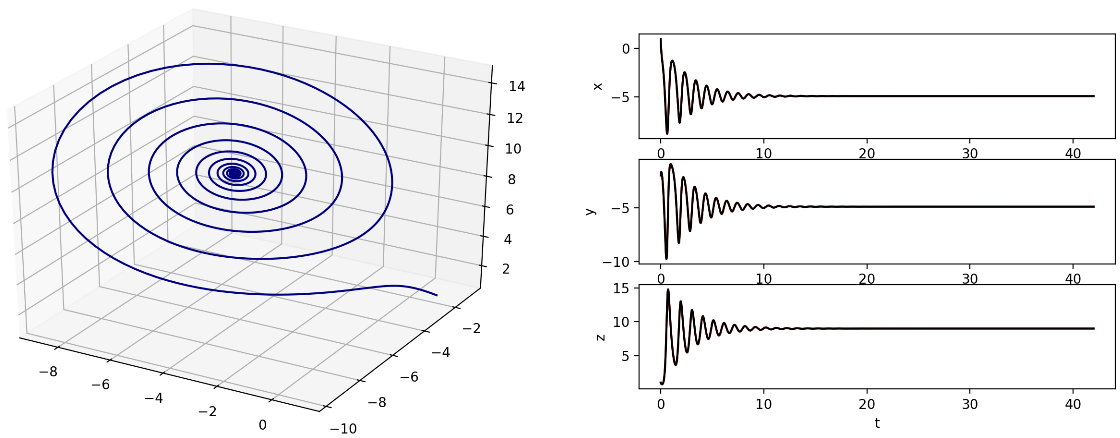

Figure 2: Phase space at $(\rho, \sigma, b) = (10, 10, 8/3), X_0 = (1, -2, 1)$ and state variables vs time graphs for $X_0 = (1, -2, 1)$ in red and $X_0 = (1.001, -2.001, 1.001)$ in black

4.3 Case 3: $\rho > \rho_c = 24.7$

Recall that since $\rho > \rho_c, E_2$ and E_3 are no longer stable. In Figure 3, one can see very erratic behaviour around two attracting points creating an interesting butterfly like composition. We can also see that the state variable vs time graphs are traced in unison until around $t = 9$ where they seem to diverge into individual and random paths. The initial conditions were 99.9% accurate, yet after nine days, the paths were completely different; thus, for $\rho > \rho_c$, we have chaos.

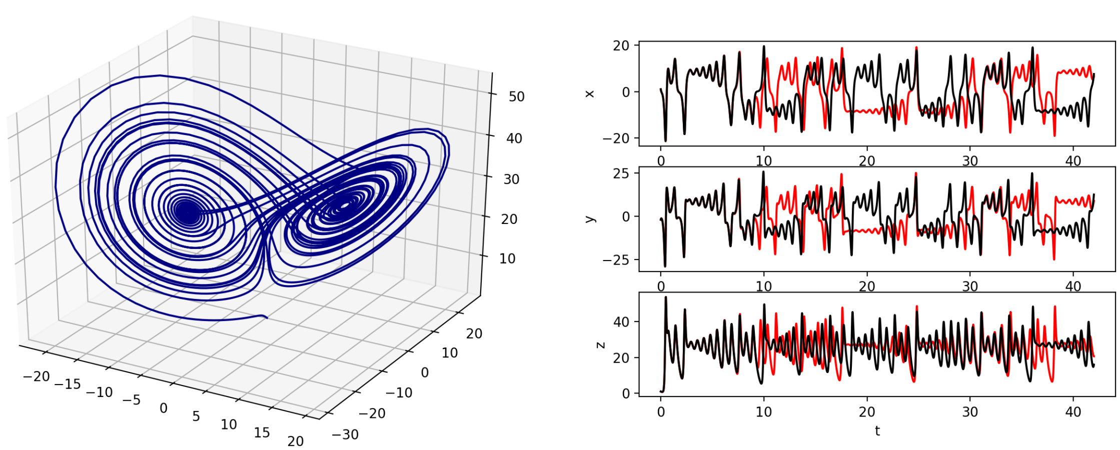

Figure 3: Phase space at $(\rho, \sigma, b) = (28, 10, 8/3), X_0 = (1, -2, 1)$ and state variables vs time graphs for $X_0 = (1, -2, 1)$ in red and $X_0 = (1.001, -2.001, 1.001)$ in black

[4]

[4] Python code for Figure 3 is provided at the end [1].

Remark 4.2. A brief summary of the previous results are compiled in Table 2.

Table 2: Chaos Classification

ρ	Location	Chaos
$\rho \in (0, 1)$	Equator	Non-Chaotic
$\rho \in (1, \rho_c)$	Neighbourhood of Equator	Non-Chaotic
$\rho > \rho_c$	Elsewhere	Chaotic

5 The Lorenz Attractor

The last property of The Lorenz System that will be discussed is a set of chaotic solutions that form an object called The Lorenz Attractor.

Definition 5.1 (Attractor)**.** Let $X' = F(X)$ be a system of differential equations in $\mathbb{R}^n$ with flow ϕ_t. A set Λ is called an *attractor* if

1. Λ is compact and invariant
2. There is an open set U containing Λ such that $\forall X \in U, \phi_t(X) \in U$ and $\bigcap_{t\geq 0} \phi_t(U) = \Lambda$
3. Given any points $Y_1, Y_2 \in \Lambda$ and any open neighbourhoods U_j about Y_j in U, there is a solution curve that begins in U_1 and later passes through U_2.

Further, an attractor is called *strange* if it has *fractal* dimension (non-integer dimension)

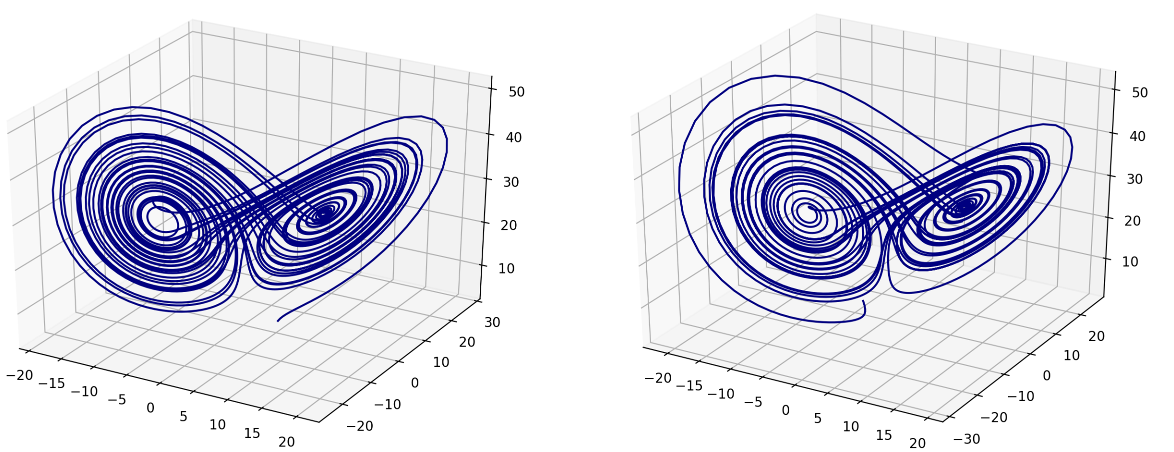

Figure 4: Lorenz Attractor for initial condition $X_0 = (4, 3, 3)$ and $X_0 = (-5, 3, 2)$

To clarify, the second condition means that solutions in a neighbourhood around the attractor converge to it, and the third condition is a transitivity requirement so that we concern ourselves with a unique attractor. Note the slight resemblance with chaos; in fact, many chaotic systems have strange attractors, but be warned, there exists non-chaotic attractors as well. The Lorenz Attractor has many interesting properties that one could study; it is a strange attractor, has zero volume, the solutions trace out a non-periodic path for all time never repeating itself, and much more, but we will not go into detail with regards to all of its properties here. To convince yourself that The Lorenz Attractor is a strange attractor, consider Figure 4 that shows two solution curves at quite different initial conditions, both following a non-periodic path around The Lorenz Attractor. If you imagine taking the limit as time goes to infinity, we would see the same butterfly like object, appearing solid, yet having zero volume as its fractal dimension is about 2.06 [1].

6 Conclusion

Despite being a simplified model of the atmosphere, we can use The Lorenz System to try to explain some of the weathers inherent structure. Most importantly, we learned that the weather is a chaotic system which makes it extremely sensitive to initial conditions. In pop-culture, this is known as the butterfly effect and is romanticized in the sense that the very flap of a butterflies wings could be responsible for a tornado in a different part of the world. Although, this is extremely glamourized, there is a cryptic truth to it. Consider the typical weather pattern shown in Case 3; even if we were able to measure the weather with 99.9% accuracy, because the weather is a chaotic system, any slight variation due to measurement error will cause drastically different atmospheres. At these parameters, we experienced nine days of extremely accurate forecasting; in real life, it is often only meticulously accurate for two to five days and roughly accurate for about 10 [5]. On the contrary, Case 1 and Case 2 experienced around two weeks of accurate weather forecasting which is slightly unrealistic, however, this hints at the idea that it one is far more likely to accurately predict the weather near places with very little temperature variation. Overall, this implies that we will not be able to feasibly predict longterm weather as we will never achieve 100% measurement accuracy. Now, if we interpret these cases even further, one can notice that the seasons are well represented in the phase space. At the equator, we have no chaos and global stability which implies relatively no seasons. Moving to a neighbourhood around the equator, we have partial instability, a spiral, and no chaos, which represents the binary seasons one would find near the equator, namely, wet and dry periods. Finally, for typical weather patterns elsewhere, we have complete instability, chaos, and the existence of a strange attractor; in this case, we see every weather solution infinitely traverse through the The Lorenz attractor just as typical weather infinitely traverses through the standard seasons. Although Lorenz stumbled upon chaos by chance, we have learned that the weather is a completely deterministic system that is attracted to the seasons and is sensitive to slight disturbances. In some sense, this knowledge is disheartening as longterm weather forecasting is not feasible, however, we should continue to model the weather as accurately as we can. Not only did Lorenz's discovery influence the way modern models of the weather are created, but it also revealed the mathematics necessary to understand chaotic systems that continue to be found in a wide variety of applied disciplines including biology, chemistry, engineering, and cryptography. As a final thought, we encourage you to embrace the romanticized notion of the butterfly effect, as in many ways, our lives are chaotic systems and there is no feasible method to predict the positive consequences of our small and insignificant actions.

Python for Figure 3

November 19, 2019

[ ]:
```
from pylab import *
from mpl_toolkits.mplot3d import Axes3D

# parameters
sigma = 10.
rho = 28
b = 8 / 3.
Dt = 0.01

# set up graph one
def initialize():
    global x, xresult, y, yresult, z, zresult, t, timesteps
    x = 1
    y = -2
    z = 1
    xresult = [x]
    yresult = [y]
    zresult = [z]
    t = 0.
    timesteps = [t]

def observe():
    global x, xresult, y, yresult, z, zresult, t, timesteps
    xresult.append(x)
    yresult.append(y)
    zresult.append(z)
    timesteps.append(t)

def update():
    global x, xresult, y, yresult, z, zresult, t, timesteps
    nextx = x + (sigma * (y - x)) * Dt
    nexty = y + (rho * x - y - x * z) * Dt
    nextz = z + (x * y - b * z) * Dt
    x, y, z = nextx, nexty, nextz
    t = t + Dt
```

```
# set up graph two
def initialize1():
    global x1, xresult1, y1, yresult1, z1, zresult1, t1, timesteps1
    x1 = 1.001
    y1 = -2.001
    z1 = 1.001
    xresult1 = [x1]
    yresult1 = [y1]
    zresult1 = [z1]
    t1 = 0.
    timesteps1 = [t1]

def observe1():
    global x1, xresult1, y1, yresult1, z1, zresult1, t1, timesteps1
    xresult1.append(x1)
    yresult1.append(y1)
    zresult1.append(z1)
    timesteps1.append(t1)

def update1():
    global x1, xresult1, y1, yresult1, z1, zresult1, t1, timesteps1
    nextx1 = x1 + (sigma * (y1 - x1)) * Dt
    nexty1 = y1 + (rho * x1 - y1 - x1 * z1) * Dt
    nextz1 = z1 + (x1 * y1 - b * z1) * Dt
    x1, y1, z1 = nextx1, nexty1, nextz1
    t1 = t1 + Dt

# iterate through states
initialize()
initialize1()
while t < 42.:
    update()
    observe()
    update1()
    observe1()

# x vs t plot
subplot(3, 1, 1)
plot(timesteps, xresult, 'red')
xlabel('t')
ylabel('x')
plot(timesteps1, xresult1, 'black')
xlabel('t')
ylabel('x')
```

```
# y vs t plot
subplot(3, 1, 2)
plot(timesteps, yresult, 'r')
xlabel('t')
ylabel('y')
plot(timesteps1, yresult1, 'black')
xlabel('t')
ylabel('y')

# z vs t plot
subplot(3, 1, 3)
plot(timesteps, zresult, 'r')
xlabel('t')
ylabel('z')
plot(timesteps1, zresult1, 'black')
xlabel('t')
ylabel('z')

# 3d phase space
figure()
ax = gca(projection='3d')
ax.plot(xresult, yresult, zresult, '#000080')

show()
```

References

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