Math 4A03 - Assignment 2

September 22, 2020

Question 1:

a) Given $x, y \in \ell^1$, we will show that $x + y \in \ell^1$ and $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$.

First fix $t \in \mathbb{N}$ with $x_t = (x(1), \ldots, x(t)), y_t = (y(1), \ldots, y(t)) \in \mathbb{R}^t$. By the triangle inequality for $\mathbb{R}^t$ we have

$$\sum_{i=1}^{t} |x(i) + y(i)| \leq \sum_{i=1}^{t} |x(i)| + \sum_{i=1}^{t} |y(i)|$$

Since each term is a partial sum of the absolute coordinates of x and y, they are monotonically increasing, and so as $t \to \infty$ we have

$$\sum_{i=1}^{t} |x(i)| + \sum_{i=1}^{t} |y(i)| \leq \sum_{i=1}^{\infty} |x(i)| + \sum_{i=1}^{\infty} |y(i)| = \|x\|_1 + \|y\|_1$$

Because $x, y \in \ell^1$, we know that $\|x\|_1 + \|y\|_1$ is a constant meaning $\sum_{i=1}^{t} |x(i) + y(i)|$ is both uniformly bounded and monotonically increasing. Then, by the monotone convergence theorem, $\sum_{i=1}^{\infty} |x(i) + y(i)|$ is finite and so $x + y \in \ell^1$. Therefore, we can safely conclude

$$\|x + y\|_1 = \sum_{i=1}^{\infty} |x(i) + y(i)| \leq \sum_{i=1}^{\infty} (|x(i)| + |y(i)|) = \sum_{i=1}^{\infty} |x(i)| + \sum_{i=1}^{\infty} |y(i)| = \|x\|_1 + \|y\|_1$$

b) We will show that $x^n \to x \in \ell^1$ as $n \to \infty$ or equivalently $d(x, x^n) \to 0$ as $n \to \infty$.

First fix some $k \in \mathbb{N}$, then we have

$$d(x, x^k) = \left\|x - x^k\right\|_1 = \sum_{i=1}^{\infty}|x(i) - x^k(i)| = \sum_{i=1}^{k}|x(i) - x^k(i)| + \sum_{i=k+1}^{\infty}|x(i) - x^k(i)|$$

Now, using the facts that $x^k(i) = 0$ when $i \leq k$ and $x^k(i) = x(i)$ when $i > k$, we can simplify our formula to the following

$$\sum_{i=1}^{k}|x(i) - x(i)| + \sum_{i=k+1}^{\infty}|x(i) - 0| = \sum_{i=k+1}^{\infty}|x(i)|$$

Similarly, for $k+1$ we have

$$d(x, x^{k+1}) = \sum_{i=k+2}^{\infty}|x(i)|$$

Notice that this means that $d(x, x^n)$ is a monotonically decreasing sequence in ℓ^1 since for any $k \in \mathbb{N}$,

$$d(x, x^k) = \sum_{i=k+1}^{\infty}|x(i)| = \sum_{i=k+2}^{\infty}|x(i)| + |x(k+1)| \geq \sum_{i=k+2}^{\infty}|x(i)| = d(x, x^{k+1})$$

Finally, using the fact that $d(x, x^n)$ is monotonically decreasing and bounded below by an infimum of 0, we have by the monotone convergence theorem that $d(x, x^n) \to 0$ as $n \to \infty$ or equivalently, $x^n \to x \in \ell^1$ as $n \to \infty$.

c) To find such an $x \in \ell^\infty$, we need its sequence $(x^n)_{n\in\mathbb{N}}$ to satisfy $d(x, x^n) \not\to 0$ as $n \to \infty$. So we can use any sequence that does not vanish so that $d(x, x^n)$ remains significant as $n \to \infty$. For example, a constant sequence say $x = (13, 13, 13, 13\ldots,)$ where $x(i) = 13, \forall i \in \mathbb{N}$ will do the job. Note that $x \in \ell^\infty$ since $\|x\|_\infty = \sup_i|x(i)| = 13$ and for each fixed $n \in \mathbb{N}$, we have $\|x^n\|_\infty = 13$ and so $x^n \in \ell^\infty$ as well, thus, we can safely describe $\lim_{n\to\infty}(x^n)_{n\in\mathbb{N}}$. Now, for all $n \in \mathbb{N}$ we have

$$d(x, x^n) = \|x - x^n\|_\infty = \sup_i|x(i) - x^n(i)| = \sup_i|(0, 0, 0, \ldots, x(n+1), x(n+2), \ldots)|$$

$$= \sup_i|(0, 0, 0, \ldots, 13, 13, \ldots)| = 13$$

So $d(x, x^n) \not\to 0$ as $n \to 0$. Thus, we have $x = (13, 13, 13, 13, \ldots,) \in \ell^\infty$ with $x^n \not\to x$ as $n \to \infty$.