Math 4A03 - Assignment 5

November 15, 2020

Question 1: Let (X, d) and (Y, ρ) be metric spaces and let $f, f_n : X \to Y$ with $f_n \rightrightarrows f$ on X. Suppose each f_n is continuous at $x \in X$ and $x_n \to x$ in X, then we show $\lim_{n\to\infty} f_n(x_n) = f(x)$.

Let $\epsilon > 0$ be given, then since $f_n \rightrightarrows f$, by the definition of uniform convergence we can choose $N_1 \in \mathbb{N}$ such that

$$\rho(f_n(x), f(x)) < \frac{\epsilon}{2}, \forall n \geq N_1$$

Now using the assumptions $x_n \to x$ in X and each f_n is continuous at $x \in X$, we have by the definition of continuity that

$$f_n(x_n) \to f_n(x) \text{ in } Y \text{ for each } f_n$$

That is, we can choose $N_2 \in \mathbb{N}$ such that $\rho(f_n(x_n), f_n(x)) < \frac{\epsilon}{2}, \forall n \geq N_2$. But then for any given $\epsilon > 0$, we can choose $N = \max\{N_1, N_2\} \in \mathbb{N}$ such that

$$\rho(f_n(x_n), f(x)) \leq \rho(f_n(x_n), f_n(x)) + \rho(f_n(x), f(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \forall n \geq N$$

or equivalently,

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

Question 2: Let (f_n) be a sequence of functions in $C[0,1]$ and that $f_n \rightrightarrows f$ on $[0,1]$, then we show that $\int_0^{1-1/n} f_n \to \int_0^1 f$.

Since each f_n is continuous and $f_n \rightrightarrows f$ on $[0,1]$, Theorem 10.4 implies that f is continuous on $[0,1]$. Thus, we may safely define the sequence of functions $F_n(x) = \int_0^x f_n$ and the function $F(x) = \int_0^x f$ on $C[0,1]$ since the continuity of each f_n and f implies the existence and continuity of each $F_n(x)$ and $F(x)$ on $[0,1]$. Now we must ensure that $F_n(x)$ converges uniformly to $F(x)$ on $[0,1]$ as follows

$$\begin{aligned} |F_n(x) - F(x)| &= \left| \int_0^x f_n(t)dt - \int_0^x f(t)dt \right| \\ &\leq \int_0^x |f_n(t) - f(t)|\, dt \\ &\leq (x-0)\, \|f_n - f\|_\infty \to 0 \end{aligned}$$

Note the last line goes to zero since we have assumed $f_n \rightrightarrows f$ on $[0,1]$ and furthermore, the sequence $(1-1/n)$ goes to 1 since

$$\forall \epsilon > 0, N = \frac{1}{\epsilon} \implies |1 - 1/n - 1| = |1/n| < \epsilon, \forall n > N$$

So by Question 1, since each each $F_n(x)$ is continuous, $F_n(x) \rightrightarrows F(x)$, and $(1-1/n) \to 1$ on $[0,1]$, we have

$$\begin{aligned} &\lim_{n\to\infty} F_n(1-1/n) = F(x) \\ &\equiv \lim_{n\to\infty} \int_0^{1-1/n} f_n = \int_0^1 f \\ &\equiv \int_0^{1-1/n} f_n \to \int_0^1 f \end{aligned}$$

as required.