Math 4A03 - Assignment 3

September 30, 2020

Question 1: (each part starts on a new page)

a) We want to show

$$F = \left\{ x = (x(1), \dots,) : |x(k)| \leq \frac{1}{k^2}, \forall k \in \mathbb{N} \right\} \subset \ell^1$$

is closed. Our strategy is to use sequences; that is, by Theorem 4.9 in the text, we have that F is closed if and only if all convergent sequences in F converge to a point in F. So let $(x^t)_{t \in \mathbb{N}}$ be any sequence in F that converges to some $x \in \ell^1$, then we need to show that $x \in F$. In other words, we need to show that

$$|x(k)| \leq \frac{1}{k^2}, \forall k \in \mathbb{N}$$

First, we establish a useful fact that we will use later. For any $y \in \ell^1$, the absolute value of any specific index $|y(i)|$ is less than or equal to the norm of the element $\|y\|_1$. This is because the series in the norm is monotonically increasing and $y \in \ell^1$, so we have the following for each fixed index $i \in \mathbb{N}$

$$|y(i)| \leq |y(1)| + \dots + |y(i)| \leq \sum_{j=1}^{\infty} |y(j)| = \|y\|_1 < \infty \tag{1}$$

Now since x^t converges to x, by definition we have that $\left\|x - x^t\right\|_1$ can be made arbitrarily small since

$$\forall \epsilon > 0, \exists T \in \mathbb{N} \text{ such that } \left\|x - x^t\right\|_1 < \epsilon \text{ whenever } t \geq T \tag{2}$$

Then using (1), (2), and transitivity, we have that any index $k \in \mathbb{N}$ can also be made arbitrarily small since

$$\left|x(k) - x^t(k)\right| \leq \left\|x - x^t\right\|_1 < \epsilon \tag{3}$$

In fact, using the definition of absolute value, monotonicity, and transitivity, we get

$$x^t(k) - \epsilon < x(k) < x^t(k) + \epsilon \tag{4}$$

But by construction, every element in the sequence (x^t) is in F, so by definition we have

$$\left|x^t(k)\right| \leq \frac{1}{k^2} \tag{5}$$

Then combining (4) and (5) gives us

$$\frac{-1}{k^2} - \epsilon \leq x^t(k) - \epsilon < x(k) < x^t(k) + \epsilon \leq \frac{1}{k^2} + \epsilon \tag{6}$$

or more succinctly

$$\frac{-1}{k^2} - \epsilon < x(k) < \frac{1}{k^2} + \epsilon \tag{7}$$

Notice that this strict inequality holds for every arbitrarily small $\epsilon > 0$ so it is equivalent to write (7) as

$$\left(\frac{-1}{k^2} \leq x(k) \leq \frac{1}{k^2}\right) \equiv \left(|x(k)| \leq \frac{1}{k^2}\right)$$

as required.

b) We want to show

$$G = \left\{ x = (x(1), \dots,) : |x(k)| < \frac{1}{k^2}, \forall k \in \mathbb{N} \right\} \subset \ell^1$$

is not open. Our strategy is to find an element in G with no neighbourhood that is contained in G. That is, we must show that

$$\nexists \epsilon > 0 \text{ such that } B_\epsilon(x) \subset G \text{ for some } x \in G$$

We will use 0 which is an element of G since all indices are 0 and are trivially less than $\frac{1}{k^2}, \forall k \in \mathbb{N}$. Now, the Archimedean property states that given any positive real number x, no matter how small, one can always find a fraction $\frac{1}{n}, n \in \mathbb{N}$ that is smaller i.e. $\frac{1}{n} < x$. So, given any $\epsilon > 0$, by the Archimedean property, we will always be able to find a fraction $\frac{1}{k^2}, k \in \mathbb{N}$ less than $\frac{\epsilon}{4}$. This allows us to construct the element

$$y_\epsilon = (0, 0, 0, \dots, \frac{\epsilon}{2}, 0, \dots) \in \ell^1 \tag{8}$$

i.e. the element in ℓ^1 with 0 for each index except for the k^{th} index which is $\frac{\epsilon}{2}$ and where k is the index such that $\frac{1}{k^2} < \frac{\epsilon}{4}$. Note that $y_\epsilon \in B_\epsilon(0)$ since

$$d(y_\epsilon, 0) = \|y_\epsilon - 0\|_1 = |0 - 0| + \dots + |\frac{\epsilon}{2} - 0| + |0 - 0| + \dots = \frac{\epsilon}{2} < \epsilon \tag{9}$$

But we also have the $y_\epsilon \notin G$ since

$$|y_\epsilon(k)| = \frac{\epsilon}{2} > \frac{\epsilon}{4} > \frac{1}{k^2} \tag{10}$$

Thus, no matter which $\epsilon > 0$ we choose, there will always be elements in $B_\epsilon(0)$ that are not in G. Therefore, $B_\epsilon(0) \not\subset G, \forall \epsilon > 0$ and so G is not open.

c) We would like to show $F = \bar{G}$. Our strategy is to show

$$F \subset \bar{G} \text{ and } \bar{G} \subset F$$

First, suppose $x \in F$, then by Proposition 4.10 in the textbook, $x \in \bar{G}$ if and only if $B_\epsilon(x) \cap G \neq \emptyset, \forall \epsilon > 0$. Given any positive ϵ, we have

$$B_\epsilon(x) \cap G = \left\{ y \in \ell^1 : \|y - x\|_1 < \epsilon \text{ and } |y(k)| < \frac{1}{k^2}, \forall k \in \mathbb{N} \right\} \tag{11}$$

We will break up the problem into two cases, $\epsilon > 1$ and $0 < \epsilon \leq 1$. For $\epsilon > 1$, we can construct the element

$$x_\epsilon = \left(\frac{x(1)}{\epsilon}, \ldots, \frac{x(k)}{\epsilon}, \ldots \right) \tag{12}$$

i.e. the element in ℓ^1 that is a scaled down version of $x \in F$. Now for x_ϵ to be in $B_\epsilon(x) \cap G$, we need to check if it satisfies both conditions. The second condition is trivial since

$$x \in F, \epsilon > 1 \implies |x_\epsilon(k)| = \left| \frac{x(k)}{\epsilon} \right| < |x(k)| \leq \frac{1}{k^2}, \forall k \in \mathbb{N} \tag{13}$$

Now for the first condition, we have

$$\begin{aligned} \|x - x_\epsilon\|_1 &= \sum_{i=1}^{\infty} \left| x(i) - \frac{x(i)}{\epsilon} \right| = \sum_{i=1}^{\infty} \left| x(i) \left(1 - \frac{1}{\epsilon}\right) \right| \\ &= \left(1 - \frac{1}{\epsilon}\right) \sum_{i=1}^{\infty} |x(i)| = \frac{\epsilon - 1}{\epsilon} \|x\|_1 \end{aligned} \tag{14}$$

Note that since $x \in F$, we also have the following

$$\|x\|_1 = \sum_{i=1}^{\infty} |x(i)| \leq \sum_{i=1}^{\infty} \left| \frac{1}{i^2} \right| = \frac{\pi^2}{6} \tag{15}$$

So (14) and (15) imply

$$\|x - x_\epsilon\|_1 = \frac{\epsilon - 1}{\epsilon} \|x\|_1 \leq \frac{\epsilon - 1}{\epsilon} \frac{\pi^2}{6} \tag{16}$$

But since $\epsilon > 1$, $\frac{\epsilon-1}{\epsilon}\frac{\pi^2}{6} < \epsilon$ is always true. To see this, consider the equivalent expression

$$\left(\frac{\epsilon - 1}{\epsilon} \frac{\pi^2}{6} < \epsilon \right) \equiv \left(0 < \frac{6}{\pi^2} \epsilon^2 - \epsilon + 1 \right) \tag{17}$$

This is a quadratic in ϵ with complex roots and positive opening thus is always greater than zero. So, x_ϵ satisfies both conditions i.e. $\|x - x_\epsilon\|_1 < \epsilon$ and $x_\epsilon(k) < \frac{1}{k^2}, \forall k \in \mathbb{N}$; therefore, $x_\epsilon \in B_\epsilon(x) \cap G$ for $\epsilon > 1$.

For the next case $0 < \epsilon \leq 1$, we construct the element

$$x^\epsilon = \left(\left(1 - \frac{6\epsilon}{2\pi^2}\right) x(1), \ldots, \left(1 - \frac{6\epsilon}{2\pi^2}\right) x(k), \ldots \right) \tag{18}$$

which is another scaled down version of x in ℓ^1. As before, the second condition is easily satisfied since

$$x \in F, 0 < \epsilon \leq 1 \implies |x^\epsilon(k)| = \left|\left(1 - \frac{6\epsilon}{2\pi^2}\right) x(k)\right| < |x(k)| \leq \frac{1}{k^2}, \forall k \in \mathbb{N} \tag{19}$$

Similarly for condition one, we get

$$\begin{aligned} \|x - x^\epsilon\|_1 &= \sum_{i=1}^{\infty} \left| x(i) - \left(1 - \frac{6\epsilon}{2\pi^2}\right) x(i) \right| = \sum_{i=1}^{\infty} \left| x(i) \left(\frac{6\epsilon}{2\pi^2}\right) \right| \\ &= \left(\frac{6\epsilon}{2\pi^2}\right) \sum_{i=1}^{\infty} |x(i)| = \left(\frac{6\epsilon}{2\pi^2}\right) \|x\|_1 \end{aligned} \tag{20}$$

As before, we have

$$\|x - x^\epsilon\|_1 = \left(\frac{6\epsilon}{2\pi^2}\right) \|x\|_1 \leq \left(\frac{6\epsilon}{2\pi^2}\right) \frac{\pi^2}{6} = \frac{\epsilon}{2} < \epsilon \tag{21}$$

So, x^ϵ satisfies both conditions i.e. $\|x - x^\epsilon\|_1 < \epsilon$ and $x^\epsilon(k) < \frac{1}{k^2}, \forall k \in \mathbb{N}$; therefore, $x^\epsilon \in B_\epsilon(x) \cap G$ for $0 < \epsilon \leq 1$. Thus, $\forall \epsilon > 0$, we have that $B_\epsilon(x) \cap G \neq \emptyset$ and so, $x \in \bar{G}$.

For the other direction, suppose that $x \in \bar{G}$, then we must show $x \in F$. Since $x \in \bar{G}$, it must be either a point of G or a limit point of G (from class notes). If it is a point a G, then it is trivially in F since

$$x \in G \implies |x(k)| < \frac{1}{k^2}, \forall k \in \mathbb{N} \implies |x(k)| \leq \frac{1}{k^2}, \forall k \in \mathbb{N} \implies x \in F \tag{22}$$

If x is a limit point of G then it must be the limit of a convergent sequence (x^t) in G. But all sequence elements of (x^t) have the property that each value at index $k \in \mathbb{N}$ is bounded by $\left|x^t(k)\right| < \frac{1}{k^2}$. So, if (x^t) converges to x, by part a) equations (1) to (7), we must have

$$\frac{-1}{k^2} - \epsilon < x(k) < \frac{1}{k^2} + \epsilon, \forall \epsilon > 0$$

or equivalently

$$|x(k)| \leq \frac{1}{k^2}$$

Thus, the limiting value of each index $k \in \mathbb{N}$ from a sequence in G is bounded by $\frac{1}{k^2}$ and so $x \in F$.

Together, this implies that $F \subset \bar{G}$ and $\bar{G} \subset F$ and so $F = \bar{G}$.