Math 4A03 - Assignment 6

December 5, 2020

Question 1:

Assume that $f : \mathbb{R} \to \mathbb{R}$ is bounded and continuous on $(-\infty, 0) \cup (0, \infty)$ with a jump discontinuity at $x = 0$. That is both limits below exist:

$$f(0-) = \lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x) = f(0+)$$

Let $\varphi_n(x), n \in \mathbb{N}$ be the family of mollifiers defined in class:

(1) $\varphi_n \in C_0^\infty(\mathbb{R})$ with $\text{supp}(\varphi_n) \subseteq [\frac{-1}{n}, \frac{1}{n}]$;

(2) $\varphi_n(x) \geq 0$, for all $x \in \mathbb{R}$;

(3) $\varphi_n(-x) = \varphi_n(x)$, for all $x \in \mathbb{R}, n \in \mathbb{N}$;

(4) $\int_{-\infty}^{\infty} \varphi_n(t)dt = 1$, for all $n \in \mathbb{N}$

Define

$$f_n(x) = \int_{-\infty}^{\infty} f(t)\varphi_n(x - t)dt, \qquad x \in \mathbb{R}, n \in \mathbb{N}, \qquad (5)$$

Then we show that

$$\lim_{n \to \infty} f_n(0) = \frac{1}{2}(f(0-) + f(0+))$$

Proof:

Let $\epsilon > 0$ be given, then since both one sided limits exist we can choose $\delta_1, \delta_2 > 0$ such that

$$x \in (0, \delta_1) \implies \left|f(x) - f(0^+)\right| < \epsilon \text{ and } x \in (-\delta_2, 0) \implies \left|f(x) - f(0^-)\right| < \epsilon$$

Now using the definitions and assumptions above, we have the following calculation

$\left|f_n(0) - \frac{1}{2}\left(f(0^-) + f(0^+)\right)\right|$

$= \left|\int_{-\infty}^{\infty} f(t)\varphi_n(0-t)dt - \frac{1}{2}\left(f(0^-) + f(0^+)\right)\int_{-\infty}^{\infty} \varphi_n(t)dt\right|, (4), (5)$

$= \left|\int_{-\infty}^{\infty} f(t)\varphi_n(t)dt - \frac{1}{2}\left(f(0^-) + f(0^+)\right)\int_{-\infty}^{\infty} \varphi_n(t)dt\right|, (3)$

$= \left|\int_{-\infty}^{\infty} \left[f(t) - \frac{1}{2}\left(f(0^-) + f(0^+)\right)\right]\varphi_n(t)dt\right|$

$= \left|\int_{-1/n}^{1/n} \left[f(t) - \frac{1}{2}\left(f(0^-) + f(0^+)\right)\right]\varphi_n(t)dt\right|, (1)$

$\leq \sup_{t\in[-1/n,1/n]/\{0\}} \left|f(t) - \frac{1}{2}\left(f(0^-) + f(0^+)\right)\right| \int_{-1/n}^{1/n} |\varphi_n(t)|dt$

$= \sup_{t\in[-1/n,1/n]/\{0\}} \left|f(t) - \frac{1}{2}\left(f(0^-) + f(0^+)\right)\right|, (2), (4)$

$= \sup_{t\in[-1/n,1/n]/\{0\}} \left|\frac{1}{2}f(t) - \frac{1}{2}f(0^-) + \frac{1}{2}f(t) - \frac{1}{2}f(0^+)\right|$

$\leq \sup_{t\in[-1/n,1/n]/\{0\}} \frac{1}{2}\left|f(t) - f(0^-)\right| + \frac{1}{2}\left|f(t) - f(0^+)\right|$

$< \epsilon/2 + \epsilon/2 = \epsilon, \forall n > \frac{1}{\lceil \max\{\delta_1,\delta_2\}\rceil}$ since $n > \frac{1}{\lceil \max\{\delta_1,\delta_2\}\rceil} \implies [-1/n, 1/n]/\{0\} \subset (-\delta_2, 0) \cup (0, \delta_2)$

Thus, for any $\epsilon > 0$, we choose $N = \frac{1}{\lceil \max\{\delta_1,\delta_2\}\rceil} \in \mathbb{N}$ such that $\left|f_n(0) - \frac{1}{2}\left(f(0^-) + f(0^+)\right)\right| < \epsilon$ for $n > N$ and so $\lim_{n\to\infty} f_n(0) = \frac{1}{2}(f(0-) + f(0+))$ as required.