Math 4A03 - Assignment 4

November 1, 2020

Question 1: Let (M, d) be any metric space.

a) Suppose $(K_t)_{t\in\mathbb{N}}$ is a decreasing sequence of nonempty compact sets in M, then we show that $\cap_{t\in\mathbb{N}} K_t \neq \emptyset$.

Let $(x_i)_{i\in\mathbb{N}}$ be a sequence such that $x_j \in K_j$, we can choose such a sequence since we have assumed that K_j is nonempty for all j. Since the sets are decreasing, the entire sequence is in the first set K_1. Now K_1 is compact so $(x_i)_{i\in\mathbb{N}}$ contains a convergent subsequence $(x_{i_q})_{q\in\mathbb{N}}$ with $(x_{i_q}) \to x \in K_1$. If we fix any positve integer z, then because the sets are decreasing, $\exists N \in \mathbb{N}$ such that $x_{i_q} \in K_z$ for $q \geq N$. But K_z is compact so we have $x \in K_z$. Since this is true for all $z \in \mathbb{N}$ we conclude that the compact intersection is nonempty since $x \in \cap_{t\in\mathbb{N}} K_t$. Note that $\cap_{t\in\mathbb{N}} K_t$ is compact because each K_j is compact and therefore closed for all j so the arbitrary intersection is also closed - then since $\cap_{t\in\mathbb{N}} K_t \subseteq K_1$, we have that $\cap_{t\in\mathbb{N}} K_t$ is a closed subset of a compact set which implies that it is also compact (From Corollary 8.3).

b) Suppose $K \subseteq M$ is compact, $f : K \to K$ is continuous, and $(A_t)_{t\in\mathbb{N}}$ is a sequence of sets such that $A_1 = f(K), A_t = f(A_{t-1}), \forall t \in \mathbb{N}$. Then we show that $A = \cap_{t\in\mathbb{N}} A_t$ is nonempty, compact, and invariant.

First, we claim that $(A_t)_{t\in\mathbb{N}}$ is a decreasing sequence of compact sets.

Base case: $n = 1 \implies A_2 \subseteq A_1 \equiv f(f(K)) \subseteq f(K)$ This clearly holds since f is a function from $K \to K$ so either $f(f(K)) = f(K)$ if f is onto or if not, then we lose points so $f(f(K)) \subset f(K)$. We also have that K is compact and f continuous so both sets are compact (Theorem 8.4).

Suppose this holds for all sets up to $n \in \mathbb{N}$, then we prove $n + 1$ holds.

Induction step: $A_{n+1} \subseteq A_n \equiv f(A_n) \subseteq f(A_{n-1})$ This holds since the induction hypothesis asserts $A_n \subseteq A_{n-1}$ and since f is continuous and the induction hypothesis asserts A_n is compact $A_{n+1} = f(A_n)$ is compact as well.

Thus, $(A_t)_{t\in\mathbb{N}}$ is a decreasing sequence of compact sets. By part a), this means that the intersection $A = \cap_{t\in\mathbb{N}} A_t$ is nonempty and compact. Now we just need to show that A is invariant.

First fix a positive integer c. Then we have

$$(A_1 \cap \cdots \cap A_C) \subseteq A_i, i = 1, \ldots, c \implies f((A_1 \cap \cdots \cap A_C)) \subseteq f(K) \cap f(A_1) \cap \cdots \cap f(A_c)$$

Since this is true for any fixed c, we take the limit $c \to \infty$ to get

$$f(A) \subseteq \cap_{i=1}^{\infty} f(A_i) \cap f(K) = \cap_{i=1}^{\infty} A_{i+1} \cap A_1 = A$$

Similarly for the other direction, fix any positive integer c. Now since the sequence of sets is decreasing, we have for any $n \in \mathbb{N}, A_n \subseteq A_{n+1} = f(A_n)$. This implies the following

$$A_1 \cap \cdots \cap A_c \subseteq A_2 \cap \cdots \cap A_{c+1} = f(A_1) \cap \cdots \cap f(A_c)$$

Since this is true for any fixed c, we take the limit $c \to \infty$ to get

$$A \subseteq f(A)$$

So since $f(A) \subseteq A$ and $A \subseteq f(A)$, we have $f(A) = A$. Therefore, A is nonempty, compact and invariant.