Math 4A03 - Assignment 1

September 15, 2020

Question 1: $\mathcal{M}_1 = (M, d_1)$ and $\mathcal{M}_2 = (M, d_2)$ are metric spaces and thus, d_1 and d_2 are metrics.

a) We prove that $d_3 = \max\{d_1, d_2\}$ is a metric.

(i) First, we need $0 \leq d_3(x, y) < \infty, \forall x, y \in M$. Now, $d_3(x, y) = \max\{d_1, d_2\}$, thus, the range of d_3 is bounded by the range of d_1 and d_2. Since d_1 and d_2 are metrics on M, we have $0 \leq d_1(x, y), d_2(x, y) < \infty, \forall x, y \in M$, i.e. the range of d_1 and d_2 is bounded to non-negative finite values for any $x, y \in M$. Therefore, for any $x, y \in M$, d_3 will take on a non-negative finite value dictated by either d_1 or d_2 (or both if they are equal), more succinctly, $0 \leq d_3(x, y) < \infty$ for any $x, y \in M$, as required.

(ii) Next, we require $d_3(x, y) = 0 \iff x = y$. Suppose that $d_3(x, y) = \max\{d_1, d_2\} = 0$. Then both $d_1(x, y)$ and $d_2(x, y)$ must be zero since they are non-negative and if one was positive, $d_3(x, y)$ would take that value instead of zero. Now, since d_1 and d_2 are metrics, we have that $d_1(x, y), d_2(x, y) = 0 \implies x = y$, as required. On the other hand, suppose that $x = y$, then we have $d_3(x, y) = d_3(x, x) = \max\{d_1(x, x), d_2(x, x)\}$. Since d_1 and d_2 are metrics, we have $d_1(x, x), d_2(x, x) = 0 \implies d_3(x, y) = \max\{d_1(x, x), d_2(x, x)\} = \max\{0, 0\} = 0$, as required.

(iii) Now, we need to show $d_3(x, y) = d_3(y, x), \forall x, y \in M$. Let x and y be any elements in M, then by the symmetry of the metrics d_1 and d_2 we have $d_3(x, y) = \max\{d_1(x, y), d_2(x, y)\} = \max\{d_1(y, x), d_2(y, x)\} = d_3(y, x)$, as needed.

(iv) Lastly, we prove $d_3(x,y) \leq d_3(x,z) + d_3(z,y), \forall x,y,x \in M$. Starting from the left with arbitrary $x,y,z \in M$ and $d_1(x,y) > d_2(x,y)$, we have $d_3(x,y) = \max\{d_1, d_2\} = d_1(x,y)$. Now since d_1 is a metric we have $d_3(x,y) = d_1(x,y) \leq d_1(x,z) + d_1(z,y) \leq \max\{d_1(x,z), d_2(x,z)\} + \max\{d_1(z,y), d_2(z,y)\} = d_3(x,z) + d_3(z,y)$. Since x,y, and z are arbitrary, we have $d_3(x,y) \leq d_3(x,z) + d_3(z,y), \forall x,y,z \in M$, as required. If $d_1(x,y)$ is equal to $d_2(x,y)$ we can use the same argument and if $d_1(x,y) < d_2(x,y)$ then we can use an analogous argument by switching the indices since d_2 is also a metric. For example, $d_3(x,y) = d_2(x,y) \leq d_2(x,z) + d_2(z,y) \leq \max\{d_1(x,z), d_2(x,z)\} + \max\{d_1(z,y), d_2(z,y)\} = d_3(x,z) + d_3(z,y)$.

b) We prove that $d_4(x,y) = \min\{d_1, d_2\}$ is not a metric.

Let $M = \mathbb{R}^2, d_1((x_1,y_1),(x_2,y_2)) = 5|x_1 - x_2| + |y_1 - y_2|, d_2((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + 5|y_1 - y_2|$. Notice that d_1 is much more efficient in the vertical direction and d_2 is much more efficient in the horizontal direction which is why the triangle inequality fails. You would much rather partition your travels into strictly vertical and horizontal segments as they would use the more efficient metrics rather than using one expensive metric to travel directly. For a specific example, let $x = (0,0), y = (5,5)$, and $z = (0,5)$. Then, we have $d_4(x,y) = \min\{30,30\} = 30, d_4(x,z) = \min\{25,5\} = 5, d_4(z,y) = \min\{5,25\} = 5$. This implies that the triangle inequality fails since $d_4(x,y) = 30 \nleq 5 + 5 = d_4(x,z) + d(z,y)$.