$$f_n(x) = \int_{-\infty}^{\infty} f(t)\varphi_n(x-t)\,dt, \quad \text{so}$$

$$f_n(0) = \underbrace{\int_{-\infty}^{0} f(t)\varphi_n(0-t)\,dt}_{} + \underbrace{\int_{0}^{\infty} f(t)\varphi_n(0-t)\,dt}_{}$$

$$=: \quad I_n^- \quad + \quad I_n^+$$

As $\varphi_n(-x) = \varphi_n(x)$, $\operatorname{supp}\varphi_n = [-\frac{1}{n}, \frac{1}{n}]$,

$$\int_0^\infty \varphi_n(t)\,dt = \int_0^{1/n} \varphi_n(t)\,dt = \frac{1}{2} = \int_{-\frac{1}{n}}^{0} \varphi_n(t)\,dt = \int_{-\infty}^{0} \varphi_n(t)\,dt.$$

Since $\lim_{x\to 0^+} f(x) = f(0+)$, $\lim_{x\to 0^-} f(x) = f(0-)$,

$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0$ with both:

$|f(t) - f(0-)| < \frac{\varepsilon}{2}$ $\forall\, t \in (-\delta, 0)$ and $|f(t) - f(0+)| < \frac{\varepsilon}{2}$ $\forall\, t \in [0, \delta)$

Now, let $n \in \mathbb{N}$ with $\frac{1}{n} < \delta$. Then

$$|I_n^+ - \tfrac{1}{2} f(0^+)| = \left| \int_0^\infty f(t)\varphi_n(0-t)\,dt - \tfrac{1}{2} f(0+)\right|$$

$$= \left| \int_0^{1/n} f(t)\varphi_n(t)\,dt - \left(\int_0^{1/n} \varphi_n(t)\,dt\right) f(0+)\right|$$

$$= \left| \int_0^{1/n} \big(f(t) - f(0+)\big)\varphi_n(t)\,dt \right|$$

$$\leq \int_0^{1/n} \underbrace{|f(t) - f(0+)|}_{<\varepsilon,\ \text{since } 0 \leq t < \frac{1}{n} < \delta} \varphi_n(t)\, dt \qquad \text{as } \varphi_n(\cdot) \geq 0$$

$$< \varepsilon \cdot \int_0^{1/n} \varphi_n(t)\, dt = \frac{\varepsilon}{2}, \quad \forall\ n > \frac{1}{\delta}.$$

Similarly,

$$\left| I_n^- - \tfrac{1}{2} f(0-) \right| < \frac{\varepsilon}{2} \quad \forall\ n > \frac{1}{\delta}$$

by exactly the same steps.

$$\therefore \left| f_n(0) - \tfrac{1}{2}\left(f(0+) + f(0-)\right) \right|$$

$$= \left| \left[I_n^+ - \tfrac{1}{2} f(0+) \right] + \left[I_n^- - \tfrac{1}{2} f(0-) \right] \right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall\ n > \frac{1}{\delta}.$$