MATH 4L03/6L03 Assignment #4

Due: Friday, November 6, 11:59pm.

Upload your solutions to the Avenue to Learn course website. Detailed instructions will be provided on the course website.

Unless otherwise noted, you may argue informally about the satisfaction (or not) of formulas by structures, rather than working through, in each case, the formal definition of satisfaction given in the textbook.

1. Let L be the first order language that has a single 2 place relation symbol E. Show that if $\mathcal{A} = \langle A, E^{\mathcal{A}} \rangle$ is an L-structure that satisfies the following two sentences:

$$\forall x\,(E(x,x))$$

$$\forall x \forall y \forall z\,((E(x,y) \wedge E(y,z)) \rightarrow E(z,x))$$

then it also satisfies the sentence

$$\forall x \forall y\,(E(x,y) \rightarrow E(y,x))\,.$$

Note that this provides a slightly shorter axiomatization of the class of equivalence relation structures.

2. (a) Using the same language L as in the previous question, show that there is no L-structure $\mathcal{A} = \langle A, E^{\mathcal{A}} \rangle$ which satisfies the sentences:

$$\exists x \forall y\,(E(x,y))$$

$$\exists x \forall y\,(\neg E(x,y))$$

$$\forall x \forall y((E(x,y) \rightarrow E(y,x))\,.$$

   (b) Is there an L-structure $\mathcal{A} = \langle A, E^{\mathcal{A}} \rangle$ that satisfies the sentences:

$$\forall x \exists y\,(E(x,y))$$

$$\forall x \forall y\,(E(x,y) \rightarrow \neg E(y,x))$$

$$\forall x \forall y \forall z\,((E(x,y) \wedge E(y,z)) \rightarrow E(x,z))?$$

this is supposed to be the usual equality symbol

3. Let $\underline{\mathbb{N}}$ be the structure $\langle \mathbb{N}, +, \cdot, 0, 1, \leq \rangle$, where the function symbols $+$ and $\cdot$, the constant symbols 0 and 1 and the predicate symbol $\leq$ have their usual interpretations on the set of natural numbers. Determine which of the following sentences are satisfied by the structure $\underline{\mathbb{N}}$:

   i) $\forall x \exists y (x \approx y + y \vee x \approx (y + y) + 1)$,

   ii) $\forall x \forall y \exists z (x + z \approx y)$

   iii) $\forall x \forall y (x \leq y \leftrightarrow \exists z (x + z \approx y))$.

I meant "universally valid"

4. Let L be the first order language with two 1-place relation symbols A and B and one 2-place relation symbol C. Which of the following sentences are logically valid?

   a) $[\forall x A(x) \to \forall x B(x)] \to [\forall x (A(x) \to B(x))]$,

   b) $\forall x (A(x) \to B(x)) \to (\forall x A(x) \to \forall x B(x))$,

   c) $\forall x \forall y \forall z [C(x,x) \wedge (C(x,z) \to [C(x,y) \vee C(y,z)])] \to \exists y \forall z C(y,z)$.

5. Let L be a first order language with equality. For each natural number n, find sentences α_n, β_n and γ_n such that for all normal L-structures $\mathcal{A}$:

   (a) $\mathcal{A} \models \alpha_n$ iff A has exactly n elements,

   (b) $\mathcal{A} \models \beta_n$ iff A has at least n elements,

   (c) $\mathcal{A} \models \gamma_n$ iff A has at most n elements.

   Find a set Σ of sentences such that a normal L-structure $\mathcal{A} \models \Sigma$ iff A is infinite. Note that the set Σ must consist of infinitely many sentences (we will prove this later).

6. Let L be a first order language, Γ a set of L-formulas and ϕ and ψ L-formulas. Note that these formulas might not be sentences, i.e., they may have free variables.

   (a) Show that $\Gamma \cup \{\phi\} \models \psi$ if and only if $\Gamma \models (\phi \to \psi)$.

   (b) Show that $\phi \models \psi$ and $\psi \models \phi$ if and only if $(\phi \leftrightarrow \psi)$ is valid. You can use part (a) for part (b).

7. Find a formula ϕ in prenex normal form that is logically equivalent to the formula

$$(\exists x \forall y R(x, y) \vee \forall x \exists y Q(x, y)).$$

8. Do exercise 4.22 on page 161 of the textbook.

9. Do exercise 4.28 (e) from page 165 of the textbook.