Math 4L03 - Assignment 4

November 6, 2020

Question 1:

Let L be a first-order language with binary relation E and let $\mathcal{A} = \left\langle A, E^{\mathcal{A}} \right\rangle$ be an L-structure that satisfies
(i) $\forall x(E(x,x))$
(ii) $\forall x \forall y \forall z((E(x,y) \wedge E(y,z)) \rightarrow E(z,x))$
then we show it also satisfies $\forall x \forall y(E(x,y) \rightarrow E(y,x))$.

By definition, (ii) states that for every $x, y, z \in A$, $((E(x,y) \wedge E(y,z)) \rightarrow E(z,x))$ holds, so in particular, when $z = y \in A$ we have that for every $x, y \in A$, $((E(x,y) \wedge E(y,y)) \rightarrow E(y,x))$ holds. But (i) asserts that for every $x \in A$, $E(x,x)$ is satisfied, so $E(y,y)$ is true in $((E(x,y) \wedge E(y,y)) \rightarrow E(y,x))$. Now true is the identity of the $\wedge$ truth table so we have for every $x, y \in A$, $(E(x,y) \rightarrow E(y,x))$ holds i.e. $\forall x \forall y(E(x,y) \rightarrow E(y,x))$ is satisfied by $\mathcal{A}$.

Question 2:

a) Using the same language from Question 1, we assume
(i) $\exists x \forall y (E(x, y))$
(ii) $\exists x \forall y (\neg E(x, y))$
and show $\forall x \forall y (E(x, y) \to E(y, x))$ cannot be satisfied.

The first formula states that there is some $x \in A$ such that for every $y \in A$, $E(x, y)$ holds, call it x_i. The second states that that there is some $x \in A$ such that for every $y \in A$, $\neg E(x, y)$ holds, call it x_{ii}. Now, the third formula insists that for any $x, y \in A$, $(E(x, y) \to E(y, x))$ holds, so in particular, it should hold for $x = x_i, y = x_{ii}$. But then we have $(E(x_i, x_{ii}) \to E(x_{ii}, x_i))$ which evaluates to true $\to$ false under our assumptions. This is of course false when considering the $\to$ truth table so $\forall x \forall y (E(x, y) \to E(y, x))$ cannot be satisfied. Therefore, there is no structure that can satisfy all three formulas.

b) An L-structure that satisfies
(i) $\forall x \exists y (E(x, y))$
(ii) $\forall x \forall y (E(x, y) \to \neg E(y, x))$
(iii) $\forall x \forall y \forall z ((E(x, y) \wedge E(y, z)) \to E(x, z))$
is $\mathcal{A} = \langle \mathbb{R}, < \rangle$ where $<$ is the usual order on the reals.

First, (i) is satisfied since the reals are unbounded so for any real, there is always one that is larger. The second is satisfied since we are using the strict inequality, so if x is less than y in the reals, it is clear that y is not less than x. Lastly, the order on the reals is transitive so if x is less than y and y is less than z, then of course x is less than z.

Question 3:

Let $\underline{\mathbb{N}}$ be the structure $\langle \mathbb{N}, +, \cdot, 0, 1, \leq \rangle$ under the usual interpretations of the natural numbers.

(i) $\forall x \exists y (x = y + y \vee x = (y + y) + 1)$
This is true for $\underline{\mathbb{N}}$ since what it really says is that every natural number is either even or odd which is of course true.

(ii) $\forall x \forall y \exists z (x + z = y)$
This is false for $\underline{\mathbb{N}}$ since if x is strictly larger than y, then $x + z$ can never equal y as it will be strictly larger for any choice of z.

(iii) $\forall x \forall y (x \leq y \leftrightarrow \exists z (x + z = y))$
This is true for $\underline{\mathbb{N}}$ as it considers both cases that arose in (ii). That is, if x is strictly larger than y then both statements will be false and in the other case, if x is less than or equal to y then we can add zero or find some natural to add to x so that it equals y as the natural numbers are unbound above so both will be true. As a result, the $\leftrightarrow$ truth table will be satisfied by both cases so the entire formula is true.

Question 4:

Let L be the first order language with two 1-place relation symbols A and B and one 2-place relation symbol C.

a) $[\forall x A(x) \to \forall x B(x)] \to [\forall x(A(x) \to B(x))]$

This is not universally valid. Consider the L-structure $\mathcal{N} = \big\langle \{0,1\}, A^{\mathcal{N}} = \{0\}, B^{\mathcal{N}} = \{1\} \big\rangle$. This would make $\forall x A(x)$ false since $1 \notin A$ and so the entire left side of the principle connective would vacuously evaluate to true by the $\to$ truth table. The right side of the principle connective would evaluate to false since $0 \in A$ but $0 \notin B$. So the main formula would evaluate to true $\to$ false which is false by the $\to$ truth table.

b) $\forall x(A(x) \to B(x)) \to (\forall x A(x) \to \forall x B(x))$

To falsify this formula, the left of the principle connective must be true and the right must be false or else it will be vacuously true. Further, for the right side to be false, $\forall x A(x)$ must be true and $\forall x B(x)$ must be false. But this is impossible since our assumptions state that every x in the domain is in A and every x that is in A is also in B i.e. it cannot be the case that there is some element in the domain that is not in B, so this formula is universally valid.

c) $\forall x \forall y \forall z[C(x,x) \land (C(x,z) \to [C(x,y) \lor C(y,z)])] \to \exists y \forall z C(y,z)$.

This is universally valid. If the left side is false then it is vacuously true. If the left side holds, then the $C(x,x)$ holds for all x which in turn generates all possible tuples of other elements through $C(x,x) \to [C(x,y) \lor C(y,x)])]$ so in some sense all tuples are generated and we get $\exists y \forall z C(y,z)$ holds.

Question 5:

Let L be a first order language with equality and let A be the domain of a normal L-structure $\mathcal{A}$.

a) First we need a sentence that is true iff A has exactly n elements. A way to express this using equality is to say that there are n elements that are not equal to each other and they are unique. So we have

$$\alpha_n = \exists x_1 \ldots \exists x_n \left[\bigwedge_{1 \leq i < j \leq n} (\neg x_j = x_j) \wedge \forall y \left[\bigvee_{i=1}^{n} y = x_i \right] \right]$$

b) Now we need a sentence that is true iff A has at least n elements. To express this using equality is to say that there are n elements that are not equal to each other. Right away we can see that this is captured in the first part of the formula in a) resulting in

$$\beta_n = \exists x_1 \ldots \exists x_n \left[\bigwedge_{1 \leq i < j \leq n} (\neg x_j = x_j) \right]$$

c) Lastly we need a sentence that is true iff A has at most n elements. We can capture this by saying that there can only be n elements equal to each other. That is

$$\gamma_n = \forall x_1 \ldots \forall x_{n+1} \left[\bigvee_{1 \leq i < j \leq n+1} x_i = x_j \right]$$

An example of a set of sentences Σ such that $\mathcal{A} \models \Sigma$ iff A is infinite is

$$\Sigma = \{\beta_n : n \in \mathbb{N}\}$$

Question 6:

Let L be a first order language, Γ a set of L-formulas and ϕ and ψ L-formulas.

a) First we show $\Gamma \cup \{\phi\} \models \psi$ iff $\Gamma \models (\phi \to \psi)$.

Assuming $\Gamma \cup \{\phi\} \models \psi$ then we have that for every L-structure $\mathcal{A}$ and interpretation $[\overrightarrow{x}/\overrightarrow{a}]$ if $\mathcal{A} \models_{[\overrightarrow{x}/\overrightarrow{a}]} \gamma$ for all $\gamma \in \Gamma \cup \{\phi\}$ then $\mathcal{A} \models_{[\overrightarrow{x}/\overrightarrow{a}]} \psi$. If it is the case that $\mathcal{A} \models_{[\overrightarrow{x}/\overrightarrow{a}]} \gamma$ for all $\gamma \in \Gamma$ but $\mathcal{A} \not\models_{[\overrightarrow{x}/\overrightarrow{a}]} \phi$ then our objective is vacuously satisfied, so assume that $\mathcal{A} \models_{[\overrightarrow{x}/\overrightarrow{a}]} \gamma$ for all $\gamma \in \Gamma$ and $\mathcal{A} \models_{[\overrightarrow{x}/\overrightarrow{a}]} \phi$. But then we have that every formula in $\Gamma \cup \{\phi\}$ is satisfied so $\mathcal{A} \models_{[\overrightarrow{x}/\overrightarrow{a}]} \psi$.

Now assume that $\Gamma \models (\phi \to \psi)$ i.e. for every L-structure $\mathcal{A}$ and interpretation $[\overrightarrow{x}/\overrightarrow{a}]$ if $\mathcal{A} \models_{[\overrightarrow{x}/\overrightarrow{a}]} \gamma$ for all $\gamma \in \Gamma$ then $\mathcal{A} \models_{[\overrightarrow{x}/\overrightarrow{a}]} (\phi \to \psi)$. Now, if it is the case that $\mathcal{A} \models_{[\overrightarrow{x}/\overrightarrow{a}]} \gamma$ for all $\gamma \in \Gamma \cup \{\phi\}$ then the formulas in Γ give us $\mathcal{A} \models_{[\overrightarrow{x}/\overrightarrow{a}]} (\phi \to \psi)$ by assumption and since ϕ is also in the set we get $\mathcal{A} \models_{[\overrightarrow{x}/\overrightarrow{a}]} \phi$ as well. So by the $\to$ truth table, the only possibility is that $\mathcal{A} \models_{[\overrightarrow{x}/\overrightarrow{a}]} \psi$.

b) Now we will show that $\phi \models \psi$ and $\psi \models \phi$ iff $(\phi \leftrightarrow \psi)$ is valid.

If $\phi \models \psi$ and $\psi \models \phi$ then by part a) we have that $\models (\phi \to \psi)$ and $\models (\psi \to \phi)$ so $(\phi \to \psi)$ is valid and $(\psi \to \phi)$ is valid. But then by the $\wedge$ truth table $(\phi \to \psi) \wedge (\psi \to \phi) \equiv (\phi \leftrightarrow \psi)$ is valid.

If $(\phi \leftrightarrow \psi)$ is valid then $(\phi \to \psi) \wedge (\psi \to \phi)$ is valid which is the case exactly when $(\phi \to \psi)$ is valid and $(\psi \to \phi)$ is valid so by part a) we have $\phi \models \psi$ and $\psi \models \phi$.

Question 7:

To find a formula ϕ in prenex normal form that is equivalent to $(\exists x \forall y R(x,y) \vee \forall x \exists y Q(x,y))$ we use the equivalences from page 177 in the textbook to get

$$\phi = \exists x_1 \forall y_1 \forall x_2 \exists y_2 [R(x_1, y_1) \vee Q(x_2, y_2)]$$

Question 8:

" Formalize the following statements about $\mathbb{N}$ using a language without equality with a 1-place relation symbol P, a 3-place relation symbol S and a constant symbol c. The intended interpretation is the structure $\mathcal{A}$ with domain $\mathbb{N}$, $P^{\mathcal{A}} = \{x : x \text{ is prime}\}, S^{\mathcal{A}} = \{(x, y, z) : x+y = z\}, c^{\mathcal{A}} = 2$."

a) x equals 0.
$S(x, 2, 2)$ since it will be interpreted as $x + 2 = 2$ so x must be zero.

b) x is greater than y.
$\exists z(S(y, z, x) \wedge \neg S(z, 2, 2))$ since x is greater than y if there is some number z that must be added to y to equal x and that number is not zero.

c) Every even number greater than two is the sum of two primes.
$\forall x[\exists y S(y, y, x) \wedge \exists z(S(2, z, x) \wedge \neg S(z, 2, 2)) \wedge \exists p_1 \exists p_2(P(p_1) \wedge P(p_2) \wedge S(p_1, p_2, x))]$ since the three conjuncts say that x is the sum of two of the same number i.e. even, x is larger than 2 from b), and x is the sum of two primes.

d) There are arbitrarily large prime pairs, where $\{a, b\}$ is said to be a prime pair if both a and b are prime and a and b differ by 2, e.g. $\{29, 31\}$ and $\{101, 103\}$.

$\forall x \exists p_1 \exists p_2[P(p_1) \wedge P(p_2) \wedge (S(p_1, 2, p_2) \vee S(p_2, 2, p_1) \wedge \exists z(S(x, z, p_1) \wedge \neg S(z, 2, 2)) \wedge \exists z(S(x, z, p_2) \wedge \neg S(z, 2, 2))]$ since this says that for any natural x there are numbers p_1, p_2 that are prime, either p_2 is two more than p_1 or vice versa, p_1 is larger than x, and p_2 is larger than x.

Question 9:

" Show that the following is universally valid. (g is a 1-place function symbol, f is a 3-place function symbol and R a 2-place relation symbol.) "

e) $\forall x R(g(x), x) \rightarrow \forall x \exists y R(y, x))$

The formula is vacuously true if $\forall x R(g(x), x)$ is false so assume that it is true, then we need to show that $\forall x \exists y R(y, x)$ holds. This is equivalent to saying for all x, there is some y such that $R(y, x)$ holds, but we have that for all x, $R(g(x), x)$ holds. So, for any x we can always find a y such that $R(y, x)$ holds, namely $y = g(x)$. Thus, this formula is universally valid.