Math 4L03 - Assignment 5

November 20, 2020

Question 1: Let $\Gamma = \{\forall x E(x,x), \forall x \forall y \forall z\,(E(x,y) \to (E(y,z) \to E(z,x)))\}$.

$\Gamma \vdash \forall x \forall y\,(E(x,y) \to E(y,x))$

Proof:

$\forall x \forall y \forall z\,(E(x,y) \to (E(y,z) \to E(z,x)))$	[1 - Ass]
$\forall x \forall y \forall z\,(E(x,y) \to (E(y,z) \to E(z,x))) \to \forall y \forall z\,(E(x,y) \to (E(y,z) \to E(z,x)))$	[2 - Ax4]
$\forall y \forall z\,(E(x,y) \to (E(y,z) \to E(z,x)))$	[3 - MP 1,2]
$\forall y \forall z\,(E(x,y) \to (E(y,z) \to E(z,x))) \to \forall z\,(E(x,x) \to (E(x,z) \to E(z,x)))$	[4 - Ax4]
$\forall z\,(E(x,x) \to (E(x,z) \to E(z,x)))$	[5 - MP 3,4]
$\forall z\,(E(x,x) \to (E(x,z) \to E(z,x))) \to (E(x,x) \to (E(x,y) \to E(y,x)))$	[6 - Ax4]
$(E(x,x) \to (E(x,y) \to E(y,x)))$	[7 - MP 5,6]
$\forall x E(x,x)$	[8 - Ass]
$\forall x E(x,x) \to E(x,x)$	[9 - Ax4]
$E(x,x)$	[10 - MP 8,9]
$(E(x,y) \to E(y,x))$	[11 - MP 7,10]
$\forall y (E(x,y) \to E(y,x))$	[12 - Gen 11]
$\forall x \forall y (E(x,y) \to E(y,x))$	[13 - Gen 12]

Question 2:

a) $\vdash \forall x(A(x) \rightarrow B(x)) \rightarrow (\forall x A(x) \rightarrow \forall x B(x))$

Using the deduction theorem twice, it suffices to show $\forall x(A(x) \rightarrow B(x)), \forall x A(x) \vdash \forall x B(x)$.

Proof:

$\forall x(A(x) \rightarrow B(x))$	[1 - Ass]
$\forall x(A(x) \rightarrow B(x)) \rightarrow (A(x) \rightarrow B(x))$	[2 - Ax4]
$(A(x) \rightarrow B(x))$	[3 - MP 1,2]
$\forall x A(x)$	[4 - Ass]
$\forall x A(x) \rightarrow A(x)$	[5 - Ax4]
$A(x)$	[6 - MP 4,5]
$B(x)$	[7 - MP 3,6]
$\forall x B(x)$	[8 - Gen 7]

b) $\vdash \forall x(A(x) \rightarrow B(x)) \rightarrow (\exists x A(x) \rightarrow \exists x B(x))$

In our system, this is defined as $\forall x(A(x) \rightarrow B(x)) \rightarrow (\neg\forall x\neg A(x) \rightarrow \neg\forall x\neg B(x))$. Then using the deduction theorem once, it suffices to show $\forall x(A(x) \rightarrow B(x)), \vdash (\neg\forall x\neg A(x) \rightarrow \neg\forall x\neg B(x))$.

Proof:

$\forall x(A(x) \rightarrow B(x))$	[1 - Ass]
$\forall x(A(x) \rightarrow B(x)) \rightarrow (A(x) \rightarrow B(x))$	[2 - Ax4]
$(A(x) \rightarrow B(x))$	[3 - MP 1,2]
$(A(x) \rightarrow B(x)) \rightarrow (\neg B(x) \rightarrow \neg A(x))$	[4 - Instance of a tautology]
$(\neg B(x) \rightarrow \neg A(x))$	[5 - MP 3,4]
$\forall x(\neg B(x) \rightarrow \neg A(x))$	[6 - Gen 5]
$(\forall x\neg B(x) \rightarrow \forall x\neg A(x))$	[7 - Question 2a]
$(\forall x\neg B(x) \rightarrow \forall x\neg A(x)) \rightarrow (\neg\forall x\neg A(x) \rightarrow \neg\forall x\neg B(x))$	[8 - Instance of a tautology]
$(\neg\forall x\neg A(x) \rightarrow \neg\forall x\neg B(x))$	[9 - MP 7,8]

c) There is no deduction for $\vdash (\forall x A(x) \rightarrow \forall x B(x)) \rightarrow (\forall x(A(x) \rightarrow B(x)))$. If there was a deduction soundness would imply that the formula is universally valid, but as we proved last assignment there are structures that falsify this formula such as $E = \langle\{0,1\}, A^E = \{0\}, B^E = \{\}\rangle$.

Question 3: $\vdash (\forall x E(g(x), x) \to \forall x \exists y E(y, x))$

The formula above is written in our system as $\vdash (\forall x E(g(x), x) \to \forall x \neg \forall y \neg E(y, x))$. Using the deduction theorem once it suffices to show $\forall x E(g(x), x) \vdash \forall x \neg \forall y \neg E(y, x)$. Notice that if we were able to deduce $\forall x E(g(x), x) \vdash \neg \forall y \neg E(y, x)$ we could safely apply generalization to the last step to get $\forall x E(g(x), x) \vdash \forall x \neg \forall y \neg E(y, x)$, so it suffices to show $\forall x E(g(x), x) \vdash \neg \forall y \neg E(y, x))$. Then using proof by contradiction, it suffices to show $\forall x E(g(x), x), \neg\neg \forall y \neg E(y, x) \vdash \psi, \neg \psi$ for some formula ψ.

Proof:

$\forall x E(g(x), x)$	[1 - Ass]
$\forall x E(g(x), x) \to E(g(x), x)$	[2 - Ax4]
$E(g(x), x)$	[3 - MP 1,2]
$\neg\neg \forall y \neg E(y, x)$	[4 - Ass]
$\neg\neg \forall y \neg E(y, x) \to \forall y \neg E(y, x)$	[5 - Instance of a tautology]
$\forall y \neg E(y, x)$	[6 - MP 4,5]
$\forall y \neg E(y, x) \to \neg E(g(x), x)$	[7 - Ax4]
$\neg E(g(x), x)$	[8 - MP 6,7]

We were able to derive $E(g(x), x)$ and $\neg E(g(x), x)$ setting off the chain of implications as required.

Question 4: Let G be the group axioms, then we show $G \vdash \forall x \forall y(((x + (-x)) + y) = y)$.

Proof:

$\forall x(0 + x = x)$	[1 - Ass]
$\forall x(0 + x = x) \to (0 + y = y)$	[2 - Ax4]
$(0 + y = y)$	[3 - MP 1,2]
$\forall x(x + (-x) = 0)$	[4 - Ass]
$\forall x(x + (-x) = 0) \to (x + (-x) = 0)$	[5 - Ax4]
$(x + (-x) = 0)$	[6 - MP 4,5]
$(x + (-x) = 0) \to [(0 + y = y) \to (((x + (-x)) + y) = y)]$	[7 - Theorem 5.10]
$[(0 + y = y) \to (((x + (-x)) + y) = y)]$	[8 - MP 6,7]
$(((x + (-x)) + y) = y)$	[9 - MP 3,8]
$\forall y(((x + (-x)) + y) = y)$	[10 - Gen 9]
$\forall x \forall y(((x + (-x)) + y) = y)$	[11 - Gen 10]

Question 5: Let L be a first order language with a binary relation symbol $<$ and let $\phi = \forall x \forall y (x < y \to \exists z (x < z \land z < y))$. This formula represents the density of a strict linear order $<$, so any structure that satisfies ϕ must necessarily be infinite as any finite domain would eventually fail to find an element in between others. Note that infinite domain is not sufficient, for example, the reals with the usual order would satisfy this and the naturals with the usual order would fail to satisfy this, but no finite structure would work.