Math 4L03 - Assignment 2

October 1, 2020

Note: As discussed in class, the proper bracketing clearly causes confusion, so I will follow Dr. Valeriote in omitting some brackets when it increases clarity. (Each question starts on a new page)

Question 1:

a)

$$(p \wedge q) \to r$$

$$\equiv \langle \text{definition of } \to \rangle$$

$$\neg(p \wedge q) \vee r$$

$$\equiv \langle \text{De Morgan} \rangle$$

$$(\neg p \vee \neg q \vee r)$$

Notice that this is both in disjunctive normal form and conjunctive normal since it can be considered as a disjunction of three literals or it can be considered as an entire entity of disjunctions that trivially satisfies conjunctive normal form.

b)

We note that $(p \vee q) \wedge (\neg p \vee r)$ is already in conjunctive normal form since it is a conjunction of disjunctions of literals, thus, we can use the formula $(\neg p \vee r) \wedge (p \vee q)$ by commutativity of $\wedge$ which is an equivalnet formula in conjunctive normal form. For disjunctive normal form, we have

$$(p \vee q) \wedge (\neg p \vee r)$$

$$\equiv \langle \text{distributivity of } \wedge \rangle$$

$$[p \wedge (\neg p \vee r)] \vee [q \wedge (\neg p \vee r)]$$

$$\equiv \langle \text{absorption} \rangle$$

$$[p \wedge r] \vee [q \wedge (\neg p \vee r)]$$

$$\equiv \langle \text{distributivity of } \wedge \rangle$$

$$(p \wedge r) \vee (q \wedge \neg p) \vee (q \wedge r)$$

as required.

c) For disjunctive normal form we have

$$(p \vee q) \leftrightarrow c$$

$$\equiv \langle \text{definition of } \leftrightarrow \rangle$$

$$[(p \vee q) \rightarrow c] \wedge [c \rightarrow (p \vee q)]$$

$$\equiv \langle \text{question 7 a) on assignment 1} \rangle$$

$$[(p \vee q) \wedge c] \vee [\neg(p \vee q) \wedge \neg c]$$

$$\equiv \langle \text{distributivity of } \wedge \rangle$$

$$(p \wedge c) \vee (q \wedge c) \vee [\neg(p \vee q) \wedge \neg c]$$

$$\equiv \langle \text{De Morgan} \rangle$$

$$(p \wedge c) \vee (q \wedge c) \vee (\neg p \wedge \neg q \wedge \neg c)$$

For conjunctive normal form we have

$$(p \vee q) \leftrightarrow c$$

$$\equiv \langle \text{definition of } \leftrightarrow \rangle$$

$$[(p \vee q) \rightarrow c] \wedge [c \rightarrow (p \vee q)]$$

$$\equiv \langle \text{definition of } \rightarrow \rangle$$

$$[\neg(p \vee q) \vee c] \wedge (\neg c \vee p \vee q)$$

$$\equiv \langle \text{De Morgan } \rangle$$

$$[(\neg p \wedge \neg q) \vee c] \wedge (\neg c \vee p \vee q)$$

$$\equiv \langle \text{distributivity of } \vee \rangle$$

$$(\neg p \vee c) \wedge (\neg q \vee c) \wedge (\neg c \vee p \vee q)$$

as required.

Question 2: We prove this by induction on the length of the formula ϕ.

Base case: Let $n = 1$, then ϕ can only be a propositional variable of the form p. So we have,

$$\phi = p \implies (\phi' \equiv \neg p) \text{ and } (\neg\phi \equiv \neg p)$$

thus, the base case holds. Now suppose that all formulas of length $\leq n$ have the property that their dual is equivalent to their negation, then we show that it holds for formulas of length $n + 1$.

Induction step: Let ϕ be a formula of length $n + 1$, then using our connectives, it must be of the form $(\theta \vee \psi)$, $(\theta \wedge \psi)$, or $\neg\theta$ for some formulas θ and ψ.

In case 1, we have

$$\phi = (\theta \vee \psi) \implies \phi' \equiv (\theta' \wedge \psi') \text{ and } \neg\phi \equiv \neg(\theta \vee \psi) \equiv (\neg\theta \wedge \neg\psi)$$

Since the length of ϕ is $n + 1$, θ and ψ must both have lengths $\leq n$, and so by the induction hypothesis we have $\theta' \equiv \neg\theta$ and $\psi' \equiv \neg\psi$. As proved in class, we can substitute subformulas for their equivalences, so we get

$$\phi' \equiv (\theta' \wedge \psi') \equiv (\neg\theta \wedge \neg\psi) \equiv \neg\phi$$

In case two, we have

$$\phi = (\theta \wedge \psi) \implies \phi' \equiv (\theta' \vee \psi') \text{ and } \neg\phi \equiv \neg(\theta \wedge \psi) \equiv (\neg\theta \vee \neg\psi)$$

Similarly, since the length of ϕ is $n + 1$, θ and ψ must both have lengths $\leq n$, and so by the induction hypothesis we have $\theta' \equiv \neg\theta$ and $\psi' \equiv \neg\psi$. So we substitute to get

$$\phi' \equiv (\theta' \vee \psi') \equiv (\neg\theta \vee \neg\psi) \equiv \neg\phi$$

Now for the final case,

$$\phi = \neg\theta \implies (\phi' \equiv \neg\theta') \text{ and } (\neg\phi \equiv \neg\neg\theta \equiv \theta)$$

Note, that we didn't replace the negation because by construction, the length of θ is n so it is not a propositional variable. This also implies that we can use the induction hypothesis to replace θ' with $\neg\theta$ so we get

$$\phi' \equiv \neg\theta' \equiv \neg\neg\theta \equiv \theta \equiv \neg\phi$$

So we have $\phi' \equiv -\phi$ for any formula ϕ of length $n \geq 1$.

Question 3:

a) We must show $\Gamma \models \tau$ i.e. we must show that for all truth assignments ν, if $\nu(\gamma) = T, \forall \gamma \in \Gamma$ then $\nu(\tau) = T$. But τ is a tautology, so it evaluates to T regardless of the truth assignment. Therefore, every truth assignment that satisfies all γ in Γ will vacuously satisfy τ, and thus, $\Gamma \models \tau$.

b) We must show that $\tau \models \rho \iff \rho$ is a tautology. First, suppose that $\tau \models \rho$, then every truth assignment that satisfies τ also satisfies ρ. But since τ is a tautology, it is satisfied by all truth assignments, so our assumption implies that all truth assignments also satisfy ρ. Therefore, ρ must also be a tautology. Now suppose that ρ is a tautology, then it is satisfied by all truth assignments. This implies that $\tau \models \rho$ since all truth assignments satisfy τ and all truth assignments satisfy ρ.

Question 4:

We will check if the truth table of $(\phi \to \psi) \to (\phi \to \theta)$ is true when $\phi \to (\psi \to \theta)$ is true over the formulas ϕ, ψ, and θ.

ϕ	ψ	θ	$\psi \to \theta$	$\phi \to (\psi \to \theta)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

ϕ	ψ	θ	$\phi \to \psi$	$\phi \to \theta$	$(\phi \to \psi) \to (\phi \to \theta)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	T	T	T

Since $(\phi \to \psi) \to (\phi \to \theta)$ is true whenever $\phi \to (\psi \to \theta)$ is true, we have that $\phi \to (\psi \to \theta) \models (\phi \to \psi) \to (\phi \to \theta)$. Furthermore, it is also true that $(\phi \to \psi) \to (\phi \to \theta) \models \phi \to (\psi \to \theta)$ since they have identical truth tables, in fact, $(\phi \to \psi) \to (\phi \to \theta) \equiv \phi \to (\psi \to \theta)$.

Question 5:

a) Suppose Σ is the set of tautologies, then by definition, all possible truth assignments satisfy all tautologies $\sigma \in \Sigma$. Now if $\Sigma \models \alpha$ that means α is satisfied by all truth assignments that make all tautologies $\sigma \in \Sigma$ true. But this implies that α is satisfied by all possible truth assignments and so α is a tautology and $\alpha \in \Sigma$.

b) Let Σ be semantically closed i.e. for any formula α if $\Sigma \models \alpha$, then $\alpha \in \Sigma$. We know from question 3 a) that any set of formulas trivially implies any tautology, that is, if τ is any tautology, then $\Sigma \models \tau$ and so $\tau \in \Sigma$ since Σ is semantically closed. Thus, Σ contains all tautologies.

c) Let $\bigcap_{i \in I} \Sigma_i$ be the intersection of semantically closed sets Σ_i. Now suppose $\bigcap_{i \in I} \Sigma_i \models \alpha$ for some formula α, then all truth assignments ν with $\nu(\sigma) = T, \forall \sigma \in \bigcap_{i \in I} \Sigma_i$ also satisfy α. But each σ is in the intersection so, more specifically, it is in $\Sigma_i, \forall i$. Therefore, the same truth assignment that satisfied $\bigcap_{i \in I} \Sigma_i$ will also satisfy each Σ_i and thus, α. This means that $\Sigma_i \models \alpha, \forall i$ and since each Σ_i is semantically closed, $\alpha \in \Sigma_i, \forall i$. If α is in all Σ_i, it is certainly in the intersection so $\alpha \in \bigcap_{i \in I} \Sigma_i$ and $\bigcap_{i \in I} \Sigma_i$ is semantically closed.

d) As demonstrated in class, if $\Gamma \models \phi$ and $\Gamma \models \neg\phi$, then $\Gamma \models \psi$ for all formulas ψ. So any set Γ that implies a contradiction can vacuously imply any other formula, $\{p, \neg p\}$ is the simplest case. That is $\{p, \neg p\} \models p$ and $\{p, \neg p\} \models \neg p$ so $\{p, \neg p\} \models \psi$ for any formula ψ. Thus, the semantic closure of $\{p, \neg p\}$ is $\text{Form}(P, S)$, i.e. all possible formulas.

e) There are three cases for the semantic closure of a set. First, if Γ can imply a contradiction, then as we have seen, it can imply all possible formulas so the unique semantic closure is the set of all possible formulas. Second, if Γ is already semantically closed, then by definition, it is already the smallest set that contains itself and is semantically closed. If Γ does not imply a contradiction and is not semantically closed, then one can generate the implications of Γ and then add them to Γ iteratively. This will produce a finite set since we have assume that $\Gamma \models \theta$ xor $\Gamma \models \neg\theta$ for some θ. This will produce the smallest unique set since we are only adding elements that are necessarily generated through implication.

Question 6:

a) Suppose that $\Sigma \models \alpha$, then by definition, all truth assignments that satisfy Σ satisfy α. But any of those truth assignments will also satisfy $(\alpha \vee \beta)$ regardless of what they make β since $\vee$ evaluates to true when at least one of its components is true. Thus, $\Sigma \models (\alpha \vee \beta)$. Similarly, if $\Sigma \models \beta$ then for any truth assignment that satisfies Σ will satisfy β so at least one of the components of $(\alpha \vee \beta)$ will evaluate to true. Thus, $\Sigma \models (\alpha \vee \beta)$.

b) For our example, let $\Sigma = \{p, q\}$ with $\alpha = (p \vee q)$ and $\beta = (\neg p \wedge q)$. Then $\Sigma \models \alpha \vee \beta$ for only the truth assignment $\nu(p) = \nu(q) = T$ but $\nu(\beta) = F$ so $\Sigma \not\models \beta$.