Math 4L03 - Assignment 1

December 13, 2020

Question 1: Note: Knights always tell the truth and knaves always lie.
Let A, B, and E be propositional variables where A is the statement "Alice is a knight", B is the statement "Bob is a knight", and E is the statement "Eve is a knight".

a) Using the given statements and the interpretations we have the formula $(A \leftrightarrow (\neg A \vee \neg B))$. Next, we consider the four possible truth assignments of A and B along with the truth tables for our connectives (from class) to see which assignment will make the formula true.

$$\nu(A) = T, \nu(B) = F \implies \nu((A \leftrightarrow (\neg A \vee \neg B))) = (T \leftrightarrow (F \vee T)) = (T \leftrightarrow T) = T$$

One can check the other truth assignments and see that this is the only one that makes our formula true, thus, we conclude that Alice is a knight and Bob is a knave.

b) Similarly, we have the formula $(A \leftrightarrow (\neg A \vee B))$.

$$\nu(A) = T, \nu(B) = T \implies \nu((A \leftrightarrow (\neg A \vee B))) = (T \leftrightarrow (F \vee T)) = (T \leftrightarrow T) = T$$

Thus, we conclude both Alice and Bob are knights.

c) Next, using the same interpretations we have the formula

$$(((A \leftrightarrow ((\neg E \wedge \neg A) \wedge \neg B)) \wedge (B \leftrightarrow (((\neg A \wedge B) \wedge E) \vee ((A \wedge \neg B) \wedge E)) \vee ((A \wedge B) \wedge \neg E))))$$

By focusing on the left of the main conjunction i.e. Alice's statement, we can instantly rule out five of the total eight possible truth assignments since if $A = T$ or if $A = B = E = F$ the left conjunction would be false. This means $\nu(A) = F$ which would reduce the right conjunction to $(B \leftrightarrow (\neg A \wedge B \wedge E))$ which is clearly only true for $\nu(B) = \nu(E) = T$. Thus, Alice is a knave, and Bob and Eve are knights.

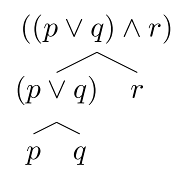
d) Again, we have

$$((A \leftrightarrow \neg B) \wedge (B \leftrightarrow ((A \wedge E) \vee (\neg A \wedge \neg E))))$$

Looking at the left of the main conjunction, we see that the truth assignments of A and B must differ which halves the total possible choices. Then, looking to the right side, we can deduce two possible truth assignments that satisfy the formula, namely, $\nu(A) = T, \nu(B) = F, \nu(E) = F$ and $\nu(A) = F, \nu(B) = T, \nu(E) = F$. Thus, we can only conclude that Eve is a knave.

Question 2:

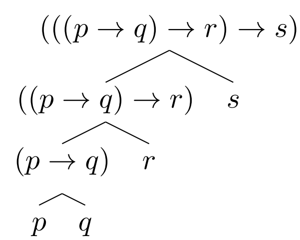
a) Valid formula - each node is a subformula.



b) Invalid formula - cannot parse unequal left and right brackets

c) Invalid formula - two binary connectives present but only one set of left and right brackets.

d) Valid formula - each node is a subformula.



e) Invalid formula - no way to parse the brackets around $\neg p_2$ with no binary connective present.

Question 3: We prove this by induction on the connective notion of length of the formula n.

Base case: Because $n = 0$ is a propositional variable, the smallest formula that can be created using one $\vee$ connective must have length $n = 1$ by connecting exactly two propositional variables.

For instance, $(p \vee q)$ where p and q are propositional variables. Thus, our base case $n = 1$ holds as any formula of length one with an $\vee$ occurrence must have at least two propositional variables.

Induction step: Suppose that there are at least two propositional variables in all formulas of length $\leq n$ that contain an $\vee$ occurrence, then we want to show that it also holds for any formula ϕ of length $n+1$ that contains an $\vee$ occurrence. Since ϕ has length $n+1$, it cannot be a propositional variable, it must be of the form of $\neg\theta, (\theta \vee \psi), (\theta \wedge \psi), (\theta \rightarrow \psi)$, or $(\theta \leftrightarrow \psi)$ where θ and ψ are formulas. In the first case, by construction, θ has length n and contains an $\vee$ occurrence so by the induction hypothesis it has at least two propositional variables. By taking the negation, we do not remove any of the propositional variables present in θ so $\neg\theta = \phi$ has at least two propositional variables. In the second case, since ϕ has length $n+1$ and an $\vee$ occurrence, θ and ψ must have length $\leq n$ and at least one of them will have an $\vee$ occurrence. By the induction hypothesis, one or both of θ and ψ will have at least two propositional variables. Connecting two formulas with $\vee$ will combine the propositional variables of each argument and so $(\theta \vee \psi)$ will have at least two propositional variable regardless of if they came from θ, ψ, or both. Notice that the other cases are exactly identical to $\vee$ as the connectives $\vee, \wedge, \rightarrow$, and $\leftrightarrow$ all combine the propositional variables of each of their arguments. Thus, the induction hypothesis is enough to ensure that at least one of the arguments in each case will have at least two propositional variables and so $(\theta \wedge \psi), (\theta \rightarrow \psi)$, and $(\theta \leftrightarrow \psi)$ all have at least two propositional variables. Since our claim is true in all cases of length $n+1$, it follows that the result holds for all formulas of length $n \geq 1$ that contain an $\vee$ occurrence.

Question 4: We prove by induction on the length of the formula n.

Base case: $n = 0 \implies$ the formula is a propositional variable say p and is satisfiable by $\nu(p) = T$.

Induction step: Assume formulas made from $\{\vee, \wedge\}$ are satisfiable for length $\leq n$ by $\nu(p) = T$ for all propositional variables $p \in P$, then we show formulas of length $n+1$ of the same type are satisfiable. Let ϕ be a formula of length $n+1$. Since ϕ is of length $n+1$ it cannot be a propositional variable, so it must be of the form $(\theta \vee \psi)$ or $(\theta \wedge \psi)$ where θ and ψ are formulas of length $\leq n$. Using the induction hypothesis, we have that both θ and ψ are satisfiable by $\nu(p) = T$ for all propositional variables p. Using our truth tables, we have that $\nu(p) = T, \forall p \in P \implies \nu(\theta) = \nu(\psi) = T \implies \nu((\theta \vee \psi)) = T$ and $\nu((\theta \vee \psi)) = T$. Since our claim is true in all cases of length $n+1$, it follows that all formulas made from the connectives $\{\vee, \wedge\}$ are satisfiable by $\nu(p) = T, \forall p \in P$ for all lengths $n \geq 0$.

Question 5:

a) This satisfiable but not a tautology. $\nu(p) = F, \nu(q) = \nu(r) = T$ makes this formula false and $\nu(p) = \nu(q) = \nu(r) = T$ makes this formula true.

b) We will show that this formula is a tautology and in fact we will show the stronger result $(p \rightarrow (r \rightarrow s)) \equiv ((p \wedge r) \rightarrow s)$ which is obviously more than enough to confirm b). Note, we use definition of $\rightarrow$ from question 6 and Theorem 2.3 from the textbook.

$(p \to (r \to s))$
$\equiv <$ Two applications of the definition of $\to >$
$(\neg p \vee (\neg r \vee s))$
$\equiv <$ Associativity of $\vee >$
$((\neg p \vee \neg r) \vee s)$
$\equiv <$ De Morgan's Law $>$
$(\neg(p \wedge r) \vee s)$
$\equiv <$ Definition of $\to >$
$((p \wedge r) \to s)$
as required.

c) This is satisfiable but not a tautology. $\nu(p) = \nu(q) = F$ makes this formula false and $\nu(p) = \nu(q) = T$ makes this formula true.

Question 6: To see that these formulas are equivalent we draw both their truth tables and check that they match.

$p \mid q \mid\mid (\neg p \vee q) \mid (p \to q)$
$T \mid T \mid\mid T \mid T$
$T \mid F \mid\mid F \mid F$
$F \mid T \mid\mid T \mid T$
$F \mid F \mid\mid T \mid T$

Since these formulas share the same truth table, we have $(\neg p \vee q) \equiv (p \to q)$.

Now, an alternative definition of $\vee$ in terms $\to$ is described in the following equivalence,
$((p \vee q) \equiv ((p \to q) \to q))$ as seen in the following truth table.

$p \mid q \mid\mid (p \vee q) \mid ((p \to q) \to q)$
$T \mid T \mid\mid T \mid T$
$T \mid F \mid\mid T \mid T$
$F \mid T \mid\mid T \mid T$
$F \mid F \mid\mid F \mid F$

Question 7:

a) Starting from the right side, we show that these formulas are equivalent. Note: I will use equivalences established from the textbook which were also covered in class (Theorem 2.3 and Theorem 2.4).

$((p \wedge q) \vee (\neg p \wedge \neg q))$
$\equiv <$two applications of distributivity of $\vee$ over $\wedge >$
$(((p \vee \neg p) \wedge (q \vee \neg p)) \wedge ((p \vee \neg q) \wedge (q \vee \neg q)))$

$\equiv$ <Law of excluded middle and identity of $\wedge$ >
$((q \vee \neg p) \wedge (p \vee \neg q))$
$\equiv$ <Definition of implication and commutativity of $\vee$ >
$((p \to q) \wedge (q \to p))$

as required.

b) To see that these formulas are equivalent, we examine their truth tables.

p	q	r	$((p \leftrightarrow q) \leftrightarrow r)$	$(p \leftrightarrow (q \leftrightarrow r))$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	T	T
F	T	T	F	F
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F

Thus, the formulas are equivalent.

Question 8:

Below, we show via truth tables that $T(\alpha, \beta, \gamma) \equiv \phi$ where

$$\phi = ((((\alpha \wedge \beta) \vee (\beta \wedge \gamma)) \vee (\alpha \wedge \gamma)) \wedge \neg((\alpha \wedge \beta) \wedge \gamma))$$

α	β	γ	$T(\alpha, \beta, \gamma)$	ϕ
T	T	T	F	F
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

as required.