# Math 4L03 - Assignment 3

October 17, 2020

(Each question starts on a new page)

**Question 1:**

a) $\neg p \vdash (p \to q)$

Proof:

| | |
|---|---|
| $\neg p$ | [1 - Assumption] |
| $(\neg p \to (\neg q \to \neg p))$ | [2 - Axiom 1] |
| $(\neg q \to \neg p)$ | [3 - Modus Ponens 1, 2] |
| $((\neg q \to \neg p) \to (p \to q))$ | [4 - Axiom 3] |
| $(p \to q)$ | [5 - Modus Ponens 3, 4] |

b) $\vdash ((\phi \to (\psi \to \theta)) \to (\psi \to (\phi \to \theta)))$

Using the deduction theorem three times, it suffices to show

$(\phi \to (\psi \to \theta)), \psi, \phi \vdash \theta$

Proof:

| | |
|---|---|
| $\psi$ | [1 - Assumption] |
| $\phi$ | [2 - Assumption] |
| $(\phi \to (\psi \to \theta))$ | [3 - Assumption] |
| $(\psi \to \theta)$ | [4 - Modus Ponens 2, 3] |
| $\theta$ | [4 - Modus Ponens 1, 4] |

c) $\vdash ((\phi \to \psi) \to (\neg\psi \to \neg\phi))$

Using the deduction theorem twice and proof by contradiction, it suffices to show

$(\phi \to \psi), \neg\psi, \neg\neg\phi \vdash \beta$ and $(\phi \to \psi), \neg\psi, \neg\neg\phi \vdash \neg\beta$ for some formula $\beta$.

Proof:

| | |
|---|---|
| $\neg\psi$ | [1 - Assumption] |
| $\neg\neg\phi$ | [2 - Assumption] |
| $(\neg\neg\phi \to (\neg\neg\neg\neg\phi \to \neg\neg\phi))$ | [3 - Axiom 1] |
| $(\neg\neg\neg\neg\phi \to \neg\neg\phi)$ | [4 - Modus Ponens 2, 3] |
| $((\neg\neg\neg\neg\phi \to \neg\neg\phi) \to (\neg\phi \to \neg\neg\neg\phi))$ | [5 - Axiom 3] |
| $(\neg\phi \to \neg\neg\neg\phi)$ | [6 - Modus Ponens 4, 5] |
| $((\neg\phi \to \neg\neg\neg\phi) \to (\neg\neg\phi \to \phi))$ | [7 - Axiom 3] |
| $(\neg\neg\phi \to \phi)$ | [8 - Modus Ponens 6, 7] |
| $\phi$ | [9 - Modus Ponens 2, 8] |
| $(\phi \to \psi)$ | [10 - Assumption] |
| $\psi$ | [11 - Modus Ponens 9, 10] |

This is a contradiction since we can derive both $\psi$ and $\neg\psi$.

d) $\vdash (\phi \to (\neg\theta \to \neg(\phi \to \theta)))$

Using the deduction theorem twice and proof by contradiction, it suffices to show

$\phi, \neg\theta, \neg\neg(\phi \to \theta) \vdash \beta$ and $\phi, \neg\theta, \neg\neg(\phi \to \theta) \vdash \neg\beta$ for some formula $\beta$.

Note: what I mean in step 3 below is instead of repeating the exact same argument as in question 1 c), notice that if we replace $(\phi \to \theta)$ for $\phi$ in steps 2 to 9 we get that $\neg\neg(\phi \to \theta)$ is enough to give us $(\phi \to \theta)$.

Proof:

| | |
|---|---|
| $\neg\theta$ | [1 - Assumption] |
| $\neg\neg(\phi \to \theta)$ | [2 - Assumption] |
| $(\phi \to \theta)$ | [3 - 2 and question 1 c)] |
| $\phi$ | [4 - Assumption] |
| $\theta$ | [5 - Modus Ponens 4, 3] |

This is a contradiction since we can derive both $\theta$ and $\neg\theta$.

e) If $\Gamma, \phi \vdash \neg\psi$ then $\Gamma, \psi \vdash \neg\phi$.

Since $\Gamma, \phi \vdash \neg\psi$, we have $\Gamma \vdash (\phi \to \neg\psi)$ by the deduction theorem, so we can use it as an assumption in the proof of $\Gamma, \psi \vdash \neg\phi$. Now, to prove $\Gamma, \psi \vdash \neg\phi$, we use proof by contradiction to show $\Gamma, \psi, \neg\neg\phi \vdash \beta$ and $\Gamma, \psi, \neg\neg\phi \vdash \neg\beta$ for some formula $\beta$.

Proof:

| | |
|---|---|
| $\psi$ | [1 - Assumption] |
| $(\phi \to \neg\psi)$ | [2 - Assumption] |
| $\neg\neg\phi$ | [3 - Assumption] |
| $\phi$ | [4 - 3 and question 1 c)] |
| $\neg\psi$ | [5 - Modus Ponens 4, 2] |

This is a contradiction since we can derive both $\psi$ and $\neg\psi$.

**Question 2:**

$$\vdash ((\phi \to (\phi \to \neg\theta)) \to (\psi \to \theta))$$

is not true for all formulas $\phi, \psi$, and $\theta$. To show this, it is enough to show that the formula is not a tautology (soundness), so consider the case where each formula is a propositional variable with the following truth assignment

$$\nu(\phi) = \nu(\psi) = T, \nu(\theta) = F$$

Then our formula evaluates to false since

$$\nu((\phi \to (\phi \to \neg\theta)) \to (\psi \to \theta))$$

$\equiv$

$$((\top \to (\top \to \neg\bot)) \to (\top \to \bot))$$

$\equiv$

$$((\top \to \top) \to \bot)$$

$\equiv$

$$(\top \to \bot)$$

$\equiv$

$$\bot$$

Thus, our formula is not a tautology and so it is not derivable by the empty set.

**Question 3:**

Let $\Gamma$ be a set a formulas then the following are equivalent:
(a) $\Gamma$ is inconsistent
(b) $\Gamma \vdash \neg(\phi \to \phi)$ for all formulas $\phi$
(c) $\Gamma \vdash \neg(\phi \to \phi)$ for some formula $\phi$

We will prove (a) $\implies$ (b) $\implies$ (c) $\implies$ (a).

(a) $\implies$ (b): Suppose $\Gamma$ is inconsistent, then we need to show $\Gamma \vdash \neg(\phi \to \phi)$ for all formulas $\phi$. Since $\Gamma$ is inconsistent, $\exists \beta$ such that $\Gamma \vdash \beta, \neg\beta$. Then for any formula $\psi$ we have the following derivation,

$\Gamma \vdash \psi$

Proof:

| | |
|---|---|
| $\beta$ | [1 - Assumption] |
| $\neg\beta$ | [2 - Assumption] |
| $(\neg\beta \to (\neg\psi \to \neg\beta))$ | [3 - Axiom 1] |
| $(\neg\psi \to \neg\beta)$ | [4 - Modus Ponens 2, 3] |
| $((\neg\psi \to \neg\beta) \to (\beta \to \psi))$ | [5 - Axiom 3] |
| $(\beta \to \psi)$ | [6 - Modus Ponens 4, 5] |
| $\psi$ | [7 - Modus Ponens 1, 6] |

Thus, if $\Gamma$ is inconsistent, it can derive any formula, so clearly $\Gamma \vdash \neg(\phi \to \phi)$ for all formulas $\phi$.

(b) $\implies$ (c): If $\Gamma$ derives $\neg(\phi \to \phi)$ for any $\phi$, it most certainly can derive $\neg(\phi \to \phi)$ for a specific $\phi$.

(c) $\implies$ (a): Suppose that $\Gamma \vdash \neg(\phi \to \phi)$ for some formula $\phi$ then we show that $\Gamma$ is inconsistent by finding a formula $\beta$ such that $\Gamma \vdash \beta, \neg\beta$. The claim is that $\beta$ is the formula $(\phi \to \phi)$, that is, if $\Gamma \vdash \neg(\phi \to \phi)$ for some $\phi$, then $\Gamma$ is inconsistent because $\Gamma \vdash (\phi \to \phi), \neg(\phi \to \phi)$. Note, using the deduction theorem, for $\Gamma \vdash (\phi \to \phi)$ it suffices to show $\Gamma, \phi \vdash \phi$, which is a one line proof since $\phi$ is an assumption and conclusion. And $\Gamma$ derives $\neg(\phi \to \phi)$ by assumption so $\Gamma \vdash (\phi \to \phi), \neg(\phi \to \phi)$. Thus, $\Gamma$ is inconsistent.

**Question 4:**

a) The formulas in question can be described by tautologies and contradictions since these are true and false regardless of the truth assignment and thus will satisfy $X_\phi = V$ and $X_\theta = \{\}$ respectively. For a specific example, one could use $(\psi \to \psi), \psi \in \text{Form}(P, S)$ for the tautology since if $\nu(\psi) = T$ then $\nu((\psi \to \psi)) = T$ and if $\nu(\psi) = F$ then $\nu((\psi \to \psi)) = T$. This also gives us a formula for the contradiction since we could just use $\neg(\psi \to \psi)$. Therefore, $X_{(\psi \to \psi)} = V$ and $X_{\neg(\psi \to \psi)} = \{\}$.

b) Let $\phi$ and $\theta$ be formulas, $\nu \in X_\phi \cap X_\theta$, and $\gamma = (\phi \wedge \theta)$. Then we have $\nu(\phi \wedge \theta) = T$ since $\nu(\phi \wedge \theta)$ is true exactly when $\nu(\phi) = \nu(\theta) = T$, but this is the case as $\nu \in X_\phi \cap X_\theta$. Thus, $\nu \in X_\gamma$. We also have $X_\gamma \subseteq X_\phi \cap X_\theta$ since if $\nu' \in X_\gamma$ then $\nu'(\gamma) = \nu'(\phi \wedge \theta) = T$ which is only possible if $\nu'(\phi) = T$ and $\nu'(\theta) = T$ i.e. $\nu' \in X_\phi \cap X_\theta$.

c) Given a formula $\phi$ we have $X_\phi = \{\nu \in V : \nu(\phi) = T\}$, so $V \backslash X_\phi = \{\nu \in V : \nu(\phi) = F\} = \{\nu \in V : \nu(\neg\phi) = T\} = X_{\neg\phi}$. Thus, $X_\theta = V \backslash X_\phi = X_{\neg\phi}$.

d) Since $\cap_{\phi \in \Sigma} X_\phi = \emptyset$ we know that there is no $\nu \in V$ that satisfies all $\phi_i \in \Sigma$ i.e. $\Sigma$ is not satisfiable. Using the contrapositive of compactness, we have if $\Gamma$ is not satisfiable then every finite subset is not satisfiable. So in our case, since $\Sigma$ is not satisfiable, by compactness, there exists a finite proper subset $s = \{\phi_0, \phi_1, \ldots, \phi_{n-1}\} \subset \Sigma$ that is not satisfiable. Now since $s$ is not satisfiable, then there exists no truth assignment $\nu \in V$ that satisfies all $\phi_i \in s$, and so $X_{\phi_0} \cap X_{\phi_1} \cap \cdots \cap X_{\phi_{n-1}} = \emptyset$.

**Question 5:**

Given $\{p, (p \to q)\} \vdash q$ by

| | |
|---|---|
| $p$ | [1 - Assumption] |
| $(p \to q)$ | [2 - Assumption] |
| $q$ | [3 - Modus Ponens 1, 2] |

we use the proof of the deduction theorem to show $\{p\} \vdash (p \to q) \to q$, that is, we will show $\{p\} \vdash (p \to q) \to \psi_i, i = 1, 2, 3$ where $\psi_i$ is a step in the derivation above.

Proof:

| | |
|---|---|
| $\psi_1 : ((p \to q) \to p)$ | |
| $p$ | [1 - Assumption] |
| $(p \to ((p \to q) \to p))$ | [2 - Axiom 1] |
| $((p \to q) \to p)$ | [3 - Modus Ponens 1, 2] |
| | |
| $\psi_2 : ((p \to q) \to (p \to q))$ | |
| $((p \to q) \to (((p \to q) \to (p \to q)) \to (p \to q)))$ | [4 - Axiom 1] |
| $(((p \to q) \to (((p \to q) \to (p \to q)) \to (p \to q)))$ | [5 - Axiom 2] |
| $\to (((p \to q) \to ((p \to q) \to (p \to q))) \to ((p \to q) \to (p \to q))))$ | [5 - Axiom 2] |
| $(((p \to q) \to ((p \to q) \to (p \to q))) \to ((p \to q) \to (p \to q)))$ | [6 - Modus Ponens 4, 5] |
| $((p \to q) \to ((p \to q) \to (p \to q)))$ | [7 - Axiom 1] |
| $((p \to q) \to (p \to q))$ | [8 - Modus Ponens 6, 7] |
| | |
| $\psi_3 : ((p \to q) \to q)$ | |
| $(((p \to q) \to (p \to q)) \to (((p \to q) \to p) \to ((p \to q) \to q)))$ | [9 - Axiom 2] |
| $((p \to q) \to p) \to ((p \to q) \to q))$ | [10 - Modus Ponens 8, 9] |
| $((p \to q) \to q)$ | [11 - Modus Ponens 3, 10] |