MATH 4L03/6L03 Assignment #3

Due: Tuesday, October 20, 11:59pm.

Upload your solutions to the Avenue to Learn course website. Detailed instructions will be provided on the course website.

1. Use the proof system S from the text and the lectures to show the following. You may not use the Completeness Theorem to solve these, i.e., you can't work with $\models$ in place of $\vdash$. You may use any meta-theorem that was proved in the lectures, in particular the Deduction Theorem and the Proof by Contradiction Theorem.

   (a) $\neg p \vdash (p \to q)$. Show this without using any meta-theorem, i.e., provide a complete derivation for this.

   (b) $\vdash ((\phi \to (\psi \to \theta)) \to (\psi \to (\phi \to \theta)))$,

   (c) $\vdash ((\phi \to \psi) \to (\neg\psi \to \neg\phi))$,

   (d) $\vdash (\phi \to (\neg\theta \to \neg(\phi \to \theta)))$,

   (e) If $\Gamma, \phi \vdash \neg\psi$ then $\Gamma, \psi \vdash \neg\phi$.

note that this question has been updated to fix a typing error.

2. Is the following true for all formulas ϕ, ψ, and θ?

$$\vdash ((\phi \to (\phi \to \neg\theta)) \to (\psi \to \theta)).$$

3. Let Γ be a set of formulas. Prove that the following statement are equivalent:

   (a) Γ is inconsistent,

   (b) $\Gamma \vdash \neg(\phi \to \phi)$ for **all** formulas ϕ,

   (c) $\Gamma \vdash \neg(\phi \to \phi)$ for **some** formula ϕ.

4. In this question, all of the usual connectives, $S = \{\neg, \wedge, \vee, \to, \leftrightarrow\}$, may appear in the formulas in question, and the propositional variables that appear come from the infinite set $P = \{p_1, p_2, \ldots\}$, i.e., we are considering formulas from $Form(P, S)$. Let V be the set of all truth assignments for the set of propositional variables from P, i.e., $V = \{\nu : P \to \{T, F\}\}$.

   For ϕ a formula, let $X_\phi = \{\nu \in V \mid \nu(\phi) = T\}$.

(a) Show that there exist formulas ϕ and θ such that $X_\phi = V$ and X_θ is the empty set.

(b) Given formulas ϕ and θ and $\nu \in X_\phi \cap X_\theta$ show that there is a formula γ such that $\nu \in X_\gamma$ and $X_\gamma \subseteq X_\phi \cap X_\theta$.

(c) Let ϕ be a formula. Show that there is some formula θ such that $X_\theta = V \setminus X_\phi$, i.e., X_θ is the complement of X_ϕ in V.

(d) Let Σ be a set of formulas such that

$$\bigcap_{\phi \in \Sigma} X_\phi = \emptyset,$$

i.e., the intersection of the X_ϕ for $\phi \in \Sigma$ is the empty set. Prove that for some natural number n, there are $\phi_0, \phi_1, \ldots, \phi_{n-1} \in \Sigma$ such that

$$X_{\phi_0} \cap X_{\phi_1} \cap \ldots \cap X_{\phi_{n-1}} = \emptyset.$$

HINT: Use the Compactness Theorem.

5. The following is a simple derivation of the formula q from the set $\{p, (p \to q)\}$:

$$\begin{array}{lcr} (1) & p & \text{Ass.} \\ (2) & p \to q & \text{Ass.} \\ (3) & q & \text{MP, 1, 2.} \end{array}$$

Use the proof of the Deduction Theorem to convert the above derivation into a derivation of

$$p \vdash (p \to q) \to q.$$

[The proof of the Deduction Theorem can be regarded as a description of a procedure that takes as input a derivation of $\Gamma, A \vdash B$ and produces as output a derivation of $\Gamma \vdash (A \to B)$.]