Assignment 6 **MATH 4NA** **29 March 2021**

This assignment is due at 6pm on Wednesday 14 April (note later due date).

Instructions:

- Carefully read and answer **all** parts of each question. Four significant figures are sufficient for numerical answers.
- Include printouts of **all** `matlab` m-files and scripts you wrote to generate your results.
- Include and fully **label** all graphs: label axes, add a title or figure caption and indicate the corresponding question.
- Submit as a **single pdf file** on the Avenue dropbox by the due date.

Review exercises NOT TO BE HANDED IN:

1. Start reviewing the course material, starting from the floating point number system.
2. What is the Lax equivalence theorem?
3. How is the von Neumann stability method used to analyze the stability of a finite difference method for a partial differential equation? Apply it to the FTCS method (without looking at the notes!).
4. What is the difference between convergence and consistency? How is the leading order consistency error term found?
5. What is the *stencil* of a finite difference method for a partial differential equation? What information does it give?
6. Can the stepsize in time and stepsize in space be chosen independently? Why or why not?
7. How do the characteristics of the wave equation constrain the stability of a finite difference approximation to the wave equation?
8. What is a major drawback of the method of lines?

Exercises TO BE HANDED IN:

1. Consider the heat equation with initial and boundary conditions

$$
\begin{aligned}
\partial u/\partial t &= \partial^2 u/\partial x^2, & 0 < x < 1, \\
u(x,0) &= f(x), & 0 \le x \le 1, \\
u(0,t) &= g_0(t), & u(1,t) = g_1(t), \quad t > 0.
\end{aligned}
$$

We divide the spatial domain $[0,1]$ into J equispaced subintervals and let $h = 1/J$ be the meshsize and $x_j = jh$, $0 \le j \le J$ be the meshpoints. By applying the method of lines, the solution values $u(x_j,t)$ can be approximated by $u_j(t)$, which satisfies the system of ODEs:

$$\begin{aligned} U'(t) &= AU(t) + S(t), \\ U(0) &= (f(x_1), f(x_2), \ldots, f(x_{J-1})), \end{aligned}$$

where $U(t) = (u_1(t), u_2(t), \ldots, u_{J-1}(t))^T$, $S(t) = (1/h^2)(g_0(t), 0, \ldots, 0, g_1(t))^T$ and $A = (1/h^2)(C - 2I)$ is a $(J-1) \times (J-1)$ matrix with

$$C = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}$$

and where I is the identity matrix.

[5] (a) What is the maximum stepsize k that can be used to solve (2) by Euler's method? (You may refer to results from the class notes.)

[8] (b) Let $f(x) = 2x$ on $[0, 0.5]$ and $f(x) = 2 - 2x$ on $(0.5, 1]$ and $g_0(t) = g_1(t) = 0$. Use the method of lines with $h = 0.1$ to obtain approximate values $u_5(t)$ of $u(0.5,t)$ for $0 \le t \le 0.6$. In order to solve system (2), use Euler's method with time stepsize $k = 0.001, 0.005, 0.0052, 0.0056$, respectively. In each case plot your numerical solution over the time interval $[0, 0.6]$. Note that the analytic solution is

$$u(x,t) = \frac{8}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)\sin(n\pi x)e^{-n^2\pi^2 t}}{n^2}.$$

Explain the anomalies that you may observe in terms of stability.

2. An equation frequently used as a model for fluid mechanics is Burgers' equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}.$$

[5] (a) Give an explicit second-order in space, first-order in time finite difference method for solving this equation.

[8] (b) Given the initial condition

$$u(x,0) = \sin \pi x \quad 0 < x < 1,$$

and the boundary conditions

$$u(0,t) = u(1,t) = 0,$$

solve the problem for $\nu = 0.1, 0.01, 0.001$ with space stepsize $h = 0.02$, time stepsize $k = 0.001$. Plot the solution for $t = 0, 0.2, 0.4, 0.6, 0.8, 1.0$ up to $t = 1$, and explain the results.

3. Consider the one-dimensional linear wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad 0 < t < 2$$

with the following boundary and initial conditions

$$u(0,t) = 0 \quad \text{and} \quad u(1,t) = 0$$
$$u(x,0) = f(x) \quad u_t(x,0) = 0$$

[6] (a) Write a `matlab` function to solve the above wave equation using the explicit finite difference method (where both the time and space derivatives are approximated using second-order central differences). State the stability criterion.

[5] (b) Use the above `matlab` function to solve the wave equation with initial condition $f(x) = \sin 2\pi x$ with space stepsize $h = 0.01$ and time stepsize $k = 0.005$. What is the exact solution? What is special about the solution at the time $t = 2/c = 2$? Plot the numerical solution at $t = 0, 0.5, 1.0, 1.5, 2.0$. Comment on the accuracy of the method.

[3] (c) Now consider the following initial condition:

$$f(x) = \begin{cases} 5(x-0.3) & 0.3 < x < 0.5 \\ 5(0.7-x) & 0.5 < x < 0.7 \\ 0 & 0 < x < 0.3 \text{ and } 0.7 < x < 1. \end{cases}$$

Plot the numerical solution at the same times as in (b). Comment on the accuracy of the method. What is the problem? [Hint: the initial condition can be thought of as being made up of waves of different amplitudes. What is happening to these waves?]

[10] 4. Consider the Poisson equation with homogeneous Dirichlet boundary conditions ($u = 0$ on the boundaries of the square domain)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \sin^2 x + \sin^2 y, \quad 0 < x < \pi, \quad 0 < y < \pi.$$

Write a matlab code to solve this Poisson boundary value problem using the SOR method with the optimal parameter on a 50×50 grid ω (see §9.6 of the textbook). Stop the iterations when the relative change in the L2 norm of the solution is less than 10^{-6}. Use the command

```
[X,Y] = meshgrid(x,y); surf(X,Y,u,'edgecolor','none');
```

to visualize the solution. How many iterations are sufficient for convergence? Repeat the computations for $n = 25, 100, 200$? How does the number of iterations change with the size n of the $n \times n$ grid?

[Total: 50]