Assignment 4 **MATH 4NA** **22 February 2021**

This assignment is due at 6pm on Monday 8 March.

Instructions:

- Carefully read and answer **all** parts of each question. Four significant figures are sufficient for numerical answers.
- Include printouts of **all** `matlab` m-files and scripts you wrote to generate your results.
- Include and fully **label** all graphs: label axes, add a title or figure caption and indicate the corresponding question.
- Submit as a **single pdf file** on the Avenue dropbox by the due date.

Review exercises NOT TO BE HANDED IN:

1. What is a *stiff* system of differential equations? What methods are suitable to solve them?
2. In general, how do the stability properties of explicit and implicit methods differ?
3. How are Neumann boundary conditions incorporated into a finite difference method for a linear boundary-value problem?
4. How can you spot numerical instability in the solution to a boundary-value problem?

Exercises TO BE HANDED IN:

[4] 1. (a) Let $g(z)$ be a user-defined function with possibly complex-valued coefficients and variables. Then the set of all such z in the complex plane that satisfy $|g(z)| = 1$ is in general a curve or consists of several curves. Write a `Matlab` script to plot the curves(s) (in the complex plane) determined by the equation $|g(z)| = 1$, where $g(z) = \sum_{k=1}^{10} z^k/k!$.

[6] (b) Recall that one of the RK3 methods is as follows:

$$\begin{aligned}
y_{n+1} &= y_n + \frac{1}{6}(k_1 + 4k_2 + k_3), \quad \text{where} \\
k_1 &= hf(x_n, y_n) \\
k_2 &= hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1), \\
k_3 &= hf(x_n + h, y_n - k_1 + 2k_2).
\end{aligned}$$

Show that the stability region of the above RK3 method is determined by $|E(\lambda h)| \leq 1$ where

$$E(\lambda h) = 1 + \lambda h + \frac{(\lambda h)^2}{2!} + \frac{(\lambda h)^3}{3!}.$$

[2] (c) In general, the stability region of a $p-$th order Runge–Kutta method is determined by $|E(\lambda h)| \leq 1$ with $E(\lambda h) = \sum_{k=0}^{p} \frac{(\lambda h)^k}{k!}$. Write `Matlab` scripts (similar to that used in (a)) to plot the stability regions for $p = 1, 2, 3, 4, 5$ by letting $\lambda h = z$. Are the stability regions larger for larger p?

[12] 2. Exercise 9.2. Hint: the exact solution is

$$y(x) = c_1 e^{(1+\sqrt{5})x/2} + c_2 e^{(1-\sqrt{5})x/2} - x^2 + 3x - 5,$$

where (c_1, c_2) solves the linear system

$$\begin{aligned} (1+\sqrt{5})c_1 + (1-\sqrt{5})c_2 &= -6, \\ (1+\sqrt{5})e^{(1+\sqrt{5})/2}c_1 + (1-\sqrt{5})e^{(1-\sqrt{5})/2}c_2 &= -2. \end{aligned}$$

[12] 3. Exercise 8.24. Also include step $h = 0.001$ and use the initial condition $x(0) = 2, y(0) = -2$ (the initial conditions in the textbook lead to unstable solutions, even for the continuous equations).

[6] 4. (a) Show that the multi-step implicit method known as the Milne method

$$y_{k+1} = y_{k-1} + \frac{h}{3}\left(f(t_{k-1}, y_{k-1}) + 4f(t_k, y_k) + f(t_{k+1}, y_{k+1})\right)$$

is unconditionally *unstable* by applying it to the model equation $y' = \lambda y$. (Hint: let $y_k = q^k$, so $y_{k+1} = q y_k$, and then plot $\max(|q_-|, |q_+|)$ for the two roots of the resulting quadratic on $[-10, 10]$).

[4] (b) Derive a semi-implicit version of Milne's method by making the approximation

$$f(t_{k+1}, y_{k+1}) \approx f(t_k, y_k) + (y_{k+1} - y_k) f'(t_k, y_k).$$

[4] (c) Apply the semi-implicit Milne's method to the initial-value problem

$$y' = y(3 - 4y + y^2),\ y(0) = 2,$$

on the interval $[0, 5]$ with step sizes $h = 0.1$ and $h = 0.05$ and comment on the results. (Hint: start with an Euler step.)

[Total: 50]