Assignment 3 **MATH 4NA** **8 February 2021**

This assignment is due at 6pm on Monday 22 February.

Instructions:

- Carefully read and answer **all** parts of each question. Four significant figures are sufficient for numerical answers.
- Include printouts of **all** `matlab` m-files and scripts you wrote to generate your results.
- Include and fully **label** all graphs: label axes, add a title or figure caption and indicate the corresponding question.
- Submit as a **single pdf file** on the Avenue dropbox by the due date.

Review exercises NOT TO BE HANDED IN:

1. What is the difference between Newton-Cotes quadrature and Gaussian quadrature? Which is better (why)?
2. How can Richardson extrapolation be used to improve the accuracy of Newton–Cotes quadrature rules?
3. What is a singular integral? What are two ways of dealing with singular integrals?
4. What does it mean for a numerical solution to an ODE to be *convergent*?
5. What does it mean for a numerical solution to an ODE to be *stable*?

Exercises TO BE HANDED IN:

[12] 1. Apply Romberg integration to evaluate the integral

$$\int_0^{\pi} \sin^3 x \, \mathrm{d}x.$$

(You can use the routine `Romberg.m` from Avenue.) Terminate the iterations when the absolute value of the difference between two successive approximations is smaller than 10^{-12}. Plot the absolute error versus the number of iterations.

[12] 2. Exercise 8.18 from the textbook.

[8] 3. (a) Write `Matlab` functions to integrate the initial value problem

$$y' = f(x, y), \quad y(a) = y_0,$$

on an interval $[a, b]$ using:

- Euler's method

- Modified Euler
- Improved Euler
- Runge–Kutta 4

Hint: implement the methods as, for example,

```
[x,y] = eulerimp('f',a,y0,b,stepsize)
```

where $(x, y) = (x_n, y_n)$ is the computed solution.

[6] (b) Use the four `Matlab` function you wrote in (a) with stepsize $h = 1/40$ and $1/80$ to solve the following ODE:

$$y' = -y^3/2, \quad y(0) = 1 \quad \text{with} \quad y_{exact}(x) = \frac{1}{\sqrt{x+1}}. \tag{1}$$

Calculate the absolute errors (i.e. $|y_{exact}(x) - y_{computed}(x)|$) at $x = 1$ to see if they are roughly reduced by a half for Euler's method, by a quarter for Modified Euler and Improved Euler, and by 1/16 for RK4, as the stepsize h is halved from 1/40 to 1/80. Discuss the performance of the methods by comparing the computation time used by each method to calculate $y(1)$ with stepsize $h = 1/40$ and $1/80$ (use the `Matlab` functions `tic` and `toc` to do the timing).

[12] 4. Analyze the stability of the explicit method

$$y_{k+1} = 4y_k - 3y_{k-1} - 2hf(t_{k-1}, y_{k-1}).$$

for $f(y) = -\lambda y$. Apply the method to the initial value ODE problem

$$y' = -y \log y, \quad y(0) = 2$$

on the interval $[0, 3]$ with different time steps h and investigate the stability of this explicit method. [Hint: use your implementation to confirm the stability analysis.]

[Total: 50]