Assignment 2 | **MATH 4NA** | **25 January 2021**

This assignment is due in class on Monday 8 February.

Instructions:

- Carefully read and answer **all** parts of each question. Four significant figures are sufficient for numerical answers.
- Include printouts of **all** `matlab` m-files and scripts you wrote to generate your results.
- Include and fully **label** all graphs: label axes, add a title or figure caption and indicate the corresponding question.
- Submit as a **single pdf file** on the Avenue dropbox by the due date.

Review exercises NOT TO BE HANDED IN:

1. Review the `matlab` tutorials available on the course website (if you haven't already done so).
2. What is the difference between conditioning and numerical stability?
3. What is Richardson extrapolation? Does it give a more accurate answer than the values on which it is based?
4. What is a good way of numerically approximating the derivative a function given only at a discrete set of data points? How do you deal with the effect of noise in the data?
5. How can you estimate the error in an integration approximation?

Exercises TO BE HANDED IN:

1. You are given the *cardinal Whittaker* function

$$f(x) = \frac{\sin(x)}{x}$$

in the interval $\Omega = [-\pi, \pi]$.

[6] (a) Using the step size $h = 0.2$ calculate approximations to $f'(x)$ in Ω using the following methods:

i. second–order *one–sided* (forward) differences,

ii. second–order central differences,

iii. complex step derivative, i.e., $f'(x) = \frac{\Im(F(x+ih))}{h} + \mathcal{O}(h^3)$ where $F(z)$ is a complex analytic extension of $f(x)$, i.e., $F : \mathbb{C} \to \mathbb{C}$ and $F(x) = f(x)$ for $x \in \mathbb{R}$,

iv. fourth–order central differences.

Plot the resulting curves using solid lines in different colors and the derivative computed analytically using a dashed line.

[4] (b) Using the four approximate methods mentioned above, calculate for $x_0 = \frac{\pi}{12}$ the relative errors of the derivatives as

$$\left|\frac{f'_{approx}(x_0) - f'_{exact}(x_0)}{f'_{exact}(x_0)}\right|$$

and plot them on a log–log graph as a function of the step size h (for h use the values obtained with `logspace(-10,0,10)`). Comment on the error convergence and appearance of round-off error in the each method.

[8] 2. Apply Richardson extrapolation to evaluate the second derivative of $f(x) = \mathrm{sech}^2(x)$ at $x = 1$. Terminate the iterations when the total distance between two successive approximations is smaller than 10^{-8}. Plot the absolute error versus the number of iterations.

[10] 3. Use the composite trapezoidal rule and midpoint rules to approximate numerically the integrals

$$\int_0^1 x^{3/2}\,\mathrm{d}x, \qquad \int_0^1 x^{5/2}\,\mathrm{d}x.$$

Plot the global error E versus the step size h and find the least-square power fits $E = Ch^p$. Explain why $p \neq 2$ for some computations. Repeat the exercise for Simpson's rule and explain why $p \neq 4$ for all computations.

4. Consider the following *improper* integral.

$$\int_0^1 \frac{e^{-x}}{x^{3/4}}\,\mathrm{d}x.$$

[4] (a) Evaluate this integral by first *removing* the singularity by a change of variables, and then applying the composite trapezoid rule. Plot the absolute error as a function of stepsize h on a log–log graph. [Hint: once you have made the change of variables, use `maple` to evaluate the integral to 16 digits precision.]

[4] (b) Evaluate this integral by using a combination of composite trapezoid rule at interior points and the mid-point rule at the first point. Plot the absolute error as a function of stepsize h on a log–log graph.

[2] (c) Comment on any differences between the results obtained in (a) and (b).

5. Consider the function $f(x) = x(1 - x^2 + 2x^4 - x^6)$.

[4] (a) Approximate the first derivative of $f(x)$ at $x = 1$ by central difference approximations of orders $m = 1, 2, 3, 4, 5, 6$ (use matlab).

[4] (b) At what order does the central difference approximation recover the exact first derivative of $f(x)$? Does the error keep decreasing for higher orders? Why or why not?

[4] (c) Repeat the same example with Richardson extrapolation and check if the algorithm is truncated at a finite number of iterations.

[Total: 50]