Math 4NA3 - Assignment 5

April 1, 2021

Question 1:

a) We are given the equation

$$y'' + Q(x)y = F(x)$$

and the central difference approximations are $y'' = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}$ and $y' = \frac{y_{n+1} - y_{n-1}}{2h}$. Together, this gives the system

$$y_{n+1}\,[1] + y_n\left[h^2 Q(x) - 2\right] + y_{n-1}\,[1] = h^2 F(x)$$

We can implement the system with each set of boundary values as follows.

```
function [x,y] = fdbvp1(qfun,ffun,a,b,ya,yb,h)
x = a:h:b;
n = length(x);
Q = arrayfun(qfun,x);
F = arrayfun(ffun,x);
C = ones(1,n);
B = (h^2)*Q - 2*ones(1,n);
A = ones(1,n);
D = (h^2)*F;
D(1,1) = D(1,1) - ya;
D(1,end) = D(1,end) - yb;
y = tridiag(A,B,C,D);
end

function [x,y] = fdbvp2(qfun,ffun,a,b,ya,yp,h)
n = 1/h;
x = linspace(a,b,n+1);
Q = arrayfun(qfun,x);
F = arrayfun(ffun,x);
C = ones(1,n+1);
C(1,1) = -2;
B = (h^2)*Q - 2*ones(1,n+1);
A = ones(1,n+1);
A(1,end) = -2;
D = (h^2)*F;
D(1,1) = D(1,1) - ya;
D(1,end) = D(1,end) - 2*h*yp;
y = tridiag(A,B,C,D);
end
```

b) We use the MATLAB functions on the given bvp's as follows.

```
[x1,y1] = fdbvp1(@(x) 1,@(x) 3*x^2,0,2,0,3.5031,0.1);
[x2,y2] = fdbvp2(@(x) 1,@(x) 3*x^2,0,2,0,6.5442,0.1);
hold on
grid on
```

```
plot(x1,y1,"o")
plot(x2,y2,"o")
fplot(@(x) 6*cos(x) + 3*(x^2-2), [0,2])
legend("(3)","(4)", "Exact")
```

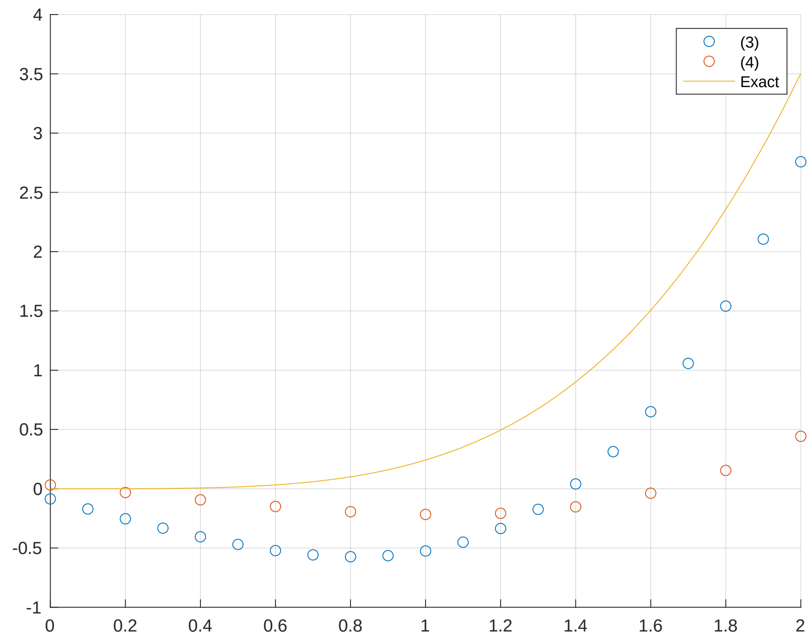

Figure 1: Numerical Solutions and Exact

From the solutions, I suspect there is a bug in my code or I wrote the matrix wrong when I was deriving it.

c) Using the above function at $h = 0.1$ and $h = 0.05$ gives the following error plot. Halving the grid spacing halved the error. Thus, I suspect the method is $O(h)$.

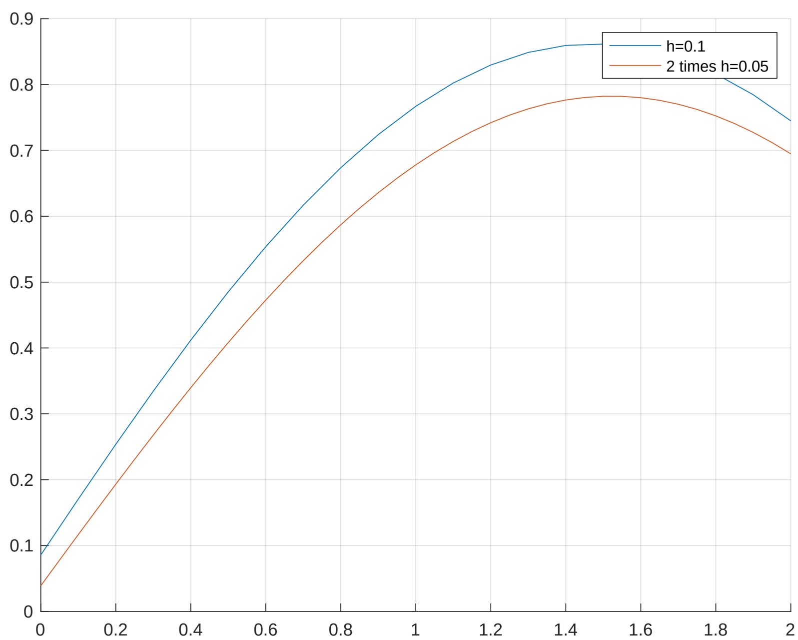

Figure 2: Error Plot

Question 2:

a) Here is the MATLAB implementation of the finite element method applied to equation (1) with the given approximations.

```
function [x,y] = fe(qfun, ffun, L, R, ya, yb, h)
x = L:h:R;
n = length(x);
for i = 1:n-1;
     a(i) = quad(qfun,x(i),x(i+1))./3;
     b(i) = quad(qfun,x(i),x(i+1))./6;
     e(i) = quad(ffun,x(i),x(i+1))./2;
end
C = b(2:end) + (1/h);
B = a(1:end-1) + a(2:end) - 2*(1/h);
A = b(1:end-1) + (1/h);
D = e(2:end) + e(2:end);
D(1) = D(1) - ((1/h) + a(1))*ya;
D(end) = D(end) - ((1/h) + a(end))*yb;
y = tridiag(A,B,C,D);
end
```

b) We apply the function to the given problem as follows.

```
[x,y] = fe(@(x) sin(x)-2,@(x) (sin(x)).^2-6,0,4,3,2.2432,0.1)
[x1,y1] = fe(@(x) sin(x)-2,@(x) (sin(x)).^2-6,0,4,3,2.2432,0.05)
size(x)
size(y)
plot(x(2:end-1),y,"o")
hold on
grid on
fplot(@(x) sin(x)+3,[0,4])
legend("Approx","Exact")
```

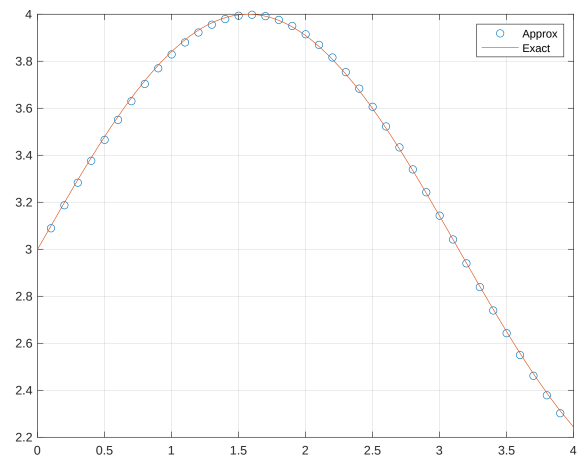

Figure 3: $h = 0.1$

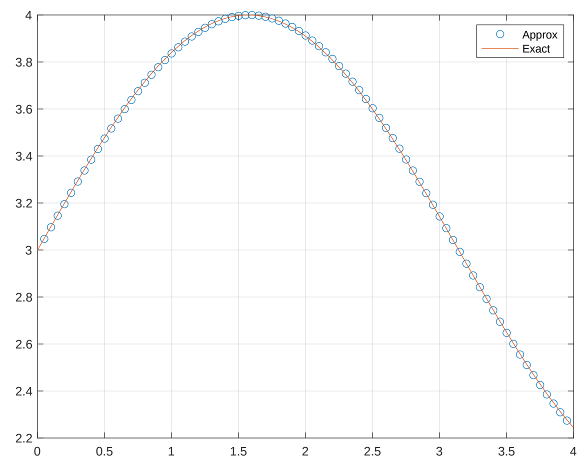

Figure 4: $h = 0.05$

Now we can plot the error as follows.

```
xnew = x(2:end-1);
xnew1 = x1(2:end-1);

for i = 1:length(xnew)
    e(i) = exact(xnew(i))-y(i);
end
for i = 1:length(xnew1)
    e1(i) = exact(xnew1(i))-y1(i);
end

plot(xnew,e)
hold on
grid on
plot(xnew1,e1)
legend("h=0.1","h=0.05")
```

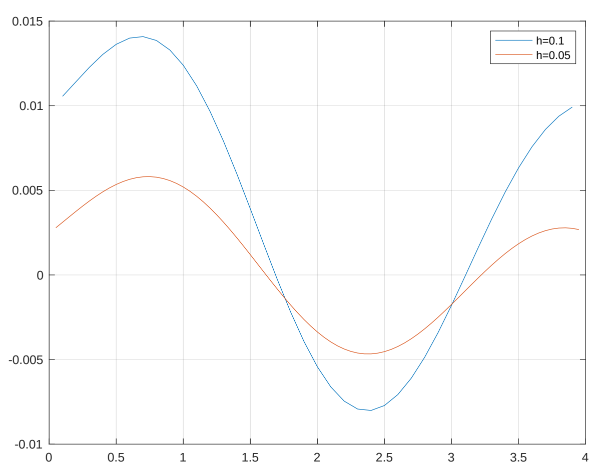

Figure 5: Error

From the error plot, we suspect that the method is $O(h)$.

Question 3:

We are given the linear bvp

$$y'' + x^2 y = 1,\ x \in (0,1),\ y(0) = y(1) = 0$$

To solve this we use the finite difference function from question 1 and the altered shooting method code as follows.

```
clear
clc

tic
[x1,y1] = fdbvp1(@(x) x^2,@(x) 1,0,1,0,0,0.001);
toc

hold on
plot(x1,y1,"r")

tic
h = 0.01;
x = 0:h:1;
F = @(x) 1;
Q = @(x) x.^2;
alpha = 0; beta = 0;
fu = @(x,y) -Q(x)*y + F(x);
fv = @(x,y) -Q(x)*y;
u1(1) = alpha; u2(1) = 0;
v1(1) = 0;     v2(1) = 1;
for k = 1 : length(x)-1
    fu1 = fu(x(k),u1(k));
    u1p = u1(k) + h * u2(k);
    u2p = u2(k) + h * fu1;
    u1(k+1) = u1(k) + 0.5*h * (u2(k) + u2p);
    u2(k+1) = u2(k) + 0.5*h * (fu1 + fu(x(k+1),u1p));
    fv1 = fv(x(k),v1(k));
    v1p = v1(k) + h * v2(k);
    v2p = v2(k) + h * fv1;
    v1(k+1) = v1(k) + 0.5*h * (v2(k) + v2p);
    v2(k+1) = v2(k) + 0.5*h * (fv1 + fv(x(k+1),v1p));
end
A = (beta - u1(end)) / v1(end);
y = u1 + A*v1;
toc

plot(x,y,'b'); xlabel('x'); ylabel('y(x)'); grid on;
```

```
axis([x(1),x(end),-0.15, 0.22]);set(gca,'fontsize',16);
legend("FD","Shoot")
hold off
```

```
Elapsed time is 0.005448 seconds.
Elapsed time is 0.011002 seconds.
```

The solutions are somewhat similar at $h = 0.01$ as seen from the solution and difference plot. Furthermore, the shooting method takes twice the time as the finite difference method at $h = 0.1$. For $h = 0.001$ we have the inverse result.

```
Elapsed time is 0.013307 seconds.
Elapsed time is 0.006602 seconds.
```

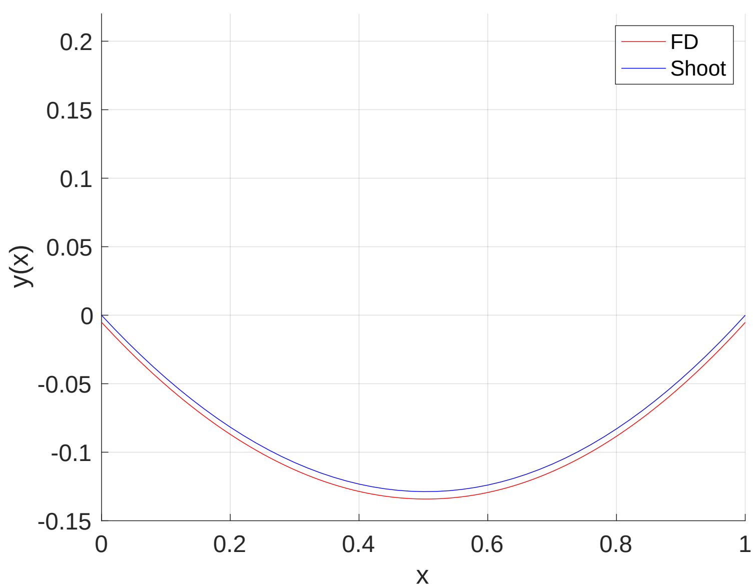

Figure 6: Solutions

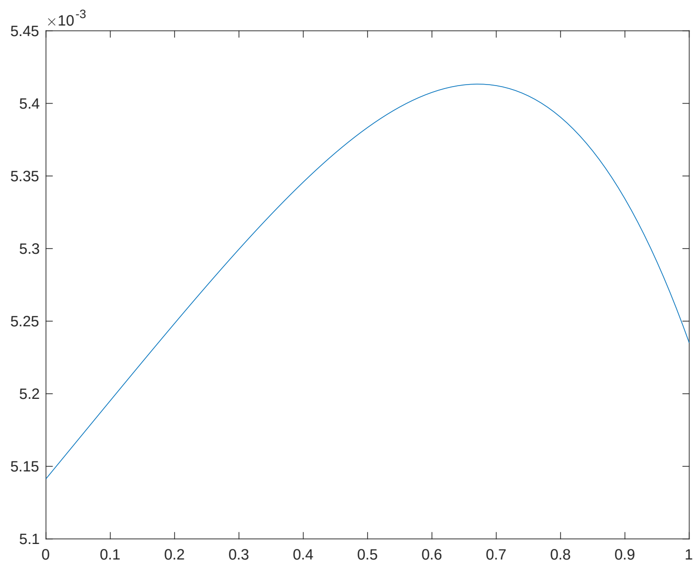

Figure 7: Difference