Assignment 5 **MATH 4NA** **15 March 2021**

This assignment is due at 6pm on Monday 29 March.

Instructions:

- Carefully read and answer **all** parts of each question. Four significant figures are sufficient for numerical answers.
- Include printouts of **all** `matlab` m-files and scripts you wrote to generate your results.
- Include and fully **label** all graphs: label axes, add a title or figure caption and indicate the corresponding question.
- Submit as a **single pdf file** on the Avenue dropbox by the due date.

Review exercises NOT TO BE HANDED IN:

1. What is the difference between direct and iterative methods for solving a linear system of equations?
2. When do the Gauss–Seidel method differ and Jacobi methods converge?
3. What is the relation between linear systems and solutions of boundary value problems?
4. Will the shooting method always converge?
5. Write down the equations for the weighted residual method for solving a linear second-order boundary value problem.
6. What are the test functions in the collocation method?
7. What is a Galerkin method? What is a finite element method?

Exercises TO BE HANDED IN:

1. Consider the following boundary value problems

$$y'' + Q(x)y = F(x), \quad y(a) = \alpha, \quad y(b) = \beta. \tag{1}$$

$$y'' + Q(x)y = F(x), \quad y(a) = \alpha, \quad y'(b) = \beta. \tag{2}$$

[10] (a) Write `Matlab` functions to solve BVPs (1) and (2) using the finite difference method. In the algorithms, use the central difference formulas to approximate y'' and $y'(b)$ (add a fictitious grid point where it is appropriate) and solve the resulting tridiagonal equations using the `Matlab` function `tridiag` from the course website on Avenue. Hint: implement the functions for (1) and (2) as follows:

```
[x,y] = fdbvp1('funcq','funcf',a,b,ya,yb,gridsize)
[x,y] = fdbvp2('funcq','funcf',a,b,ya,yprimeb,gridsize)
```

where

```
gridsize = the length of each subinterval of [a,b];
x=a:gridsize:b; y=the computed solution array;
funcq = the function name of the function Q(x);
funcf = the function name of the function F(x);
a and b are, respectively, the left and right endpoints;
ya=y(a), yb=y(b), yprimeb=y'(b) are the boundary conditions.
```

[4] (b) Using a uniform gridsize $h = 0.1$, apply the `Matlab` functions written in (a) to solve

$$y'' + y = 3x^2, \quad y(0) = 0, \quad y(2) = 3.5031, \tag{3}$$

$$y'' + y = 3x^2, \quad y(0) = 0, \quad y'(2) = 6.5442. \tag{4}$$

Plot the computed solutions of (3) and (4).

[6] (c) Now use the reduced gridsize $h = 0.05$ to solve (3). Note that the exact (analytic) solution of (3) is $y_{exact}(x) = 6\cos(x) + 3(x^2 - 2)$. Graph the error function $e(x) = y_{exact}(x) - y_{computed}(x)$ with $h = 0.1$ and 0.05 respectively, on the same plot. Determine the approximation order of the finite difference method.

[10] 2. (a) Write a `Matlab` function to solve (1) by the finite element method using linear splines with a uniform gridsize h. In the algorithm, you will have to evaluate the integrals a_i, b_i, d_i, e_i and f_i (following the notation introduced in class). They can be computed using the following approximation formulas:

$$a_i = d_i = \frac{h}{3} Q_{av,i} \quad b_i = \frac{h}{6} Q_{av,i}, \quad e_i = f_i = \frac{h}{2} F_{av,i},$$

where $N = \texttt{floor}((b-a)/h)$, $i = 1, 2, \ldots, N$ and

$$Q_{av,i} = \frac{1}{h} \int_{x_{i-1}}^{x_i} Q(x)\,\mathrm{d}x, \quad F_{av,i} = \frac{1}{h} \int_{x_{i-1}}^{x_i} F(x)\,\mathrm{d}x$$

are average values of $Q(x)$ and $F(x)$ over the $i-$th element: $[x_{i-1}, x_i]$. To compute $Q_{av,i}$ and $F_{av,i}$, you may use the `Matlab` function `quad`. To solve the resulting tridiagonal equation, use the function `tridiag` from the course website on Avenue.

[8] (b) Use the `Matlab` function you wrote in (a) to solve the following boundary value problem:

$$y'' + (\sin x - 2)y = \sin^2 x - 6, \quad y(0) = 3, \quad y(4) = 2.2432,$$

with gridsize $h = 0.1$ and 0.05 respectively. Plot the computed solutions. Compare them with the exact solution $y_{exact}(x) = \sin(x) + 3$ by plotting the error function $e(x) = y_{exact}(x) - y_{computed}(x)$ with $h = 0.1$ and 0.05, respectively, on the same plot. Determine the approximation order of the finite element method.

[12] 3. Consider the linear BVP

$$y'' + x^2 y = 1, \quad 0 < x < 1, \quad y(0) = y(1) = 0.$$

Solve this BVP using the second-order finite difference method and the linear shooting method with step size $h = 0.01$.

Plot the two solutions on the same graph, and the difference between the two solutions on a second graph. Are the results sufficiently close? Why or why not?

Compare the timing of the two methods (using the matlab commands `tic, toc`). Which method is faster and by how much? Repeat the timings for $h = 0.001$. Are the results the similar?

[Hint: you can adapt `shooting_linear_ODE.m` from Avenue.]

[Total: 50]