Bayesian Inference and Decision

Instructions

1. The student is allowed two, double-sided note sheets
2. Previously proved results should be clearly cited and properly invoked

1. Suppose y_i, for $i = 1, \ldots, n$, is a binary outcome defined by the following representation:

$$y_i = \begin{cases} 0 & z_i < 0 \\ 1 & z_i \geq 0 \end{cases}$$

where $z_i \sim N(x_i'\boldsymbol{\beta}, \lambda^{-1})$ for the $p \times 1$ vectors of covariates x_i and coefficients $\boldsymbol{\beta}$. Using the joint prior $\pi(\boldsymbol{\beta}, \lambda) \propto \lambda^{-1}$, answer the following.

(a) Determine the likelihood and the posterior.

(b) Find the full conditionals for the binary model and determine the best sampling strategy.

(c) Now suppose that y_i is an ordinal outcome with three levels defined by the following latent variable representation:

$$y_i = \begin{cases} 0 & c_0 < w_i < c_1 \\ 1 & c_1 \leq w_i < c_2 \\ 2 & c_2 \leq w_i < c_3 \end{cases}$$

where $w_i \sim N(x_i'\boldsymbol{\phi}, \eta^{-1})$, $c_0 = -\infty$, and $c_3 = \infty$. Place the ridge prior on $\boldsymbol{\phi}$ and let $\pi(\eta) \propto \eta^{-1}$. Derive the likelihood and find the posterior.

(d) Find the full conditionals for the ordinal model and determine a sampling strategy.

2. Let $\boldsymbol{\theta}_i \sim Multinom(\boldsymbol{\alpha}_i)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, J, J > 2$, total categories.

   (a) Find the conjugate prior for $\boldsymbol{\alpha}_i$.

(b) Define a transformation of $\boldsymbol{\alpha}_i$, call it $\boldsymbol{\gamma}_i$, such that $\boldsymbol{\gamma}_i \overset{iid}{\sim} MVN\left(\boldsymbol{\mu}, \sigma^2 I_{J\times J}\right)$ is a valid prior; where $\boldsymbol{\mu}$ is a $J \times 1$ vector of population means and $I_{J\times J}$ is the $J \times J$ identity matrix. Determine the resulting posterior for this model.

(c) Develop a strategy for drawing samples from the posterior in (b).

(d) Explain how to make the prior on $\boldsymbol{\gamma}_i$ informative as well as how to make it weakly informative.