CSCI 335 Software Design and Analysis III

Splay Trees/B-trees

Chapter 4

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Amortized cost

- Consider a sequence of M operations(insert/delete/find)
  - What is the total cost?
  - What is the average cost per operation?

Amortized cost

- Consider a sequence of M operations (insert/delete/find):
- Example: binary search tree (regular)
  - M operations can cost in the worst case **O(M * N)**
  - Each operation will **not** cost more than **O(N)**
  - On average each operation costs **O(N)**

Amortized cost

- Consider a sequence of M operations (insert/delete/find):
- AVL tree:
  - A sequence of M operations will cost **O(M * logN)**
  - Each operation will **not** cost more than **O (logN)**
  - On average each operation costs **O (logN)**

Amortized cost

- Consider a sequence of M operations (insert/delete/find):
- Suppose that total cost is **O(M * f(N))**

  irrespective of the actual sequence of operations.
- The average cost is **O(f(N))** for each operation.
- This is called **amortized running time**.
- Caveat:

Individual operations in the sequence can be expensive though !

Splay tree

- A tree with amortized running time of O (logN)
- Some operations could be more expensive
- How can we achieve this?

Splay tree

- The trick is to rebalance the tree after a **find()** operation
- Bring the item returned by find() to the root while applying rotations on the way to the root.

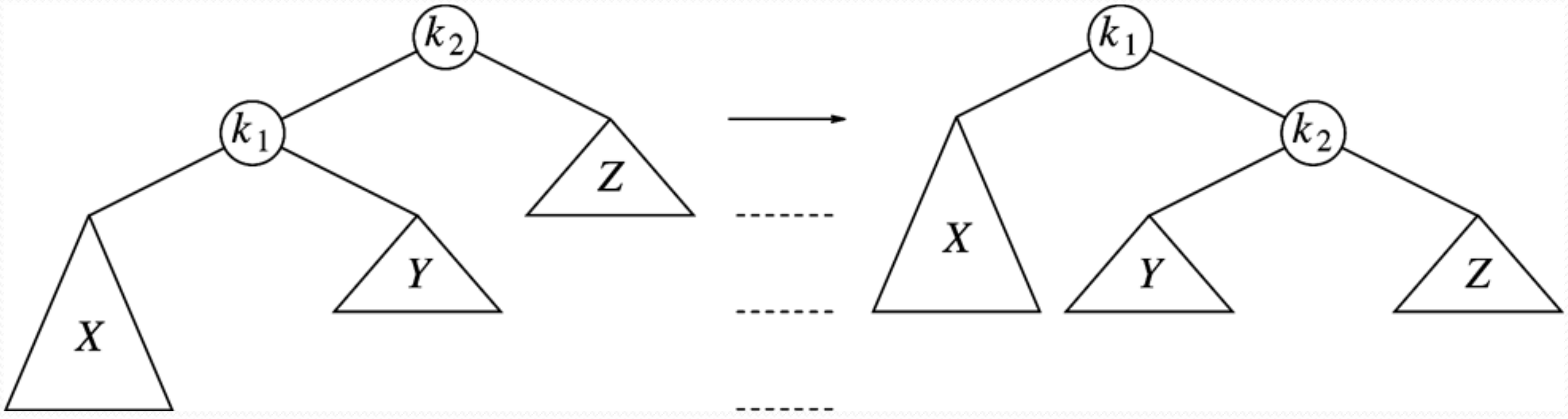
Splay tree

- Successive **finds()** of same element will be pretty fast
- Other items are coming **closer** to the root
- No need to store height information at each node
- **Amortized cost** of sequence of M operations is O(logN)

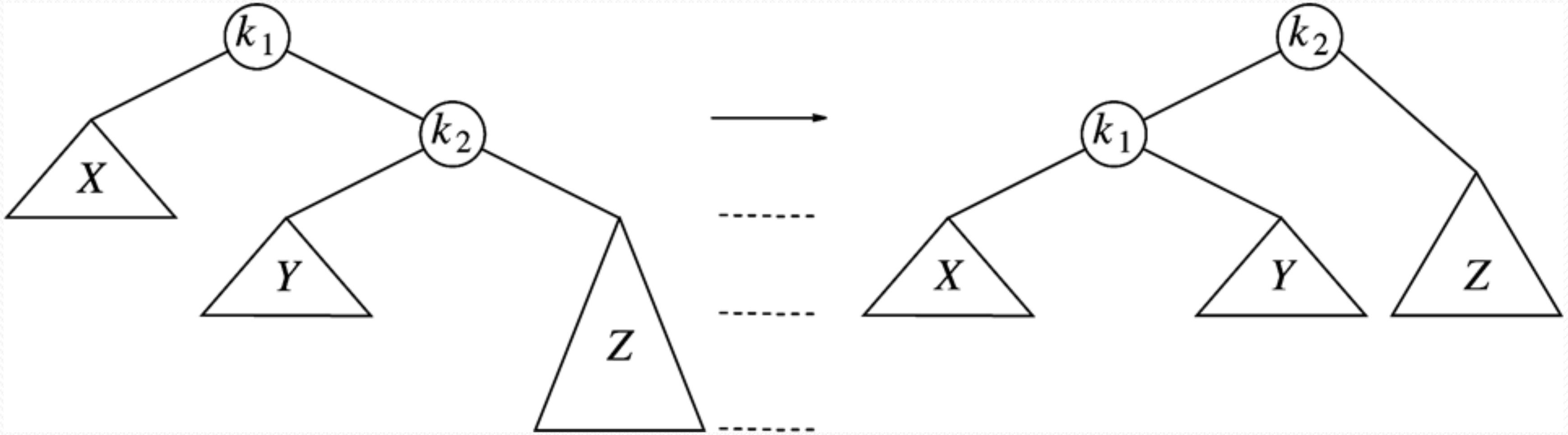
Splay Trees: a simple idea that does not work

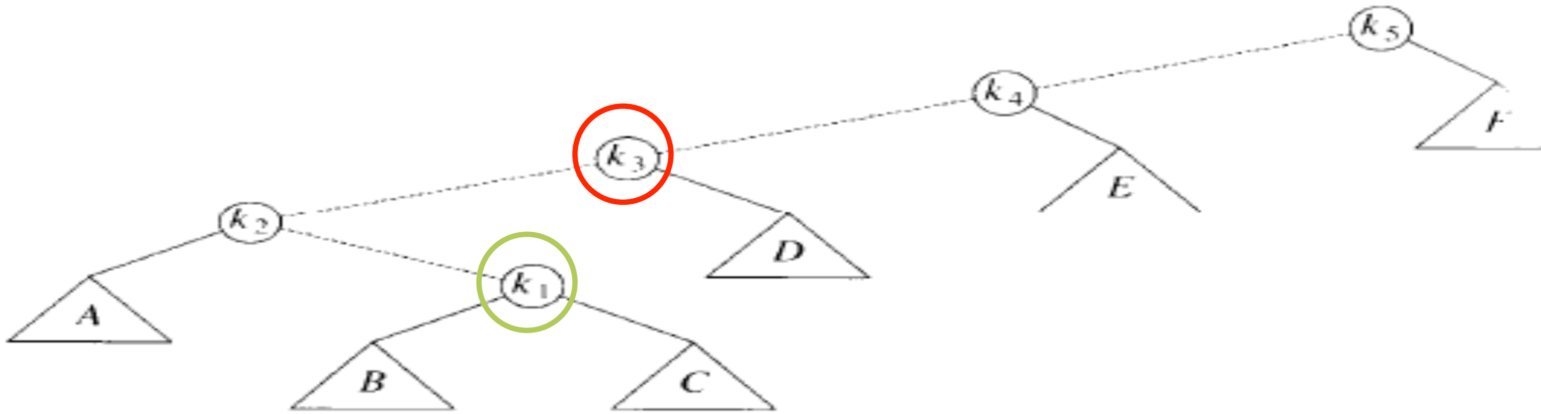
- After find() perform single-rotations bottom-up

Single rotation (case 1)

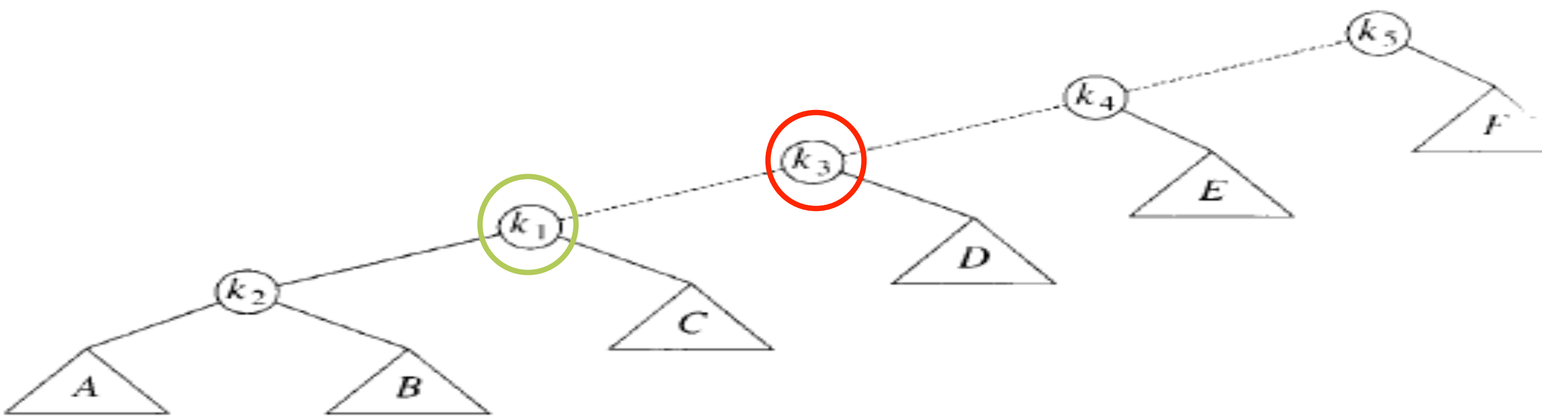


Single rotation (case 4)

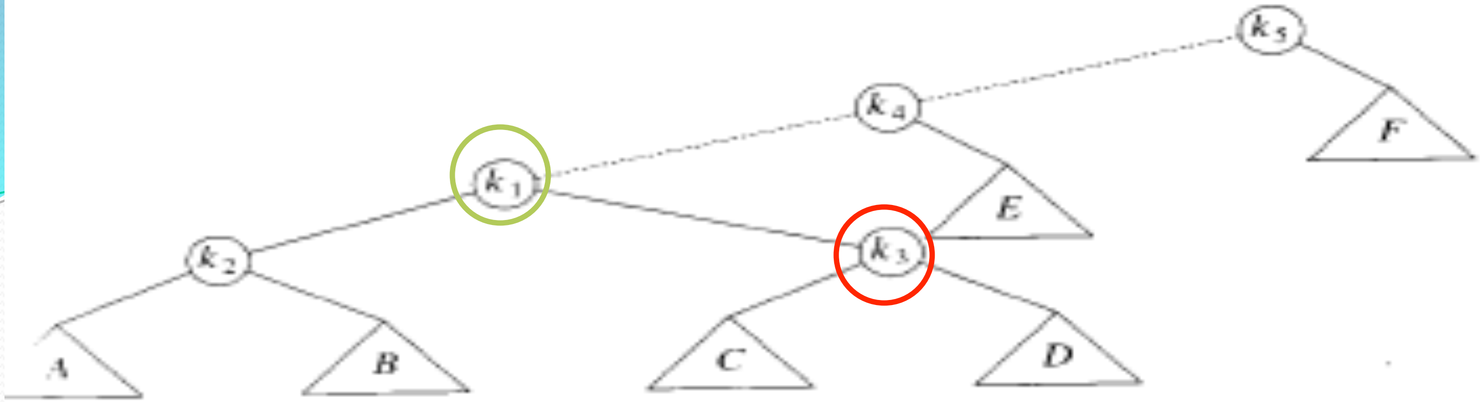




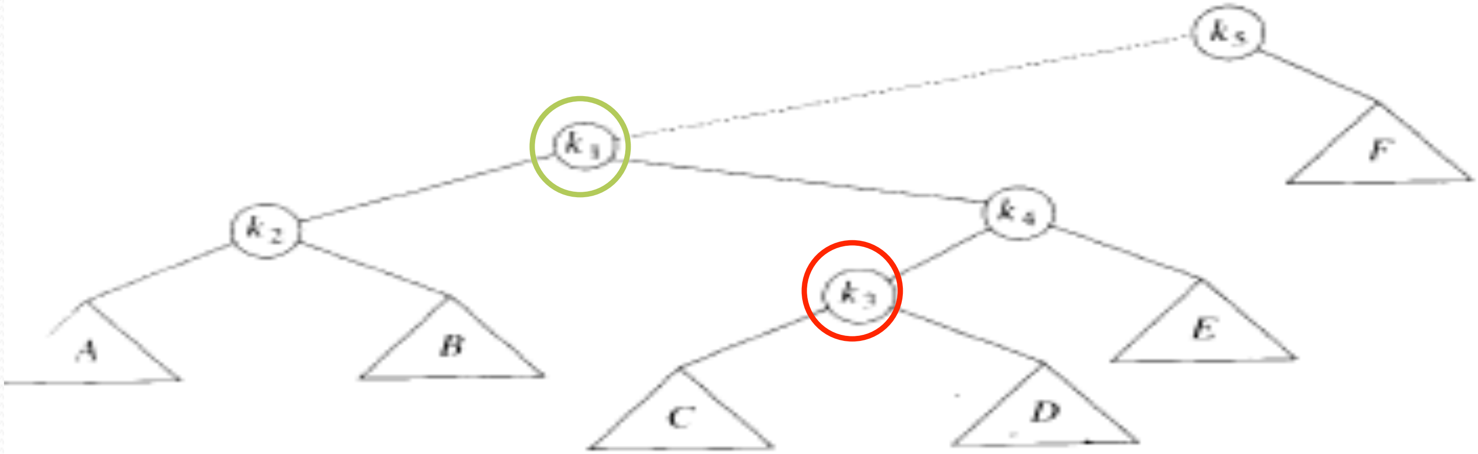
The access path is dashed. First, we would perform a single rotation between k_1 and
parent, obtaining the following tree.

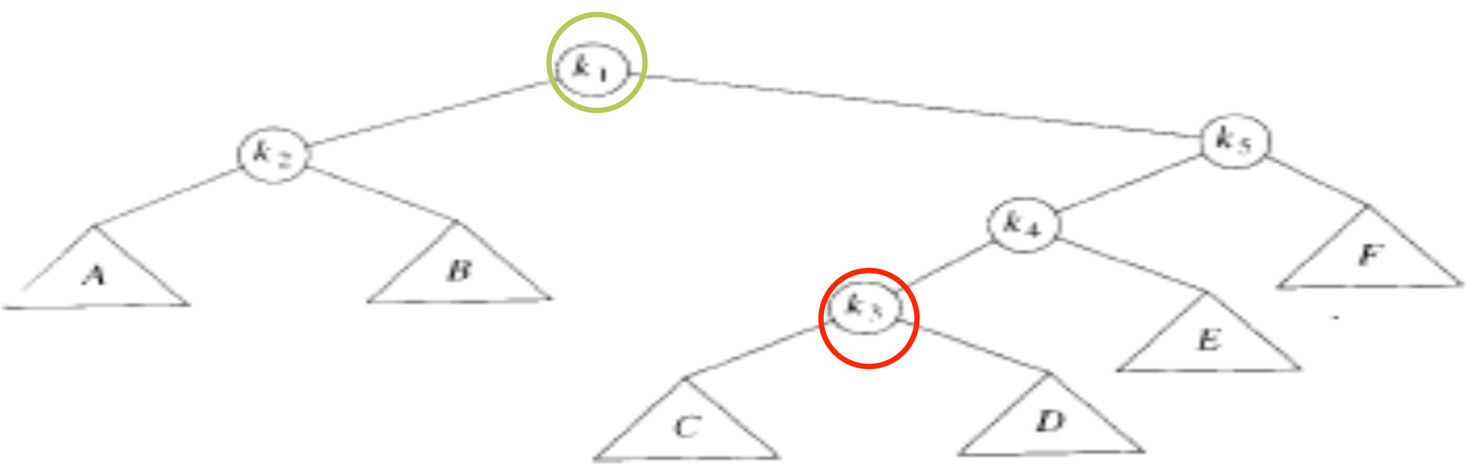


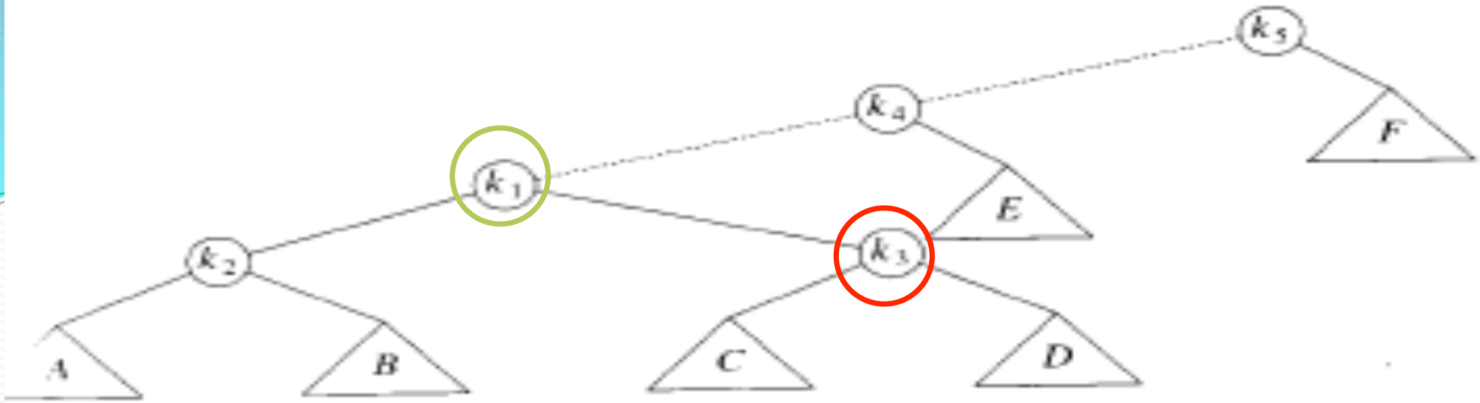
Then, we rotate between k_1 and k_3, obtaining the next tree.



en two more rotations are performed until we reach the root.







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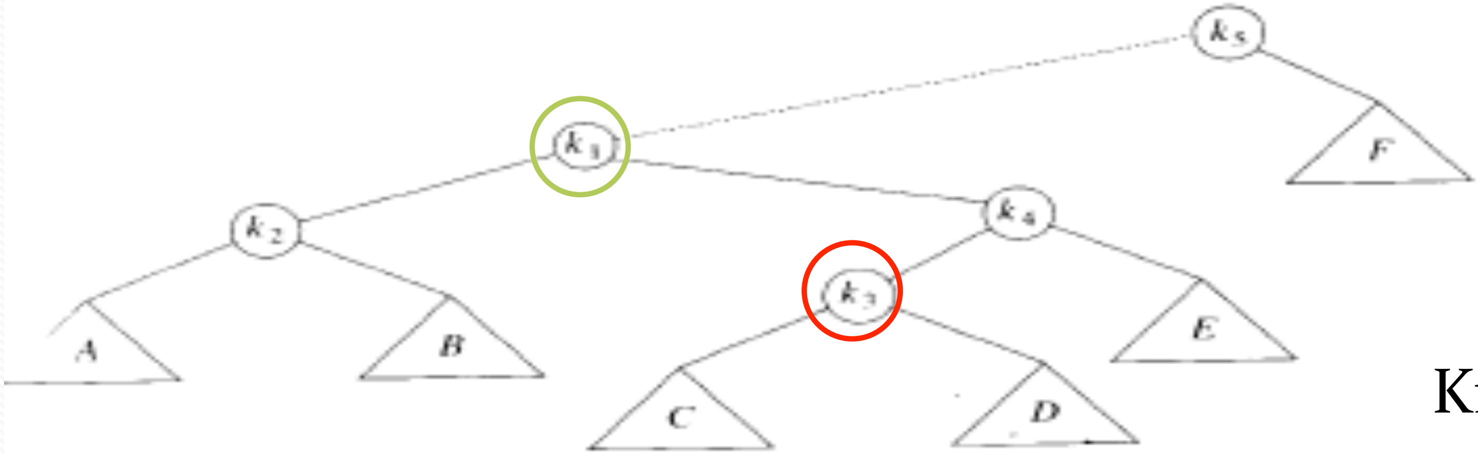
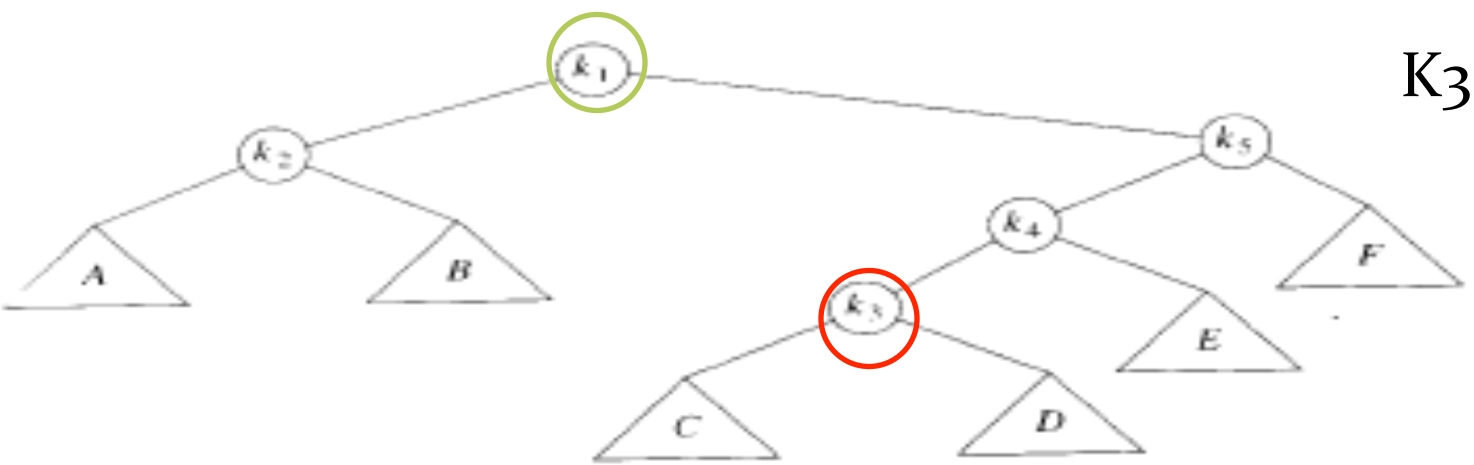




K1 went up

K3 went down

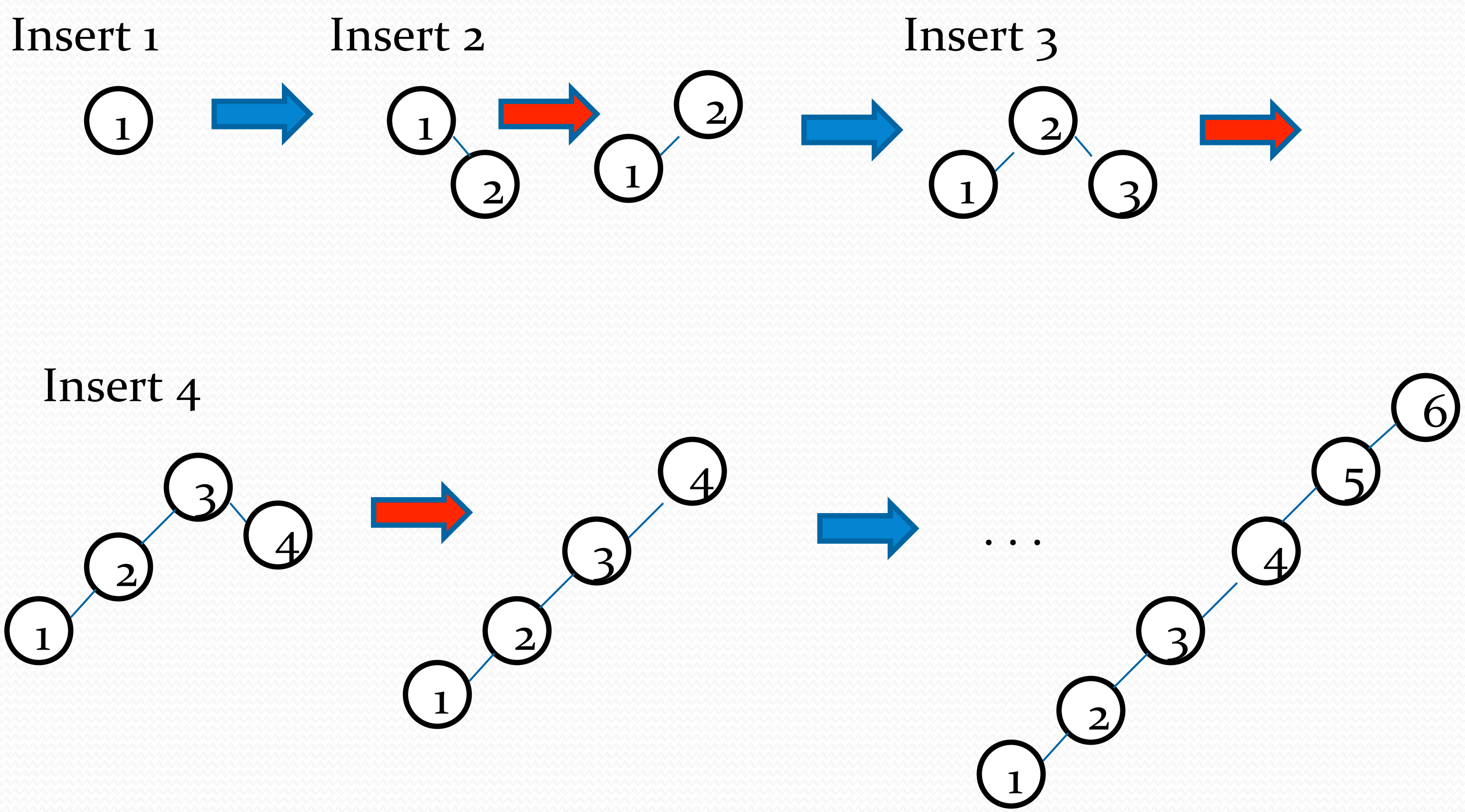
Splay Trees: a simple idea that does not work

- There is a sequence of M operations requiring $\Omega(N)$ time (**amortized**).
- => we want logarithmic amortized time

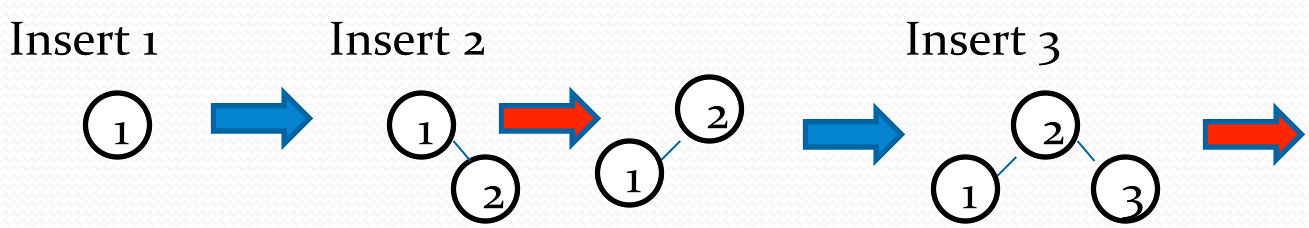
Splay Trees: M operations requiring $\Omega(N)$ time

- Consider inserting 1,2,3, … , N into an initially empty tree
  - Note that you splay on insertion (i.e. single AVL rotation)=>
  - Only left children
- Total time to build tree is O (N) (not bad)

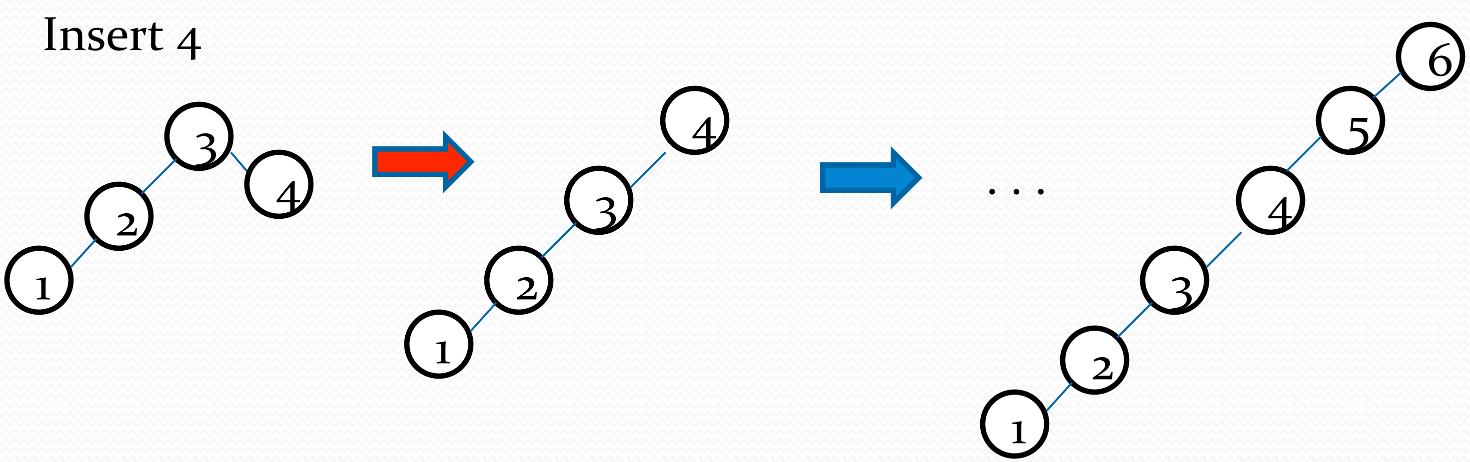
Insert 1, 2, 3, 4, 5, 6, ...



Insert 1, 2, 3, 4, 5, 6, … , N

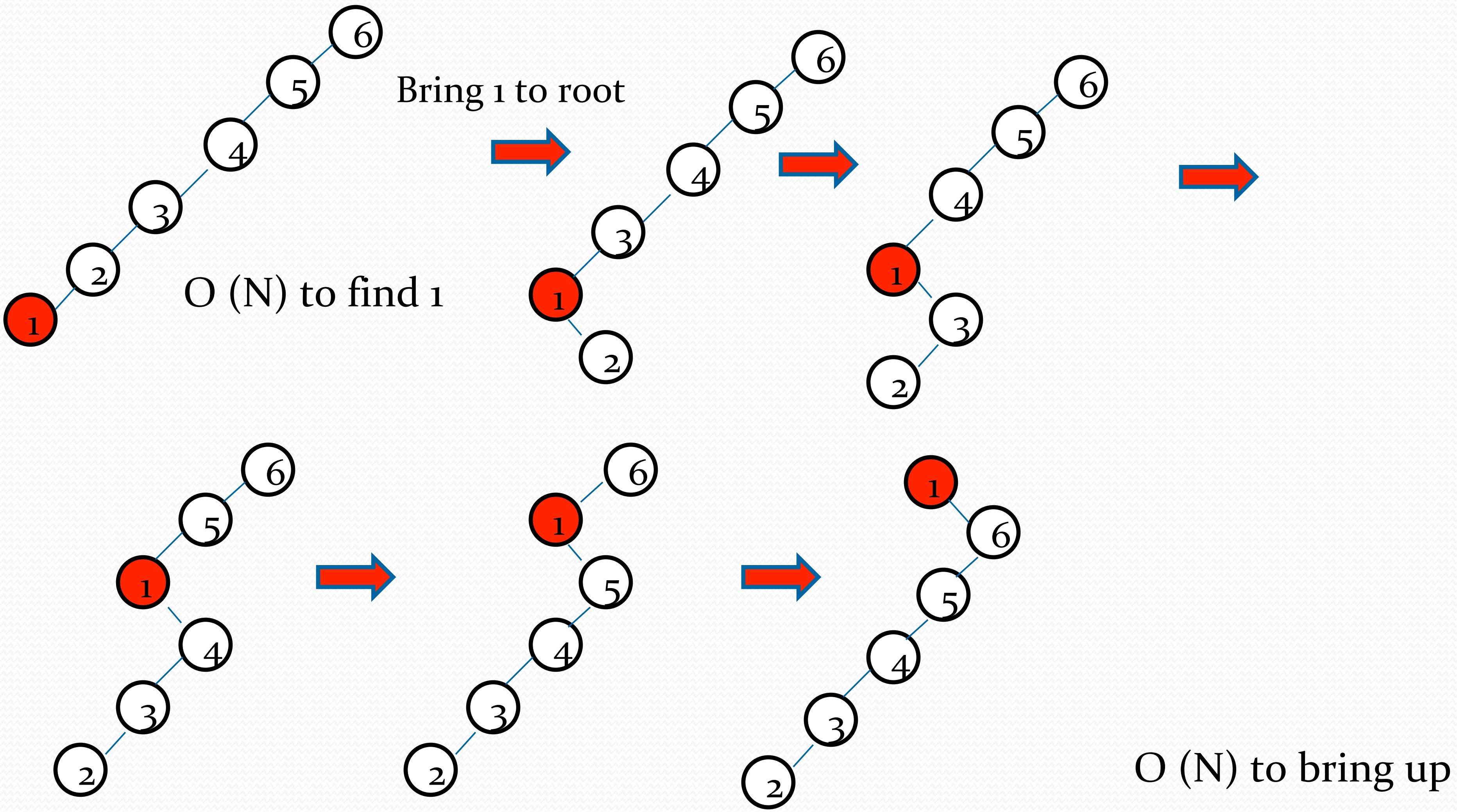


Total time to build tree is O (N)



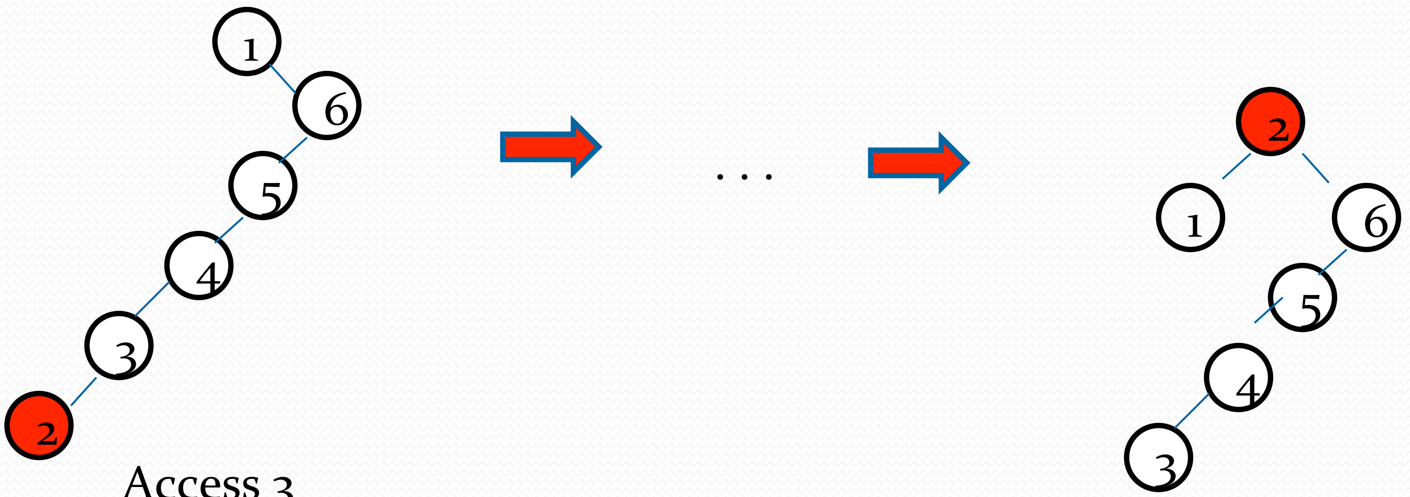
Access 1, 2, 3, 4, 5, 6

Access 1

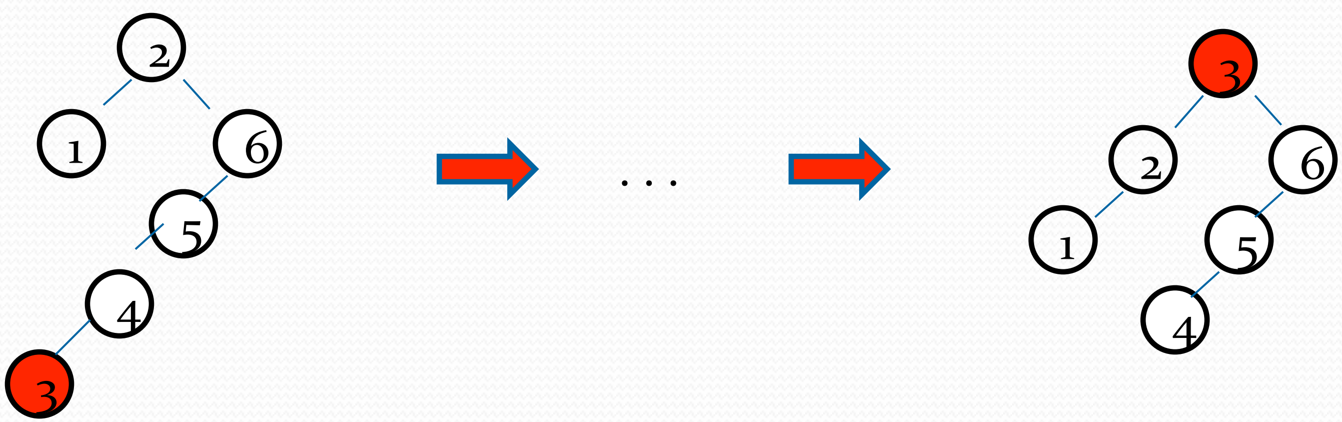


Access 1, 2, 3, 4, 5, 6

Access 2



Access 3



Splay Trees:

M operations requiring $\Omega(N)$ time

- Accessing the sequence 1,2,... is Ω (N^2) though...
- Run example:
  - Access 1 (time O(N)), then perform sequence of single rotations (time O(N))
  - Access 2 time O(N-1)...

Splaying – the correct way!

Rotate bottom-up on access, along access path

Let **X** a non-root node on access path at which we are rotating

if (parent of **X** is the root) { single rotation with root }

else {

Let **P** be parent of **X**, and **G** be the grandparent

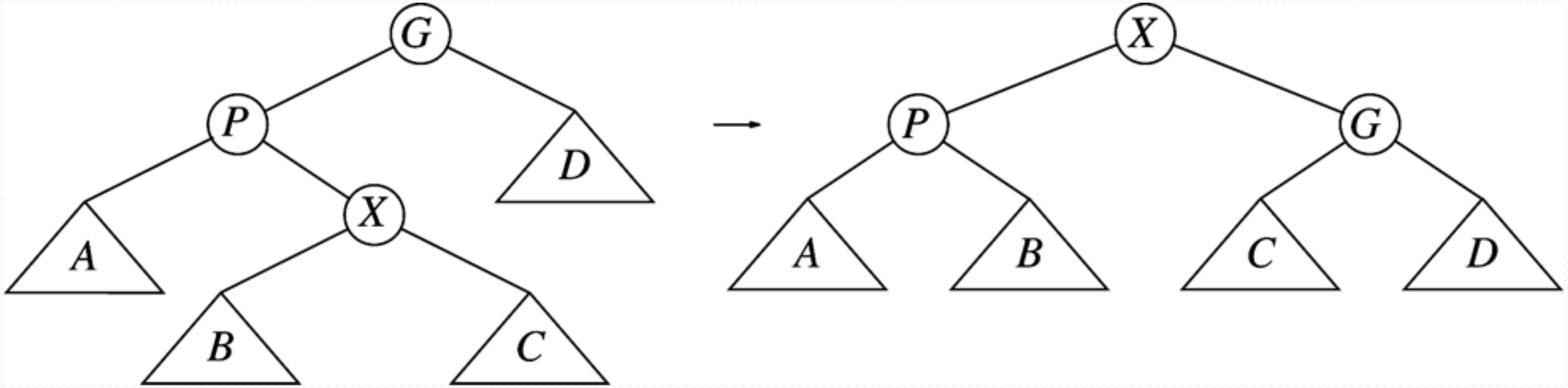
Two cases (plus symmetries) to consider:

- Case 1 (zig-zag): double rotation (see figure)
- Case2 (zig-zig): (see figure)

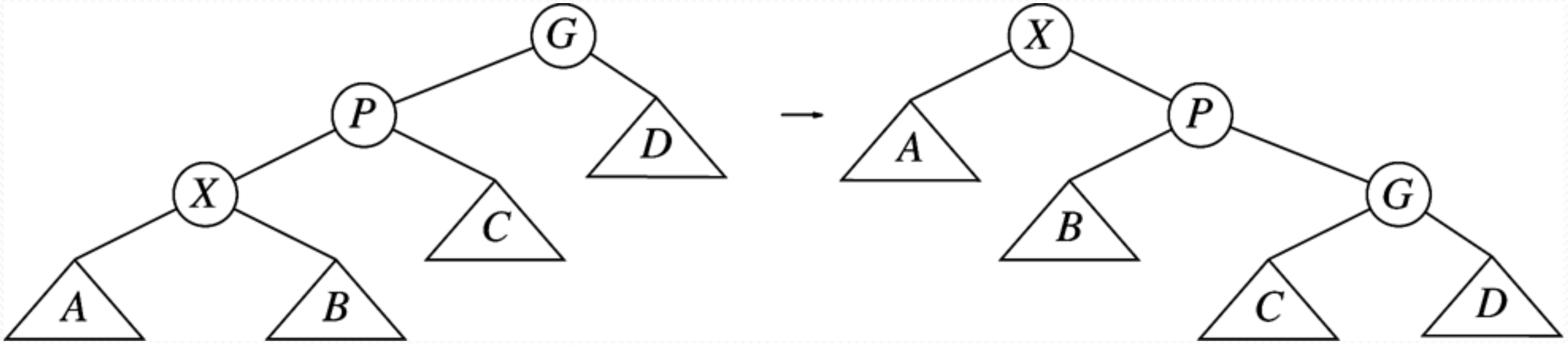
}

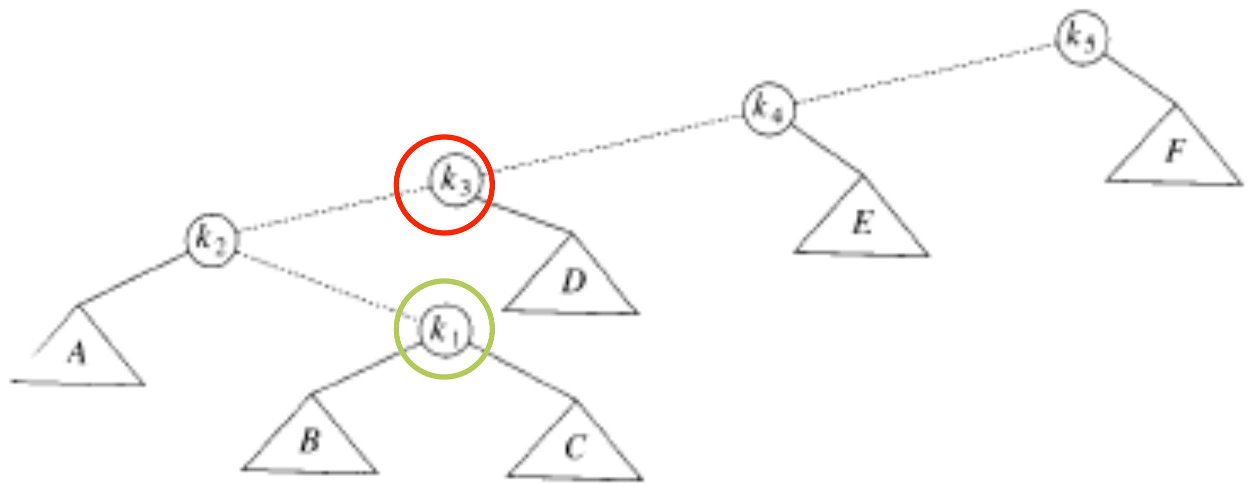
Applet: https://www.cs.usfca.edu/~galles/visualization/SplayTree.html

Zig-zag

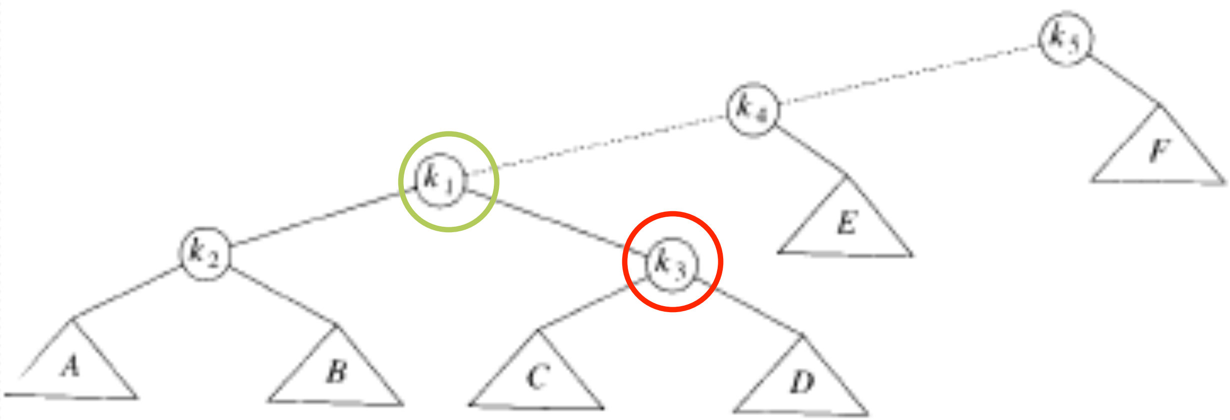


Zig-Zig

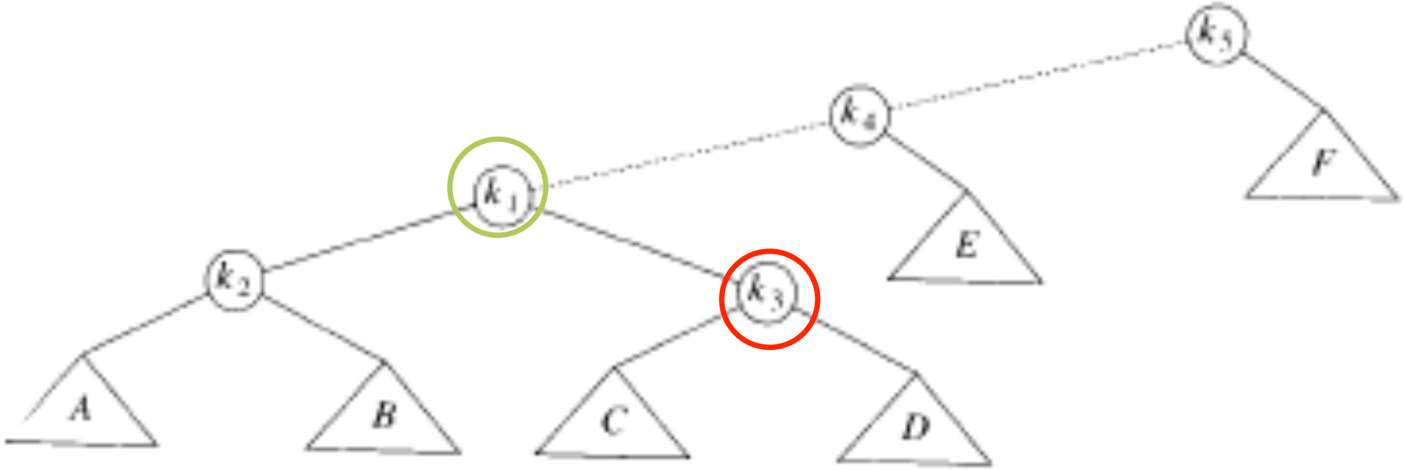




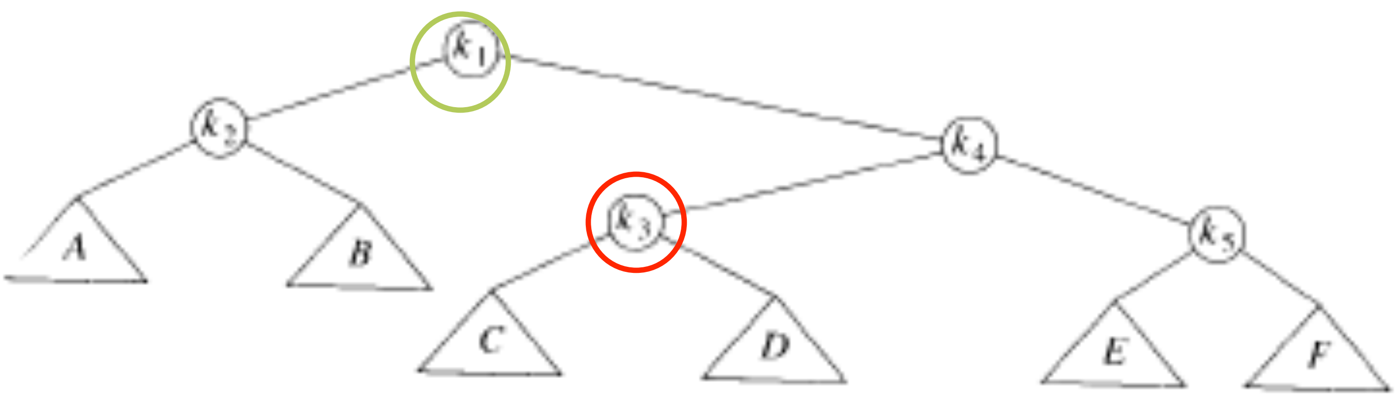
The first splay step is at k_1, and is clearly a *zig-zag*, so we perform a standard AVL double rotation using k_1, k_2, and k_3. The resulting tree follows.

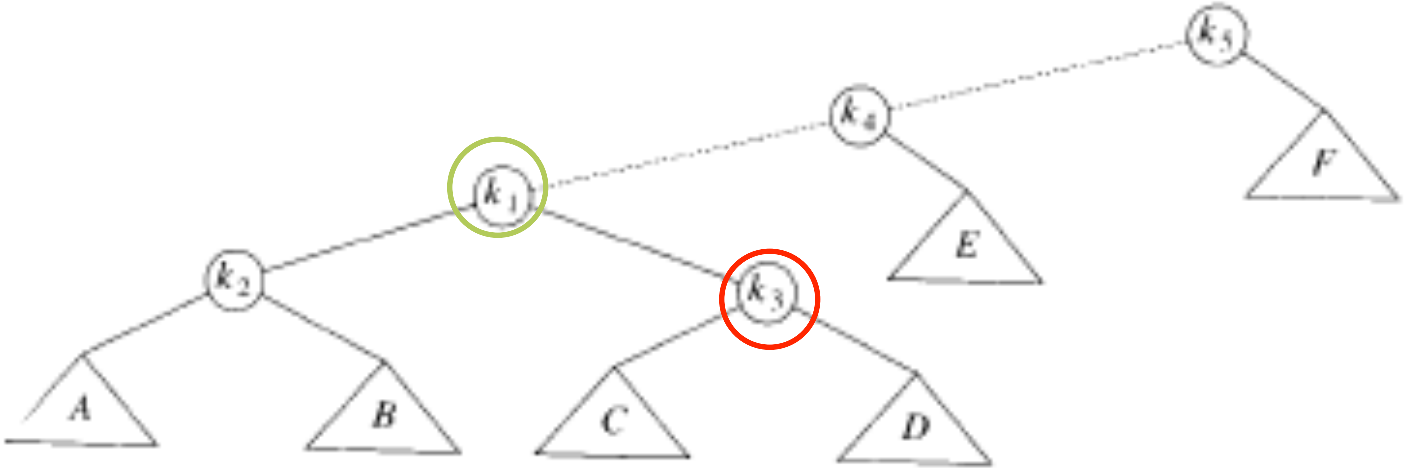


The next splay step at k_1 is a zig-zig



The next splay step at k_1 is a zig-zig, so we do the zig-zig rotation with k_1, k_4, and k_5, obtaining the final tree.





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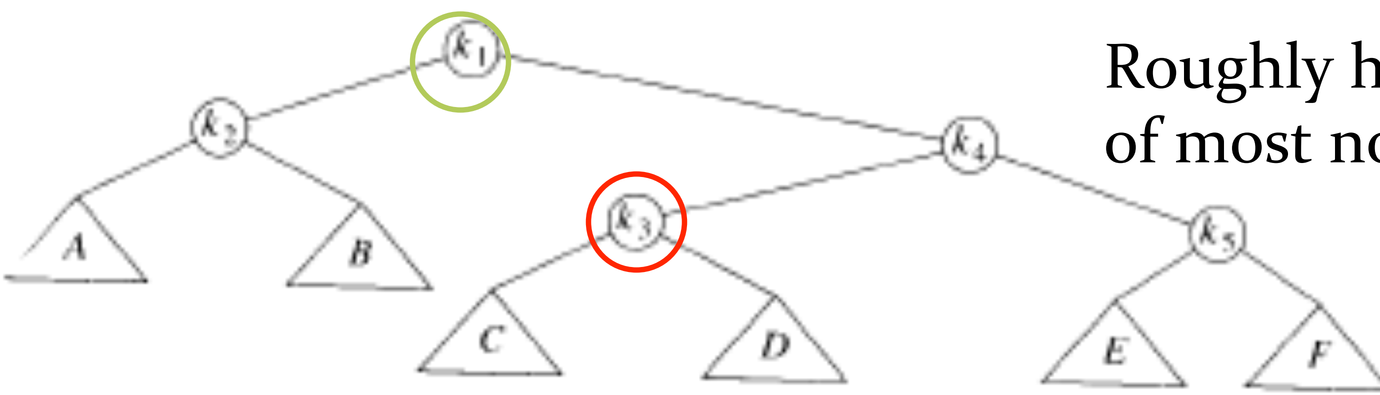
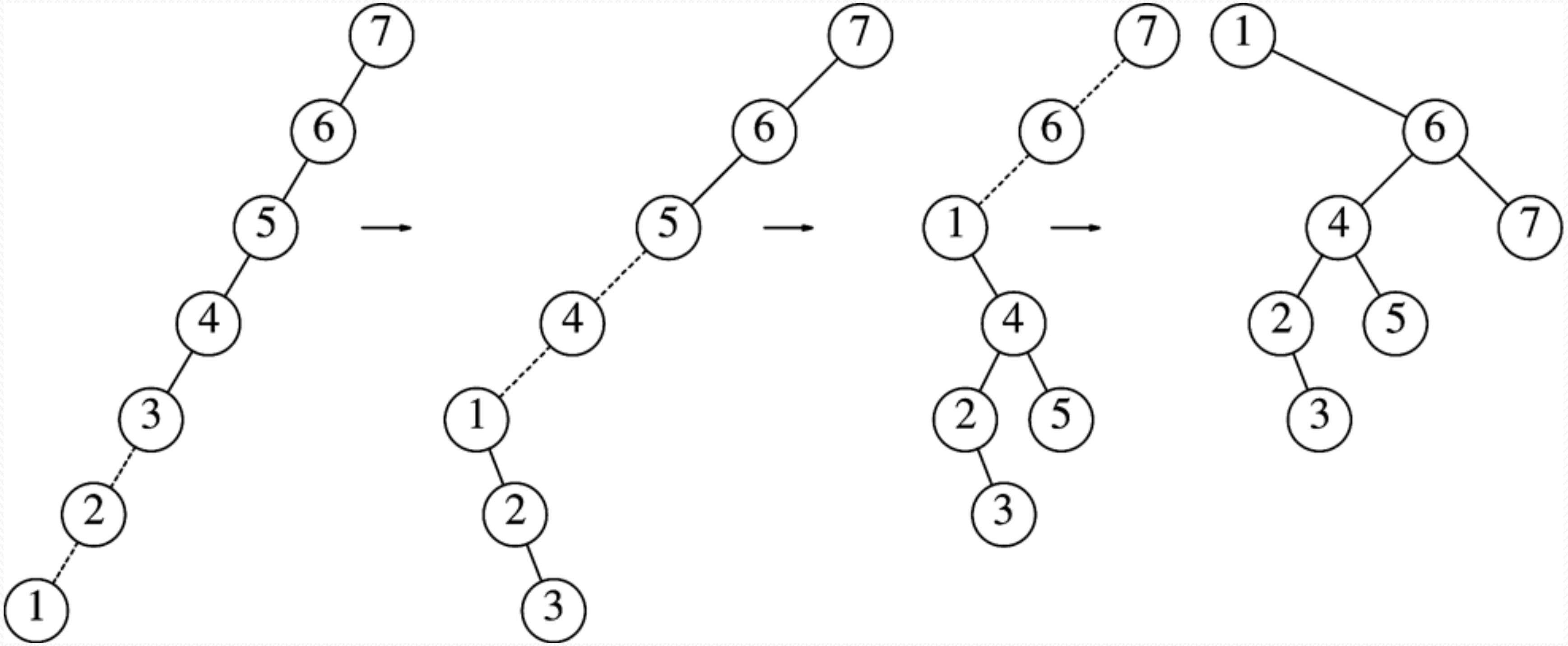
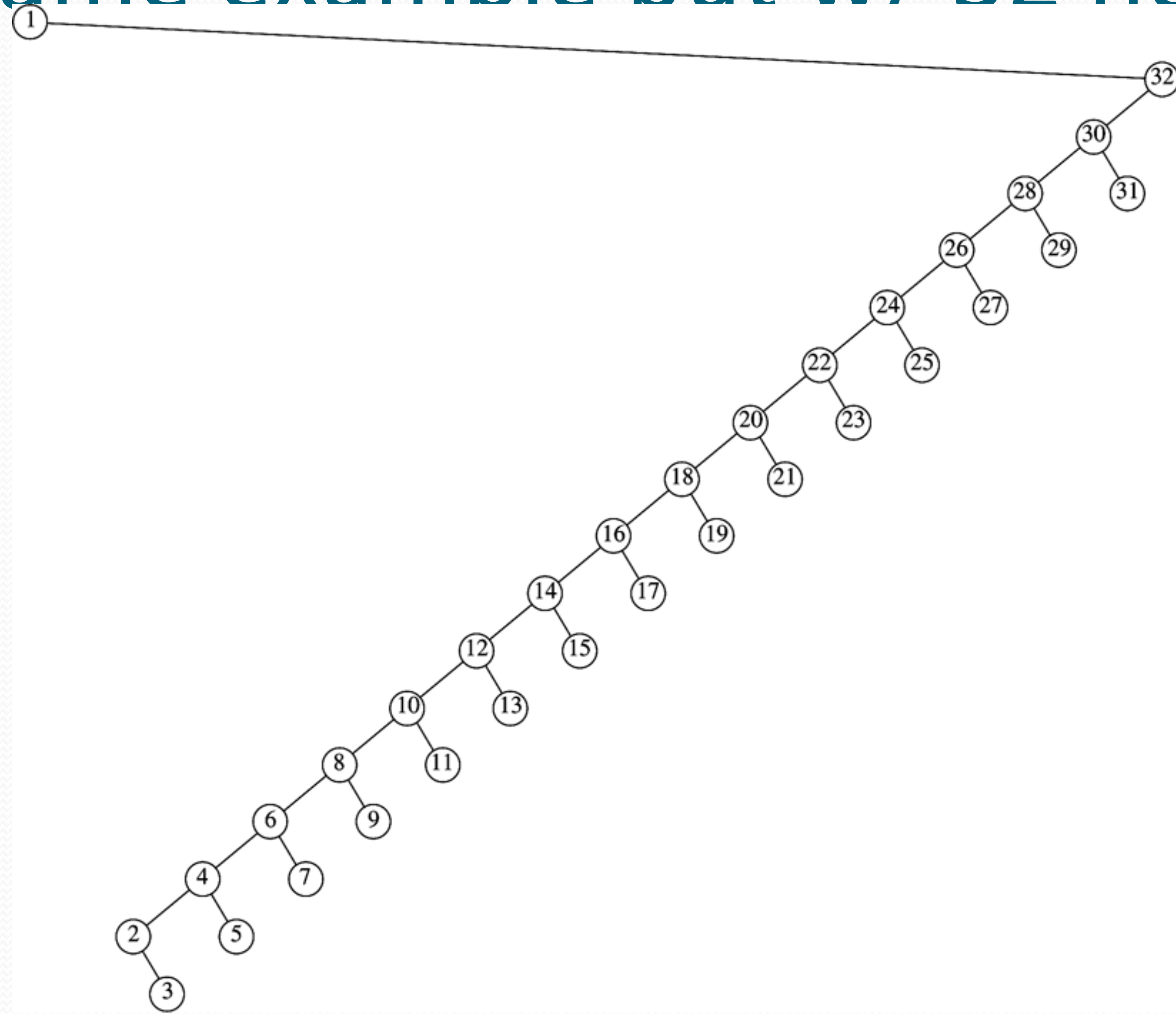


Roughly halving depth of most nodes…

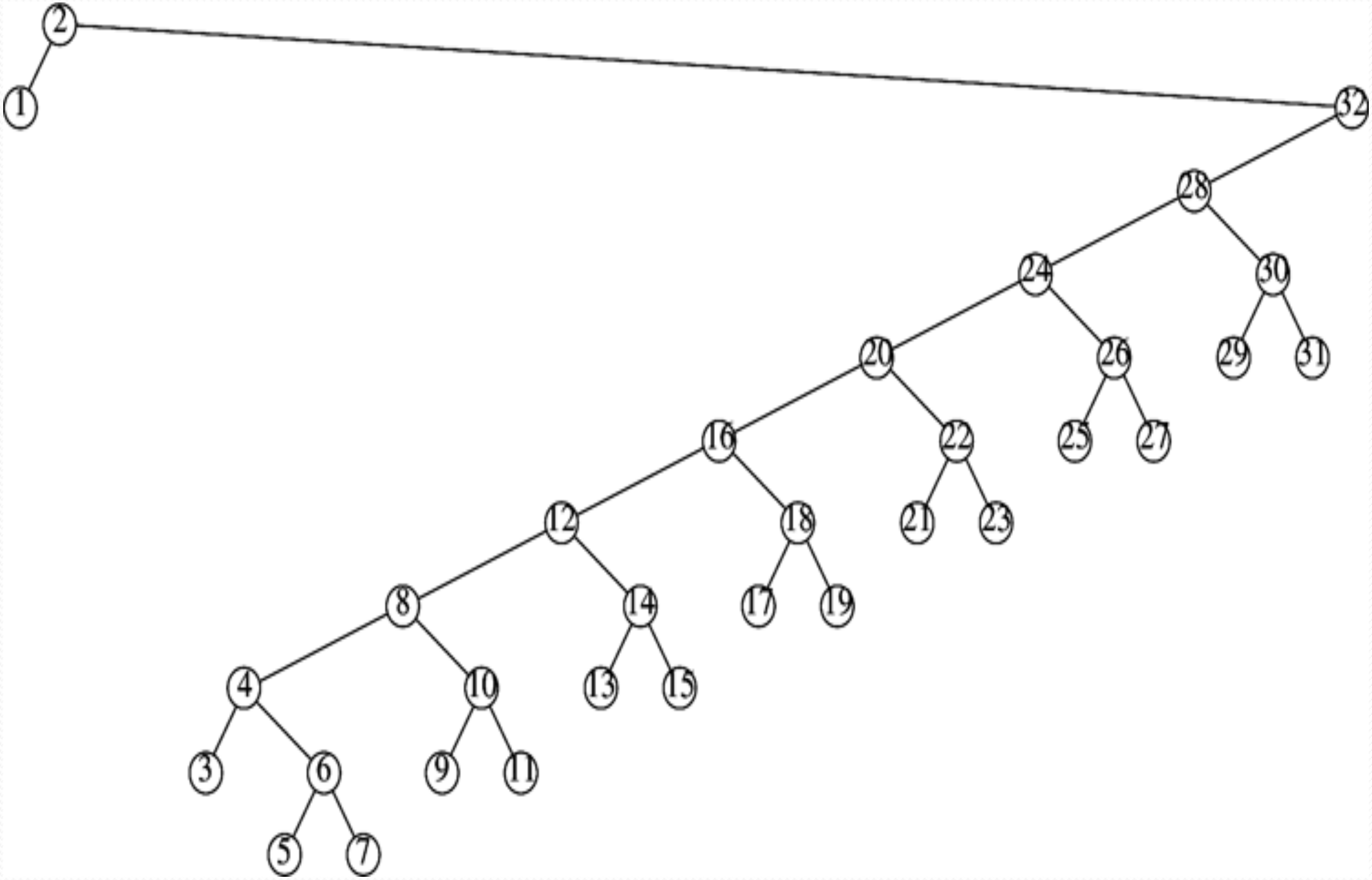
Example
Insert 1,…,7 -> then access 1



Same example but w/ 32 nodes



Same example but w/ 32 nodes



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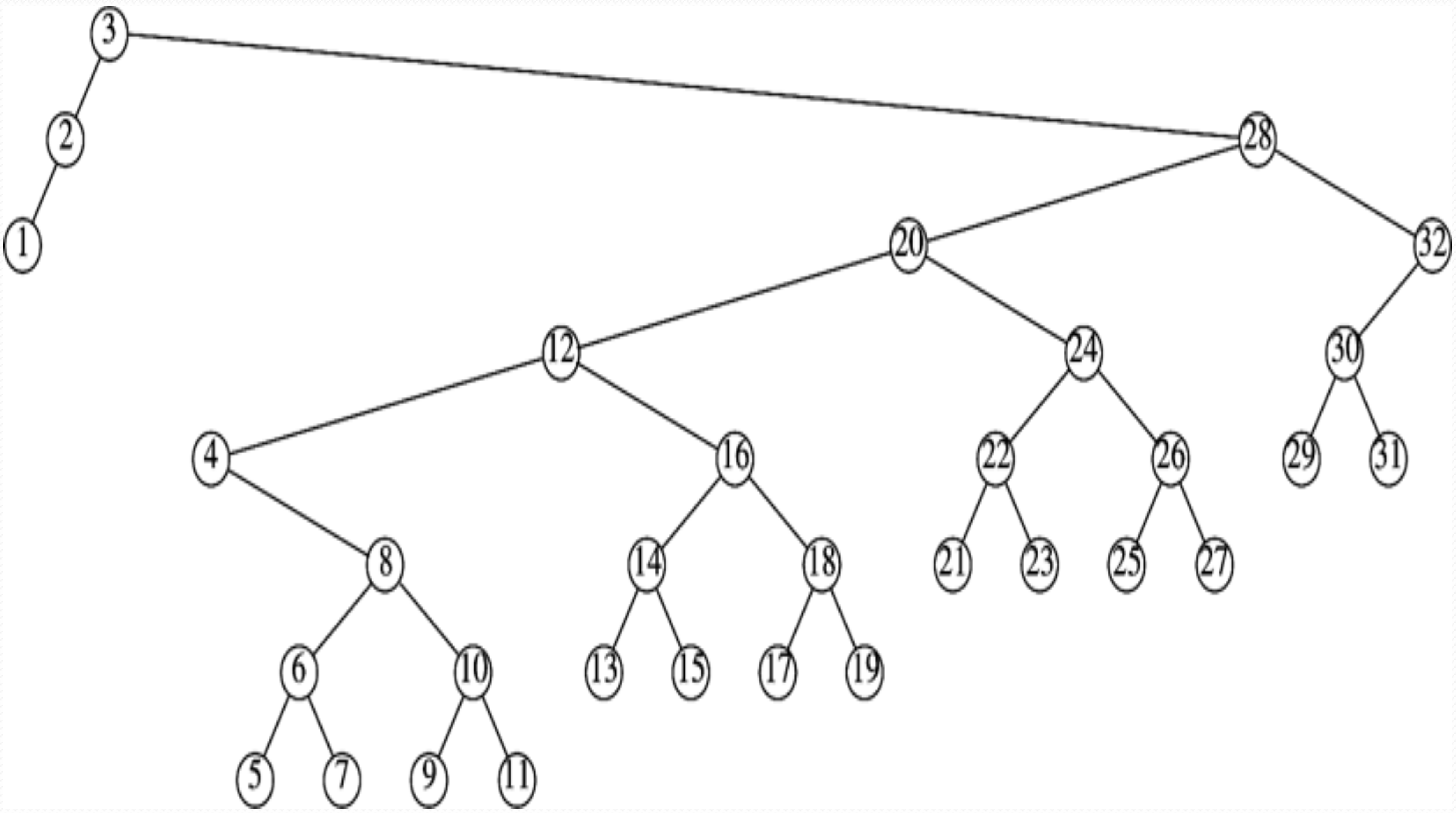

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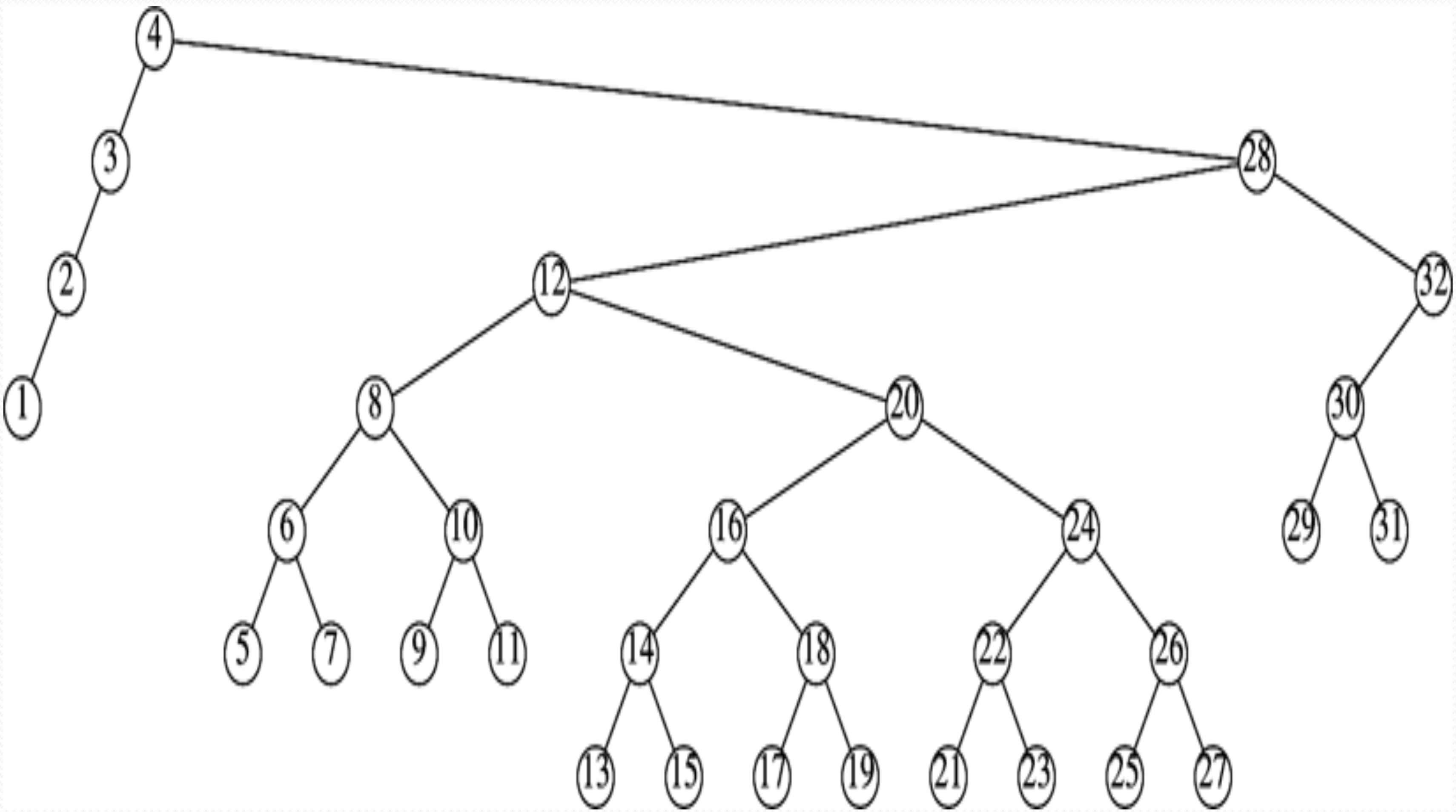

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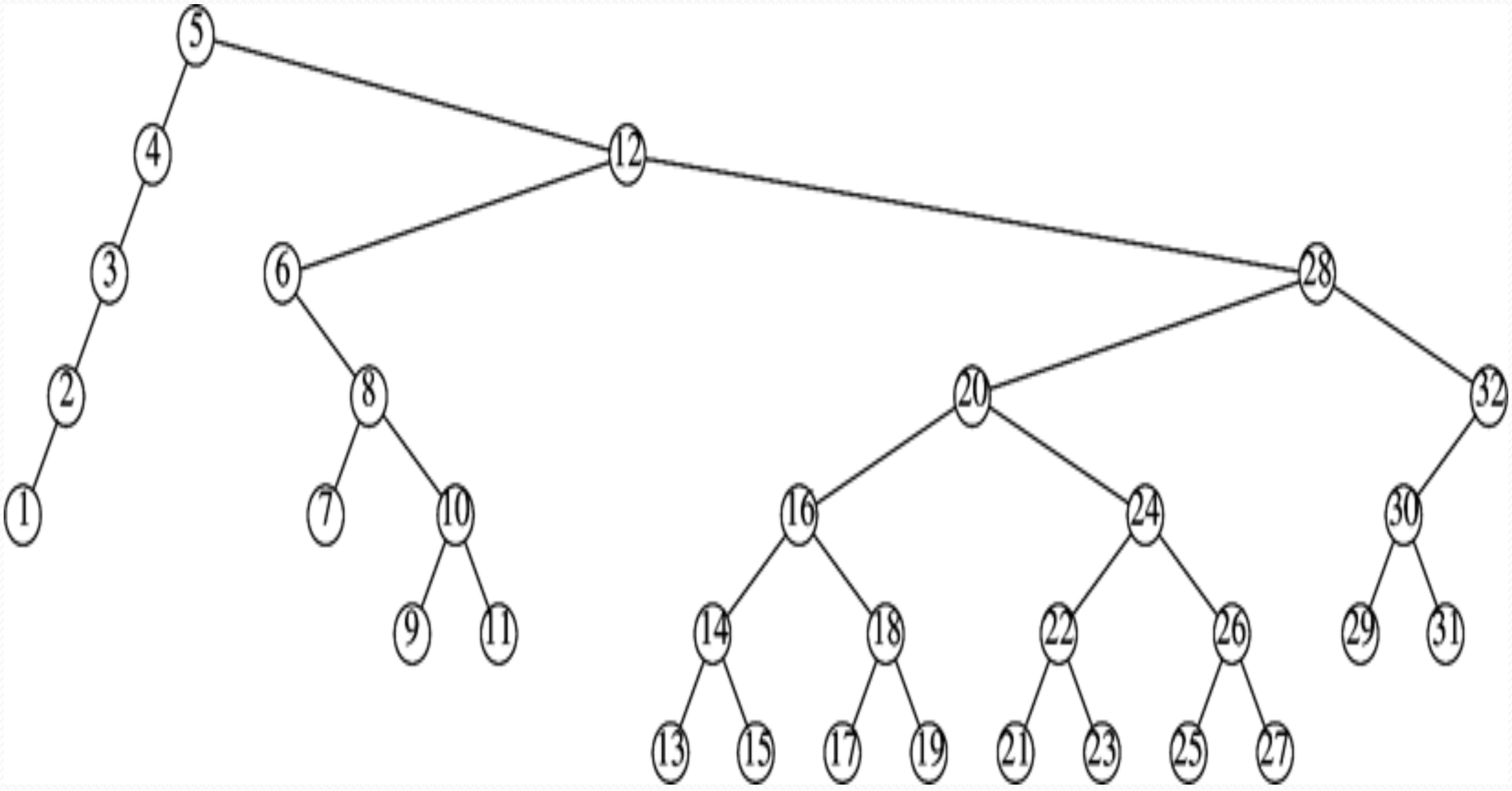

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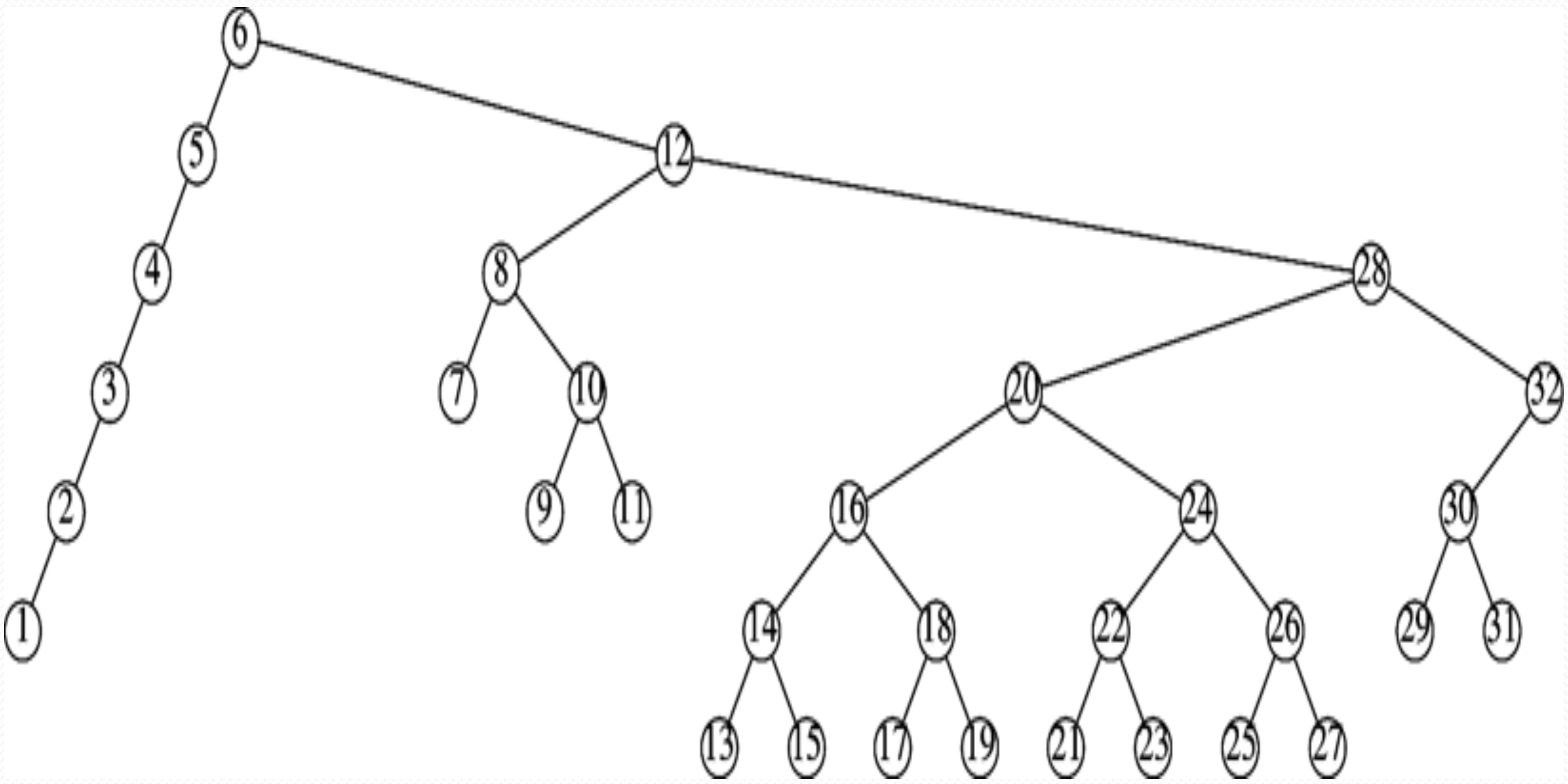

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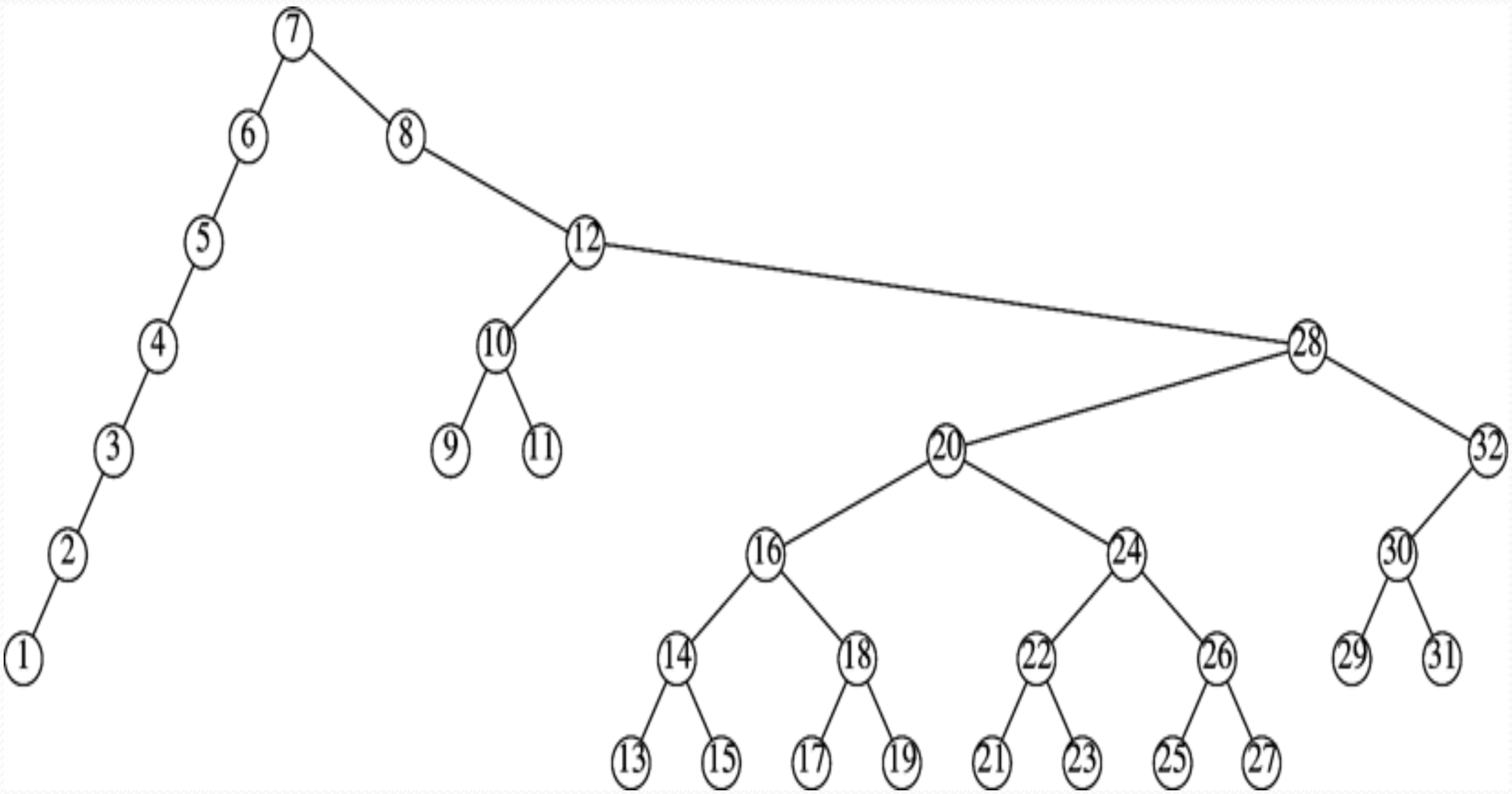

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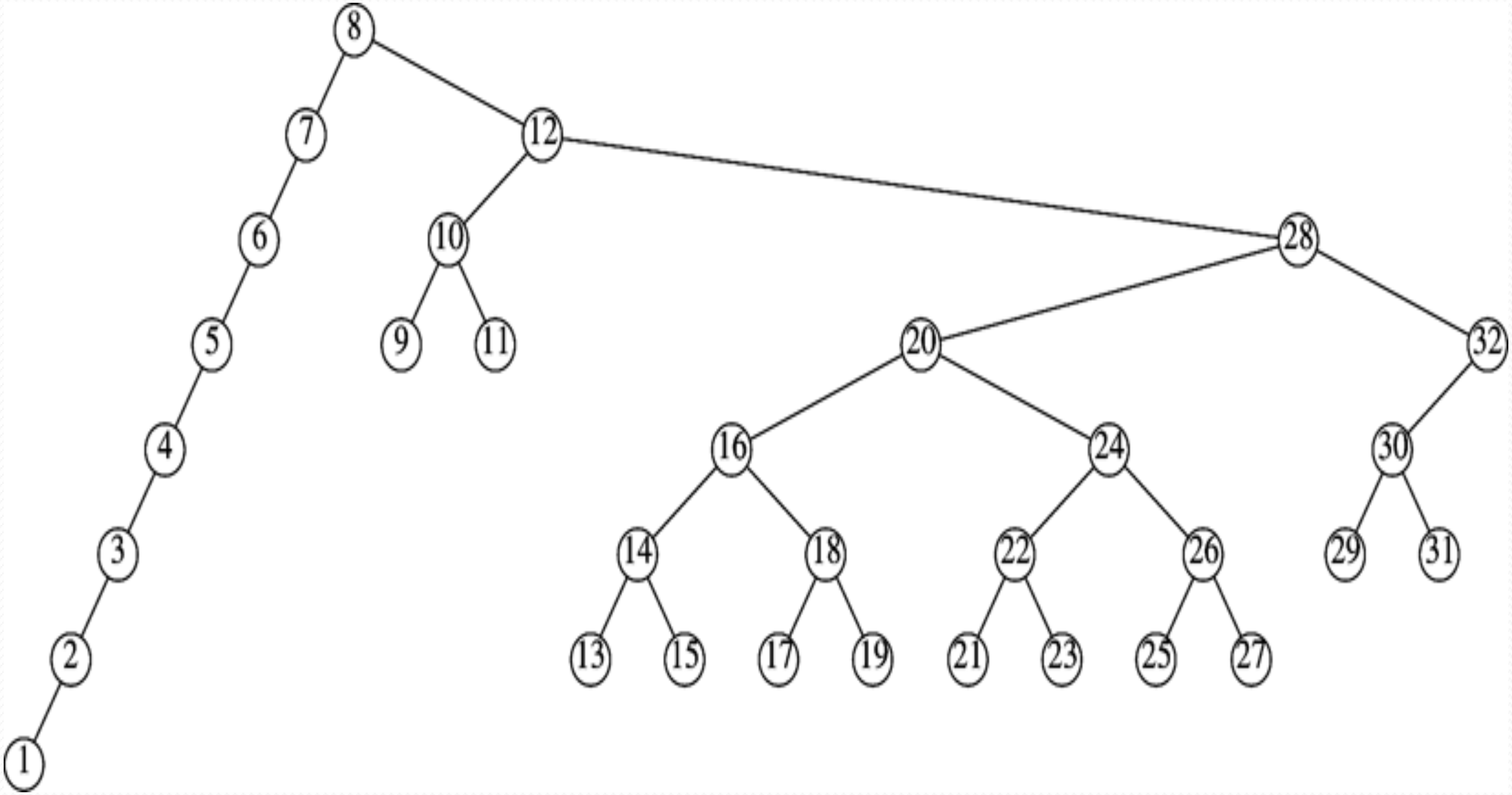

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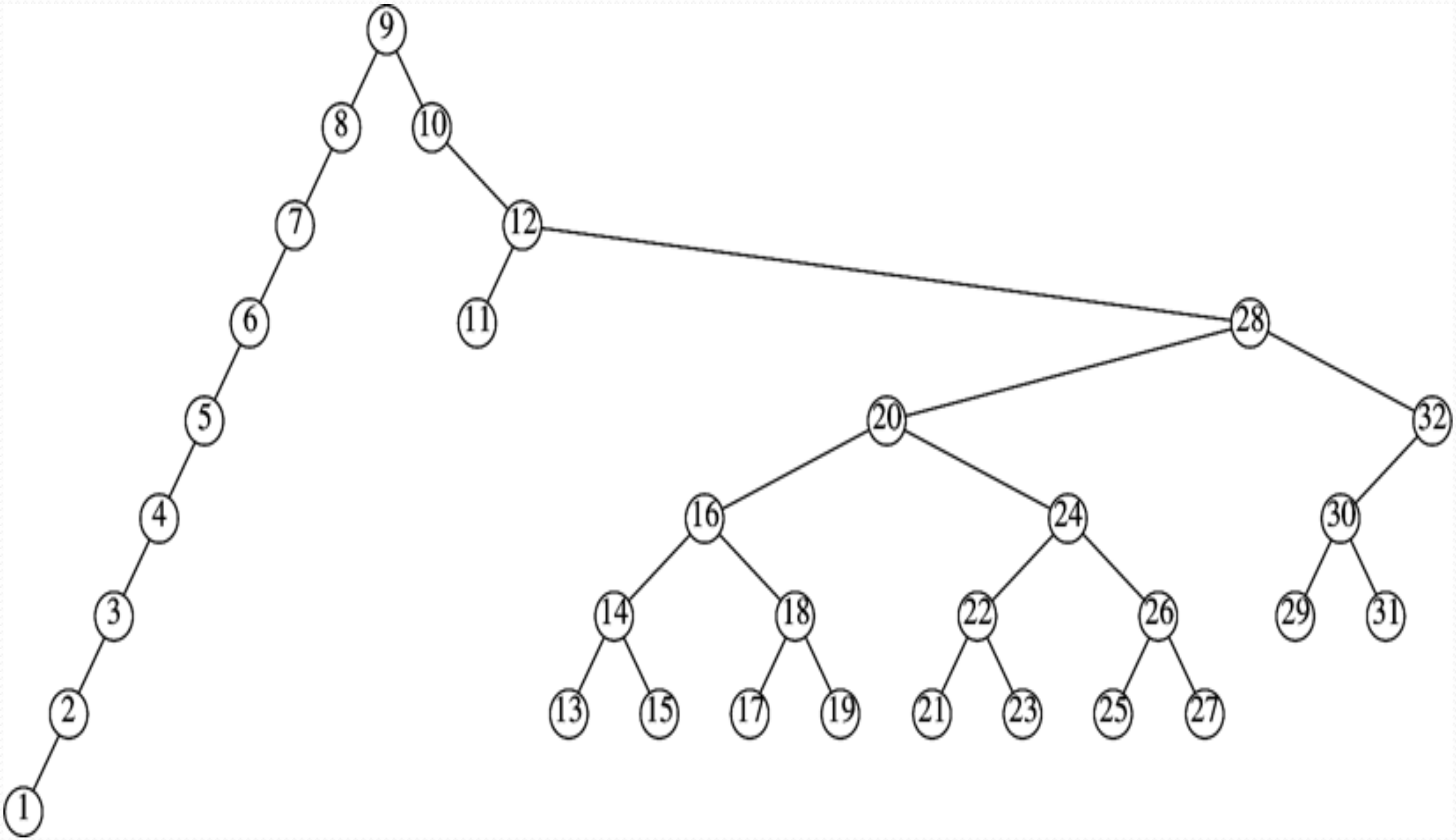


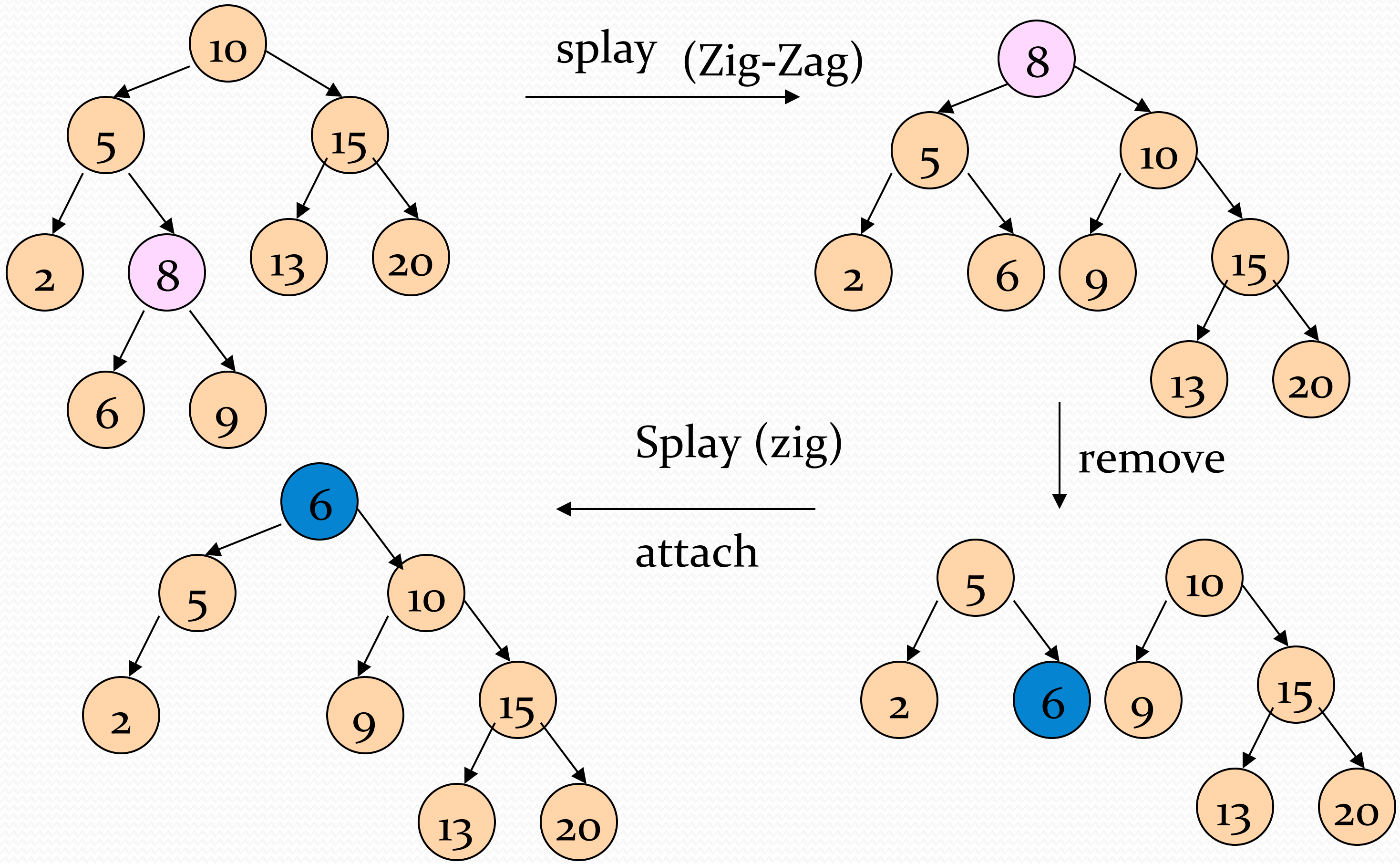
Analysis

- Hard (see later chapter)
- In the example: Access of 2 -> N/4 of root, …, up-to logN of root
- Fundamental properties:
  - When access paths are long, longer search time, but rotations good for future operations
  - When access is cheap, rotations not as good.
  - It can be proved that time is O(logN) per operation (amortized)
- Deletion?
- Much simpler to program with fewer cases
- No need to store balance information

Deletion

- Access, bring node to top
- Delete creating two subtrees
- Access largest element in left tree, bring it to root with no right child
- Patch in right tree as right child

Example Deletion of 8



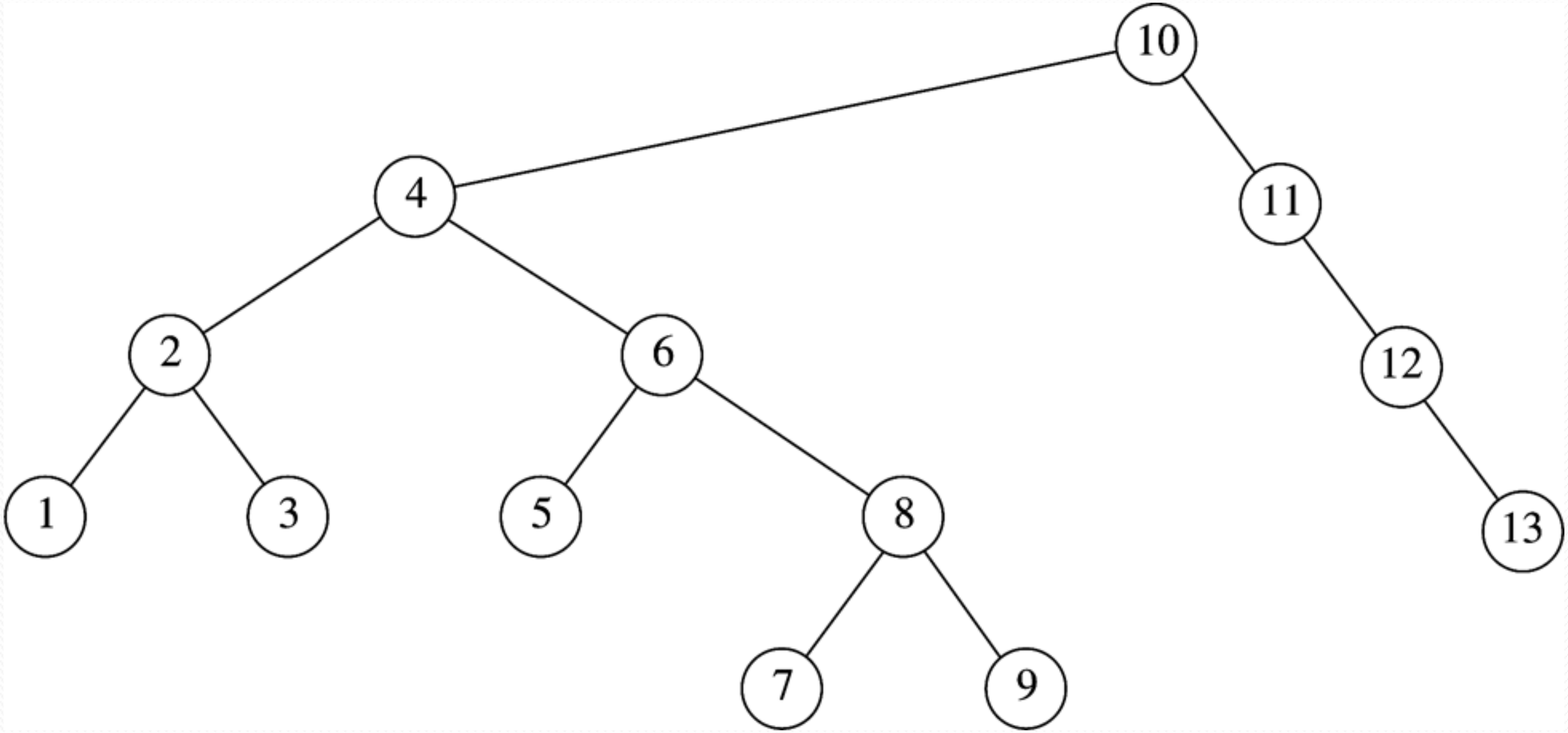
From Univ. of Washington

Exercise from book

- 4.26 (Double rotation implementation without two single rotations)
- 4.27, 4.28

For exercises

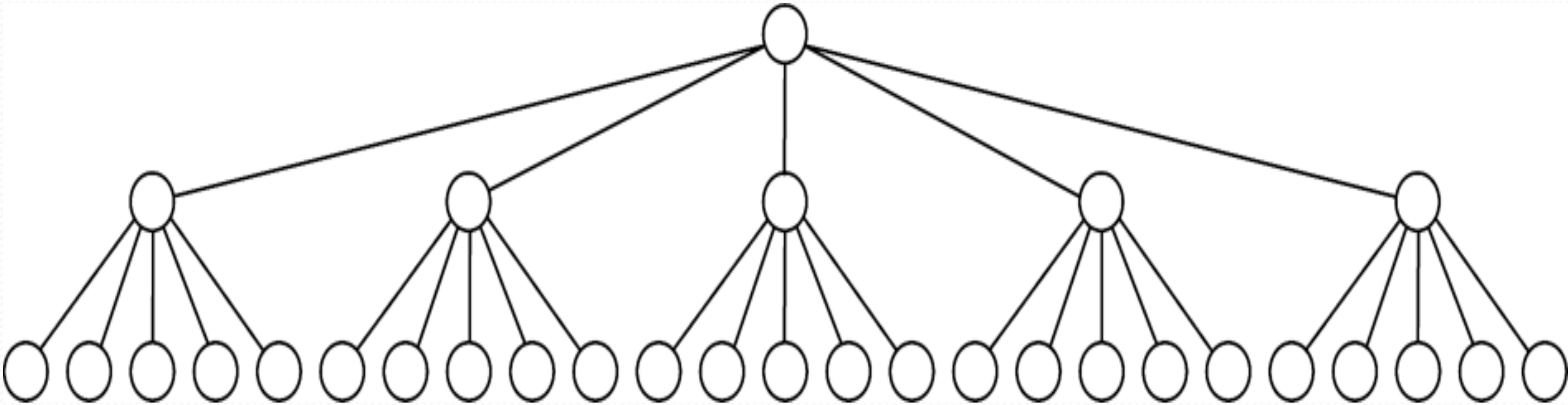
Find: 3, 9, 1, 5
Delete: 6



B-trees

- Scenario: Tree is large and can't fit into memory
- Fact: Disk I/O is much slower than machine instruction (one disk access ~ 4M instructions)
- Assume that we have 10M records, each of 256 bytes, and key is 32 bytes
- Solution?
  - AVL? (worst case is **1.44 logN** , but each operation is costly)
  - We need even smaller trees
    - **M-ary tree has height $\log_M(N)$**

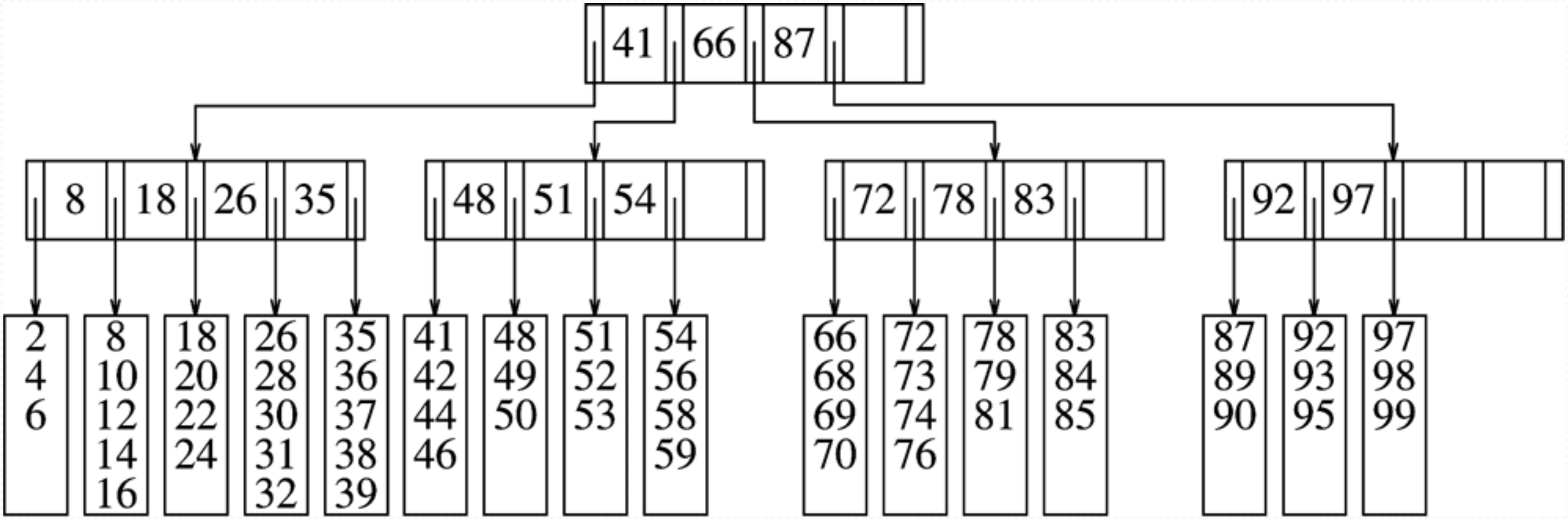
5-ary tree of 31 nodes



Definition of B-tree

- M-ary tree such that:
  - Data is stored at leaves
  - The nonleaf nodes store up to M-1 keys to guide searching. <u>Key i represents smallest key in subtree i+1</u>
  - Root is either a leaf or has between two and M children
  - All nonleaf nodes (except root) have between M/2 and M children (up to half full).
  - All leaves are at same depth and have between L/2 and L data items for some L (up to half full).
- M, and L are determined based on disk block (one access should load a whole node).

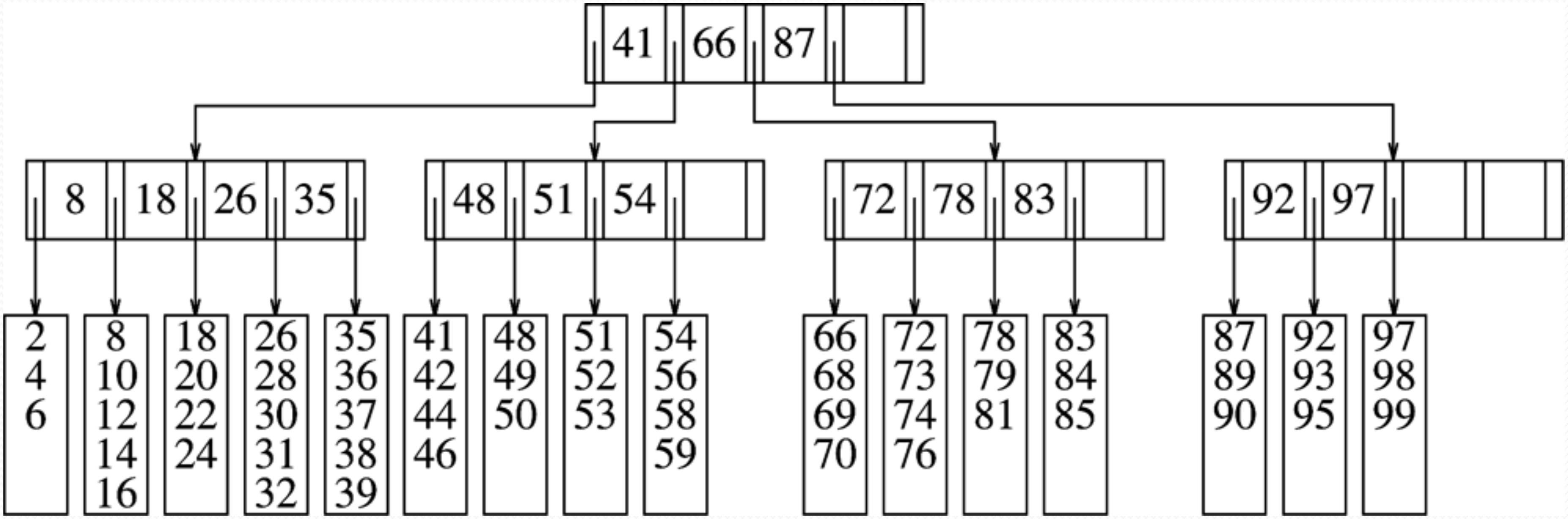
Example B-tree of order 5 (M=5)



Example

- Block size is 8,192 bytes
- Key is 32 bytes
- Internal nodes hold M-1 keys => 32(M-1) bytes plus M branches (4 bytes per branch) => total is 36M-32 bytes.
- Since 8,192 <= 36M-32 => M = 228
- Data record is 256 bytes => 32 records on a block => L=32
- So each leaf should hold between 16 to 32 records
- Each internal node should have at least 114 branches
- 10 million records => 625,000 leafs (10 million/16). In worst case leaves on level 4 (why?)
  - On average number of accesses is **$\log_{M/2}$ (N)**
  - Root and first levels could also be cached in memory…

Example B-tree of order 5

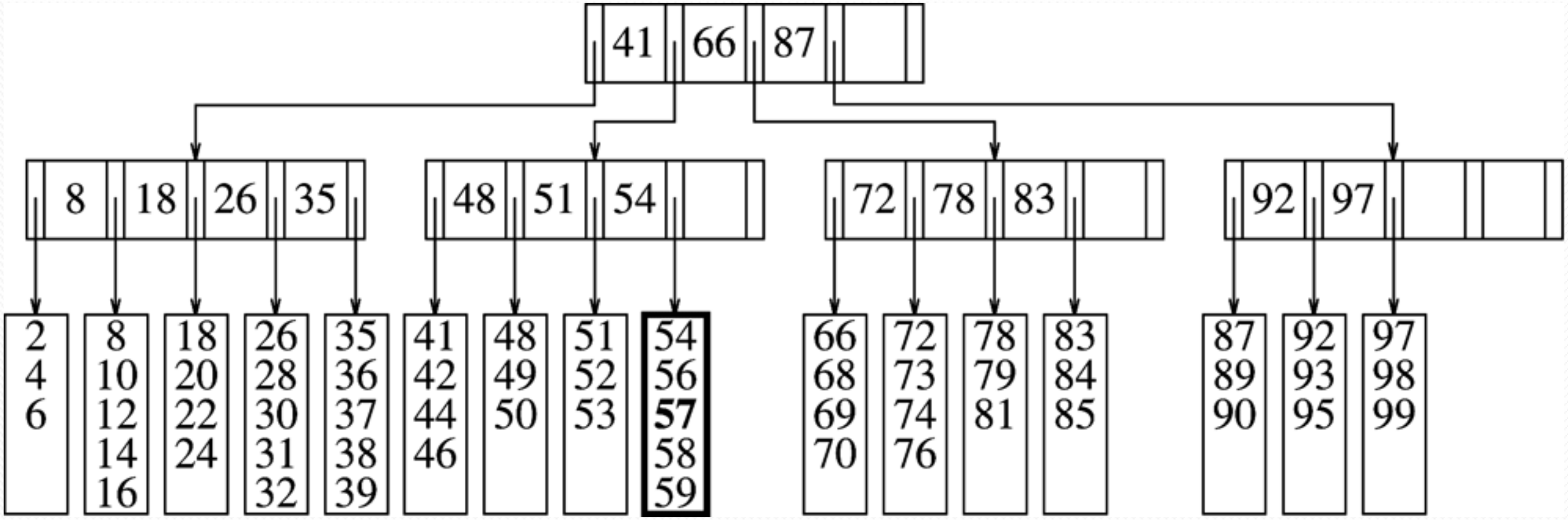


Insertion

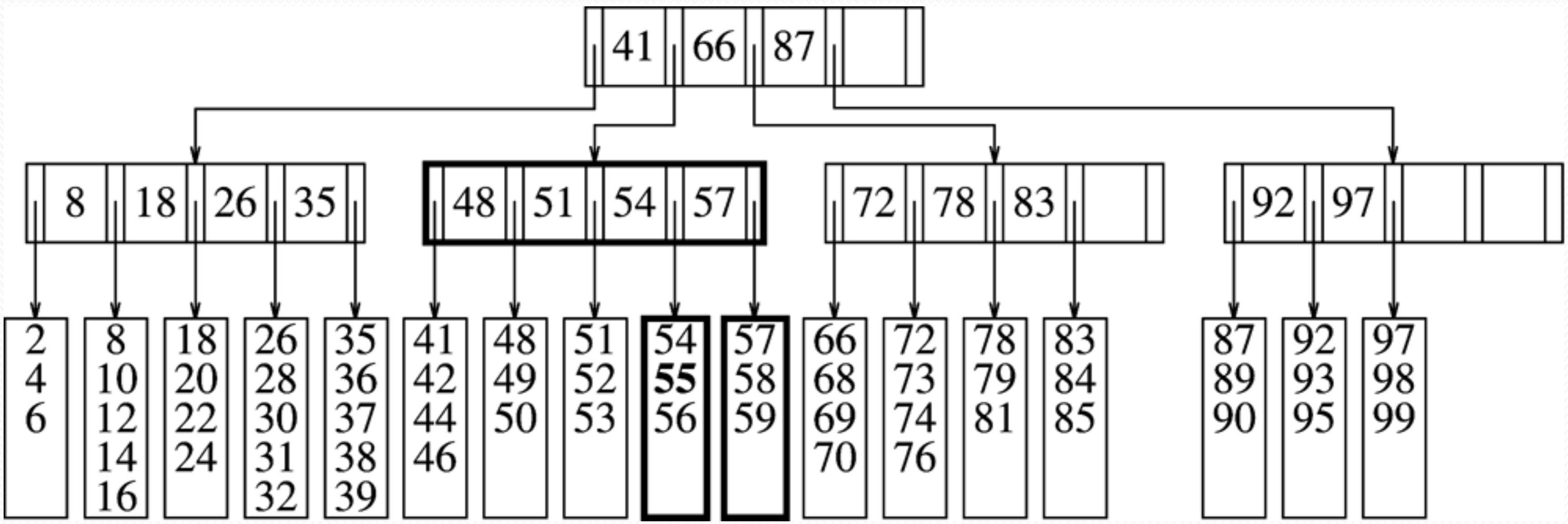
To insert:

- Put it in the appropriate leaf.
- If the leaf is full, break it in two, adding a child to the parent.
- If this puts the parent over the limit, split upwards recursively.
- If you need to split the root, add a new one with two children. This is the only way you add depth.

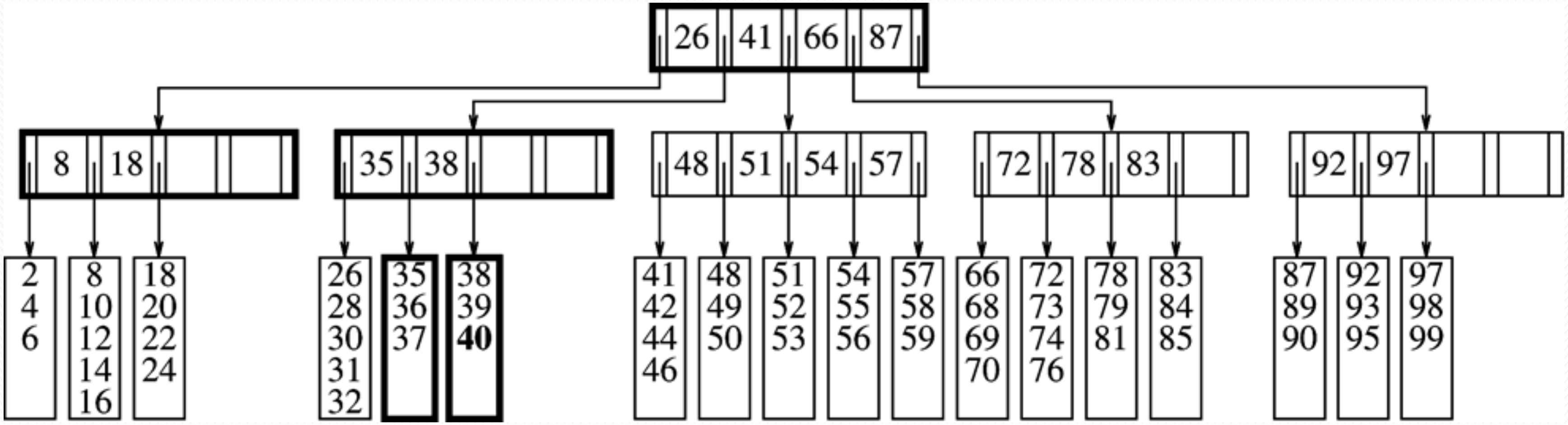
Example of insertion (57)



Example of insertion (55)



Example of insertion (40)



deletion

Delete from appropriate leaf.

- If the leaf is below its minimum, adopt from a neighbor if possible.
- If that's not possible, you can merge with the neighbor. This causes the parent to lose a branch and you continue upward recursively.

Example of deletion (99)