Geometry

Tommy Petersen
tp@aia.dk

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Abstract

These notes are about geometry.

1 Right or left sided vectors

For vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ with $b, d > 0$, consider the two situations I and II in figure 1.

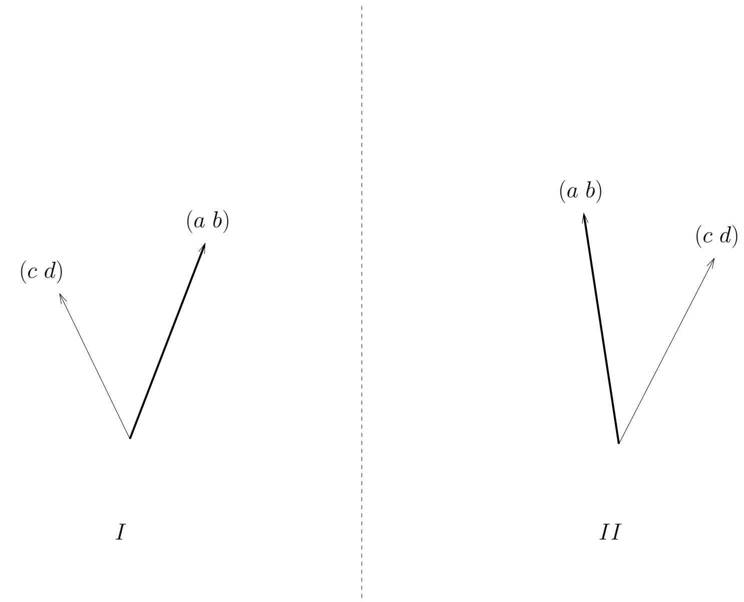

Figure 1: Right or left sided vectors

Due to technicalities, the vectors in the figure are printed in their transpose form, for example $(a\ b)$. In the text, the upright form is used, like $\begin{pmatrix} a \\ b \end{pmatrix}$.

In figure 1, situation I, the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is right sided relative to the vector $\begin{pmatrix} c \\ d \end{pmatrix}$ and it is left sided in situation II. But how can we tell from the vector components a, b, c and d if the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is right or left sided relative to the vector $\begin{pmatrix} c \\ d \end{pmatrix}$? We can use the determinant, $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, of $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The determinant is equal to the dot product of the vector $\begin{pmatrix} c \\ d \end{pmatrix}$ and the counter-clockwise perpendicular vector of $\begin{pmatrix} a \\ b \end{pmatrix}$ given by $\begin{pmatrix} -b \\ a \end{pmatrix}$:

$$\begin{aligned} \begin{pmatrix} c \\ d \end{pmatrix} \cdot \begin{pmatrix} -b \\ a \end{pmatrix} &= -cb + da \\ &= ad - bc \end{aligned}$$

In figure 2, the perpendicular vector $\begin{pmatrix} -b \\ a \end{pmatrix}$ has been added in both of the situations, I and II.

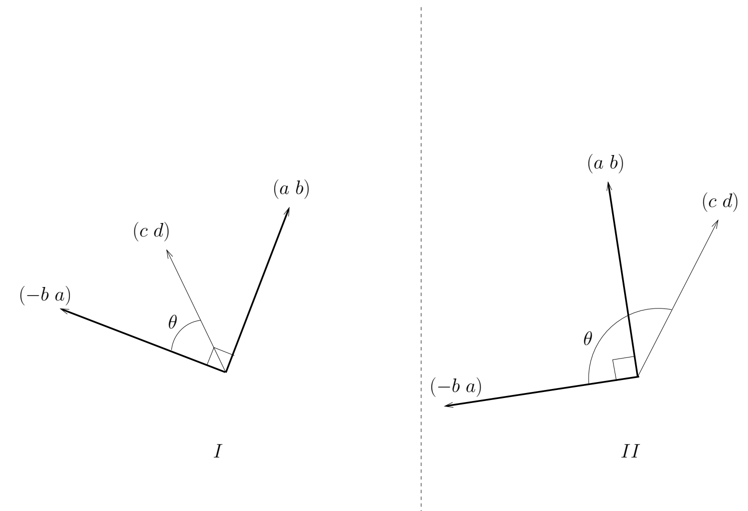

Figure 2: Right or left sided vectors

For the angle θ, the cosine is given by

$$\cos(\theta) \;=\; \frac{\begin{pmatrix} c \\ d \end{pmatrix} \cdot \begin{pmatrix} -b \\ a \end{pmatrix}}{\left\| \begin{pmatrix} c \\ d \end{pmatrix} \right\| \left\| \begin{pmatrix} -b \\ a \end{pmatrix} \right\|}$$

In the two situations in figure 2 we have the following:

- **Situation** I: The angle θ lies between 0 and $\pi/2$. Hence, $\cos(\theta)$ lies between 0 and 1 and therefore $\cos(\theta) > 0$.
- **Situation** II: The angle θ lies between $\pi/2$ and π. Hence, $\cos(\theta)$ lies between 0 and -1 and therefore $\cos(\theta) < 0$.

Note, that if θ is equal to $\pi/2$ then $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ coincide which is a situation we do not consider here.

Since the sign of $\cos(\theta)$ is determined by the dot product $\begin{pmatrix} c \\ d \end{pmatrix} \cdot \begin{pmatrix} -b \\ a \end{pmatrix}$ which is equal to the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, we can now conclude the following:

- If $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is **positive**, then the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is **right** sided.
- If $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is **negative**, then the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ is **left** sided.

2 Trigonometric relations in a right angle triangle

In the following sections, line names, line directions and line lengths are used interchangeably but it should be clear from context what is meant.

2.1 Cosine relation

The unit circle and the right angle triangle are shown in figure 3. We want an expression for the length *cosine*.

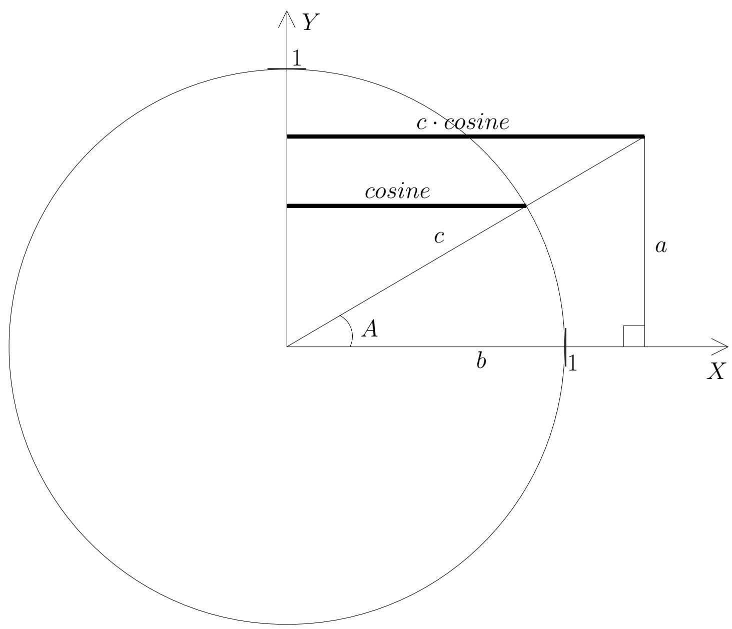

Figure 3: Cosine relation

The intersection of the line c and the unit circle is a line of length 1. So figure 3 shows us that when we go one unit in the direction of c then we must go *cosine* units in the direction of *cosine* in order to move linearly up the Y-axis.
So when we go c units in the direction of c then we must go $c \cdot cosine$ units in the direction of *cosine* in order to move linearly up the Y-axis. But we know that the quantity $c \cdot cosine$ is b because of the triangle. So we can now solve for the unknow *cosine*:

$$b = c \cdot cosine$$
$$cosine = \frac{b}{c}$$

Using the terms *adjacent* and *hypotenuse* for b and c respectively we get the commonly known relation that

$$\cos(A) = \frac{adjacent}{hypotenuse}$$

2.2 Sine relation

The unit circle and the right angle triangle are shown in figure 4. We want an expression for the length *sine*.

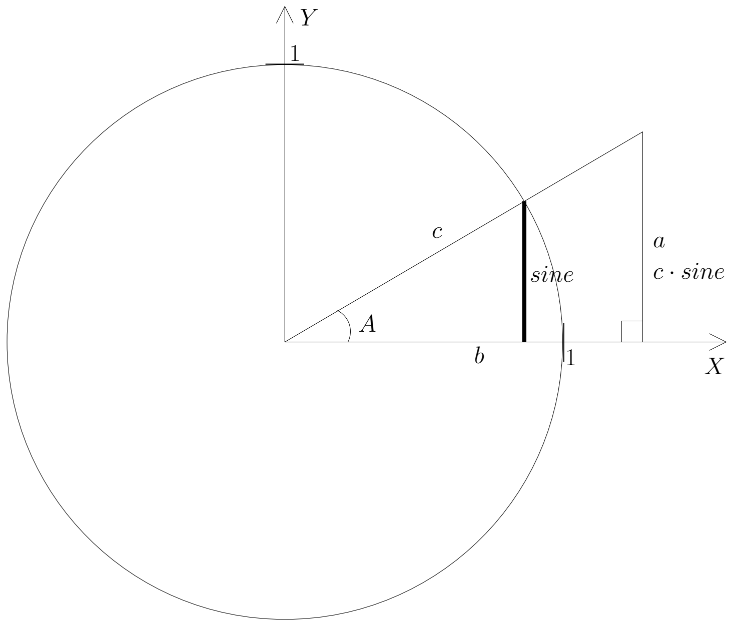

Figure 4: Sine relation

The intersection of the line c and the unit circle is a line of length 1. So figure 4 shows us that when we go one unit in the direction of c then we must go *sine* units in the direction of *sine* in order to move linearly along the X-axis.
So when we go c units in the direction of c then we must go $c \cdot sine$ units in the direction of *sine* in order to move linearly along the X-axis. But we know that the quantity $c \cdot sine$ is a because of the triangle. So we can now solve for the unknow *sine*:

$$a = c \cdot sine$$
$$sine = \frac{a}{c}$$

Using the terms *opposite* and *hypotenuse* for a and c respectively we get the commonly known relation that

$$\sin(A) = \frac{opposite}{hypotenuse}$$

2.3 Tangent relation

The unit circle and the right angle triangle are shown in figure 5. We want an expression for the length *tangent*.
The intersection of the line b and the unit circle is a line of length 1. So figure 5 shows us that when we go one unit in the direction of b then we must go *tangent*

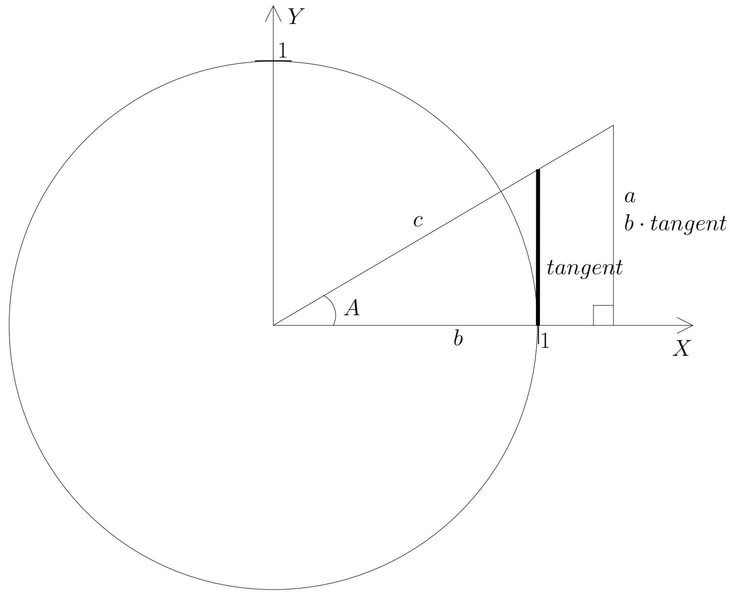

Figure 5: Tangent relation

units in the direction of *tangent* in order to move linearly along c.
So when we go b units in the direction of b then we must go $b \cdot tangent$ units in the direction of *tangent* in order to move linearly along c. But we know that the quantity $b \cdot tangent$ is a because of the triangle. So we can now solve for the unknow *tangent*:

$$a = b \cdot tangent$$
$$tangent = \frac{a}{b}$$

Using the terms *opposite* and *adjacent* for a and b respectively we get the commonly known relation that

$$\tan(A) = \frac{opposite}{adjacent}$$