Math 466 - Numerical Methods: Project 2

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My full `.jl` file can be found on my Github at the following link:

1a

Let $A \in \mathbb{R}^{n \times n}$ and consider the iteration

$$X_{k+1} = \frac{1}{2}\left(X_k + \left(X_k^{-1}\right)^{\top}\right) \quad \text{where} \quad X_0 = A.$$

Write a program to perform this iteration and test your program with the input

$$A = \begin{bmatrix} -0.49 & -0.21 & -0.40 & -0.21 \\ 0.36 & 0.10 & 0.29 & -0.04 \\ 0.12 & -0.01 & 0.48 & -0.47 \\ 0.09 & 0.09 & -0.41 & 0.22 \end{bmatrix}$$

Verify the Frobenius matrix norm $||X_0 - X_1||_F \approx 16.37054203598731$.

Find first iteration

```
########################################
# 1a.

# function that calculates next iteration
pol! = (A::Matrix{Float64}) -> (A + (inv(A))') / 2;

# driver function that finds the n'th iteration
function pol_driver(A::Matrix{Float64}, n::Int64)::Matrix{Float64}
    B = copy(A)
    for i = 0:n
        B = pol!(B)
    end
    return B;
end
```

Julia REPL for first iteration

```
julia> include("polar.jl")
||X_0 - X_1|| = 16.37054203598732
```

1b

Define $\Delta_k = X_{k+1} - X_k$. If X_k converges then it follows that $\Delta_k \to 0$ as $k \to \infty$. Compute $||\Delta_k||_F$ for $k = 0, \ldots, 9$.

Find and print Delta k

```
########################################
# 1b.

# compute delta_k
delta_k = (X, k) -> pol_driver(X, k) - X;

# store all the delta_k's in a list for k = 0:9
delta_ks = [delta_k(copy(A), k) for k = 0:9];

# display the norm of each delta_k
map(x -> "||Delta_k|| = $(norm(x))", delta_ks);

```

Julia REPL for Delta k

```
10-element Vector{String}:
 "||Delta_k|| = 16.37054203598732"
 "||Delta_k|| = 8.218875822891379"
 "||Delta_k|| = 4.207915072995769"
 "||Delta_k|| = 2.325403875547138"
 "||Delta_k|| = 1.5563638515455411"
 "||Delta_k|| = 1.3312012593101599"
 "||Delta_k|| = 1.3023069956996867"
 "||Delta_k|| = 1.301756540894667"
 "||Delta_k|| = 1.3017563374673147"
 "||Delta_k|| = 1.3017563374672871"
```

1c

Suppose X_k is α-order convergent such that $||\Delta_{k+1}||_F \approx M||\Delta_k||_F^{\alpha}$. Then

$$\log||\Delta_{k+1}||_F \;\approx\; \log M + \alpha \log||\Delta_k||_F$$

would show $\log||\Delta_{k+1}||_F$ is a linear function of $\log||\Delta_k||_F$. Plot the points

$$(\log||\Delta_k||_F\,,\, \log||\Delta_{k+1}||_F) \quad \text{for} \quad k = 1, \ldots, 8.$$

Do all the points fall on a line? Find the slope between the last two points by computing

$$\alpha \approx \frac{\log||\Delta_9||_F - \log||\Delta_8||_F}{\log||\Delta_8||_F - \log||\Delta_7||_F}$$

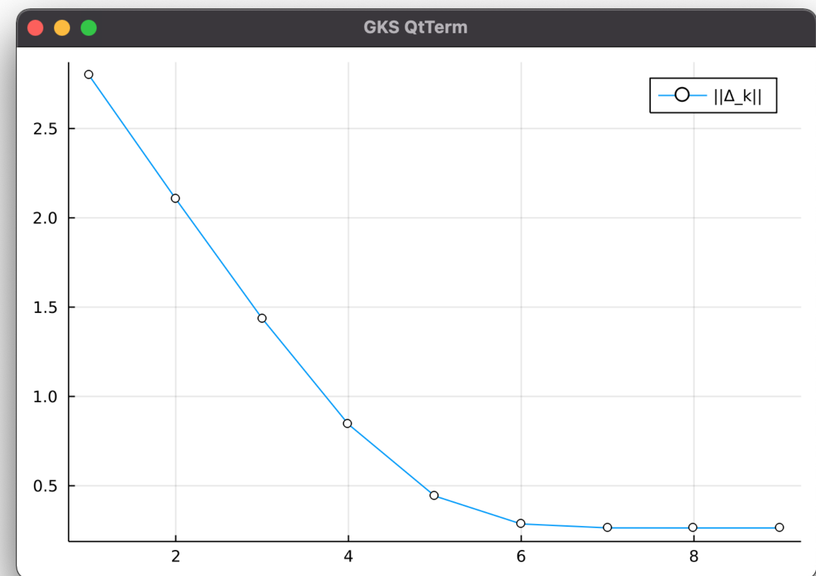
Plot Delta k Points

```
#########################################
# 1c.

# plot log of delta_ks
plot(log_delta_k, markershape = :circle, markercolors = :white, label="||Delta_k||")

# find slope of last two points
alpha = log_delta_k[9] - log_delta_k[8] / log_delta_k[8] - log_delta_k[7]

# print alpha
println("Slope between last two points: Alpha = $alpha")
```



Julia REPL for finding Alpha

```
julia> include("polar.jl")
Slope between last two points: Alpha = -1.0004229223103496
```

. . . .

1d

Let W be the limit of X_k as $k \to \infty$. Numerically check whether W is an orthogonal matrix by computing $X_T^\top X_k$ for $k = 8, 9, 10$. What are your conclusions?

Check orthogonality of W

```
#######################################
# 1d.

# calculate X_k for k = 8, 9, 10
X_8_9_10 = [pol_driver(copy(A), k) for k = 8:10]

# helper method to calculate X^T * X
check_orth = (X) -> X' * X;

# display the resulting matrices
map(x -> display(check_orth(x)), X_8_9_10)
```

Julia REPL for resulting matrices

```
julia> include("polar.jl")
4×4 Matrix{Float64}:
  1.0          -2.57131e-14  -1.41068e-15   2.46364e-15
 -2.57131e-14   1.0           3.54173e-15  -6.04376e-15
 -1.41068e-15   3.54173e-15   1.0          -3.81e-16
  2.46364e-15  -6.04376e-15  -3.81e-16      1.0
4×4 Matrix{Float64}:
 1.0           1.34121e-16   3.76323e-17   9.75894e-18
 1.34121e-16   1.0          -3.37948e-17  -4.60842e-17
 3.76323e-17  -3.37948e-17   1.0           2.65522e-17
 9.75894e-18  -4.60842e-17   2.65522e-17   1.0
4×4 Matrix{Float64}:
  1.0           2.34459e-17  -4.79178e-17  -1.14298e-17
  2.34459e-17   1.0          -3.4489e-17    3.71826e-17
 -4.79178e-17  -3.4489e-17    1.0           7.58685e-17
 -1.14298e-17   3.71826e-17   7.58685e-17   1.0
```

Because the resulting matrices after multiplying X_8, X_9, and X_{10} by their transpose are very, very close to the identity matrix, clearly each matrix is orthogonal.

1e

Define $P = W^{-1}A$ so that $A = WP$. This is called the polar decomposition of A. Use the built-in Julia function `eigvals` to find the eigenvlues of P and A. Are the eigenvalues of A positive? What about the eigenvalues of P?

Find and display eigenvalues of P and A

```
#######################################
# 1e.

W = X_8_9_10[3]
P = inv(W) * copy(A)
println("Eigenvalues of P = $(eigvals(P))");
println("Eigenvalues of A = $(eigvals(copy(A)))");
```

Julia REPL for eigenvalues of P and A

```
Eigenvalues of P = [0.03086137273259231, 0.19890329211534932,
                    0.664894706530955, 0.964105847552981]

Eigenvalues of A = ComplexF64[-0.20699017509306217 - 0.2067450022872031im,
                              -0.20699017509306217 + 0.2067450022872031im,
                              -0.05873738107811822 + 0.0im,
                               0.7827177312642422 + 0.0im]
```

For the matrix P, we have all positive eigenvalues. But for A we have 3 negative eigenvalues and 1 positive one.