Math 132H Homework 8

Tom Slavonia

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10.

Proof. If we consider $G : \mathbb{H} \to \mathbb{D}$ where
$$z \mapsto \frac{z-i}{z+i}.$$
We then have that $|G(z)| \le 1$ always, and any function that maps from $\mathbb{H} \to \mathbb{C}$ such that $|F(z)| \le 1$ it must be that F maps to some subset of the closed unit disc. Also, with G we have that $G(i) = 0$. We have that G and G^{-1} are conformal maps, and the composition of two holomorphic maps is holomorphic. Therefore, $F \circ G^{-1} : \mathbb{H} \to \mathbb{C}$ is a holomorphic map such that $F(i) = 0$ and by Schwarz lemma we have
$$|F(G^{-1}(z))| \le z \le 1 \text{ as } z \in \mathbb{D}.$$
Thus, we have that
$$|F(G^{-1}(G(z)))| = |F(z)| \le |G(z)| = \left|\frac{z-i}{z+i}\right|.$$
$\square$

11.

Proof. Consider the maps $\varphi : D(0,R) \to \mathbb{D}$ and $\phi : D(0,M) \to \mathbb{D}$ where $z \mapsto \frac{z}{R}$ and $z \mapsto \frac{z}{M}$. We then have that $\varphi^{-1} : \mathbb{D} \to D(0,R)$ maps $z \mapsto Rz$ and thus we define $F = \phi \circ f \circ \varphi^{-1}$. Therefore, we have
$$F(z) = \phi(f(zR)) = \frac{f(zR)}{M}.$$
Note that with $z \in \mathbb{D}$ we have $zR \in D(0,R)$. Then,
$$\begin{aligned}
\left|\frac{F(z)-F(0)}{1-\overline{F(0)}F(z)}\right| &= \left|\frac{\frac{f(zR)}{M} - \frac{f(0)}{M}}{1 - \overline{\frac{f(0)}{M}}\frac{f(zR)}{M}}\right| \\
&= \left|\frac{\frac{f(zR)-f(0)}{M}}{\frac{M^2-\overline{f(0)}f(zR)}{M^2}}\right| \\
&= \left|\frac{Mf(zR) - Mf(0)}{M^2 - \overline{f(0)}f(zR)}\right| \le z
\end{aligned}$$

with the last step being justified by the Schwartz lemma. Rearranging we get

$$\left| \frac{f(zR) - f(0)}{M^2 - \overline{f(0)} f(zR)} \right| \leq \frac{z}{M}.$$

Then if we set $w = zR \in D(0, R)$ we obtain

$$\left| \frac{f(w) - f(0)}{M^2 - \overline{f(0)} f(w)} \right| \leq \frac{w}{MR}$$

and the result is proven. □

12.

a.

Proof. Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic with two fixed points. Suppose α, α' are the two fixed points. Consider the map $\varphi_\alpha : \mathbb{D} \to \mathbb{D}$ such that for $|\alpha| < 1$

$$\varphi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}$$

and since $|\alpha| < 1$ the map will never have a pole but is equal to 0 when $z = \alpha$. The book proves that φ_α is indeed an automorphism of $\mathbb{D}$. With the choice of φ_α as proven in the book we also have $\varphi_\alpha^{-1} = \varphi_\alpha$. Define

$$\tilde{f} = \varphi_\alpha \circ f \circ \varphi_\alpha$$

and we clearly have

$$\tilde{f}(0) = \varphi_\alpha(f(\varphi_\alpha(0))) = \varphi_\alpha(f(\alpha)) = \varphi_\alpha(\alpha) = 0$$

using that $\varphi_\alpha(0) = \alpha$, $f(\alpha) = \alpha$ as it is a fixed point of f, and $\varphi_\alpha(\alpha) = 0$. But, we also obtain a second fixed point for $\tilde{f}$, namely $\varphi_\alpha(\alpha')$:

$$\varphi_\alpha(f(\varphi_\alpha(\varphi_\alpha(\alpha')))) = \varphi_\alpha(f(\alpha')) = \varphi_\alpha(\alpha')$$

as $\varphi_\alpha \circ \varphi_\alpha(z) = z$ and α' is a fixed point of f by assumption. Hence $\tilde{f}$ has two fixed points, but since $\tilde{f}(0) = 0$ and clearly $\tilde{f} : \mathbb{D} \to \mathbb{D}$ is holomorphic as φ_α and f are holomorphic, we may apply the Schwartz lemma. Since we have a nonzero $z_0 = \varphi_\alpha(\alpha')$ such that

$$|\tilde{f}(\varphi_\alpha(\alpha'))| = |\varphi_\alpha(\alpha')|$$

by Schwartz's lemma it must be that $\tilde{f}$ is a rotation. But it can't be a rotation that isn't the identity; otherwise, every point will be rotated by the same amount, meaning we can't have two fixed points. Thus, $\tilde{f} = id$ and it must be that $f = id$. □

b.

Proof. We have already looked at the maps $G : \mathbb{D} \to \mathbb{H}$ where $G(z) = i\frac{1-z}{1+z}$ and $F : \mathbb{H} \to \mathbb{D}$ with $F(w) = \frac{i-w}{i+w}$. Therefore, composing these conformal maps with the conformal map $f : \mathbb{H} \to \mathbb{H}$ with $f(z) = z + 1$ we have $F \circ f \circ G : \mathbb{D} \to \mathbb{D}$ a conformal map from the unit disc to itself, but with no fixed points as in the upper half plane every point is shifted by 1 and then remapped into the disc. □

13.

a.

Proof. We can begin by defining $\psi_\alpha : \mathbb{D} \to \mathbb{D}$ by $\psi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$. Note that the book has already discussed that this is an automorphism of the unit disc. The inverse that arises from the definition of ψ_α is $\psi_\alpha^{-1}(z) = \frac{z+\alpha}{1+z\overline{\alpha}}$ which is once again an automorphism of the unit disc. Therefore, $\psi_{f(w)} \circ f \circ \psi_w^{-1}$ is a holomorphic function from the unit disc to itself as it is the composition of holomorphic funcitions from the unit disc to itself. Note that we then have

$$\psi_{f(w)}(f(\psi_w^{-1}(0))) = \psi_{f(w)}(f(w)) = \frac{f(w) - f(w)}{1 - |f(w)|^2} = 0$$

as $|f(w)| < 1$. Thus, we meet the necessary conditions to apply Schartz's lemma and attain the result

$$|\psi_{f(w)}(f(\psi_w^{-1}(z)))| \leq |z|$$

and setting $z = \psi_w(z)$ we have that

$$|\psi_{f(w)}(f(\psi_w^{-1}(\psi_w(z))))| = |\psi_{f(w)}(f(z))| \leq |\psi_w(z)|$$

thus giving us the result

$$\rho(f(z), f(w)) \leq \rho(z, w).$$

If f is an automorphism of $\mathbb{D}$, then f^{-1} exists as a conformal map from $\mathbb{D} \to \mathbb{D}$. Now consider

$$\psi_{f^{-1}(w)} \circ f^{-1} \circ \psi_w^{-1}(0) = \psi_{f^{-1}(w)}(f^{-1}(w)) = \frac{f^{-1}(w) - f^{-1}(w)}{1 - |f^{-1}(w)|^2} = 0$$

as $f^{-1}(w) \in \mathbb{D}$. Clearly the composition of automorphisms of the disc above is holomorphic and therefore the conditions of Schwarz lemma are satisfied and we have

$$|\psi_{f^{-1}(w)}(f^{-1}(\psi_w^{-1}(z)))| \leq |z|.$$

Set $z = \psi_w(z)$ and we ahve

$$|\psi_{f^{-1}(w)}(f^{-1}(z))| \leq |\psi_w(z)|$$

and once again we can set $z = f(z)$ which implies $w = f(w)$ and see

$$|\psi_w(z)| \leq |\psi_{f(w)}(f(z))|$$

and thus

$$\rho(z, w) \leq \rho(f(z), f(w)).$$

Given that f is an automorphism we have

$$\rho(z, w) \leq \rho(f(z), f(w)) \text{ and } \rho(f(z), f(w)) \leq \rho(z, w)$$

implying by squeeze lemma

$$\rho(z, w) = \rho(f(z), f(w)).$$

□

b.

Proof. From the previous result we know

$$\left|\frac{f(z)-f(w)}{1-\overline{f(w)}f(z)}\right| \le \left|\frac{z-w}{1-\overline{w}z}\right|$$

with $z, w \in \mathbb{D}$. Rearranging we gather

$$\left|\frac{f(z)-f(w)}{(z-w)(1-\overline{f(w)}f(z))}\right| \le \left|\frac{1}{1-\overline{w}z}\right|.$$

Taking the limit as $z \to w$:

$$\left|\lim_{z\to w}\frac{f(z)-f(w)}{z-w}\cdot\frac{1}{1-\overline{f(w)}f(z)}\right| \le \frac{1}{|1-|w|^2|},$$

as f is holomorphic we may more the limit inside f, therefore

$$\left|\frac{f'(w)}{1-|f(w)|^2}\right| \le \frac{1}{|1-|w|^2|}$$

and

$$\frac{|f'(w)|}{1-|f(w)|^2} \le \frac{1}{1-|w|^2}$$

with

$$|1-|f(w)|^2| = 1-|f(w)|^2 \text{ and } |1-|w|^2| = 1-|w|^2$$

because $w, f(w) \in \mathbb{D}$. $\square$

14.

Proof. Suppose $f : \mathbb{H} \to \mathbb{D}$ is a conformal map. Recall we have the conformal map $g : \mathbb{D} \to \mathbb{H}$ by

$$g(z) = i\frac{1-z}{1+z}.$$

Hence, $f \circ g$ is a conformal map and thus an automorphism of $\mathbb{D}$. Then we have for $z \in \mathbb{D}$

$$f(g(z)) = f\left(i\frac{1-z}{1+z}\right).$$

By a theorem from the book it must be that any automorphism of the disc is of the form

$$e^{i\theta}\frac{\alpha-z}{1-\overline{\alpha}z}$$

for $\alpha \in \mathbb{D}$ and $\theta \in \mathbb{R}$, so

$$f(g(z)) = f\left(i\frac{1-z}{1+z}\right) = e^{i\theta}\frac{\alpha-z}{1-\overline{\alpha}z}.$$

The motivation of the next step comes from noting that as previously mentioned in the book

$$g^{-1}(z) = \frac{i-z}{i+z}.$$

Take $z = \frac{i-z}{i+z}$ and we have

$$\begin{aligned}
f(z) &= e^{i\theta}\frac{\alpha - \frac{i-z}{i+z}}{1-\overline{\alpha}\frac{i-z}{i+z}} \\
&= e^{i\theta}\frac{\alpha i + \alpha z - i + z}{i + z - \overline{\alpha} i + \overline{\alpha} z} \\
&= e^{i\theta}\frac{z(1+\alpha) + \alpha i - i}{z(1+\overline{\alpha}) + i - \overline{\alpha} i} \\
&= e^{i\theta}\frac{z + \frac{\alpha i - i}{1+\alpha}}{z + \frac{i - \overline{\alpha} i}{1+\overline{\alpha}}} \\
&= e^{i\theta}\frac{z - \frac{i - \alpha i}{1+\alpha}}{z - \frac{\overline{\alpha} i - i}{1+\overline{\alpha}}} \\
&= e^{i\theta}\frac{z - i\frac{1-\alpha}{1+\alpha}}{z - i\frac{\overline{\alpha}-1}{1+\overline{\alpha}}}.
\end{aligned}$$

Notice

$$\overline{i\frac{1-\alpha}{1+\alpha}} = i\frac{\overline{\alpha}-1}{1+\overline{\alpha}}$$

and that by the known mapping $g(z) = i\frac{1-z}{1+z} \in \mathbb{H}$ and with $\alpha \in \mathbb{D}$ we have

$$i\frac{1-\alpha}{1+\alpha} \in \mathbb{H}.$$

Taking

$$\beta = i\frac{1-\alpha}{1+\alpha} \text{ and then } \overline{\beta} = i\frac{\overline{\alpha}-1}{1+\overline{\alpha}}$$

we get the result

$$f(z) = e^{i\theta}\frac{z - i\frac{1-\alpha}{1+\alpha}}{z - i\frac{\overline{\alpha}-1}{1+\overline{\alpha}}} = e^{i\theta}\frac{z-\beta}{z-\overline{\beta}}.$$

□

15.

a.

Proof. We know by previous results that all automorphisms of the upper half-plane are of the form

$$z \mapsto \frac{az+b}{cz+d}.$$

The identity map fixes all points so it certainly fixes three different points. Suppose z is a fixed point by an automorphism of $\mathbb{H}$, then

$$z = \frac{az+b}{cz+d}.$$

Rearranging the above implies

$$\frac{az+b-cz^2-dz}{cz+d} = 0$$

and thus that

$$-z^2 + z(a-d) + b = 0.$$

According to the Fundamental theorem of Algebra, the polynomial above has two solutions, so three points can't be fixed unless the map is the identity. □

b.

Proof. The idea here is that we build a map that assigns x_1, x_2, and x_3 to three manageable points. Define

$$f(z) = \frac{(z-x_1)(x_2-x_3)}{(z-x_3)(x_2-x_1)}$$

here $f(x_1) = 0$, $f(x_2) = 1$, and $\lim_{z\to x_3} f(z) = \infty$. This is an automorphism of $\mathbb{H}$, as the determinant of the coefficients is greater than 0 which is sufficient as $z \in \mathbb{H}$. Also, define

$$g(z) = \frac{zy_2y_3 - zy_1y_3 - y_1y_2 + y_1y_3}{zy_2 - zy_1 - y_2 + y_3}$$

which is the inverse of

$$\frac{(z-y_1)(y_2-y_3)}{(z-y_3)(y_2-y_1)},$$

but I will spare the reader that long and tedious computation. We then have that

$$g \circ f(x_1) = \frac{-y_1y_2 + y_1y_3}{-y_2 + y_3} = y_1$$

$$g \circ f(x_2) = \frac{y_2y_3 - y_1y_2}{y_3 - y_1} = y_2$$

$$\lim_{z\to x_3} g \circ f(z) = \frac{\infty \cdot y_2y_3 - \infty \cdot y_1y_3}{\infty \cdot y_2 - \infty \cdot y_1} = y_3.$$

Suppose f and g are two conformal maps such that $f(x_1) = g(x_1) = y_1$, $f(x_2) = g(x_2) = y_2$ and $f(x_3) = g(x_3) = x_3$ with all variables as defined by the problem. By the first part of the problem, it must be that $f \circ g^{-1} = id$ as the map would fix three points, and thus $f = g$. □

17.

Proof. We begin by using the hint provided for the first integral, since ψ_α is holomorphic and bijective, we may use that integrals of the form

$$\frac{1}{\pi}\iint_{\mathbb{D}} |\psi_\alpha'|^2 dxdy = \frac{1}{\pi}\cdot \text{Area}(\psi_\alpha(\mathbb{D}))$$

but ψ_α is a automorphism of $\mathbb{D}$, so

$$\psi_\alpha(\mathbb{D}) = \mathbb{D}$$

and since the radius of the unit disc is 1

$$\frac{1}{\pi}\iint_{\mathbb{D}} |\psi_\alpha'|^2 dxdy = \frac{1}{\pi}\cdot \text{Area}(\psi_\alpha(\mathbb{D})) = \frac{1}{\pi}\cdot \text{Area}(\mathbb{D}) = \frac{1}{\pi}\cdot \pi = 1.$$

The second integral must be handled with more care. If we parametrize $z = e^{i\theta}$. Now, we can calculate the derivative of

$$\psi_\alpha(z) = \frac{\alpha - z}{1-\overline{\alpha}z}.$$

Using the quotient rule we calculate the derivative to be

$$\psi_\alpha'(z) = \frac{-1+|\alpha|^2}{(1-\overline{\alpha}z)^2}.$$

Plugging into the integral

$$\frac{1}{\pi}\iint_{\mathbb{D}} |\psi_\alpha'| dxdy = \frac{1}{\pi}\iint_{\mathbb{D}} \left|\frac{-1+|\alpha|^2}{(1-\overline{\alpha}z)^2}\right| dxdy.$$

With our parametrization of z in hand, we will tackle this integral over contour of the unit disc. Note that since $\alpha \in \mathbb{D}$ we have

$$|1-|\alpha|^2| = 1-|\alpha|^2.$$

Therefore, with $d\theta = \frac{1}{iz}dz$ our integral becomes

$$\begin{aligned}\frac{1}{\pi}\iint_{\mathbb{D}} |\psi_\alpha'| dxdy &= \frac{1-|\alpha|^2}{\pi}\int_0^1\int_0^{2\pi}\frac{1}{|1-\overline{\alpha}re^{i\theta}|^2} r d\theta dr\\ &= \frac{1-|\alpha|^2}{\pi}\int_0^1\int_{|z|=1}\frac{r}{iz|1-\overline{\alpha}rz|^2} dzdr.\end{aligned}$$

With $|z| = 1$ we have $z\cdot\overline{z} = 1$ and so $\overline{z} = \frac{1}{z}$ and the integral is now

$$\begin{aligned}\frac{1}{\pi}\iint_{\mathbb{D}} |\psi_\alpha'| dxdy &= \frac{1-|\alpha|^2}{\pi}\int_0^1\int_{|z|=1}\frac{r}{iz|1-\overline{\alpha}rz|^2} dzdr\\ &= \frac{1-|\alpha|^2}{\pi}\int_0^1\int_{|z|=1}\frac{r}{iz(1-\overline{\alpha}rz)\left(1-\alpha r\frac{1}{z}\right)} dzdr\end{aligned}$$

$$
\begin{aligned}
&= \frac{1-|\alpha|^2}{i\pi}\int_0^1 \int_{|z|=1} \frac{r}{z(1-\overline{\alpha}rz)\left(\frac{z-\alpha r}{z}\right)} dzdr \\
&= \frac{1-|\alpha|^2}{i\pi}\int_0^1 \int_{|z|=1} \frac{rz}{z(1-\overline{\alpha}rz)\left(z-\alpha r\right)} dzdr \\
&= \frac{1-|\alpha|^2}{i\pi}\int_0^1 \int_{|z|=1} \frac{r}{(1-\overline{\alpha}rz)\left(z-\alpha r\right)} dzdr \\
&= \frac{1-|\alpha|^2}{i\pi}\int_0^1 r\int_{|z|=1} \frac{1}{(1-\overline{\alpha}rz)(z-\alpha r)} dzdr.
\end{aligned}
$$

Now we must find the residues of the poles. The poles are $z = \frac{1}{\alpha r}$ and $z = \alpha r$. Note that since $r < 1$ and $\alpha \in \mathbb{D}$ we have that the pole $\frac{1}{\alpha r}$ is not in the disc therefore we need only consider the pole $z_1 = \alpha r$. Clearly, this pole only has an order of 1. Calculating the residue we get

$$
res_{z_1} f = \lim_{z\to\alpha r}(z-\alpha r)\frac{1}{(1-\overline{\alpha}rz)(z-\alpha r)} = \frac{1}{1-|\alpha|^2r^2}.
$$

Thus, by the residue integration formula

$$
\begin{aligned}
\frac{1}{\pi}\iint_{\mathbb{D}} |\psi'_\alpha| dxdy &= \frac{1-|\alpha|^2}{i\pi}\int_0^1 r\int_{|z|=1} \frac{1}{(1-\overline{\alpha}rz)(z-\alpha r)} dzdr \\
&= \frac{1-|\alpha|^2}{i\pi}\int_0^1 r\frac{2\pi i}{1-|\alpha|^2r^2} dr \\
&= (2-2|\alpha|^2)\int_0^1 \frac{r}{1-|\alpha|^2r^2} dr.
\end{aligned}
$$

Performing classical u–substitution with $u = 1-|\alpha|^2r^2$ we have

$$
\frac{du}{dr} = -2r|\alpha|^2
$$

and thus

$$
-\frac{du}{2r|\alpha|^2} = dr.
$$

We can then simplify the integral into

$$
\begin{aligned}
\frac{1}{\pi}\iint_{\mathbb{D}} |\psi'_\alpha| dxdy &= (2-2|\alpha|^2)\int_0^1 \frac{r}{1-|\alpha|^2r^2} dr \\
&= (2-2|\alpha|^2)\int_1^{1-|\alpha|^2} \frac{r}{u(-2r|\alpha|^2)} du \\
&= -\frac{1-|\alpha|^2}{|\alpha|^2}\int_1^{1-|\alpha|^2} \frac{1}{u} du \\
&= \frac{1-|\alpha|^2}{|\alpha|^2}\int_{1-|\alpha|^2}^1 \frac{1}{u} du \\
&= -\frac{1-|\alpha|^2}{|\alpha|^2}\left[\ln(|u|)\right]_{1-|\alpha|^2}^1
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1-|\alpha|^2}{|\alpha|^2}\left(-\ln\big(|1-|\alpha|^2|\big)\right) \\
&= \frac{1-|\alpha|^2}{|\alpha|^2}\ln\big((1-|\alpha|^2)^{-1}\big) \\
&= \frac{1-|\alpha|^2}{|\alpha|^2}\ln\left(\frac{1}{1-|\alpha|^2}\right)
\end{aligned}
$$

with the last step true by basic logarithm rules. Hence, the last proof of my undergraduate career is complete.

$\square$