Math 110B Homework 7

Tom Slavonia

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1.

Proof. By the first Sylow theorem G has a Sylow p-subgroup of order H of order p^k. The number of Sylow p-subgroups divides the order of G and $n_p \equiv 1(mod\ p)$. Since $(m,p) = 1$, $n_p | m$ and thus since $m < p$ there must be only one Sylow p-subgroup and by a corollary, since there is only one Sylow p-subgroup it must be a normal subgroup implying the group is not simple. □

2.

Proof. Let $|G| = 48 = 2^4 \cdot 3$. Therefore, we have a at least one Sylow $2-$subgroup of order 2^4 and a Sylow 3-subgroup of order 3. The number of Sylow $2-$subgroups must be either $n_2 \in \{1,3\}$ otherwise Sylow theorem 3 would be broken.

Claim: For any two subgroups $H, K \leq G$ we have

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

Note that $HK = \{hk : h \in H,\ k \in K\}$. For the selection of $h \in H$ we have $|H|$ choices and for the selection of $k \in K$ we have $|K|$ choices. Consider if $h_1k_1 = h_2k_2$ for $h_1, h_2 \in H$ and $k_1, k_2 \in K$. This implies that $k_1k_2^{-1} = h_1^{-1}h_2 \in K, H$ so when we have this relationship, these elements are in the intersection of H and K. Therefore, we have

$$|HK| = \frac{|H||K|}{|H \cap K|}.$$

Returning to the original problem, for the sake of contradiction, suppose $n_2 = 3$. Take H_1 and H_2 to be distinct Sylow $2-$subgroups, both of these subgruops will have $|H_1| = |H_2| = 2^4 = 16$. Note that $H_1 \cap H_2$ is a subgroup of H_1 and thus by Lagrange's theorem $|H_1 \cap H_2|$ divides $|H_1|$. Since the subgroups must be distinct they can't have all the same elements, so $|H_1 \cap H_2| = 1, 2, 4,$ or 8. We know that we must satisfy

$$|H_1H_2| = \frac{|H_1||H_2|}{|H_1 \cap H_2|} \leq |G| = 60$$

as every element of H_1H_2 is in G. The only choice of $|H_1 \cap H_2|$ that satisfies this relation is $|H_1 \cap H_2| = 8$. This subgroup is index 2 in H_1 and H_2 by Lagrange's theorem, and is thus normal in

H_1 and H_2. Since $|H_1| = |H_2| = 16$ and $|H_1 \cap H_2| = 8$, we have $|N_G(H \cap K)| \geq 24$ but since $N_G(H \cap K)$ is a subgroup of G it must be that $|N_G(H \cap K)| = 24$ or 48 in order to divide the order of G. If $|N_G(H \cap K)| = 24$ it is index 2 and thus normal in G, otherwise $|N_G(H \cap K)| = 48$ which implies $N_G(H \cap K) = G$ and thus $H \cap K$ is a normal non-trivial subgroup of G. □

3.

Proof. Let $|G| = pqr$ where p, q, r are all primes. Without loss of generality, suppose $p < q < r$. Look at the Sylow q−subgroups, and Sylow r−subgroups which we know exist by the first Sylow theorem. Consider n_p we know that $n_p | qr$ but since q, r are coprime we ahve $n_p | q$ and $n_p | r$. Since q, r, p are all distinct primes it must be that $n_p = 1$ which implies that the Sylow p−subgroup is a normal subgroup. □

4.

Proof. By definition we have $K \leq N(K) \leq N(N(K))$. Suppose $x \in N(N(K))$ and therefore $xKx^{-1} \leq xN(K)x^{-1} = N(K)$. The second Sylow theorem ensures that for every Sylow p−subgroup $\exists y \in N(K)$ such that

$$xKx^{-1} = yKy^{-1} = K$$

as $y \in N(K)$, and thus $x \in N(P)$. Thus, we have both inclusions and so $N(K) = N(N(K))$. □

5.

Proof. Using the third Sylow theorem we have that $n_5 \equiv 1 \ (mod\ 5)$ and $n_5 | 60$ implies that $n_5 =$ 1 or 6. We have that $(12345) \in A_5$ and $(12354) \in A_5$ as unique elements that are 5−cycles. The cyclic groups $\langle (12345) \rangle$ and $\langle (12354) \rangle$ are both of order 5 and are unique as

$$(12354) \notin \langle (12345) \rangle = \{e, (12345), (13524), (14253), (15432)\}$$

so the cyclic subgroups are unique and thus there are at least 2 Sylow 5−subgroups, which implies that there are 6 Sylow 5−subgroups. □

6.

Proof. By the class equation, we have that for $a_i \in G$, $i \in [n]$,

$$|G| = |Z(G)| + [G : C(a_1)] + \cdots + [G : C(a_n)].$$

For the sake of contradiction, suppose $Z(G) = \{e\}$. Then, $[G : C(a_i)] \neq 1$ for all $i \in [n]$ as otherwise they would've been included in the total for $Z(G)$. Since $|G| = p^n$, the smallest nontrivial integer that divides $|G|$ is p, and since $[G : C(a_i)]$ divides $|G|$ by Lagrange's theorem, we must have that $p | [G : C(a_i)]$ for all $i \in [n]$. Therefore,

$$[G : C(a_i)] = pq_i, \ \ q_i \in \mathbb{Z}.$$

Using the class equation, we have

$$|Z(G)| = |G| - [G : C(a_1)] - \cdots [G : C(a_n)] = p^n - pq_1 - \cdots - pq_n = p(p^{n-1} - q_1 - \cdots - q_n)$$

implying p divides $|Z(G)|$, a contradiction. Thus, $Z(G)$ is a nontrivial subgroup of G and $Z(G)$ is normal. If $Z(G) \neq G$ then we are done as then $Z(G)$ is a nontrivial normal subgroup. If $Z(G) = G$, then the group is abelian, but by the first Sylow theorem since $|G| = p^n = p \cdot p^{n-1}$ and $n \geq 2$, there exists a nontrivial Sylow p–subgroup of order p which will be an abelian group, since G is abelian under this assumption and thus the subgroup would be normal in G. $\square$