Math 110B Homework 6

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1.

a.

Let G be a group and let $D = \{(a,a,a) : a \in G\}$.

Proof. Let $x \in D$, then $x = (a,a,a)$ for some $a \in G$. Then, since G is a group $a^{-1} \in G$, so $x(a^{-1},a^{-1},a^{-1}) = (a,a,a)(a^{-1},a^{-1},a^{-1}) = (e,e,e)$ so $x \in D$ has $x^{-1} \in D$. Let $x, y \in D$. Then, $x = (a,a,a)$ and $y = (b,b,b)$ for $a, b \in G$. The product is then $xy = (ab,ab,ab)$ but since G is a group, we have $ab \in G$, so $xy = (ab,ab,ab) \in D$. Thus, D is a subgroup of $G \times G \times G$. $\square$

b.

Proof. $\Rightarrow$) Suppose that D is normal in $G \times G \times G$. Then, for any $(a,b,c) \in G \times G \times G$ for $a, b, c \in G$ we have $(a,b,c)D = D(a,b,c)$, so then for every $(d,d,d) \in D$ with $d \in G$ we have that there exists $(d_1,d_1,d_1) \in D$ with $d_1 \in G$ such that $(a,b,c)(d,d,d) = (ad,bd,cd) = (d_1,d_1,d_1)(a,b,c) = (d_1a,d_1b,d_1c)$. We then have that

$$(ada^{-1}, bdb^{-1}, cdc^{-1}) = (d_1,d_1,d_1) \in D$$

and thus

$$ada^{-1} = bdb^{-1} = cdc^{-1}$$

and if we set $d = b$ we have

$$aba^{-1} = bbb^{-1} \Rightarrow aba^{-1} = b \Rightarrow ab = ba$$

and the claim is proven in this direction.
$\Leftarrow$) Supposing the contrary, if we have that G is abelian, then for any $(a,b,c) \in G \times G \times G$ we have that

$$(a,b,c)D$$

has elements

$$(a,b,c)(d,d,d)$$

for $d \in G$. Then, since G is abelian we have

$$(a,b,c)(d,d,d) = (ad,bd,dc) = (da,db,dc) = (d,d,d)(a,b,c)$$

which gives the implication

$$(a,b,c)D = D(a,b,c)$$

and that D is a normal subgroup of $G \times G \times G$. $\square$

2.

Proof. Suppose N, K are subgroups of G such that $G = N \times K$ with M a normal subgroup of N. Take $g \in G$. Then, $g = nk$ for some $n \in N$ and $k \in K$. Then,

$$g^{-1}Mg = k^{-1}n^{-1}Mnk$$

but M is normal in N, so

$$n^{-1}Mn \subset M.$$

Because $G = N \times K$ we have that $nk = kn$, and with $M \subset N$ normal we have that

$$g^{-1}Mg = k^{-1}n^{-1}Mnk = k^{-1}Mk = Mk^{-1}k = M.$$

□

3.

a.

Proof. Let $a, b \in G$. Then, first showing that f^* is a homomorphism using that f_i is a homomorphism

$$f^*(a+b) = (f_1(a_1+b_1), f_2(a_2+b_2), \dots, f_n(a_n+b_n)) = (f_1(a_1)+f_1(b_1), f_2(a_2)+f_2(b_2), \dots, f_n(a_n)+f_n(b_n))$$
$$= (f_1(a_1), f_2(a_2), \dots, f_n(a_n)) + (f_1(b_1), f_2(b_2), \dots, f_n(a_n)) = f^*(a) + f^*(b)$$

$$f^*(ab) = (f_1(a_1b_1), f_2(a_2b_2), \dots, f_n(a_nb_n)) = (f_1(a_1)f_1(b_1), f_2(a_2)f_2(b_2), \dots, f_n(a_n)f_n(b_n))$$
$$= (f_1(a_1), f_2(a_2), \dots, f_n(a_n))(f_1(b_1), f_2(b_2), \dots, f_n(a_n)) = f^*(a)f^*(b).$$

Now, we can show that

$$\pi_i \circ f^*(a) = \pi_i(f^*(a)) = \pi_i((f_1(a_1), f_2(a_2), \dots, f_n(a_n))) = f_i(a_i)$$

so the claim holds for all i. □

b.

Proof. Suppose that g is another homomorphism from G to $G_1 \times \cdots \times G_n$ such that $\pi_i \circ g = f_i$. We then have that $\pi_i(a_1, a_2, \dots, a_n) = a_i$, so this implies $\pi_i((g(a_1), g(a_2), \dots, g(a_n))) = f_i(a_i)$ which implies $g(a_i) = f_i(a_i)$ for all i and thus $g = f^*$. □

4.

a.

Proof.

$$\mathbb{Z}_{12},\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$$

□

b.

Proof.

$$\mathbb{Z}_{15}$$

□

c.

Proof.

$$\mathbb{Z}_{30}$$

□

d.

Proof.

$$\mathbb{Z}_{72},\ \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3,\ \mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3,\ \mathbb{Z}_8\times\mathbb{Z}_3\times\mathbb{Z}_3,\ \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_9,\ \mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_9 i$$

□

e.

Proof.

$$\mathbb{Z}_{90},\ \mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_5$$

□

f.

Proof.

$$\mathbb{Z}_{144},\ \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3,\ \mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3,\ \mathbb{Z}_8\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3,\ \mathbb{Z}_{16}\times\mathbb{Z}_3\times\mathbb{Z}_3,$$
$$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_9,\ \mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_9,\ \mathbb{Z}_8\times\mathbb{Z}_2\times\mathbb{Z}_9$$

□

g.

Proof.

$$\mathbb{Z}_{600},\ \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_5,\ \mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5\times\mathbb{Z}_5,\ \mathbb{Z}_8\times\mathbb{Z}_3\times\mathbb{Z}_5,\ \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_{25},\ \mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_2 5$$

□

h.

Proof.

$$\mathbb{Z}_{1160},\ \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_5\times\mathbb{Z}_{29}, \mathbb{Z}_4\times\mathbb{Z}_2\times\mathbb{Z}_5\times\mathbb{Z}_{29}.$$

□

5.

Proof. Suppose G is a finite abelian group and p a prime that divides $|G|$. By the fundamental theorem we know that G is the direct sum of cycle groups each of prime power order. Therefore, we have a cycle group of order p^k for some p. By theorem 7.9 that element p is in the group. □

6.

Let G, H, K be finite abelian groups.

a.

Proof. Put

$$G \cong \mathbb{Z}/p_1^{e_1} \bigoplus \cdots \bigoplus \mathbb{Z}/p_r^{e_r}$$

where $p_1, \ldots, p_r$ are primes each $e^i \geq 1$

$$H \cong \mathbb{Z}/q_1^{f_1} \bigoplus \cdots \bigoplus \mathbb{Z}/q_s^{f_r}$$

where $q_1, \ldots, q_s$ are primes and $f^i \geq 1$. Classification theorem says $G \cong H \iff (p_1^{e_1}, \ldots, p_r^{c_r}) = (q_1^{f_1}, \ldots, q_s^{f_s})$ up to some ordering. We know

$$G \bigoplus G \cong (\mathbb{Z}/{p_1}^{e_1} \bigoplus \mathbb{Z}/p_1^{e_1}) \bigoplus \cdots \bigoplus (\mathbb{Z}/p_r^{q_r} \bigoplus \mathbb{Z}/p_r^{q_r})$$

and similarly for $H \oplus H$.

$$(p_1^{e_1}, p_1^{e_1}, \ldots p_r^{e_r}, p_r^{e_r})$$

$$(q_1^{f_1}, q_1^{f_1}, \ldots, q_s^{f_s}, q_s^{f_s}).$$

Using classification we have that $r = s$, so we can reorder the q's such that

$$p_1^{e_1} = q_1^{f_1}, \ldots, p_r^{e_r} = q_r^{f_r}$$

but $p_i^{e_i}$s and $q_j^{f_j}$s are element divisiors for G adn H and thus they are equal, so

$$G \cong H.$$

□

b.

Proof. Let

$$K \cong \mathbb{Z}/m_1^{g_1} \bigoplus \cdots \bigoplus \mathbb{Z}/m_t^{g_t}$$

for m's prime and $g_i \geq 1$ for all i. We then have that

$$G \bigoplus H \cong (\mathbb{Z}/p_1^{e_1} \bigoplus \cdots \bigoplus \mathbb{Z}/p_r^{e_r}) \bigoplus \left(\mathbb{Z}/q_1^{f_1} \bigoplus \cdots \bigoplus \mathbb{Z}/q_s^{f_s}\right)$$

$$G \bigoplus K \cong (\mathbb{Z}/p_1^{e_1} \bigoplus \cdots \bigoplus \mathbb{Z}/p_r^{e_r}) \bigoplus (\mathbb{Z}/m_1^{g_1} \bigoplus \cdots \bigoplus \mathbb{Z}/m_t^{g_t})$$

with the two isomorphic. We can reorder $(p_1^{e_1}, \ldots, p_n^{e_n}, m_1^{g_1}, \ldots, m_t^{g_t})$ to get $(p_1^{e_1}, \ldots, p_r^{e_r}, q_1^{k_1}, \ldots, q_s^{f_s})$ so $s = t$. Assume $p_i^{e_i} = q_j^{f_j}$ in the reordering. Then $p_i^{e_i}$ in $G \bigoplus H$ matches with either $p_j^{e_j}$ or $m_k^{e_k}$. If $p_n^{e_n}$ in $G \bigoplus H$ matches with $m_i^{e_i}$, then change pairing so that $p_i^{e_i}$ swaps with $p_j^{e_j}$ and $q_i^{p_i}$ swaps with $m_i^{p_i}$. If $p_i^{e_i}$ matches with another $p_j^{f_j}$ then the same is true for $p_j^{e_j}$. Eventually we will have swapped all $p_i^{e_i}$ and get that every q^f and m^g agree and $H \cong K$. $\square$