Math 132H Homework 5

Tom Slavonia

May 13, 2024

12.

Proof. Consider the circle of radius $(N+\frac{1}{2})$. Take $f(z) = \frac{\pi\cot(\pi z)}{(u+z)^2}$. Note that since $\cot(\pi z) = \frac{\cos(\pi z)}{\sin(\pi z)}$ we have

$$f(z) = \frac{\pi\cot(\pi z)}{(u+z)^2} = \frac{\pi\cos(\pi z)}{(u+z)^2\sin(\pi z)}.$$

Recall that the residue integral corollary states that

$$\int_C f(z)dz = 2\pi i\sum_{k=0}^{n}\operatorname{res}_{z_k} f$$

for $z_0, z_1, \ldots, z_n$ poles of $f(z)$. From our definition of $f(z)$ we can look at $\frac{1}{f(z)} = 0$ to see

$$\frac{1}{f(z)} = (u+z)^2\sin(\pi z)$$

and we see that we have poles at $z = -u$ and whenever $z \in \mathbb{Z}$ such that $|z| < N + \frac{1}{2}$, this set of poles is exactly $\{-N, -(N-1), \ldots, N-1, N\}$. We can write

$$f(z) = (u+z)^{-2}\frac{\pi\cos(\pi z)}{\sin(\pi z)}$$

and since $u \notin \mathbb{Z}$ we have $\frac{\pi\cos(\pi z)}{\sin(\pi z)}$ is holomorphic at $z = -u$, thus the order of the pole $-u$ is 2. The order of the poles $\{-N, -(N-1), \ldots, N-1, N\}$ is one because they each appear once in the calculation of $\sin(\pi z)$ for z in the interior of the circle of radius $N + \frac{1}{2}$. Denote $z_0 = -u$ and $z_1 = -N, z_2 = -(N-1), \ldots, z_{2N} = N-1, z_{2N+1} = N$. Using the residue formula

$$\begin{aligned}
\operatorname{res}_{z_0} f &= \lim_{z\to -u}\frac{d}{dz}(z+u)^2\frac{\pi\cos(\pi z)}{(u+z)^2\sin(\pi z)} \\
&= \lim_{z\to -u}\frac{d}{dz}\frac{\pi\cos(\pi z)}{\sin(\pi z)} \\
&\overset{(a)}{=} \lim_{z\to -u}\frac{-\pi^2\cos^2(\pi z) - \pi^2\sin^2(\pi z)}{\sin^2(\pi z)} \\
&= \lim_{z\to -u} -\frac{\pi^2\cos^2(\pi z)}{\sin^2(\pi z)} - \pi^2
\end{aligned}$$

$$
\begin{aligned}
&\overset{(b)}{=} -\pi^2 \left(\cot^2(\pi u) + 1\right) \\
&= -\pi^2 \frac{1}{\sin^2(\pi u)} \\
&= -\frac{\pi^2}{(\sin(\pi u))^2}
\end{aligned}
$$

with steps $(a) - (b)$ justified

(a) division rule for derivatives

(b) trigonometric identity $\csc^2(x) = \cot^2(x) + 1$.

Now, to solve for the residue for an arbitrary z_k for $k \in \{1, \ldots, 2N+1\}$ and suppose $z_k = m \in \{z \in \mathbb{C} : z \in \mathbb{Z}, |z| < N + \frac{1}{2}\}$ we have

$$
\begin{aligned}
\operatorname{res}_{z_k} f &= \lim_{z \to m} (z - m) \frac{\pi \cos(\pi z)}{(u+z)^2 \sin(\pi z)} \\
&\overset{(a)}{=} \lim_{z \to m} \frac{\pi \cos(\pi z) - z\pi^2 \sin(\pi z) + m\pi^2 \sin(\pi z)}{2(u+z)\sin(\pi z) + (u+z)^2 \pi \cos(\pi z)} m \\
&= \frac{\pi(-1)^m}{(u+m)^2 \pi (-1)^m} \\
&= \frac{1}{(u+m)^2}
\end{aligned}
$$

with step (a) justified by using L'Hopital's rule. Now, we inspect the integral

$$
\int_C f(z) dz = \int_C \frac{\pi \cot(\pi z)}{(u+z)^2} dz.
$$

Taking the norm of this integral, we see with $z = x + iy$

$$
\begin{aligned}
\left| \int_C \frac{\pi \cot(\pi z)}{(u+z)^2} dz \right| &\le \int_C \left| \frac{\pi \cot(\pi z)}{(u+z)^2} \right| dz \\
&= \int_C \frac{|\pi \cos(\pi z)|}{|u+z|^2 |\sin(\pi z)|} dz \\
&\overset{(a)}{=} \pi \int_C \frac{1}{|u+z|^2} \cdot \left(\frac{\cos^2(\pi x) + \sinh^2(\pi y)}{\sin^2(\pi x) + \sinh^2(\pi y)} \right)^{\frac{1}{2}} dz \\
&\overset{(b)}{=} \pi \int_C \frac{1}{|u+z|^2} \cdot \left(\frac{\cos(2\pi x) + \cosh(2\pi y)}{\cosh(2\pi y) - \cos(2\pi x)} \right)^{\frac{1}{2}} dz \\
&= \pi \int_C \frac{1}{|u+z|^2} \cdot \left(1 + \frac{2\cos(2\pi x)}{\cosh(2\pi y) - \cos(2\pi x)} \right)^{\frac{1}{2}} dz
\end{aligned}
$$

with step (a) being the hint given in the homework and step (b) being an extremely long derivation of the equality through the exponential definitions of the functions that I worked out on paper but are probably on Wikipedia. Note that $\cosh(2y) \geq 1$ always, and $\lim\limits_{|y|\to\infty} \cosh(2y) = \infty$ and by our constaint that $\sqrt{x^2+y^2} = N + \frac{1}{2}$ the term

$$\frac{2\cos(2\pi x)}{\cosh(2\pi y) - \cos(2\pi x)} \leq 1.$$

Therefore, the integral is bounded by

$$\pi \int_C \frac{1}{|u+z|^2} \cdot \left(1 + \frac{2\cos(2\pi x)}{\cosh(2\pi y) - \cos(2\pi x)}\right)^{\frac{1}{2}} dz \leq \pi \cdot \frac{1}{\left(N+\frac{1}{2}\right)^2 - u\left(N+\frac{1}{2}\right) - u^2} \cdot 2\pi\left(N + \frac{1}{2}\right) \xrightarrow[N\to\infty]{} 0$$

Thus, the integral

$$\int_C f(z)dz \xrightarrow[N\to\infty]{} = 0$$

and so by noticing that by the poles we found $\lim\limits_{N\to\infty} \sum\limits_{k=1}^{2N+1} \operatorname{res}_{z_k} f = \sum\limits_{N=-\infty}^{\infty} \frac{1}{(u+N)^2}$ we have

$$\lim_{n\to\infty} \int_C f(z)dz = 0 = \lim_{N\to\infty} 2\pi i \sum_{k=0}^{n} \operatorname{res}_{z_k} f = \lim_{N\to\infty} \left(-\frac{\pi^2}{(\sin(\pi u))^2} + \sum_{k=1}^{2N+1} \operatorname{res}_{z_k} f\right) = -\frac{\pi^2}{(\sin(\pi u))^2} + \sum_{N=-\infty}^{\infty} \frac{1}{(u+N)^2}$$

which implies

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin(\pi u))^2}.$$

□

13.

Proof. The inequality we are given also implies

$$|z - z_0||f(z)| \leq A|z - z_0|^\epsilon.$$

since f is holomorphic in $D_r(z_0) - \{z_0\}$, z_0 is the only possible pole of f and with our new bound

$$\operatorname{res}_{z_0} f = \lim_{z\to z_0} |z - z_0||f(z)| \leq \lim_{z\to z_0} A|z - z_0|^\epsilon = 0$$

so the residue of the only possible pole is 0. Therefore, $f(z)$ must be bounded and the singularity removable. □

14.

Proof. Consider $F(z) = f\left(\frac{1}{z}\right)$ which has a singularity at 0. First, we consider if 0 is a removable singularity F. If 0 is a removable singularity of F, then F is bounded near 0. Because f is entire and $F(z) = f\left(\frac{1}{z}\right)$ is bounded, by Liouville's theorem, f is constant, but this is a contradiction as we assumed f to not be constant. Now, suppose F has a pole at 0, implying f has a pole at ∞. By the hint, this implies $f(z)$ is a polynomial. But, if f is injective, we cannot have multiple values map to 0. Because f is non-constant (we can't have $f(z) = 0$), by a corollary in the book, every polynomial of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$. Therefore, since f is injective, f must be of the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ such that $a \neq 0$. If F has an essential singularity at 0, then for a disc of an arbitrary size $r \in \mathbb{R}_{>0}$ by Casorati-Weierstrass the image of $D_r(0) - \{0\}$ under F is dense in $\mathbb{C}$. This statement is equivalent to asying $f(\{z : |z| > r\})$ is dense in $\mathbb{C}$. By the Open Mapping theorem, since f is non-constant and entire, we have that $f(\{z : |z| < r\})$ is open. But that would imply

$$f(\{z : |z| < r\}) \cap f(\{z : |z| > r\}) \neq \varnothing$$

which contradicts that the function is injective. Therefore, the only valid case is where $F(z)$ has a pole at 0 and that implies that $f(z) = az + b$. □

16.

a.

Proof. Suppose f, g are holomorphic in a region containing D such that $|z| \leq 1$ and f has a simple zero at $z = 0$ and vanishes nowhere else in $|z| \leq 1$. Define

$$f_\epsilon(z) = f(z) + \epsilon g(z).$$

Since f vanishes nowhere else in $|z| \leq 1$, we can make ϵ small enough such that $|f(z)| > \epsilon |g(z)|$ for all $z \in C$. Thus, f has the same number of zeros as $f + \epsilon g$ by Rouche's theorem. Since f has only one zero, the zero of $f + \epsilon g$ must be unique. □

b.

Proof. Suppose z_ϵ is the zero of f_ϵ. Fix ϵ small. For the sake of contradiction suppose that the sequence $\{\epsilon_n\}_{n=1}^\infty$ converges to ϵ, but $\{z_{\epsilon_n}\}_{n=1}^\infty$ does not converge to z_ϵ. The disc $|z| \leq 1$ is closed and bounded, so by Bolzano Weierstrass we have that z_{ϵ_n} has a convergent subsequence suppose this is $\{z_{\epsilon_k}\}_{k=1}^\infty$ which converges to $\tilde{z}$. Then, we have that $f_{\epsilon_k}(z_{\epsilon_k}) \to f_\epsilon(\tilde{z})$ but $\lim_{k\to\infty} f_{\epsilon_k}(z_{\epsilon_k}) = 0$ as $\epsilon \to \epsilon_k$, so that would imply $f_\epsilon(\tilde{z}) = 0$ but f_ϵ has only one unique zero z_ϵ. So, the claim must be false. Because sequential continuity is equivalent to continuity, the claim is proven. □

17.

a.

Proof. Let $f(z) = w_0$ for $w_0 \in \mathbb{D}$. Note that f is holomorphic in $\mathbb{D}$, and since w_0 is just a constant, it is also holomorphic in $\mathbb{D}$. Because $w_0 \in \mathbb{D}$ we have that $|w_0| < 1 = |f(z)|$ for $|z| = 1$ (z in the boundary of the unit disc), and therefore we may apply Rouche's theorem and state that the number of roots of $f(z) - w_0$ is equal to the number of roots of f. Therefore, it suffices to show $f(z) = 0$ has a root as it will equal the number of roots of $f(z) - w_0$. For the sake of contradiction, suppose $f(z)$ does not have a root. Without a root, $\frac{1}{f(z)}$ is non-constant and holomorphic in $\mathbb{D}$. Importantly, since f is holomorphic in an open set containing the closed unit disc $\bar{\mathbb{D}}$, then we have that f and, with our assumption that f does not have a root, $\frac{1}{f}$ are continuous on $\bar{\mathbb{D}}$. Therefore, we can apply the corollary to the maximum modulus principle to state

$$\inf_{z\in\mathbb{D}}\left|\frac{1}{f(z)}\right| \le \sup_{z\in\mathbb{D}}\left|\frac{1}{f(z)}\right| \le \sup_{z\in\bar{\mathbb{D}}-\mathbb{D}}\left|\frac{1}{f(z)}\right| = \sup_{z\in\partial\mathbb{D}}\left|\frac{1}{f(z)}\right| = 1$$

and thus

$$1 \le \inf_{z\in\mathbb{D}}|f(z)|.$$

Also, once again, using the corollary to the maximum modulus principle, we can get the result

$$\sup_{z\in\mathbb{D}}|f(z)| \le \sup_{z\in\partial\mathbb{D}}|f(z)| = 1$$

and thus the implication that

$$1 \le \inf_{z\in\mathbb{D}}|f(z)| \le \sup_{z\in\mathbb{D}}|f(z)| \le 1$$

implying

$$\inf_{z\in\mathbb{D}}|f(z)| = \sup_{z\in\mathbb{D}}|f(z)|$$

and that f is constant. But, this is a contradiction, as f was assumed to be non-constant. Thus, $f(z) = 0$ has a root proving the claim that the image of f contains the unit disc by the hint. □

b.

Proof. Once again, we only have to prove that $f(z) = 0$ has a root, and we can also use the corollary to reach the maximum modulus principle again. We proceed by contradiction, assuming $f(z) = 0$ has no roots. Then, $\frac{1}{f(z)}$ is holomorphic in $\bar{\mathbb{D}}$, and therefore, using the maximum modulus principle

$$\inf_{z\in\mathbb{D}}\left|\frac{1}{f(z)}\right| \le \sup_{z\in\mathbb{D}}\left|\frac{1}{f(z)}\right| \le \sup_{z\in\partial\mathbb{D}}\left|\frac{1}{f(z)}\right| \le 1$$

which implies

$$\inf_{z\in\mathbb{D}}|f(z)| \ge 1$$

but $z_0 \in \mathbb{D}$ and

$$|f(z_0)| < 1.$$

This is a contradiction, and therefore $f(z) = 0$ does have a zero, and the previous hint proves the claim. □

19.

a.

Proof. For the sake of contradiction, suppose u does attain its local maximum at some $z_0 \in \Omega$. Suppose $u = \mathrm{Re}(f)$ with $f = u + iv$. If u attains a maximum in Ω, then u attains a maximum in some disc of radius r, $D_r(z_0)$, around z_0, which exists because Ω is an open set. Note that f cannot be constant on Ω as u would be constant. By the Open Mapping theorem, we have $f(D_r(z_0))$ as an open set that contains $f(z_0)$. But, since $f(D_r(z_0))$ is an open set, there exists $z \in f(D_r(z_0))$ such that $z > f(z_0)$ which contradicts that $u = \mathrm{Re}(f)$ attains a maximum. □

b.

Proof. Since function is continuous it will attain a maximum in $\bar{\Omega}$. By the first part of the problem we have that u does not attain its maximum in the set Ω. Therefore, the maximum must be on the boundary of Ω, so

$$\sup_{z\in\Omega}|u(z)| \leq \sup_{z\in\bar{\Omega}-\Omega}|u(z)|.$$

$\square$