Math 110B Homework 8

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1.

Proof. Consider the Sylow 3−subgroup of S_4 $\langle(123)\rangle$. We need to find the normalizers of the subgroup in order to have that the conjugacy class is preverved using the definition of conjugacy class. Note that if we have $x \in G$ such that the conjugacy class is preserved, by the second Sylow theorem we have that all Sylow p−subgroups are conjugate to one another. Therefore, every other Sylow 3−subgroup is conjugate to this Sylow 3−subgroup. The identity element can be trivially included in the conjugates as well. $\square$

2.

Proof. For $C_a = \{gag^{-1} : g \in G\}$ take $b \in C_a$. We then have $b = gag^{-1}$ for some $g \in G$. Therefore, $f(b) = f(gag^{-1}) = f(g)f(a)f(g^{-1}) = f(g)f(a)f(g)^{-1} \in f(C_a)$ and thus $f(C_a)$ is also a conjugacy class of G. $\square$

3.

Proof. Let G be an infinite group and $H \subset G$ be all elements that have finite conjugates. Consider $a, b \in H$. Then, if a in H implies that a has finite conjugates and therefore every conjugate of a can be written gag^{-1}, so every conjugate of a^{-1} is $(gag^{-1})^{-1} = ga^{-1}g$ and therefore $a^{-1} \in H$. The product ab has conjugates $gabg^{-1} = gag^{-1}gbg^{-1} = (gag^{-1})(gbg^{-1})$ and a, b have finite conjugages, so ab has finite conjugate as we can write it in terms of conjugates of a and b. Thus, H is closed under inverses and multiplication and is thus a subgroup. $\square$

4.

Proof. Let H be a proper subgroup of G. By theorem 9.25 we know that the number of H conjugates of G must divide the order of G. Therefore, if H is a proper subset of G, then since H is a normal subgroup of $N(H)$, then $[H : H \cap N(H)]$ is strictly less than the order of G implying G cannot be the union of all the conjugates of H. $\square$

5.

Proof. To begin, by viewing the multiplication table we can see that there are exactly 2 generators, same as in D_4, then we can see that all elements correspond to multiplication just as the same as in D_4. Similarly the claim holds for G_2 and Q_8. Hence, we get the desired result. $\square$

6.

Proof. Take $(n_1, k_1), (n_2, k_2), (n_3, k_3) \in N \rtimes_\phi K$. We then have

$$\begin{aligned}
((n_1, k_1)(n_2, k_2))(n_3, k_3) &= ((n_1\phi_{k_1}(n_2), k_1k_2))(n_3, k_3) \\
&= (n_2\phi_{k_1}(n_2), k_1k_2)(n_3, k_3) \\
&= (n_3\phi_{k_1k_2}(n_3)\phi_{k_1}(n_2), k_1k_2k_3) \\
&= (n_1\phi_{k_1}(n_2\phi_{k_2}(n_3)), k_1k_2k_3) \\
&= (n_1, k_1)(n_2\phi_{k_2}(n_3), k_2k_3) \\
&= (n_1, k_1)((n_2\phi_{k_2}(n_3), k_2k_3)) \\
&= (n_1, k_1)((n_2, k_2)(n_3, k_3))
\end{aligned}$$

using that automorphisms are themselves group homomorphisms. Thus, we have the associative property and the semidirect product is a group. $\square$