Math 132H Homework 1

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April 2024

2.

Proof. Let $z, w \in \mathbb{C}$ where $z = x_1 + iy_1$ and $w = x_2 + iy_2$. Now, expanding the relevant equation, we get:

$$\begin{aligned}(z,w) + (w,z) &= z\bar{w} + w\bar{z} \\ &= (x_1 + iy_1)(x_2 - iy_2) + (x_2 + iy_2)(x_1 - iy_1) \\ &= x_1x_2 - ix_1y_2 + ix_2y + y_1y_2 + x_1x_2 - ix_2y_1 + ix_1y_2 + y_1y_2 \\ &= 2x_1x_2 + 2y_1y_2 \\ &= 2\langle z, w\rangle.\end{aligned}$$

Thus, we can conclude that

$$\frac{(z,w) + (w,z)}{2} = \langle z, w\rangle = \mathrm{Re}(z,w).$$

□

7.

a.

Proof. Let $z = r_1e^{i\theta_1}$ and $w = r_2e^{i\theta_2}$. Using the fact that $\left|e^{i\theta_1}\right| = 1$,

$$\begin{aligned}\left|\frac{w - z}{1 - \bar{w}z}\right| &= \left|\frac{r_2e^{i\theta_2} - r_1e^{i\theta_1}}{1 - r_1r_2e^{i(-\theta_2+\theta_1)}}\right| \\ &= \left|e^{i\theta_1}\frac{r_2e^{i(\theta_2-\theta_1)} - r_1}{1 - r_1r_2e^{i(-\theta_2+\theta_1)}}\right| \\ &= \left|e^{i\theta_1}\right|\left|\frac{r_2e^{i(\theta_2-\theta_1)} - r + 1}{1 - r_1r_2e^{i(-\theta_2+\theta_1)}}\right| \\ &= \left|\frac{r_2e^{i(\theta_2-\theta_1)} - r + 1}{1 - r_1r_2e^{i(-\theta_2+\theta_1)}}\right|.\end{aligned}$$

Define $w' = r_2e^{i(\theta_2-\theta_1)}$ and note that

$$|w'| = |r_2e^{i(\theta_2-\theta_1)}| = r_2|e^{i(\theta_2-\theta_1)}| = r_2 = |w| < 1.$$

Therefore, we can now look at the equation

$$\left|\frac{w' - r_1}{1 - r_1\bar{w}'}\right|$$

where now $z \in \mathbb{R}$. Now, by the hint given in the textbook, we have to show that

$$(r_1 - w)(r_1 - \bar{w}) \le (1 - r_1w)(1 - r_1\bar{w})$$

is true. Thus,

$$\begin{aligned}
&(r_1 - w)(r_1 - \bar{w}) \le (1 - r_1 w)(1 - r_1 \bar{w}) \\
\iff &r_1^2 - r_1\bar{w} - r_1 w + |w|^2 \le 1 - r_1\bar{w} - r_1 w + r_1^2|w|^2 \\
\iff &r_1^2 + |w|^2 \le 1 + r_1^2|w|^2 \\
\iff &|w|^2(1 - r_1^2) \le 1 - r_1^2
\end{aligned}$$

which holds because $|w|^2 < 1$ and $r_1 < 1$.

□

b.

i)

Proof. Fix $w \in \mathbb{D}$ the unit disk. Take $z \in \mathbb{D}$. We want to show that $F(z) \in \mathbb{D}$. Since both $z, w \in \mathbb{D}$ we have that $|z|, |w| < 1$ and thus, by the previous part of the problem, we have that $\left|\frac{w-z}{1-\bar{w}z}\right| < 1$ which implies that $F(z) = \left|\frac{w-z}{1-\bar{w}z}\right| \in \mathbb{D}$, so $F : \mathbb{D} \to \mathbb{D}$.
To prove F is holomorphic, note that $F(z) = \frac{w-z}{1-\bar{w}z}$ is of the form $\frac{f}{g}$ where $f(z) = w - z$ and $g(z) = 1 - \bar{w}z$. First, we will show that $f(z)$ is holomorphic:

$$\begin{aligned}
f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\
&= \lim_{h\to 0} \frac{w - z - h - w + z}{h} \\
&= \lim_{h\to 0} \frac{-h}{h} \\
&= -1.
\end{aligned}$$

Thus, $f(z)$ is holomorphic. Now we will show that $g(z)$ is holomorphic:

$$\begin{aligned}
g'(z) &= \lim_{h\to 0} \frac{g(z+h) - g(z)}{h} \\
&= \lim_{h\to 0} \frac{1 - \bar{w}(z+h) - 1 + \bar{w}z}{h} \\
&= \lim_{h\to 0} \frac{1 - \bar{w}z - \bar{w}h - 1 + \bar{w}z}{h} \\
&= \lim_{h\to 0} \frac{\bar{w}h}{h} \\
&= \bar{w}.
\end{aligned}$$

Thus, $g(z)$ is holomorphic, and since $|z| < 1$ we know $g(z) \neq 0$. Thus, by the division rule of holomorphic functions, we have that $F = \frac{f}{g}$ is holomorphic. □

ii)

Proof. We can quickly show that

$$F(0) = \frac{w - 0}{1 - \bar{w} \cdot 0} = \frac{w}{1} = w$$

and, since $|w| < 1$

$$F(w) = \frac{w - w}{1 - \bar{w}w} = \frac{0}{1 - |w|^2} = 0$$

.

□

iii)

Proof. If $z = 1$, then

$$|F(z)| = \left|\frac{w - z}{1 - \bar{w}z}\right| = 1$$

by the first part of the problem. □

iv)

Proof. We will prove that F is bijective by proving that F is its inverse. Thus, we will show $F \circ F(z) = z$ for $z \in \mathbb{D}$:

$$\begin{aligned}
F \circ F(z) &= \frac{w - \frac{w-z}{1-\bar{w}z}}{1 - \bar{w}\left(\frac{w-z}{1-\bar{w}z}\right)} \\
&= \frac{\frac{w-|w|^2z-w+z}{1-\bar{w}z}}{1 - \frac{|w|^2-\bar{w}z}{1-\bar{w}z}} \\
&= \frac{\frac{w-|w|^2z-w+z}{1-\bar{w}z}}{\frac{1-\bar{w}z-|w|^2+\bar{w}z}{1-\bar{w}z}} \\
&= \frac{z-|w|^2z}{1-|w|^2} = z\left(\frac{1-|w|^2}{1-|w|^2}\right) \\
&= z.
\end{aligned}$$

□

9.

Proof. Let $z = x+iy = x(r,\theta)+iy(r,\theta) = re^{i\theta} = \cos\theta+i\sin\theta$. Let $f(z) = u(z(r,\theta),y(r,\theta))+iv(x(r,\theta),y(r,\theta))$. We find that,

$$\begin{aligned}
\frac{\partial u}{\partial r} &\overset{(a)}{=} \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\
&= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \\
&\overset{(b)}{=} \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta \\
&= \frac{1}{r}\left(\frac{\partial v}{\partial y}r\cos\theta - \frac{\partial v}{\partial x}r\sin\theta\right) \\
&= \frac{1}{r}\left(\frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta}\right) \\
&= \frac{1}{r}\frac{\partial v}{\partial \theta}.
\end{aligned}$$

The steps $(a)-(b)$ are justified:

(a) chain rule of partial derivates

(b) Cauchy Riemann equations.

To prove the other Cauchy Riemann equation, we use a similar process:

$$\begin{aligned}
\frac{1}{r}\frac{\partial u}{\partial \theta} &\overset{(a)}{=} \frac{1}{r}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right) \\
&= \frac{1}{r}\left(-\frac{\partial u}{\partial x}r\sin\theta + \frac{\partial u}{\partial y}r\cos\theta\right) \\
&= -\frac{\partial u}{\partial x}\sin\theta + \frac{\partial u}{\partial y}\cos\theta \\
&= -\frac{\partial u}{\partial x}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial y}\frac{x}{\partial r} \\
&\overset{(b)}{=} = -\frac{\partial v}{\partial y}\frac{\partial y}{\partial r} - \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} \\
&= -\frac{\partial v}{\partial r}.
\end{aligned}$$

The steps $(a)-(b)$ are justified:

(a) chain rule of partial derivates

(b) Cauchy Riemann equations.

To check that $\log(z) = \log(r) + i\theta$ is holomorphic in $r > 0$ and $-\pi < \theta < \pi$ we can use our newly developed Cauchy Riemann equations. Take $u = \log(r)$ and $v = \theta$. We have that

$$\frac{1}{r}\frac{\partial u}{\partial \theta} = 0 = -\frac{\partial v}{\partial \theta}$$

$$\frac{\partial y}{\partial r} = \frac{1}{r} = \frac{1}{r} \cdot 1 = \frac{1}{r}\frac{\partial v}{\partial \theta}.$$

Thus, the Cauchy Riemann equations are satisfied, and because u, v are clearly continuously differentiable for $r > 0$ and $-\pi < \theta < \pi$ we can apply the theorem stating that if a function satisfies the Cauchy Riemann equations and has u, v continuously differentiable on a set, then it is holomorphic on the set to conclude that $\log(z)$ is holomorphic on $r > 0$ and $-\pi < \theta < \pi$. $\square$

13.

a.

Proof. If $\mathrm{Re}(f) = c$ a constant, then that implies $u(x, y) = c$ and thus, using the Cauchy Riemann equations,

$$\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}.$$

Thus, $v(x, y) = c'$ a constant, and thus f is a constant.

$\square$

b.

Proof. In this case, we have that $v(x, y) = c$ a constant, and this assertion can be proved in the same way as part (a) to show that $u(x, y) = c$ and hence f is constant.

$\square$

c.

Proof. If $|f| = \sqrt{u(x,y)^2 + v(x,y)^2} = c$ a constant, then $|f|^2 = u(x, y)^2 + v(x, y)^2 = c^2$ by properties of complex numbers. We can take the partial derivative of $|f|^2$ with respect to x and y and use the chain rule for partial derivatives to get two equations:

$$0 = 2u(x,y)\frac{\partial u}{\partial x} + 2v(x,y)\frac{\partial v}{\partial x}$$
$$0 = 2u(x,y)\frac{\partial u}{\partial y} + 2v(x,y)\frac{\partial v}{\partial y}.$$

Multiplying by 2 on each side, we can simplify our equations to

$$0 = u(x,y)\frac{\partial u}{\partial x} + v(x,y)\frac{\partial v}{\partial x}$$
$$0 = u(x,y)\frac{\partial u}{\partial y} + v(x,y)\frac{\partial v}{\partial y}.$$

Now, we can multiply the first equation by $u(x, y)$ and the second equation by $v(x, y)$, which will help us simplify it shortly:

$$0 = u(x,y)^2\frac{\partial u}{\partial x} + u(x,y)v(x,y)\frac{\partial v}{\partial x}$$
$$0 = u(x,y)v(x,y)\frac{\partial u}{\partial y} + v(x,y)^2\frac{\partial v}{\partial y}.$$

Now, we can use the Cauchy-Riemann equations to simplify further

$$\begin{aligned}
0 &= u(x,y)\frac{\partial u}{\partial x} + v(x,y)\frac{\partial v}{\partial x} \\
0 &= -u(x,y)\frac{\partial v}{\partial x} + v(x,y)\frac{\partial v}{\partial y}.
\end{aligned}$$

Combining the two equations, we obtain

$$0u(x,y)^2\frac{\partial u}{\partial x} + v(x,y)^2\frac{\partial v}{\partial y}.$$

Once again, using the Cauchy-Riemann equation, we get that

$$0 = u(x,y)^2\frac{\partial u}{\partial x} + v(x,y)^2\frac{\partial u}{\partial x}$$

this implies that

$$0 = \frac{\partial u}{\partial x}(u(x,y)^2 + v(x,y)^2).$$

Either $u(x,y)^2$ and $v(x,y)^2 = 0$ in which case $f = 0$ or $\frac{\partial u}{\partial x} = 0$. Similarly, it can be shown that $\frac{\partial u}{\partial y}$, $frac\partial v\partial x$, and $\frac{\partial v}{\partial y}$ must be 0 or the function must be 0, and hence f is a constant.

□

14.

Proof. Assume that $0 \leq M \leq N$ as otherwise, the equation will always equal 0. We will manipulate the series in several ways to show the equality:

$$\begin{aligned}
\sum_{n=M}^{N} a_n b_n &= \sum_{n=M}^{N} a_n \left(\sum_{i=1}^{n} b_i - \sum_{i=1}^{n-1} b_i\right) \\
&= \sum_{n=M}^{N} a_n (B_n - B_{n-1}) \\
&= \sum_{n=M}^{N} a_n B_n - \sum_{n=M}^{N} a_n B_{n-1} \\
&= a_N B_N + \sum_{n=M}^{N-1} a_n B_n - a_M B_{M-1} - \sum_{n=M+1}^{N} a_n B_{n-1} \\
&\overset{(a)}{=} a_N B_N - a_M B_{M-1} + \sum_{n=M}^{N-1} a_n B_n - \sum_{n=M}^{N-1} a_{n+1} B_n \\
&= a_N B_N - a_M B_{M-1} + \sum_{n=M}^{N-1} (a_n - a_{n+1}) B_n \\
&= a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.
\end{aligned}$$

Step (a) is justified as with the convention that $B_0 = 0$, we can shift the indices to get that $\sum_{n=M+1}^{N} a_n B_{n-1} = \sum_{n=M}^{N-1} a_{n+1} B_n$.

□

19.

a.

Proof. Let $a_n = nz^n$. For a series to converge, from real analysis, we know that we need $\lim_{n\to\infty} |a_n| \to 0$. But,

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} |nz^n| = \lim_{n\to infty} n|z|^n = \lim_{n\to\infty} n = \infty.$$

Thus, the series does not converge for $|z| = 1$.

□

b.

Proof. We will use the fact that if a series converges absolutely, it converges. Thus, $\sum \left|\frac{1}{n^2}\right| |z|^n = \sum \frac{1}{n^2}$ which we know converges. Thus, our series converges for $|z| = 1$. $\square$

c.

Proof. Let $|z| = 1$. If $z = 1$, then $\sum \frac{z^n}{n} = \sum \frac{1}{n}$ which we know diverges. So, suppose $z \neq 1$. Let $S_N = \sum_{n=1}^{N} a_n b_n$, where $a_n = \frac{1}{n}$ and $b_n = z^n$. Also, let $B_N = \sum_{n=1}^{N} b_n$. By the summation by parts proven earlier we have that

$$S_N = a_N B_N - a_1 B_0 - \sum_{n=1}^{N-1} (a_{n+1} - a_n) B_n.$$

Since, $B_0 = 0$ and $-\sum_{n=1}^{N-1} (a_{n+1} - a_n) B_n = \sum_{n=1}^{N-1} (a_n - a_{n+1}) B_n$, we get

$$S_N = a_N B_N + \sum_{n=1}^{N-1} (a_n - a_{n+1}) B_n.$$

We need to show that B_N is bounded for every N. But, this is true, as $B_N = \sum_{n=1}^{N} e^{in\theta} = \frac{1-e^{i\theta(N+1)}}{1-e^{i\theta}}$ as shown in the textbook, and this is bounded for all N. Because $|B_N|$ is bounded for every N, as $|z| = 1$ let the bound for an arbitrary B_n be denoted by M. Therefore, as $a_n = \frac{1}{n} \to 0$ we know that $a_N B_N = \frac{1}{n} \sum_{n=1}^{N} z^n \to 0$ becuase B_N is bounded for all N by M. Our sequence is monotone decreasing, so

$$\sum_{n=1}^{N-1} |(a_n - a_{n+1}) B_n| \leq \sum_{n=1}^{N-1} |(a_n - a_{n+1})| M \overset{(a)}{=} \sum_{n=1}^{N-1} (a_n - a_{n+1}) M$$

where step (a) is true because the sequence is monotone decreasing. Thus, because

$$\sum_{n=1}^{N-1} (a_n - a_{n+1}) M$$

is a telescoping sum

$$\lim_{n\to\infty} \sum_{n=1}^{N-1} (a_n - a_{n+1}) M = a_1 M.$$

We can conclude that

$$\lim_{n\to\infty} S_N = \lim_{n\to\infty} \left(a_N B_N + \sum_{n=1}^{N-1} (a_n - a_{n+1}) B_n \right) = 0 + \lim_{n\to\infty} \sum_{n=1}^{N-1} (a_n - a_{n+1}) B_n \leq \lim_{n\to\infty} \sum_{n=1}^{N-1} (a_n - a_{n+1}) M = a_1 M.$$

$\square$

23.

Proof. Let $p(x)$ be a polynomial. Our claim is that for $x > 0$, where $f(x) = e^{-\frac{1}{x^2}}$ that $f^{(n)}(x)$ is of the form. We will prove this assertion by induction. For $f^{(1)}(x)$ we have

$$f^{(1)}(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}}$$

which is of the form $p(x)e^{-\frac{1}{x^2}}$ and is defined since $x > 0$. Now, assume the claim holds for all n, and we will prove the $n+1$ case. For $f^{(n+1)}(x)$ we have that by our hypothesis we know that $f^{(n)} = p(x)e^{-\frac{1}{x^2}}$, so we then have that $f^{(n+1)}(x) = p(x)\frac{2}{x^3}e^{-\frac{1}{x^2}}$. So, the function is indefinitely differentiable for $x > 0$, and if $x \leq 0$, then $f(x) = 0$, so clearly, every derivative will be just 0. Thus, f is indefinitely differentiable for any $x \in \mathbb{R}$. We know that $f^{(n)}(x) = 0$ for any $x \leq 0$, so it follows that $f^{(n)}(0) = 0$. Therefore, looking at the power series expansion for $e^{-\frac{1}{x^2}}$ we see

$$e^{-\frac{1}{x^2}} = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{x^2}\right)^n}{n!} = 1 - \frac{1}{x^2} + \frac{1}{2!}\frac{1}{x^4} - \frac{1}{3!}\frac{1}{x^6} + \cdots$$

which is undefined for the value $x = 0$, and hence, we should expect no converging power series expansion near the origin to exist. $\square$