Math 121 Homework 9

Thomas Slavonia; UID: 205511702

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3.3.5.

a.

($\Leftarrow$) If a subset $W \subseteq \mathbb{R}^n$ is convex, then for $w, y \in W$, we have that there is a straight line interval that joins w, y in W which is exactly the definition of W being star-shaped with respect to w.
($\Rightarrow$) Now, if W is a star-shaped set with respect to every point in W, then for any $x, y \in W$ we have a line segment between x and y thus proving convexity.

$\square$

b.

With respect to the point $(0,0)$. Let $W = \{(x,y) : x = 0 \text{ or } y = 0\} \subseteq \mathbb{R}^2$. For the point $(0,0)$, we have that if $x \neq 0$, then $t(x,0) + (1-t)(0,0) = (tx, 0) \in W$ for $t \in [0,1]$ is a line segment from $(0,0)$ to $(x,0)$, and similarly, if $y \neq 0$, then $t(0,y)+(1-t)(0,0) = (0,ty) \in W$ is a line segment from $(0,0)$ to $(0,y)$. But, there is no line segment in W from $(x,0)$ to $(0,y)$, if $x, y \neq 0$, as $t(x,0)+(1-t)(0,y) = (tx, (1-t)y)$ and if $t = \frac{1}{2}$, then $(\frac{1}{2}x, \frac{1}{2}y) \notin W$ if we have that $x, y \neq 0$.

c.

Suppose $W \subseteq \mathbb{R}^n$ is a star-shaped set. Let $x_0 \in W$. Let $F : W \times [0,1] \to W$ where $(x,t) \mapsto (1-t)x_0 + tx$. Then, we have that $F(x,0) = x_0$, $F(x,1) = x$, and $F(x_0, t) = x_0 - tx_0 + tx_0 = x_0$. Thus, the conditions of contractible to x_0 have been met.

$\square$

d.

Let $W \subseteq \mathbb{R}^n$ be a star-shaped set. In Problem 3 of this section, we proved that any set contractible to a point has a trivial fundamental group. We have shown that a star-shaped set is contractible to a point. Thus, it has a trivial fundamental group. A star-shaped set is path-connected; by definition, there exists a line segment between a given point and the point that the star-shaped set is with respect to. Thus, for $x, y \in W$ and if W is with respect to w, then there exists line segment $\alpha : [0,1] \to W$ such that $\alpha(t) = (1-t)x + tw$ and line segment $\beta : [0,1] \to W$ with $\beta(t) = (1-t)w + ty$. Then, $\gamma = \alpha\beta$ is a path from x to y. Thus, W is simply connected.

□

3.3.6.

For a loop γ based at (x_0, y_0), we have that it is equivalent to (α, β) where α is based at x_0 and β is based at y_0. This assertion is true because, by the properties of the product topology, $\gamma : [0,1] \to (X, x_0) \times (Y, y_0)$ is continuous if and only if $\alpha : [0,1] \to (X, x_0)$ and $\beta : [0,1] \to (Y, y_0)$ are continuous, $\gamma(t) = (\alpha(t), \beta(t))$, which they are. Similarly, a homotopy γ_t is equivalent to a pair of homotopies α_t and β_t. Thus, we have a bijection $\pi_1(X \times Y, (x_0, y_0))$ to $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ with $[\gamma] \mapsto ([\alpha], [\beta])$.

□

3.3.7.

Suppose X Y are simply connected spaces. By the previous problem, we have that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y) \cong 0 \times 0$, and thus the fundamental group is trivial in the product space. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. We know there exists path $\alpha : [0,1] \to X$ a path from x_1 to x_2, $\alpha(t) = (1-t)x_1 + tx_2$, and $\beta : [0,1] \to Y$ a path from y_1 to y_2 with $\beta(t) = (1-t)y_1 + ty_2$. Hence, $\gamma(t) = \alpha\beta(t)$ where $\gamma(t) = ((1-t)x_1, (1-t)y_1) + (tx_2, ty_2) = ((1-t)x_1 + tx_2, (1-t)y_1 + y_2)$ is a path from (x_1, y_1) to (x_2, y_2). Thus, $X \times Y$ is path-connected and thus is simply connected. □

3.4.1.

Suppose X is a simply connected topological space, and suppose $f : X \to Y$ is a homeomorphism from X to a topological space Y. We have previously shown that path connectedness is a topological property. Thus, we need only show that the property of a trivial fundamental group is a topological property. By assumption, $\pi_1(X) \cong 0$ by Corollary 4.4, we have that since X and Y are homeomorphic, there exists an isomorphism f_* from $\pi_1(X)$ to $\pi_1(Y)$ which implies $\pi_1(Y) \cong \pi_1(X) \cong 0$.

□

3.4.2.

Recall that $S^n \subset \mathbb{R}^{n+1}$. Let $f : \mathbb{R}^{n+1} \backslash \{0\} \to S^n$ map $x \in \mathbb{R}^{n+1} \backslash \{0\}$ to $\frac{x}{||x||} \in S^n$. Then, for all $x \in S^n$ we have $||x|| = 1$, so $f(x) = \frac{x}{||x||} = \frac{x}{1} = x$. We can conclude f

is a retraction and S^n is a retract of $\mathbb{R}^{n+1}\backslash\{0\}$. $\square$

3.4.3.

a.

Let $f : X \to A$ be a retraction and $x_0 \in A$. Let $j : A \hookrightarrow X$ be the inclusion map. Then, $j_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is a homomorphism by Theorem 4.3. To prove that we have injectivity, let $j_*([\alpha]) = j_*([\beta])$ for $[\alpha], [\beta] \in \pi_1(A, x_0)$. Therefore, we have $[\alpha] = [\beta]$ both in $\pi_1(X, x_0)$ and thus $[\alpha] = [\beta]$ in $\pi_1(A, x_0)$ so we have injectivity.

$\square$

b.

To prove the surjectivity of $f_* : \pi_1(X, x_0) \to \pi_1(A, x_0)$, suppose we have $[\alpha] \in \pi_1(A, x_0)$. Then, there exists $[\alpha] \in \pi_1(X, x_0)$ such that $f_*([\alpha]) = [f \circ \alpha] - [\alpha] \in \pi_1(A, x_0)$. Therefore, we have surjectivity. $\square$

c.

Now, suppose X is simply connected. Therefore, X is path connected and $\pi_1(X) \cong 0$. Let $a, b \in A$. Because X is path connected, in X there exists $\gamma : [0, 1] \to X$ such that $\gamma(t) = (1 - t)a + tb$ is a path from a to b. With f being a retraction of X, we have $f(x) = x \in A$ for all $x \in A$.

Note that $f \circ j = id_A$ and thus $f_* \circ j_*$ is identity homomorphism for the fundamental group. We know $\pi_1(X) \cong 0$, and thus for any $[\alpha] \in \pi_1(A)$, we have $f_* \circ j_*(\alpha) = f_*(j_*([\alpha]))$ and since f_* is onto, we have $f_*(j_*([\alpha])) \cong 0$ and thus A is simply connected.

$\square$

2.13.8.

a.

Because P^n is derived from S^n, which is compact, we have that P^n is compact. Take $x, y \in P^n$ such that $x \neq y$. Then, define a map $f : P^n \to S^n$ such that $x \in P^n$ maps to $f(x) = \{-a, a\} \subset S^n$ for $-a, a \in S^n$. Then, we have $f(x) = \{-a, a\}$ and $f(y) = \{-b, b\}$. Take $\epsilon = min\{||a - b||, ||a + b||\}$ and then we have $x \in B(x, \epsilon)$ and $y \in B(y, \epsilon)$ that are disjoint. Thus, $f^{-1}(f(x)) = x \in f^{-1}(B(x, \epsilon))$ and $f^{-1}(f(y)) = y \in f^{-1}(B(y, \epsilon))$ which are disjoint open sets in P^n by choice of ϵ. Thus, we have that P^n has the Hausdorff property.

$\square$

b.

Let $\pi : S^n \to P^n$ and take $x \in S^n$. Let $x \in U$ be an open neighborhood of X in S^n. Now, look at the antipodal of x $-x$, which will have an open set $-U = \{-y : y \in U\}$ as an open neighborhood of equal size U. Therefore, $\pi(U \cup -U)$ is open as π is continuous. Moreover, we have $\pi(x) \in \pi(U \cup -U)$ as $x \equiv x$ and $x \equiv -x$ and $x, -x \in U \cup U$. □

c.

To show that P^1 is homeomorphic to S^1, consider $f : S^1 \to S^1$ where $f(x) = x^2$ for $x \in S^1$. The map is surjective, as for any $x \in S^1$ there exists $\sqrt{x} \in S^1$ such that $f(\sqrt{x}) = x$. To prove continuity, note $\forall \epsilon\ \exists \delta = \min\{1, \frac{\epsilon}{1+2|y|}\}$ such that for $|x - y| < \delta$ we have,

$$|f(x)-f(y)| = |x^2-y^2| = |x-y||x+y| = |x-y||x-y+2y| \leq (|x-y|+|2y|)\delta < (1+2|y|)\delta < \epsilon.$$

Now, S^1 is a compact Hausdorff space, and f is a continuous surjective function from S^1 to itself, thus by Theorem 13.4 we have $S^1/\sim \cong S^1$ where the equivalence relation is defined as $x \sim y$ if $f(x) = f(y)$. But this is exactly the equivalence relation we want, as $f(x) = x^2$ which implies $f(x) = f(-x) = x^2$, and thus $S^1 \cong P^1$. □

d.

To find the homeomorphism between P^n and the quotient space obtained from looking at the antipodal points of the boundary of B^n, let $f : B^n \to P^n$ where for $x \in B^n$, $f(x) = \pi(x)$ where $\|x\| = 1$. Because both are compact Hausdorff spaces and since f is a continuous function, we have previously shown (Theorem 13.4) that $B^n/\sim \cong P^n$, thus we are done. □

3.5.1.

For $m \in \mathbb{Z}$, let $\alpha_m(s) = e^{2\pi i m s}$ be a loop in S^1 for $s \in [0,1]$. Let β be a loop in S^1 based at 1. Then, if β has the same index as α_m, then we have by Theorem 5.6 that they are in the same homotopy class. □

3.5.3.

Let $p : \mathbb{R}^n \to T^n$ be the exponential map where for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we have $p(x_1, \dots, x_n) = (e^{2\pi i x_1}, \dots, e^{2\pi i x_n})$. We know that p is a covering map as it is the product of covering maps and $T^n = S^1 \times \cdots \times S^1$ $n - times$. To show the desired isomorphism, note that in the previous exercise, we know that for identity element $e = (1, 1, \dots, 1)$ we have $\pi_1(T^n, e) \cong p^{-1}(e)$. Therefore, using the previously established fact $\pi_1(S^1, 1) \cong \mathbb{Z}$,

$$\pi_1(T^n, e) = \pi_1(S^1 \times \cdots \times S^1, e) = \pi_1(S^1, 1) \times \cdots \times \pi_1(S^1, 1) \cong \mathbb{Z} \times \cdots \times \mathbb{Z}$$

We can get the desired loop for the n-tuple $(m_1, \dots, m_n)$, $\gamma : [0,1] \to T^n$, by setting $\gamma(t) = (e^{2\pi i m_1 t}, \dots, e^{2\pi i m_n t})$. This will be a loop based at $(1, 1, \dots, 1)$, as for $t = 0$ we have $\gamma(0) = (e^0, \dots, e^0) = (1, 1, \dots, 1)$. For $\gamma(1)$ we have

$$\gamma(1) = (e^{2\pi i m_1}, \dots, e^{2\pi i m_n}) = (e^{2\pi i 1}, \dots, e^{2\pi i 1}) = (e^{2\pi i}, \dots, e^{2\pi i}) = (1, 1, \dots, 1).$$

□

3.5.4.

The map is surjective, as for any $z \in mathbbC\backslash\{0\}$ we have $x \in \mathbb{C}$ such that $e^x = z$. Take $x_0 \in \mathbb{C}\backslash\{0\}$. Then, let $U = \{y \in \mathbb{C} : |x_0 - y| < \epsilon\}$ which is the disk of radius $\epsilon > 0$ in the complex plane without 0. The exponential function being continuous and invertible implies that $p^{-1}(U)$ is open, and moreover, we have y_0 map to x_0, and so do all elements with period $2\pi i$.

For any $z \in \mathbb{C}$, we have $z = a + ib$. The, we have $e^z = e^{a+ib} = e^a e^{ib}$. We have previously shown that e^{ib} will be a covering map onto S^1, and it is seen that e^a is a covering map onto $\mathbb{R}_{>0}$. Therefore, we have that e^z is the product of two covering maps and is thus a covering map. If we take a look at p^{-1} for the point 1, we have $p^{-1}(\{1\}) = 2\pi i t$ for $t \in \mathbb{Z}$ and thus, $C\backslash\{0\} \cong \mathbb{Z}$.

□

3.5.5.

Let $A = \{w : e^c < |w| < e^d\}$ and take $w \in A$. Then, $\exists z \in E$ such that by the monotonicity of e^x we have $e^c < e^z < e^d$ where w can map to e^z. Now, the covering space argument is the same as in the previous problem The fundamental group of the open annulus, $\pi_1(\{w : e^c < |w| < e^d\})$ is isomorphic to that of a circle S^1 as any loop on the annulus will operate similarly to that of the circle but only with width. Therefore, $\pi_1(\{e^c < |w| < e^d\}) \cong \pi_1(S^1) \cong \mathbb{Z}$. Nothing notable changes with the closed annulus, and thus we have $\pi_1(\{e^c \le |w| \le e^d\}) \cong \pi_1(S^1) \cong \mathbb{Z}$ □