Math 110B Homework 9

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June 6, 2024

1.

Proof. $\square$

2.

Proof. Classifying the groups of order 21 the only abelian group is $\mathbb{Z}/21\mathbb{Z}$ as $21 = 7*3$ and $gcd(7,3) = 1$. For the nonabelian groups, we have a Sylow 7−subgroup of G with $|G| = 21$ is normal.

$$n_7 \equiv 1 \ (mod\ 7),\ n_7 | 3$$

so $n_7 = 1$. We also have a Sylow 3−subgroup K. Write N =Sylow 7−subgroup. We have $gcd(7,3) = 1$ which implies $N \cap K = \{e\}$ and

$$G = N \rtimes K.$$

Structure of group specfied by map $K \cong \mathbb{Z}/3\mathbb{Z}$ and $Aut(N) \cong (\mathbb{Z}/7\mathbb{Z})^{\times} \cong \mathbb{Z}/6\mathbb{Z}$. Have 2 nontrivial maps

$$[1] \mapsto [2]\ \varphi_1$$

$$[1] \mapsto [4]\ \varphi_2.$$

Check that

$$\mathbb{Z}/7\mathbb{Z} \rtimes_{\varphi_1} \mathbb{Z}/3\mathbb{Z}$$

$$\mathbb{Z}/7\mathbb{Z} \rtimes_{\varphi_2} \mathbb{Z}/3\mathbb{Z}$$

are the same. These are isomorphic, map $\varphi_2 : \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$ is given by composing $\mathbb{Z}/3\mathbb{Z} \xrightarrow[{[1}]{\mapsto} [2]]\mathbb{Z}/3\mathbb{Z}$ with φ_1, so they define isomorphic semidirect products. $\square$

3.

Proof. The matrix is invertible if and only if $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ are linearly independent over $\mathbb{Z}/p\mathbb{Z}$. This is true if and only $\begin{bmatrix} a \\ c \end{bmatrix} \neq e \begin{bmatrix} b \\ d \end{bmatrix}$ for all $e \in \mathbb{Z}/p\mathbb{Z}$. For the first vector we have $(p^2 - 1)$ choices as we can't have $a = c = 0$. The number of choices of the second vector is $p^2 - p$ as we have to get rid of p choices for the scalar multiples that would remove the linear independence. $\square$

4.

Proof. Note that $75 = 3 * 5^2$. With G a group of order 75, we have $n_5 = 1$ as $n_5 \equiv 1 \ (mod\ 5)$ and divides 3. So we want nontrivial map $\mathbb{Z}/3\mathbb{Z} \to Aut(N)$, where N is the unique Sylow 5−subgroup. If $N \cong \mathbb{Z}/25\mathbb{Z}$ then we need a map $\mathbb{Z}/3\mathbb{Z} \to Aut(\mathbb{Z}/25\mathbb{Z}) \cong (\mathbb{Z}/25\mathbb{Z})^\times$. Note

$$|(\mathbb{Z}/25\mathbb{Z})^\times| = \varphi(25) = 20.$$

No nontrivial homomorphisms from $\mathbb{Z}/3\mathbb{Z}$ to group of order 20. If $n \cong \mathbb{Z}/5\mathbb{Z} \bigoplus \mathbb{Z}/5\mathbb{Z}$, $Aut(N) \cong GL_2(\mathbb{Z}/5\mathbb{Z})$ and $|GL_2(\mathbb{Z}/5\mathbb{Z})| = (25-1)(20) = 480$. We do have that $3|480$, so has element A of order 3. Get nontrivial $\varphi : \mathbb{Z}/3\mathbb{Z} \underset{[1}{\overset{\mapsto}{\longrightarrow}} A]GL_2(\mathbb{Z}/5\mathbb{Z})$ and thus $N \rtimes_\varphi \mathbb{Z}/3\mathbb{Z}$ is a nonabelian group of order 75. $\square$