Math 132H Homework 2

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April 15, 2024

24.

Proof. Assume we are given $[a,b] \subset \mathbb{R}$, an interval, $z : [a,b] \xrightarrow[t \mapsto z(t)]{} \mathbb{C}$. We also then have the parametrization $z^-(t) = z(b+a-t)$, which is the reverse orientation of $z(t)$, and we have the bijection $t(s) = (b+a-s)$. Therefore, we have that z and z^- are equivalent parametrizations. Hence, we have that

$$\begin{aligned}
\int_\gamma f(z)\mathrm{d}z &\overset{(a)}{=} \int_a^b f(z(t))z'(t)\mathrm{d}t \\
&\overset{(b)}{=} \int_b^a f(z^-(s)){z^-}'(s)\mathrm{d}s \\
&\overset{(c)}{=} -\int_a^b f(z^-(s)){z^-}'(s)\mathrm{d}s \\
&\overset{(d)}{=} -\int_{\gamma^-} f(z)\mathrm{d}z.
\end{aligned}$$

Steps $(a)-(d)$ are justified below:

(a) Equivalent definition of the integral over γ

(b) Using that $z(t(s))$ and $z^-(s)$ are equivalent parametrizations

(c) Swapping bounds of integration requires us to multiply by negative one

(d) Reverse of justification (a).

□

25.

a.

Proof. Let γ be any circle with counterclockwise orientation centered at the origin. Here we have that $z(t) = re^{it} + c$ for $|c| < r$. We begin by integrating when $n \neq 1$:

$$\int_\gamma z^n \mathrm{d}z = \int_0^{2\pi} z(t)^n z'(t)\mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^{2\pi} r^n e^{itn} i r e^{it} \mathrm{d}t \\
&= r^{n+1} i \int_0^{2\pi} e^{it(n+1)} \mathrm{d}t \\
&\overset{(a)}{=} r^{n+1} i \int_0^{2\pi} \left(\frac{1}{i(n+1)} e^{it(n+1)} \right)' \mathrm{d}t \\
&\overset{(b)}{=} r^{n+1} i \int_0^{2\pi} \frac{d}{dt} \left(-\frac{i}{n+1} e^{it(n+1)} \right) \mathrm{d}t \\
&\overset{(c)}{=} r^{n+1} i \left(-\frac{i}{n+1} e^{i2\pi(n+1)} + \frac{i}{n+1} e^{i0(n+1)} \right) \\
&= r^{n+1} i \left(-\frac{i}{n+1} (\cos(2\pi(n+1)) + i\sin(2\pi(n+1))) + \frac{i}{n+1} \cdot 1 \right) \\
&= r^{n+1} i \left(-\frac{i}{n+1} + \frac{i}{n+1} \right) \\
&= 0.
\end{aligned}
$$

The steps $(a) - (c)$ are justified:

(a) $\frac{1}{i(n+1)} e^{it(n+1)}$ is holomorphic on the set with derivative $e^{it(n+1)}$

(b) $\frac{1}{i} = \frac{i}{i^2} = -i$

(c) Fundamental theorem of calculus.

In the case where $n = -1$ the solution differs:

$$
\begin{aligned}
\int_\gamma z^{-1} \mathrm{d}z &= \int_0^{2\pi} z(t)^{-1} z'(t) \mathrm{d}t \\
&= \int_0^{2\pi} r^{-1} e^{-it} i r e^{it} \mathrm{d}t \\
&= i \int_0^{2\pi} e^{-it+it} \mathrm{d}t \\
&= i \int_0^{2\pi} \mathrm{d}t \\
&= i(2\pi - 0) \\
&= 2\pi i.
\end{aligned}
$$

□

b.

Proof. Focusing on when γ, a circle, does not contain the origin, therefore $z(t) = re^{it} + c$ for $|c| > r$ is a valid parametrization of γ. Note that $z'(t) = rie^{it}$. For $n \neq -1$, we have that $f(z) = z^n$ has a primitive, namely $\frac{1}{n+1} z^{n+1}$ on γ, and thus by a corollary in the book, since a circle γ is a closed curve, and we can enclose that closed curve inside a larger open disc, we have that, by Cauchy's theorem

$$
\int_\gamma z^n \mathrm{d}z = 0.
$$

□

c.

Proof. Suppose $|a| < r < |b|$, and γ is a circle centered around the origin with radius r and oriented in the positive direction. The integral

$$\begin{aligned}\int_\gamma \frac{1}{(z-a)(z-b)}\mathrm{d}z &\overset{(a)}{=} \int_\gamma \frac{1}{a-b}\left(\frac{1}{z-a} - \frac{1}{z-b}\right)\mathrm{d}z \\ &= \frac{1}{a-b}\left(\int_\gamma \frac{1}{z-a}\mathrm{d}z - \int_\gamma \frac{1}{z-b}\mathrm{d}z\right) \\ &= \frac{2\pi i}{a-b}\end{aligned}$$

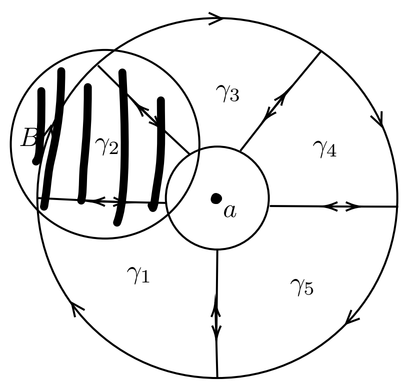
this is true as we can create the picture below and use Cauchy's theorem and the local existence of primitives. We also have that $|a| < r < |b|$, so the integral $\int_\gamma \frac{1}{z-a}\mathrm{d}z$ is inside γ and the integral $\int_\gamma \frac{1}{z-b}\mathrm{d}z$ is outside of γ, so we can apply the earlier parts of the problem to solve. We can use partial fraction decomposition to break apart the integral like so:

$$\frac{1}{(z-a)(z-b)} = \frac{A}{z-a} + \frac{B}{z-b}$$

$$\Rightarrow 1 = A(z-b) + B(z-a)$$

solving for A and B we get

$$\frac{1}{a-b} = A = B.$$



□

26.

Proof. Suppose $F(z)$ and $G(z)$ are primitives of $f(z)$. We then have that

$$0 = f(z) - f(z) = F'(z) - G'(z) = (F(z) - G(z))'$$

by the linearity of the derivative. □

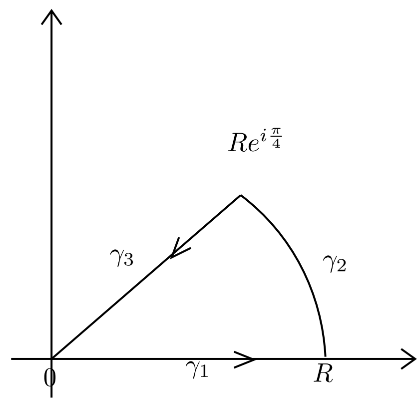
1.

a.

Proof. Take $f(z) = e^{-z^2}$. Using Cauchy's theorem, because if γ is the entire path in the figure, then it is closed and can be surrounded by an open disc, and since f is holomorphic, we have

$$0 = \int_\gamma f(z)\mathrm{d}z = \int_\gamma e^{-z^2}\mathrm{d}z.$$

Let γ_1, γ_2, γ_3 be curves shown in the figure below



We will integrate over $\gamma_1, \gamma_2, \gamma_3$ to solve for the desired integral while also using that $\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t$:

$$\begin{aligned}
0 &= \int_\gamma e^{-z^2}\mathrm{d}z \\
&= \int_{\gamma_1} e^{-z^2}\mathrm{d}z + \int_{\gamma_2} e^{-z^2} + \int_{\gamma_3} e^{-z^2}\mathrm{d}z \\
&= \int_0^R e^{-x^2}\mathrm{d}x + \int_0^{\frac{\pi}{4}} e^{-\left(Re^{i\theta}\right)^2} iRe^{i\theta}\mathrm{d}\theta - \int_0^R e^{-\left(xe^{i\frac{\pi}{4}}\right)^2} e^{i\frac{\pi}{4}}\mathrm{d}x.
\end{aligned}$$

Evaluating these integrals separately, let's begin with the easiest one as we take $R \to \infty$:

$$\begin{aligned}
\lim_{R\to\infty} \int_0^R e^{-x^2}\mathrm{d}x &= \int_0^\infty e^{-x^2}\mathrm{d}x \\
&\overset{(a)}{=} \frac{1}{2}\int_{-\infty}^\infty e^{-x^2}\mathrm{d}x \\
&\overset{(b)}{=} \frac{\sqrt{\pi}}{2}
\end{aligned}$$

with step $(a) - (b)$ justifed:

(a) Using that e^{-x^2} is an even function

(b) The hint provided in the book.

We will now view the integral

$$\int_0^{\frac{\pi}{4}} e^{-\left(Re^{i\theta}\right)^2} iRe^{i\theta}\mathrm{d}\theta$$

and observe the behavior of its norm as $R \to \infty$:

$$\begin{aligned}
\left|\int_0^{\frac{\pi}{4}} e^{-\left(Re^{i\theta}\right)^2} iRe^{i\theta}\mathrm{d}\theta\right| &= \left|iR\int_0^{\frac{\pi}{4}} e^{-\left(Re^{i\theta}\right)^2} Re^{i\theta}\mathrm{d}\theta\right| \\
&= R\left|\int_0^{\frac{\pi}{4}} e^{-\left(Re^{i\theta}\right)^2} Re^{i\theta}\mathrm{d}\theta\right| \\
&\overset{(a)}{\leq} R\int_0^{\frac{\pi}{4}} \left|e^{-\left(Re^{i\theta}\right)^2} Re^{i\theta}\right|\mathrm{d}\theta \\
&= R\int_0^{\frac{\pi}{4}} \left|e^{-R^2e^{i2\theta}} e^{i\theta}\right|\mathrm{d}\theta \\
&= R\int_0^{\frac{\pi}{4}} \left|e^{-R^2(\cos(2\theta)+i\sin(2\theta))} e^{i\theta}\right|\mathrm{d}\theta \\
&= R\int_0^{\frac{\pi}{4}} \left|e^{-R^2\cos(2\theta)}\right|\left|e^{-R^2 i\sin(2\theta)}\right|\left|e^{i\theta}\right|\mathrm{d}\theta \\
&\overset{(b)}{=} R\int_0^{\frac{\pi}{4}} \left|e^{-R^2\cos(2\theta)}\right|\mathrm{d}\theta \\
&\overset{(c)}{=} R\int_0^{\frac{\pi}{4}} e^{-R^2\cos(2\theta)}\mathrm{d}\theta \\
&\overset{(d)}{\leq} R\int_0^{\frac{\pi}{4}} e^{-R^2(1-\frac{2}{\pi}2\theta)}\mathrm{d}\theta \\
&= R\int_0^{\frac{\pi}{4}} e^{-R^2} e^{\frac{4R^2}{\pi}\theta}\mathrm{d}\theta \\
&= Re^{-R^2}\int_0^{\frac{\pi}{4}} e^{\frac{4R^2}{\pi}\theta}\mathrm{d}\theta \\
&\overset{(e)}{=} Re^{-R^2}\frac{\pi}{4R^2}\left[e^{\frac{4R^2}{\pi}\theta}\right]_0^{\frac{\pi}{4}} \\
&= Re^{-R^2}\frac{\pi}{4R^2}\left(e^{R^2}-1\right) \\
&= \frac{\pi}{4R} - \frac{e^{-R^2}\pi}{4R} \xrightarrow[R\to\infty]{} 0
\end{aligned}$$

with steps $(a) - (e)$ justfied:

(a) Integral triangle inequality

(b) Use of the fact that $\left|e^{it}\right| = 1$ for any $t \in \mathbb{R}$

(c) $e^{-R^2\cos(2\theta)}$ always is positive valued

(d) Inequality given in the hint for the homework that $\cos\theta \geq 1 - \frac{2}{\pi}\theta$ for $\theta \in \left[0, \frac{\pi}{2}\right]$

(e) Real valued integral, so we may use normal integration tools (u-substitution) to find this integral.

Now, for the final integral $\int_0^R e^{-\left(xe^{i\frac{\pi}{4}}\right)^2} e^{i\frac{\pi}{4}}\,\mathrm{d}x$ we must break it apart to see

$$\begin{aligned}
\int_0^R e^{-\left(xe^{i\frac{\pi}{4}}\right)^2} e^{i\frac{\pi}{4}}\,\mathrm{d}x &= \int_0^R e^{-x^2 e^{2i\frac{\pi}{4}}} e^{i\frac{\pi}{4}}\,\mathrm{d}x \\
&= \int_0^R e^{-x^2\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)} e^{i\frac{\pi}{4}}\,\mathrm{d}x \\
&= \int_0^R e^{-ix^2} e^{i\frac{\pi}{4}} \\
&= \int_0^R e^{-ix^2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)\mathrm{d}x \\
&= \int_0^R e^{-ix^2}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)\mathrm{d}x \\
&= \left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\int_0^R e^{-ix^2}\,\mathrm{d}x \\
&= \left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\left(\int_0^R \cos\left(x^2\right) - i\sin\left(x^2\right)\mathrm{d}x\right)
\end{aligned}$$

Piecing everything back together and taking the limit as $R \to \infty$, we have

$$\begin{aligned}
0 &= \lim_{R\to\infty}\int_\gamma e^{-z^2}\,\mathrm{d}z \\
&= \int_{\gamma_1} e^{-z^2}\,\mathrm{d}z + \int_{\gamma_2} e^{-z^2} + \int_{\gamma_3} e^{-z^2}\,\mathrm{d}z \\
&= \lim_{R\to\infty}\left(\int_0^R e^{-x^2}\,\mathrm{d}x + \int_0^{\frac{\pi}{4}} e^{-\left(Re^{i\theta}\right)^2} iRe^{i\theta}\,\mathrm{d}\theta - \int_0^R e^{-\left(xe^{i\frac{\pi}{4}}\right)^2} e^{i\frac{\pi}{4}}\,\mathrm{d}x\right) \\
&= \frac{\sqrt{\pi}}{2} - \left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\left(\int_0^\infty \cos\left(x^2\right) - i\sin\left(x^2\right)\mathrm{d}x\right)
\end{aligned}$$

We must now satisfy the equation

$$\frac{\sqrt{\pi}}{2} = \left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\left(\int_0^\infty \cos\left(x^2\right) - i\sin\left(x^2\right)\mathrm{d}x\right)$$

but by multiplying each side by the complex conjugate of $\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}$ we discern

$$\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)\frac{\sqrt{\pi}}{2} = \int_0^\infty \cos\left(x^2\right)\mathrm{d}x - i\int_0^\infty \sin\left(x^2\right)\mathrm{d}x.$$

Simplifying we get

$$\frac{\sqrt{2\pi}}{4} - i\frac{\sqrt{2\pi}}{4} = \int_0^\infty \cos\left(x^2\right)\mathrm{d}x - i\int_0^\infty \sin\left(x^2\right)\mathrm{d}x.$$

Take the real and imaginary parts to see the desired result:

$$\frac{\sqrt{2\pi}}{4} = \int_0^\infty \cos\left(x^2\right)\mathrm{d}x = \int_0^\infty \sin\left(x^2\right)\mathrm{d}x.$$

$\square$

2.

Proof. Here, we will use the indented semicircle described in lecture, the book, and pictured below:

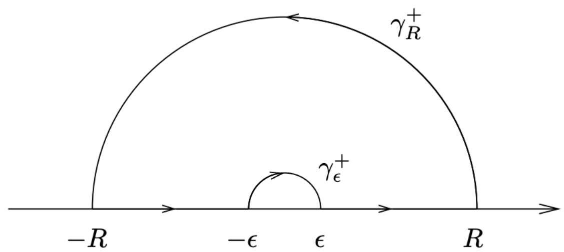


Figure 9. The indented semicircle of Example 2

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Take $f(z) = \frac{e^{iz}}{z}$ and we have that, by Cauchy's theorem,

$$0 = \int_\gamma \frac{e^{iz}}{z} = \int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\mathrm{d}x + \int_{\gamma_\epsilon^+} \frac{e^{iz}}{z}\mathrm{d}z + \int_\epsilon^R \frac{e^{ix}}{x}\mathrm{d}x + \int_{\gamma_R^+} \frac{e^{iz}}{z}\mathrm{d}z.$$

Observe

$$\begin{aligned}
\left|\int_{\gamma_R} \frac{e^{iz}}{z}\mathrm{d}z\right| &= \left|\int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} Rie^{i\theta}\mathrm{d}\theta\right| \\
&\overset{(a)}{\leq} \int_0^\pi \left|e^{iRe^{i\theta}}\right| \mathrm{d}\theta \\
&= \int_0^\pi \left|e^{i\cos(\theta)}\right|\left|e^{-R\sin(\theta)}\right| \mathrm{d}\theta \\
&= \int_0^\pi e^{-R\sin(\theta)}\mathrm{d}\theta \\
&\overset{(b)}{\leq} \int_0^\pi e^{-R\left(1-\frac{2}{\pi}\left|\theta-\frac{\pi}{2}\right|\right)}\mathrm{d}\theta \\
&= e^{-R}\int_0^\pi e^{\frac{2R}{\pi}\left|\theta-\frac{\pi}{2}\right|}\mathrm{d}\theta \\
&= e^{-R}\left(\int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}\left(\theta-\frac{\pi}{2}\right)}\mathrm{d}\theta + \int_{\frac{\pi}{2}}^\pi e^{\frac{2R}{\pi}\left(\theta-\frac{\pi}{2}\right)}\mathrm{d}\theta\right) \\
&= \int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}\theta}\mathrm{d}\theta + e^{-2R}\int_{\frac{\pi}{2}}^\pi e^{\frac{2R}{\pi}\theta}\mathrm{d}\theta \\
&\overset{(c)}{=} \left[-\frac{\pi e^{-\frac{2R}{\pi}\theta}}{2R}\right]_0^{\frac{\pi}{2}} + e^{-2R}\left[\frac{\pi e^{\frac{2R}{\pi}\theta}}{2R}\right]_{\frac{\pi}{2}}^\pi
\end{aligned}$$

$$= -\frac{\pi e^{-R}}{2R} + \frac{\pi}{2R} + \frac{\pi}{2R} - \frac{\pi e^{-R}}{2R} \xrightarrow[R\to\infty]{} 0$$

with steps $(a) - (c)$ justified as thus:

(a) Triangle inequality for integrals

(b) hint provided in homework that $\sin\theta \geq 1 - \frac{2}{\pi}\left|\theta - \frac{\pi}{2}\right|$ for $\theta \in \left[0, \frac{\pi}{2}\right]$

(c) real-valued integral so we can use known integral tricks to solve.

We will now tackle the integral $\int_{\gamma_\epsilon} \frac{e^{iz}}{z}\mathrm{d}z$. In class, we were given a lemma that states since f is continuous on an open disk around the indented semicircle with $0 \leq arg(z) \leq \pi$, and $\lim_{z\to 0} zf(z) = 1$, then we have that

$$\lim_{\epsilon\to 0^+} \int_{\gamma_\epsilon} f(z)\mathrm{d}z = i(0-\pi) = -i\pi$$

as the curve travels from $\theta = \pi$ to $\theta = 0$. Rearranging the equation, we get that

$$\int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\mathrm{d}x + \int_{\epsilon}^{R} \frac{e^{ix}}{x}\mathrm{d}x = -\int_{\gamma_\epsilon^+} \frac{e^{iz}}{z}\mathrm{d}z - \int_{\gamma_R^+} \frac{e^{iz}}{z}\mathrm{d}z.$$

Taking $\epsilon \to 0$ and $R \to \infty$ see

$$\lim_{\epsilon\to 0, R\to\infty} \left(\int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\mathrm{d}x + \int_{\epsilon}^{R} \frac{e^{ix}}{x}\mathrm{d}x\right) = \lim_{\epsilon\to 0, R\to\infty} \left(-\int_{\gamma_\epsilon^+} \frac{e^{iz}}{z}\mathrm{d}z - \int_{\gamma_R^+} \frac{e^{iz}}{z}\mathrm{d}z\right)$$

implies

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\mathrm{d}x = i\pi.$$

We now decompose to our desired integral

$$\begin{aligned} i\pi &= \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{\cos(x) + i\sin(x)}{x}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{\cos(x)}{x}\mathrm{d}x + i\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\mathrm{d}x. \end{aligned}$$

Looking at only the imaginary part, it can be seen that we must then have

$$\int_{-\infty}^{\infty} \frac{cos(x)}{x}\mathrm{d}x = 0$$

and thus, we have that

$$\pi = \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\mathrm{d}x$$

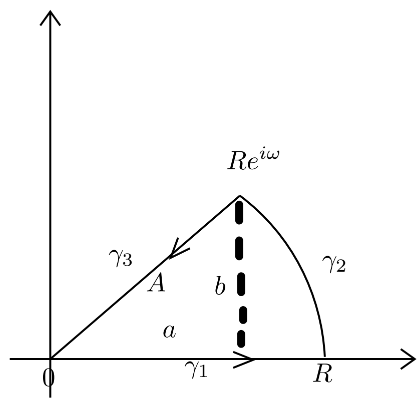
and using the fact that $\frac{\sin(x)}{x}$ is even we get the desired result

$$\frac{\pi}{2} = \int_{0}^{\infty} \frac{\sin(x)}{x}\mathrm{d}x$$

$\square$

3.

Proof. Let $A = \sqrt{a^2 + b^2}$, $\cos(\omega) = \frac{a}{A}$ which implies $\sin(\omega) = \frac{b}{A}$. Note, that we must take a sector where $a > 0$ always, so we must have that $\omega \in \left[0, \frac{\pi}{2}\right)$ making our sector look like the image below:



By Cauchy's theorem, we have that

$$\begin{aligned}
0 &= \int_\gamma e^{-Az}\mathrm{d}z \\
&= \int_{\gamma_1} e^{-Az}\mathrm{d}z + \int_{\gamma_2} e^{-Az}\mathrm{d}z + \int_{\gamma_3} e^{-Az}\mathrm{d}z \\
&\overset{(a)}{=} \int_0^R e^{-Ax}\mathrm{d}x + \int_0^\omega e^{-ARe^{i\theta}} iRe^{i\theta}\mathrm{d}\theta - \int_0^R e^{-Axe^{i\omega}}e^{i\omega}\mathrm{d}x
\end{aligned}$$

where step (a) is justfied using the fact that $\int_\gamma f(z)\mathrm{d}z = \int_a^b f(z(t))z'(t)\mathrm{d}t$. We begin by looking at the integral over γ_2:

$$\begin{aligned}
\left|\int_0^\omega e^{-ARe^{i\theta}} iRe^{i\theta}\mathrm{d}\theta\right| &\overset{(a)}{\le} \int_0^\omega \left|e^{-ARe^{i\theta}} iRe^{i\theta}\right|\mathrm{d}\theta \\
&= R\int_0^\omega e^{-AR\cos(\theta)}\mathrm{d}\theta \\
&\overset{(b)}{\le} R\int_0^\omega e^{-AR\left(1-\frac{2}{\pi}\theta\right)}\mathrm{d}\theta \\
&= Re^{-AR}\int_0^\omega e^{\frac{2}{\pi}AR\theta}\mathrm{d}\theta \\
&\overset{(c)}{=} Re^{-AR}\left[\frac{\pi}{2AR}e^{\frac{2}{\pi}AR\theta}\right]_0^\omega \\
&= \frac{\pi e^{AR\left(\frac{2}{\pi}\omega-1\right)}}{2A} - \frac{\pi e^{-AR}}{2A}
\end{aligned}$$

with steps $(a) - (c)$ justified:

(a) Triangle inequality for integrals

(b) Inequality given in the hint for the homework that $\cos\theta \ge 1 - \frac{2}{\pi}\theta$ for $\theta \in \left[0, \frac{\pi}{2}\right]$

(c) Real-valued integral can be solved with usual techniques.

Notice now that since $\omega \in \left[0, \frac{\pi}{2}\right)$ that we will always have that $AR\left(\frac{2}{\omega} - 1\right) < 0$ as $A, R > 0$. Therefore, we must have

$$\lim_{R\to\infty} \frac{\pi e^{AR\left(\frac{2}{\pi}\omega - 1\right)}}{2A} - \frac{\pi e^{-AR}}{2A} = 0.$$

Analyzing the integral over γ_2, we can quickly see that

$$\begin{aligned}\int_0^R e^{-Ax}\mathrm{d}x &= -\frac{1}{A}\left[e^{-Ax}\right]_0^R \\ &= -\frac{1}{A}\left(e^{-AR} - 1\right) \xrightarrow[R\to\infty]{} \frac{1}{A}.\end{aligned}$$

Lastly, we take a look at the integral over γ_3:

$$\begin{aligned}\int_0^R e^{-Axe^{i\omega}} e^{i\omega}\mathrm{d}x &= e^{i\omega}\int_0^R e^{-Ax(\cos(\omega)+i\sin(\omega))}\mathrm{d}x \\ &\overset{(a)}{=} e^{i\omega}\int_0^R e^{-Ax\frac{a}{A}} e^{-iAx\frac{b}{A}}\mathrm{d}x \\ &= (\cos(\omega) + i\sin(\omega))\int_0^R e^{-xa}(\cos(bx) - i\sin(bx))\mathrm{d}x \\ &\overset{(b)}{=} \left(\frac{a}{A} + i\frac{b}{A}\right)\left(\int_0^R e^{-xa}\cos(bx)\mathrm{d}x - i\int_0^R e^{-xa}\sin(bx)\mathrm{d}x\right).\end{aligned}$$

Steps $(a) - (b)$ are both justfied by that $\cos(\omega) = \frac{a}{A}$ and $\sin(\omega) = \frac{b}{A}$. Stitching everything back together and taking the limit of $R \to \infty$, we ascertain

$$0 = \frac{1}{A} - \left(\frac{a}{A} + i\frac{b}{A}\right)\left(\int_0^R e^{-xa}\cos(bx)\mathrm{d}x - i\int_0^R e^{-xa}\sin(bx)\mathrm{d}x\right)$$

which implies

$$\frac{1}{A} = \left(\frac{a}{A} + i\frac{b}{A}\right)\left(\int_0^R e^{-xa}\cos(bx)\mathrm{d}x - i\int_0^R e^{-xa}\sin(bx)\mathrm{d}x\right).$$

Multiplying each side by the complex conjugate of $\frac{a}{A} + i\frac{b}{A}$ we get

$$\left(\frac{a}{A} - i\frac{b}{A}\right)\frac{a}{A} = \int_0^R e^{-xa}\cos(bx)\mathrm{d}x - i\int_0^R e^{-xa}\sin(bx)\mathrm{d}x$$

and thus

$$\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2} = \int_0^R e^{-xa}\cos(bx)\mathrm{d}x - i\int_0^R e^{-xa}\sin(bx)\mathrm{d}x.$$

Looking at the real and imaginary parts, we see

$$\frac{a}{a^2+b^2} = \int_0^R e^{-xa}\cos(bx)\mathrm{d}x \text{ and } \frac{b}{a^2+b^2} = \int_0^R e^{-xa}\sin(bx)\mathrm{d}x.$$

$\square$

5.

Proof. Suppose f is continuously complex differentiable on Ω, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω. Denote F and G as the real and imaginary parts of f, respectively. Recall that for a complex-valued function to be continuously differentiable, both the real and imaginary parts of the function must be continuously differentiable. Let $z(t) : [a,b] \to T$ be a parametrization of the triangle with $z = x + iy$. Therefore,

$$\begin{aligned}
\int_T f(z)\mathrm{d}z &= \int_a^b f(z(t))z'(t)\mathrm{d}t \\
&= \int_a^b f(z(t))\left(\frac{\partial x}{\partial t} + i\frac{\partial y}{\partial t}\right)\mathrm{d}t \\
&= \int_a^b f(z(t))\left(\frac{\partial x}{\partial t} + i\frac{\partial y}{\partial t}\right)\mathrm{d}t \\
&= \int_a^b (F(z(t)) + iG(z(t)))\left(\frac{\partial x}{\partial t} + i\frac{\partial y}{\partial t}\right)\mathrm{d}t \\
&\overset{(a)}{=} \int_a^b F(z(t))\mathrm{d}x + iF(z(t))\mathrm{d}y + iG(z(t))\mathrm{d}x - G(z(t))\mathrm{d}y \\
&= \int_a^b F(z)\mathrm{d}x - G(z)\mathrm{d}y + i\int_a^b F(z)\mathrm{d}y + G(z)\mathrm{d}x \\
&= \int_T F\mathrm{d}x - G\mathrm{d}y + i\int_T F\mathrm{d}y + G\mathrm{d}x \\
&\overset{(b)}{=} \int_{\text{Interior of } T}\left(\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\right)\mathrm{d}x\mathrm{d}y + \int_{\text{Interior of } T}\left(\frac{\partial F}{\partial x} - \frac{\partial G}{\partial y}\right)\mathrm{d}x\mathrm{d}y
\end{aligned}$$

where step $(a) - (b)$ are justified:

(a) $\frac{\partial x}{\partial t}dt = dx$, $\frac{\partial y}{\partial t}dt = dy$

(b) Green's theorem.

By the Cauchy-Riemann equations, we have that

$$\frac{\partial F}{\partial y} = -\frac{\partial G}{\partial x}, \quad \frac{\partial F}{\partial x} = \frac{\partial G}{\partial y}$$

and thus

$$\int_{\text{Interior of } T}\left(\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y}\right)\mathrm{d}x\mathrm{d}y + \int_{\text{Interior of } T}\left(\frac{\partial F}{\partial x} - \frac{\partial G}{\partial y}\right)\mathrm{d}x\mathrm{d}y = 0.$$

□