Math 133 Homework 3

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1.

a.

Proof. Note that as x approaches $\pm\pi$, the function term $(\pi - x)^2$ and $(\pi + x)$ will become very large with the function being undefined at $x = \pm\pi$, but we define the function to be 0 there filling in the gaps. Note that the product of differentiable functions is differentiable, and thus the product of C^∞ functions is C^∞. Therefore, we must show that

$$e^{-\frac{1}{(\pi-x)^2}} \text{ and } e^{-\frac{1}{(\pi+x)^2}}$$

are C^∞. But, on the previous homework, we proved that

$$F(x)\begin{cases} 0, & x \le 0 \\ e^{-\frac{1}{x^2}}, & x > 0 \end{cases}$$

is C^∞ and our current function is simply a change of variables away from $F(x)$. Therefore,

$$e^{-\frac{1}{(\pi-x)^2}} \text{ and } e^{-\frac{1}{(\pi+x)^2}}$$

are C^∞ and so f is C^∞. To extend the function to $\mathbb{R}$ we want to shift the input of f into $[-\pi, \pi]$ from anywhere on $\mathbb{R}$ and we can do this with the function

$$f(x) = f\left(x - 2\pi\left\lfloor \frac{x}{2\pi} + \frac{1}{2}\right\rfloor\right).$$

□

b.

Proof. Integrating a_n with integration by parts, and taking $u = f(x)$ and $dv = e^{-inx}$ we have

$$\begin{aligned} a_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \\ &= \frac{1}{2\pi}\left[-f(x)\frac{e^{-inx}}{in}\right]_0^{2\pi} + \frac{1}{2\pi}\frac{1}{in}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx \end{aligned}$$

$$= \frac{1}{2\pi}\frac{1}{in}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx$$

as we have that

$$-e^{-in\pi} + e^{in\pi} = -\cos(-n\pi) - i\sin(-n\pi) + \cos(n\pi) + i\sin(n\pi) = -\cos(n\pi) + \cos(n\pi) = 0.$$

Repeating the process $k-$times we get that

$$a_n = \frac{1}{2\pi}\frac{1}{i^k n^k}\int_{-\pi}^{\pi} f^{(k)}(x)e^{-inx}dx.$$

Taking the modulus we have

$$|a_n| = \left|\frac{1}{2\pi}\frac{1}{i^k n^k}\int_{-\pi}^{\pi} f^{(k)}(x)e^{-inx}dx\right| \leq \frac{1}{2\pi}\frac{1}{n^k}\int_{-\pi}^{\pi} \left|f^{(k)}(x)e^{-inx}\right|dx$$

but, f is C^∞, so

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f^{(k)}(x)e^{-inx}dx\right| < \infty$$

say

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^{(k)}(x)e^{-inx}dx = C_k$$

for some $C_k \in \mathbb{R}$. Therefore, we have

$$|a_n| \leq \frac{1}{2\pi}\frac{1}{n^k}\int_{-\pi}^{\pi} \left|f^{(k)}(x)e^{-inx}\right|dx = C_k n^{-k}.$$

□

2.

Proof. Suppose $A_k = \{a_{k,n}\}_{n\in\mathbb{Z}}$ $k = 1, 2, \ldots$ is a Cauchy sequence. For all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\forall k, k' > N$

$$|a_{k,n} - a_{k',n}| \leq \left(\sum_{j=-\infty}^{\infty} |a_{k_j} - a_{k'_j}|^2\right)^{\frac{1}{2}} = ||A_k - A'_k||_{l^2(\mathbb{Z})} < \epsilon.$$

We know $(a_k)_{k=1}^\infty$ is a Cauchy sequence of complex numbers and it converges to a limit, denote this as b_n. Therefore $a_{k_j} \to b_j$ for some b_j as $k \to \infty$

$$b = (\ldots, b_{-1}, b_0, b_1, \ldots).$$

Since, $(a_k)_{k=1}^\infty$ is a Cauchy sequence

$$\sum_{n\in\mathbb{Z}} |a_{k,n} - a_{k',n}| < \epsilon$$

for sufficiently large $N \in \mathbb{N}$. Fix $\epsilon > 0$ and $M \in \mathbb{N}$, and

$$\sum_{j=-M}^{M} |b_j - a_{k_j}| = \sum_{j=-M}^{M} |\lim_{k'\to\infty}(a_{k'_j}) - a_{k_j}|^2 = \lim_{k'\to\infty}\sum_{j=-M}^{M} |a_{k'_j} - a_{k_j}|^2$$

$$\leq \limsup_{k'\to\infty} \sum_{j=-\infty}^{\infty} |a_{k'_j} - a_{k_j}|^2 \leq \limsup_{k'\to\infty} \epsilon = \epsilon.$$

If we take $n \to \infty$ we have that $a_k \to b$. Due to this convergence, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall k > N$

$$\begin{aligned}\left(\sum_{j=-N}^{N} |b_n|^2\right)^{\frac{1}{2}} &= \left(\sum_{j=-N}^{N} |b_n - a_{k,n} + a_{k,n}|^2\right)^{\frac{1}{2}} \\ &\overset{(a)}{=} \left(\sum_{j=-N}^{N} |b_n - a_{k,n}|^2\right)^{\frac{1}{2}} + \left(\sum_{j=-N}^{N} |a_{k,n}|^2\right)^{\frac{1}{2}} < \epsilon + M\end{aligned}$$

as A_k is in $l^2(\mathbb{Z})$, so $\left(\sum_{j=-N}^{N} |a_{k,n}|^2\right)^{\frac{1}{2}} < M$ and step (a) is justified by the triangle inequality. Thus, $b \in l^2(\mathbb{Z})$.

□

3.

Proof. Begin by noting that we have previously found that

$$\int_{-\pi}^{\pi} D_N(\theta) d\theta = \int_{-\pi}^{\pi} \frac{\sin\left(\left(N + \frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)} d\theta = 2\pi.$$

Note that the difference

$$\frac{1}{\sin\left(\frac{\theta}{2}\right)} - \frac{2}{\theta}$$

is continuous on $[-\pi, \pi]$. Look at the integral

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{\sin\left(\left(N + \frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)} d\theta - 2\int_{-\pi}^{\pi} \frac{\sin\left(\left(N + \frac{1}{2}\right)\theta\right)}{\theta} d\theta &= \int_{-\pi}^{\pi} \frac{\sin\left(\left(N + \frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)} - \frac{2\sin\left(\left(N + \frac{1}{2}\right)\theta\right)}{\theta} d\theta \\ &= \int_{-\pi}^{\pi} \sin\left(\left(N + \frac{1}{2}\right)\theta\right)\left(\frac{1}{\sin\left(\frac{\theta}{2}\right)} - \frac{2}{\theta}\right) d\theta\end{aligned}$$

and thus we can apply Riemann-Lebesgue lemma to state that

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \sin\left(\left(N + \frac{1}{2}\right)\theta\right)\left(\frac{1}{\sin\left(\frac{\theta}{2}\right)} - \frac{2}{\theta}\right) d\theta = 0.$$

Therefore,

$$\lim_{N\to\infty}\left(\int_{-\pi}^{\pi} \frac{\sin\left(\left(N + \frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)} d\theta - 2\int_{-\pi}^{\pi} \frac{\sin\left(\left(N + \frac{1}{2}\right)\theta\right)}{\theta} d\theta\right) = \lim_{N\to\infty}\left(2\pi - 2\int_{-\pi}^{\pi} \frac{\sin\left(\left(N + \frac{1}{2}\right)\theta\right)}{\theta} d\theta\right) = 0$$

and since $\frac{\sin\left(\left(N + \frac{1}{2}\right)\theta\right)}{\theta}$ is an even function we have

$$2\int_{-\pi}^{\pi} \frac{\sin\left(\left(N + \frac{1}{2}\right)\theta\right)}{\theta} d\theta = 4\int_{0}^{\pi} \frac{\sin\left(\left(N + \frac{1}{2}\right)\theta\right)}{\theta} d\theta$$

so now

$$\lim_{N\to\infty}\left(2\pi - 2\int_{-\pi}^{\pi}\frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{\theta}d\theta\right) = \lim_{N\to\infty}\left(2\pi - 4\int_{0}^{\pi}\frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{\theta}d\theta\right) = 0.$$

Therefore, we must have

$$\lim_{N\to\infty}\left(2\pi - 4\int_{0}^{\pi}\frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{\theta}d\theta\right) = 2\pi - 4\lim_{N\to\infty}\int_{0}^{\pi}\frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{\theta}d\theta = 0$$

which implies

$$\lim_{N\to\infty}\int_{0}^{\pi}\frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{\theta}d\theta = \frac{\pi}{2}.$$

Using the change of variables $x = \left(N+\frac{1}{2}\right)\theta$, we have that

$$\lim_{N\to\infty}\int_{0}^{\left(N+\frac{1}{2}\right)\pi}\frac{\sin(x)}{x}dx = \int_{0}^{\infty}\frac{\sin(x)}{x}dx = \frac{\pi}{2}.$$

□

4.

Proof. Begin with

$$\hat{f}(n) = \frac{1}{2\pi}\int_{0}^{2\pi} f(x)e^{-inx}dx$$

Taking the norm we get that

$$|\hat{f}(n)| = \left|\frac{1}{2\pi}\int_{0}^{2\pi} f(x)e^{-inx}dx\right| \le \frac{1}{2\pi}\int_{0}^{2\pi}\left|f(x)e^{-inx}\right|dx = \frac{1}{2\pi}\int_{0}^{2\pi}|f(x)|dx$$

as $\left|e^{-inx}\right| = 1$. Since $f(x) \in C^k$ it is also differentiable, implying $\hat{f}(n) \in C^1$. Using integration by parts with $dv = e^{-inx}$ and $u = f(x)$ we have that

$$\hat{f}(n) = \frac{1}{2\pi}\left[-f(x)\frac{e^{-inx}}{in}\right]_0^{2\pi} + \frac{1}{2\pi}\frac{1}{in}\int_{0}^{2\pi} f'(x)e^{-inx}dx.$$

Becuase $e^{-in2\pi} = \cos(2\pi n) - \sin(2\pi n) = 1 = e^0$ we have

$$\hat{f}(n) = \frac{1}{2\pi}\left[-f(x)\frac{e^{-inx}}{in}\right]_0^{2\pi} + \frac{1}{in}\int_{0}^{2\pi} f'(x)e^{-inx}dx = \frac{1}{2\pi}\frac{1}{in}\int_{0}^{2\pi} f'(x)e^{-inx}dx.$$

Perform this process k times and we will obtain

$$\hat{f}(n) = \frac{1}{2\pi}\frac{1}{i^k n^k}\int_{0}^{2\pi} f^{(k)}(x)e^{-inx}dx.$$

Then,

$$\begin{aligned}|\hat{f}(n)| &= \left|\frac{1}{2\pi}\frac{1}{i^k n^k}\int_0^{2\pi} f^{(k)}(x)e^{-inx}dx\right| \\ &= \frac{1}{2\pi}\frac{1}{n^k}\left|\int_0^{2\pi} f^{(k)}(x)e^{-inx}dx\right| \\ &\leq \frac{1}{2\pi}\frac{1}{n^k}\int_0^{2\pi}\left|f^{(k)}(x)e^{-inx}\right|dx \\ &= \frac{1}{2\pi}\frac{1}{n^k}\int_0^{2\pi}\left|f^{(k)}(x)\right|dx.\end{aligned}$$

With $f \in C^k$ we have that

$$\int_0^{2\pi}\left|f^{(k)}(x)\right|dx < \infty$$

and thus, as we send $n \to \infty$

$$\lim_{n\to\infty}\hat{f}(n) = \lim_{n\to\infty}\frac{1}{2\pi}\frac{1}{n^k}\int_0^{2\pi}\left|f^{(k)}(x)\right|dx = 0.$$

Thus, $\hat{f}(n)$ is $o\left(\frac{1}{n^k}\right)$. $\square$

5.

Suppose f is a $2\pi-$periodic function and satisfies the Lipschitz condition with constant K.

a.

Proof. To solve this problem we will use Parseval's identity that

$$\sum_{n=-\infty}^{\infty}|a_n|^2 = ||f||^2 = \frac{1}{2\pi}\int_0^{2\pi}|f(x)|^2dx.$$

Using the identity, we have the result

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}|g_h(x)|^2dx &\overset{(a)}{=} \sum_{n=-\infty}^{\infty}|\hat{g}_h(n)|^2 \\ &\overset{(b)}{=} \sum_{n=-\infty}^{\infty}\left|\frac{1}{2\pi}\int_0^{2\pi}(f(x+h)-f(x-h))e^{-inx}dx\right|^2 \\ &= \sum_{n=-\infty}^{\infty}\left|\frac{1}{2\pi}\int_0^{2\pi}f(x+h)e^{-inx}dx - \frac{1}{2\pi}\int_0^{2\pi}f(x-h)e^{-inx}dx\right|^2 \\ &\overset{(c)}{=} \sum_{n=-\infty}^{\infty}\left|\frac{1}{2\pi}e^{inh}\int_0^{2\pi}f(u)e^{-inu}du - \frac{1}{2\pi}e^{-inh}\int_0^{2\pi}f(v)e^{-inv}dv\right|^2 \\ &= \sum_{n=-\infty}^{\infty}\left|e^{inh}\hat{f}(n) - e^{-inh}\hat{f}(n)\right|^2\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty} \left|\hat{f}(n)\left(e^{inh} - e^{-inh}\right)\right|^2 \\
&\overset{(d)}{=} \sum_{n=-\infty}^{\infty} \left|\hat{f}2i\sin(nh)\right|^2 \\
&= \sum_{n=-\infty}^{\infty} 4|\sin(nh)|^2|\hat{f}(n)|^2
\end{aligned}
$$

with steps $(a) - (d)$ justified:

(a) Parseval's identity

(b) $g_h(x) = f(x+h) - f(x-h)$

(c) using change of variable $u = x + h$ in the first integral, and $v = x - h$ in the second integral

(d) using that cosine is even and sine is odd

$e^{inh} - e^{-inh} = \cos(nh) + i\sin(nh) - \cos(-nh) - i\sin(-nh) = \cos(nh) + i\sin(nh) - \cos(nh) + i\sin(nh) = 2i\sin(nh).$

Using our previous result we have that

$$\frac{1}{8\pi}\int_0^{2\pi} |g_n(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\sin(nh)|^2|\hat{f}(n)|^2$$

and therefore

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} |\sin(nh)|^2|\hat{f}(n)|^2 &= \frac{1}{8\pi}\int_0^{2\pi} |g_n(x)|^2 dx \\
&\overset{(a)}{=} \frac{1}{8\pi}\int_0^{2\pi} |f(x+h) - f(x-h)|^2 dx \\
&\overset{(b)}{\leq} \frac{1}{8\pi}\int_0^{2\pi} \left(K|x+h-(x-h)|\right)^2 dx \\
&= \frac{1}{8\pi}\int_0^{2\pi} 4K^2h^2 dx \\
&= K^2h^2
\end{aligned}
$$

with steps $(a) - (b)$ justified:

(a) $g_h(x) = f(x+h) - f(x-h)$

(b) Lipschitz continuity of f.

□

b.

Proof. By the previous part

$$\sum_{2^{p-1}<|n|<2^p} \left|\sin\left(n\frac{\pi}{2^{p+1}}\right)\right|^2 |\hat{f}(n)|^2 \leq \frac{K^2\pi^2}{2^{2p+2}}.$$

Since $2^{p-1} < |n| < 2^p$

$$\frac{1}{\sqrt{2}} = \sin\left(\frac{\pi}{4}\right) = \sin\left(2^{p-1}\frac{\pi}{2^{p+1}}\right) \leq \sin\left(n\frac{\pi}{2^{p+1}}\right) \leq \sin\left(2^p\frac{\pi}{2^{p+1}}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

Therefore,

$$\frac{1}{2}\sum_{2^{p-1}<|n|<2^p}|\hat{f}(n)|^2 = \sum_{2^{p-1}<|n|<2^p}\left|\frac{1}{\sqrt{2}}\right|^2|\hat{f}(n)|^2 = \sum_{2^{p-1}<|n|<2^p}\left|\sin\left(2^{p-1}\frac{\pi}{2^{p+1}}\right)\right|^2|\hat{f}(n)|^2 \leq \sum_{2^{p-1}<|n|<2^p}\left|\sin\left(n\frac{\pi}{2^{p+1}}\right)\right|^2|\hat{f}(n)|^2$$

and thus

$$\frac{1}{2}\sum_{2^{p-1}<|n|<2^p}|\hat{f}(n)|^2 \leq \frac{K^2\pi^2}{2^{2p+2}}$$

implying

$$\sum_{2^{p-1}<|n|<2^p}|\hat{f}(n)|^2 \leq \frac{K^2\pi^2}{2^{2p+1}}.$$

□

c

Proof. With the Cauchy-Schwarz inequality we have

$$\left(\sum_{2^{p-1}<|n|<2^p}|\hat{f}(n)|\right)^2 \leq \sum_{2^{p-1}<|n|<2^p}|\hat{f}(n)|^2\sum_{2^{p-1}<|n|<2^p}1 \leq 2^p\frac{K^2\pi^2}{2^{2p+1}} = \frac{K^2\pi^2}{} < \infty.$$

□

Now we have

$$\sum_{2^{p-1}<|n|<2^p}|\hat{f}(n)| < \infty$$

and so we may apply the abosolute convergence corollary (Corollary 2.3) to state that the Fourier series converges uniformly to f.