Math 110B Homework 3

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1.

a.

Proof. From Theorem 8.2 of the book we have that $a \equiv b \ (mod\ K) \iff Ka = Kb$ and we can find if $a \equiv b \ (mod\ K)$ by whether or not $ab^{-1} \in K$. For this question we ask if $[17][19]^{-1} \in \langle [9] \rangle$. After some brief calculations it can be found that $[19]^{-1} = [27]$. Then, $[17][27] = [11]$. Note that $\langle [9] \rangle = \{[9], [81], [729], [6,561]\} = \{[9], [17], [25], [1]\}$. So, we can conclude that $[17][27] \notin \langle [9] \rangle$ and thus $K[17] \neq K[19]$.

□

b.

Proof. We will solve this problem exactly how we solved the previous problem. Note that $[25]^{-1} = [9]$, so $[9][25]^{-1} = [9][9] = [81] = [17] \in \langle [9] \rangle$ and thus, $K[9] = K[25]$. □

2.

Proof. The order of the elements must divide the order of the group G, therefore the smallest possible order of G is

$$lcm(1,2,3,4,5,6,7,8,9,10,11,12) = 27,720.$$

□

3.

Proof. Suppose $H, K \leq G$ are subgroups of a finite group G such that $K \leq H$. Since the group is finite the number of left cosets $[G:H]$, $[H:K]$ is finite. By Lagrange's theorem

$$[G:K] = \frac{|G|}{|K|},\ |G| = |H|[G:H],\ |K| = \frac{|H|}{[H:K]}.$$

Thus,

$$[G:K] = \frac{|G|}{|K|} = \frac{|H|[G:H]}{\frac{|H|}{[H:K]}} = [G:H][H:K].$$

□

4.

a.

Proof. Suppose G is a nonabelian group of order 10. Note that for a finite group every element has an order, as for $a \in G$, there must be $n \neq m$ such that $a^n = a^m$ by the finiteness of G, and thus $a^{n-m} = e$. Also, the order of an element in the group must divide the order of a group, and thus the possible orders of elements in this group are 1, 2, 5, and 10. Since G is nonablian it implies G is not cyclic, as all cyclic groups are abelian, as if for $a \in G$ a generator $a^n = c$ and $a^m = b$ for $b, c \in G$ and $n \neq m$, then $bc = a^m a^n = a^n a^m = cb$. Thus, there can be no element of order 10 in the group. We only have one element of order 1, so we must deal with the case where all other elements in the group have order 2. If all but the identity element in a group have order 2, then for every $a \in G$ we have $a^2 = e$ and thus $a = a^{-1}$. Hence, for $a, b \in G$ we have $ab \in G$ by closedness of multiplication and for this group $(ab)^2 = e$ implies $ab = (ab)^{-1}$. Thus,

$$ab = (ab)^{-1} = b^{-1}a^{-1} = ba$$

implying that the group is abelian and thus a contradiction. Therefore, an element of order 5 must exist. □

b.

Proof. Every nonidentity element has order 2 or 5 and by the previous problem we know an element of order 5 exists. Let $a \in G$ such that $|a| = 5$. Look at the subgroup $\langle a \rangle = \{e, a, a^2, a^3, a^4\}$. Then, for $b \in G \backslash \langle a \rangle$,

$$\langle a \rangle e = \langle a \rangle \neq \langle a \rangle b = \{b, ab, a^2 b, a^3 b, a^4 b\}.$$

By the fact that two right cosets are either disjoint or identitcal, since $b \notin \langle a \rangle$ but, $b \in \langle a \rangle$, then $\langle a \rangle \cap \langle a \rangle b = \varnothing$. Notice that then $|\langle a \rangle \cup \langle a \rangle b| = |\langle a \rangle| + |\langle a \rangle b| = 5 + 5 = 10$ and each element in the cosets is in G. Thus, $\langle a \rangle \cup \langle a \rangle b = G$. Note that a, a^2, a^3, a^4 all have order 5. Suppose b has order 5. Since $|\langle a \rangle| = 5$ we have that by Lagrange's theorem $\frac{|G|}{|\langle a \rangle|} = [G : \langle a \rangle] = 2$ there are two cosets. One of the cosets is simply $\langle a \rangle$ implying

$$\langle a \rangle b^2 = \langle a \rangle b^3.$$

But then,

$$\langle a \rangle b^2 b^2 = \langle a \rangle b^3 b^2$$

and since $b^5 = e$ this implies

$$\langle a \rangle b^4 = \langle a \rangle.$$

Thus, this is saying that $b^4 \in \langle a \rangle$. If $b^4 \in \langle a \rangle$, since subgroups are closed under multiplication we have $b^4 b^4 = b^5 b^3 = e b^3 = b^3 \in \langle a \rangle$. Similary, we then have that $b^3 b^3 = b^5 b = eb = b \in \langle a \rangle$, but this is a contradiction as we chose $b \notin \langle a \rangle$. Thus, the only elements of order 5 are the 4 elements of order 5 in $\langle a \rangle$. □

5.

a.

Proof. Let $N, K \leq G$ be subgroups of G a group and N be normal in G. Let $a \in NK = \{nk : n \in N, k \in K\}$. Then, $a = nk$ for some $n \in N$ and some $k \in K$. Both N and K are subgroups implying $n^{-1} \in N$ and $k^{-1} \in K$. Then, $a^{-1} = (nk)^{-1} = k^{-1}n^{-1}$, but since N is a normal subgroup there exists $n' \in N$ such that $a^{-1} = k^{-1}n^{-1} = n'k^{-1} \in NK$. Hence, NK contains inverses. Now, take $a, b \in NK$. Then, $a = n_1k_1$ and $b = n_2k_2$ for $n_1, n_2 \in N$ and $k_1, k_2 \in K$. Then, $ab = n_1k_1n_2k_2$ but N is a normal group, so $\exists n_2' \in N$ such thta $k_1n_2 = n_2'k_1$ and so

$$ab = n_1k_1n_2k_2 = n_1n_2'k_1k_2$$

and since N and K are subgroups they are closed under multiplication, and thus $n_1n_2' \in N$ and $k_1k_2 \in K$ implying $ab = n_1n_2'k_1k_2 \in NK$. Thus, NK is a subgroup. □

b.

Proof. Now let N and K be normal. Let $g \in G$. Then, by the normality of N and K

$$gNKg^{-1} = Ngg^{-1}K = NK.$$

Thus, NK is a normal subgroup. □

6.

Proof. Let H be a subgroup of order n in a group G. Suppose H is the only subgroup of order n in G. Take $g \in G$. First we will show that gHg^{-1} is a subgroup. Note that for $h \in H$,

$$(ghg^{-1})(gh^{-1}g^{-1}) = ghg^{-1}gh^{-1}g^{-1} = e$$

so $(ghg^{-1})^{-1} = gh^{-1}g^{-1}$. Suppose $a \in gHg^{-1}$, then $a = ghg^{-1}$ for some $h \in H$. Therefore,

$$a^{-1} = (ghg^{-1})^{-1} = gh^{-1}g^{-1}$$

and we can clearly see that $gh^{-1}g^{-1} \in H$ as H is a subgroup, so $h^{-1} \in H$. Thus, gHg^{-1} is closed under inverses. Now take $a, b \in gHg^{-1}$ with $a = gh_1g^{-1}$ and $b = gh_2g^{-1}$ for $h_1, h_2 \in H$. Then,

$$ab = gh_1g^{-1}gh_2g^{-1} = gh_1h_2g^{-1}$$

and since H is a subgroup, we have $h_1h_2 \in H$ and thus $ab = gh_1h_2g^{-1} \in gHg^{-1}$. Thus, gHg^{-1} is a subgroup of G. Now, take $h \in H$. Then,

$$(ghg^{-1})^n = \underbrace{ghg^{-1}ghg^{-1}\cdots ghg^{-1}}_{n-times} = gh^ng^{-1}$$

and since H has order n we have that

$$gh^ng^{-1} = gg^{-1} = e$$

using Corollary 8.6 which states if $|G| = k$, then $a^k = e \;\forall a \in G$. Thus

$$|gHg^{-1}| = |H| = n.$$

Since H is the only group of order n we have that

$$gHg^{-1} = H$$

and by Theorem 8.11 this is one of the equivalent definitions for H being a normal subgroup. $\square$