Math 133 Homework 1

Thomas Slavonia

April 16, 2024

1.

Proof. Let $0 < r < 1$ and we look that the integral

$$\int_0^r \frac{1}{1+x^2} \mathrm{d}x.$$

Since $x^2 < 1$ always for $x \in [0,r]$ using power series expansion we have that

$$\int_0^r \frac{1}{1+x^2} \mathrm{d}x = \int_0^r \sum_{n=0}^{\infty} (-1)^n (x^2)^n \mathrm{d}x.$$

We can swap the integral and the sum if the series is uniformly convergent, and to find whether or not the series is uniformly convergent, we can use the Weierstrass M-test. So,

$$\sum_{n=0}^{\infty} \left|x^{2n}\right|$$

will converge as $x < 1$, and then the ratio test shows convergence clearly. Thus, the series passes the Weierstrass M-test and is uniformly convergent. Hence,

$$\int_0^r \sum_{n=0}^{\infty} (-1)^n x^{2n} \mathrm{d}x = \sum_{n=0}^{\infty} \int_0^r (-1)^n x^{2n} \mathrm{d}x = \sum_{n=0}^{\infty} (-1)^n \frac{r^{2n+1}}{2n+1}.$$

Consider the statement of Abels Theorem: if sup $\sum\limits_{n=0}^{\infty} a_n r^n$ is absolutely convergent and $\sum\limits_{n=0}^{\infty} a_n$ is convergent, then

$$\lim_{r \to 1^-} \sum_{n=0}^{\infty} a_n r^n = \sum_{n=0}^{\infty} a_n \text{ exists.}$$

Proof of Abel's:
Define $S_N = \sum\limits_{j=0}^{N} a_j$ as the partial sum. Then,

$$\begin{aligned}\sum_{k=0}^{n} a_k r^k &= \sum_{k=0}^{n} (S_k - S_{k-1}) r^k \\ &= \sum_{k=0}^{n} S_k r^k - \sum_{k=0}^{n} S_{k-1} r^k\end{aligned}$$

$$= \sum_{k=0}^{n-1} S_k r^k - \sum_{k=0}^{n-1} S_k r^{k+1} + S_n r^n$$
$$= (1-r) \sum_{k=0}^{n-1} S_k r^k + S_n r^n.$$

as $n \to \infty$,

$$\lim_{n\to\infty} \left((1-r) \sum_{k=0}^{n-1} S_k r^k + S_n r^n \right) = \sum_{n=0}^{\infty} a_k r^k$$

converges and $\lim_{n\to\infty} S_n r^n = 0$ as $r < 1$ and since S_n is bounded we must have the convergence. Thus, for $|r| < 1$ the partial sums must converge, so

$$\sum_{k=0}^{\infty} a_k r^k = (1-r) \sum_{k=0}^{\infty} S_k r^k.$$

Since S_k converges, $\forall \epsilon > 0 \ \exists N \in \mathbb{N}$ such that

$$\left| (1-r) \sum_{k=N}^{\infty} S_k r^k \right| \le \epsilon (1-r) \sum_{k=N}^{\infty} r^n = \epsilon (1-r) \frac{1}{1-r} = \epsilon$$

so we have controlled the latter part of the sum. Note that

$$(1-r) \sum_{k=0}^{N-1} S_k r^k \to 0 \text{ as } r \to 1^-$$

and thus

$$\lim_{r\to 1^-} \sum_{k=0}^{\infty} a_k r^k = \sum_{k=0}^{\infty} a_k = \lim_{r\to 1^-} (1-r) \sum_{k=0}^{\infty} S_k r^k < 2\epsilon.$$

So, Abel's theorem is true, and

$$\sum_{n=0}^{\infty} (-1)^n \frac{r^{2n+1}}{2n+1}$$

converges by the alternating series test and meets all of the qualifications for Abel's theorem. Thus, $\lim_{r\to 1^-}$ exists and

$$\lim_{r\to 1^-} \sum_{n=0}^{\infty} (-1)^n \frac{r^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots = \tan^{-1}(1) = \frac{\pi}{4}.$$

$\square$

2.

For all these problems $f \in [-\pi, \pi]$.

(1)

Proof. Let $f(x) = x$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\mathrm{d}x = 0$$

as $f(x)$ is an odd function. For $n \neq 0$

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}\mathrm{d}x.$$

Pick $u = x$, $du = 1$, $dv = e^{-inx}$, and $v = -\frac{1}{in}e^{inx} = -\frac{i}{i^2 n}e^{-inx} = \frac{i}{n}e^{-inx}$. Now,

$$\begin{aligned}
a_n &= \frac{1}{2\pi}\left(\left[x\cdot\frac{i}{n}e^{-inx}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{i}{n}e^{-inx}\mathrm{d}x\right)\\
&= \frac{1}{2\pi}\left(\frac{\pi i}{n}e^{-in\pi} + \frac{\pi i}{n}e^{in\pi}\right) - \frac{i}{2\pi n}\left[-\frac{1}{in}e^{-inx}\right]_{-\pi}^{\pi}\\
&= \frac{i}{2n}\left(\cos(-n\pi) + i\sin(-n\pi) + \cos(n\pi) + i\sin(n\pi)\right) + \frac{1}{2\pi n^2}\left(e^{-in\pi} - e^{in\pi}\right)\\
&\overset{(a)}{=} \frac{i}{2n}(2\cos(n\pi)) + \frac{1}{2\pi n^2}\left(\cos(-n\pi) + i\sin(-n\pi) - \cos(n\pi) - i\sin(n\pi)\right)\\
&= \frac{i}{n}(-1)^n
\end{aligned}$$

and for step (a) and the next step, we use the fact that cos is an even function and sin is an odd function (this will be used implicitly in many later calculations). Therefore, we check for the summation between terms of n and $-n$:

$$\begin{aligned}
a_n e^{inx} + a_{-n}e^{-inx} &= \frac{i}{n}(-1)^n e^{inx} - \frac{i}{n}(-1)^n e^{-inx}\\
&= \frac{i}{n}(-1)^n\left(e^{inx} - e^{-inx}\right)\\
&= \frac{i}{n}(-1)^n(2i\sin(x)) = -\frac{2}{n}(-1)^n\sin(nx) = \frac{2}{n}(-1)^{n+1}\sin(nx).
\end{aligned}$$

Thus,

$$\begin{aligned}
f(x) &\sim \sum_{n=-\infty}^{-1} a_n e^{inx} + \sum_{n=1}^{\infty} a_n e^{inx}\\
&= \sum_{n=1}^{\infty} a_n e^{inx} + a_{-n}e^{-inx}\\
&= \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin(nx).
\end{aligned}$$

□

(2)

Proof. Let $f(x) = x^2$. The first coefficient is

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\mathrm{d}x = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{\pi^2}{2}$$

For $n \neq 0$ pick $u = x^2$, $du = 2x$, $dv = e^{-inx}$, and $v = \frac{i}{n}e^{-inx}$:

$$\begin{aligned}
a_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\mathrm{d}x \\
&= \frac{1}{2\pi}\left[x^2 \cdot \frac{i}{n}e^{-inx}\right]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} 2x\frac{i}{n}e^{-inx}\mathrm{d}x \\
&= \frac{1}{2\pi}\left(\frac{\pi^2 i}{n}e^{-in\pi} - \frac{\pi^2 i}{n}e^{in\pi}\right) - \frac{i}{n\pi}\int_{-\pi}^{\pi} xe^{-inx}\mathrm{d}x \\
&= \frac{\pi i}{2n}\left(e^{-in\pi} - e^{in\pi}\right) - \frac{i}{n\pi}\left(2\pi\frac{i}{n}(-1)^n\right) \\
&= 0 + \frac{2}{n^2}(-1)^n \\
&= \frac{2}{n^2}(-1)^n
\end{aligned}$$

where we use the fact that we already solved for the integral $\int_{-\pi}^{\pi} xe^{-inx}\mathrm{d}x$. Therefore,

$$\begin{aligned}
a_n e^{inx} + a_{-n}e^{-inx} &= \frac{2}{n^2}(-1)^n e^{inx} + \frac{2}{n^2}(-1)^n e^{-inx} \\
&= \frac{2}{n^2}(-1)^n\left(e^{inx} + e^{-inx}\right) \\
&= \frac{4}{n^2}(-1)^n\cos(nx)
\end{aligned}$$

and thus

$$f(x) \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos(nx).$$

□

(3)

Proof. Let $f(x) = x^3$. The first coefficient is

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3\mathrm{d}x = 0$$

as x^3 is an odd function. For $n \neq 0$:

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^3 e^{-inx}\mathrm{d}x$$

pick $u = x^3$, $du = 3x^2$, $dv = e^{-inx}$, and $v = \frac{i}{n}e^{-inx}$ and we get

$$\begin{aligned}
a_n &= \frac{1}{2\pi}\left(\left[x^3\frac{i}{n}e^{-inx}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 3x^2\frac{i}{n}e^{-inx}\mathrm{d}x\right) \\
&= \frac{1}{2\pi}\left(\frac{\pi^3 i}{n}e^{-in\pi} + \frac{\pi^3 i}{n}e^{in\pi}\right) - \frac{3i}{2\pi n}\int_{-\pi}^{\pi} x^2 e^{-inx}\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\pi^2 i}{2n} 2 \cdot (-1)^n - \frac{6i}{n^3}(-1)^n \\
&= \frac{\pi^2 i n^2 - 6i}{n^3}(-1)^n.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
a_n e^{inx} + a_{-n} e^{-inx} &= \frac{\pi^2 i n^2 - 6i}{n^3}(-1)^n e^{inx} - \frac{\pi^2 i n^2 - 6i}{n^3}(-1)^n e^{-inx} \\
&= \frac{12 - 2\pi^2 n^2}{n^3}(-1)^n \sin(nx).
\end{aligned}
$$

Thus,

$$
f(x) \sim \sum_{n=1}^{\infty} \frac{12 - 2\pi^2 n^2}{n^3}(-1)^n \sin(nx).
$$

□

3.

(1)

Proof. Let $f(x) = x$. Applying the given formula, we get the result

$$
\begin{aligned}
\|f(x)\|^2 = \|x\|^2 \\
&= 2\pi \cdot 0^2 + \pi \sum \left(\frac{2}{n}(-1)^{n+1} \right)^2 \\
&= \pi \sum \frac{4}{n^2}(-1)^{2n+2} \\
&= 4\pi \frac{1}{n^2} \\
&= 4\pi\zeta(2).
\end{aligned}
$$

We have that

$$
\|x\|^2 = \int_{-\pi}^{\pi} x^2 \mathrm{d}x = \frac{2\pi^3}{3}.
$$

Solving for $\zeta(2)$ we get

$$
\begin{aligned}
\zeta(2) &= \frac{2\pi^3}{2} \cdot \frac{1}{4\pi} \\
&= \frac{2\pi^3}{12} \\
&= \frac{\pi^2}{6}.
\end{aligned}
$$

□

(2)

Proof. Let $f(x) = x^2$. Applying the given formula, we get the result

$$\begin{aligned}
\|f(x)\|^2 &= \|x^2\|^2 \\
&= 2\pi\left(\frac{\pi^2}{3}\right)^2 + \pi\sum\left(\frac{4}{n^2}(-1)^n\right)^2 \\
&= \frac{2\pi^5}{9} + \pi\sum\frac{16}{n^4} \\
&= \frac{2\pi^5}{9} + 16\pi\sum\frac{1}{n^4} \\
&= \frac{2\pi^5}{9} + 16\pi\zeta(4).
\end{aligned}$$

Solving for $\zeta(4)$ we get

$$\zeta(4) = \frac{\|f(x)\|^2}{16\pi} - \frac{\pi^4}{72}.$$

Now, we can solve

$$\|x^2\|^2 = \int_{-\pi}^{\pi} x^4 \mathrm{d}x = \frac{2\pi^5}{5}$$

and we can see that $\zeta(4)$ is

$$\zeta(4) = \frac{2\pi^5}{516\pi} - \frac{\pi^4}{72} = \frac{\pi^4}{90}.$$

$\square$

(3)

Proof. Let $f(x) = x^3$, then

$$\begin{aligned}
\|f(x)\|^3 &= \|x^3\|^2 \\
&= 2\pi\cdot 0^2 + \pi\sum\left(\frac{12-2\pi^2n^2}{n^3}(-1)^n\right)^2 \\
&= \pi\sum\frac{144-48\pi^2n^2+4\pi^4n^4}{n^6}
\end{aligned}$$

and solving for $\zeta(6)$ we get

$$\frac{\|f(x)\|^2}{144\pi} + \frac{\pi^2}{3}\zeta(4) - \frac{\pi^4}{36}\zeta(2) = \zeta(6).$$

We have that

$$\|f(x)\|^2 = \|x^3\|^2 = \int_{-\pi}^{\pi} x^6 \mathrm{d}x = \frac{2\pi^7}{7}.$$

and thus

$$\zeta(6) = \frac{\pi^6}{504} + \frac{\pi^2}{3}\cdot\frac{\pi^4}{90} - \frac{\pi^4}{36}\cdot\frac{\pi^2}{6}$$

$$= \frac{\pi^6}{945}.$$

□

4.

Proof. Let $f(x) = x^4$ for $f \in [-\pi, \pi]$. The first coefficient

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^4 \mathrm{d}x = \frac{2\pi^5}{10\pi} = \frac{\pi^4}{5}.$$

For $n \neq 0$

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^4 e^{-inx} \mathrm{d}x$$

pick $u = x^4$, $du = 4x^3$, $dv = e^{-inx}$, and $v = \frac{i}{n} e^{-inx}$, then by integration by parts

$$\begin{aligned} a_n &= \frac{1}{2\pi} \left(\left[x^4 \cdot \frac{i}{n} e^{-inx} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 4x^3 \cdot \frac{i}{n} e^{-inx} \mathrm{d}x \right) \\ &= \frac{1}{2\pi} \left(\pi^4 \cdot \frac{i}{n} e^{-in\pi} - \pi^4 \cdot \frac{i}{n} e^{in\pi} \right) - \frac{4i}{2\pi n} \int_{-\pi}^{\pi} x^3 e^{-inx} \mathrm{d}x \\ &= \frac{\pi^3 i}{2n} \left(e^{-in\pi} - e^{in\pi} \right) - \frac{4i}{2\pi n} \left(\frac{2\pi(\pi^2 in^2 - 6i)}{n^3} (-1)^n \right) \\ &= \frac{24 - 4\pi^2 n^2}{n^4} (-1)^{n+1}. \end{aligned}$$

Then,

$$\begin{aligned} a_n e^{inx} + a_{-n} e^{-inx} &= \frac{24 - 4\pi^2 n^2}{n^4} (-1)^{n+1} \left(e^{inx} + e^{-inx} \right) \\ &= \frac{48 - 8\pi^2 n^2}{n^4} (-1)^{n+1} \cos(nx). \end{aligned}$$

Thus, we get the result

$$f(x) \sim \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{48 - 8\pi^2 n^2}{n^4} (-1)^{n+1} \cos(nx).$$

□

5.

Proof. For $f(x) = x^4$, if we set $x = 0$ we get

$$f(0) = 0 = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{48 - 8\pi^2 n^2}{n^4} (-1)^{n+1}$$

and rearranging, we get

$$-\frac{\pi^4}{5} = \sum_{n=1}^{\infty}\left(\frac{48}{n^4} - \frac{8\pi^2 n^2}{n^4}\right)(-1)^{n+1}$$

$$\Rightarrow -\frac{\pi^4}{5} = 48\sum_{n=1}^{\infty}\frac{1}{n^4}(-1)^{n+1} - \sum_{n=1}^{\infty}\frac{8\pi^2 n^2}{n^4}(-1)^{n+1}.$$

If we set $x = \pi$, then

$$f(\pi) = \pi^4 = \frac{\pi^4}{5} + \sum_{n=1}^{\infty}\frac{48 - 8\pi^2 n^2}{n^4}(-1)^{n+1}(-1)^{n+1}$$

$$\Rightarrow \frac{4\pi^4}{5} = \sum_{n=1}^{\infty}\frac{48 - 8\pi^2 n^2}{n^4} = 48\zeta(4) - 8\pi^2\zeta(2)$$

and now we have a formula to solve for $\zeta(4)$ or $\zeta(2)$ knowing one or the other. □

6.

Proof. For the space $C([-\pi,\pi])$ of continuous functions on $[-\pi,\pi]$, to show it is not complete look at

$$f_n(x) = \begin{cases} 0, & 0 \le |x| \le \frac{1}{n} \\ f(x), & \frac{1}{n} < |x| \le \pi \end{cases}$$

where

$$f(x) = \begin{cases} 0, & x = 0 \\ \log\left(\left|\frac{1}{x}\right|\right), & 0 < |x| \le \pi \end{cases} .$$

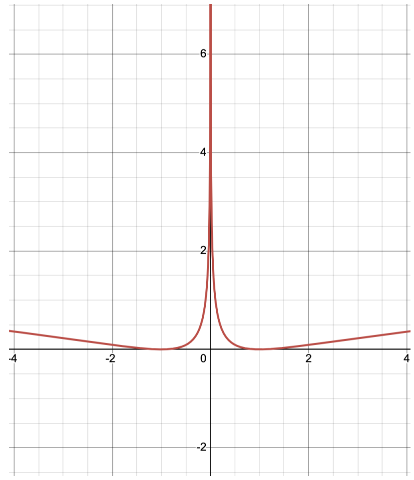
$f(x)$ is unbounded as x approaches 0 it will be huge, and we will have a discontinuity at 0. But we can always build a Cauchy sequence, as $\forall\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n, m \ge N$ without loss of generality assume $m > n$. If $|x| \le \frac{1}{m}$, then $f_n(x) = f_m(x) = 0$, so $\|f_n(x) - f_m(x)\|_2 = 0$. If $|x| > \frac{1}{n}$, then $f_n(x) = f_m(x) = \log\left(\left|\frac{1}{x}\right|\right)$, so $\|f_n(x) - f_m(x)\|_2 = 0$. So, we only care about the scenario when $\frac{1}{m} < |x| \le \frac{1}{n}$. In this scenario we have $f_n(x) = 0$ and $f_m(x) = \log\left(\left|\frac{1}{x}\right|\right)$. Then, the integral is

$$\int_{-\pi}^{\pi}\left(-\log\left(\left|\frac{1}{x}\right|\right)\right)^2 = \int_{-\frac{1}{m}}^{\frac{1}{m}}\log\left(\left|\frac{1}{x}\right|\right)^2 \mathrm{d}x.$$

The function is an even function, so our integral is equal to

$$2\int_0^{\frac{1}{m}}\log\left(\frac{1}{x}\right)^2 \mathrm{d}x$$

but as $m \to \infty$, the integral goes to 0 as the sliver you integrate of the function becomes smaller. Thus, the sequence is Cauchy.



∎

7.

Proof. Use the same function as the previous question. Once again, we have a discontinuity near 0, and again, we can have a Cauchy sequence. The same as before, the only scenario of concern is when the scenario when $\frac{1}{m} < |x| \leq \frac{1}{n}$ and as $m \to \infty$ the integral will go to 0, so we have that the sequence is Cauchy. ∎

8.

Proof. We will prove the claim by induction on k. We have already completed the base case where $k = 1$ as

$$\zeta(2) = \frac{\pi^2}{6} = rational * \pi^2.$$

For the inductive hypothesis, assume the claim holds for $2k$, and we will prove the $2(k+1) = 2k+2$ case. We begin by calculating the first Fourier coefficient

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^{2k+2} \cos(nx) \mathrm{d}x$$

to solve the integration by parts pick $u = x^{2k+2}$, $du = (2k+2)x^{2k+1}$, $dv = \cos(nx)$ and $v = \frac{1}{n}\sin(nx)$, and we get

$$\begin{aligned} a_n &= \left[x^{2k+2} \cdot \frac{1}{n}\sin(nx)\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} (2k+2)x^{2k+1} \frac{\sin(nx)}{n} \mathrm{d}x \\ &= -\frac{2k+2}{n\pi} \int_{-\pi}^{\pi} x^{2k+1} \sin(nx) \mathrm{d}x. \end{aligned}$$

Once again we can apply integration by parts by picking $u = x^{2k+1}$, $du = (2k+1)x^{2k}$, $dv = \sin(nx)$, and $v = -\frac{1}{n}\cos(nx)$, we get

$$\begin{aligned} a_n &= -\frac{2k+2}{n\pi}\left(\left[-\frac{1}{n}x^{2k+1}\cos(nx)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} -\frac{1}{n}\cos(nx)(2k+1)x^{2k}\right) \\ &= -\frac{2k+2}{n\pi}\left(-\frac{1}{n}\pi^{2k+1}(-1)^n - \frac{1}{n}\pi^{2k+1}(-1)^n\right) - \frac{(2k+2)(2k+1)}{n^2\pi}\int_{-\pi}^{\pi} x^{2k}\cos(nx)\mathrm{d}x. \end{aligned}$$

By our inductive hypothesis we have that $\zeta(2k)$ is a rational times π^{2k}. We can express the coefficients of the Fourier series of x^{2k+2} as rational numbers time primes subtracting the coefficients of x^{2k} and the first coefficient of x^{2k+2} has the term π^{2k+2}. Thus, we have that $\zeta(2k+2)$ will be a rational times π^{2k+2}. □

9.

Proof. We have already solved for $\zeta(6)$ in problem 3:

$$\zeta(6) = \frac{\pi^6}{945}.$$

□