Math 110B Homework 4

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1.

Proof. Let $Nx \in G/N$ for $x \in G$. Then, since N is a normal subgroup and $x^2 \in N$ we have

$$NxNx = NxxN = Nx^2N = NN = N.$$

Thus, implying all elements of G/N have order 2. □

2.

Proof. We look at $T \leq G$ the subgroup of elements of G of finite order. The elements of G/T are Tx where $x \in G$. Note, since the group is abelian, we have that $xT = Tx$. If x has finite order, then $x \in T$, so clearly $Tx = T$ has finite order but is also the idenity. Now, suppose x has infinite order. Suppose for sake of contradiction Tx has finite order n. Then,

$$\underbrace{Tx \cdot Tx \cdots Tx}_{n-\text{times}} = T$$

but since the group is abelian so is G/T by theorem 8.13 in the book

$$\underbrace{Tx \cdot Tx \cdots Tx}_{n-\text{times}} == Tx^n T \cdots T = T$$

implying x has finite order, but this is a contradiction as we assumed x to have infinite order. Therefore, every nontrivial element of G/T has infinite order. □

3.

Proof. Suppose N is a normal subgroup of G such that N and G/N are finitely generated. Since N is normal G/N is a subgroup and is finitely generated, so $G/N = \langle Ng_1, Ng_2, \ldots, Ng_n \rangle$ for n finite. The, we have

$$G = \cap_{i=1}^{n} Ng_i$$

but, N is finitely generated, and so it must be that G is finitely generated. □

4.

a.

Proof. Suppose $a \in K$. Therefore,

$$\left(a^{-1}\right)^2 = a^{-2} = \left(a^2\right)^{-1} = e$$

so K is closed under inverses. Take $a, b \in K$. Then, we have that, since the group is abelian

$$(ab)^2 = abab = aabb = a^2b^2 = ee = e.$$

Thus, K is closed under multiplication and inverses which is sufficient for it to be a subgroup. □

b.

Proof. Suppose $a \in H$. We then have that $a = x_1^2$ for some $x_1 \in G$. Therefore,

$$a^{-1} = \left(x_1^2\right)^{-1} = x_1^{-2} = (x_1^{-1})^2$$

and so $a^{-1} = \left(x^{-1}\right)^2$ implying $a^{-1} \in H$. Now, look at $a, b \in H$. Hence $a = x_1^2$ and $b = x_2^2$ for some $x_1, x_2 \in G$. Therefore, using that G is abelian

$$ab = x_1^2x_2^2 = x_1x_1x_2x_2 = x_1x_2x_1x_2 = (x_1x_2)^2$$

and thus $ab \in G$ by closure of groups under multiplication and H is a subgroup of G. □

c.

Proof. Let $f : G \to H$ be a map where $x \mapsto x^2$ for $x \in G$. First to show this is a homomorphism look at $a, b \in G$, and since G is abelian

$$f(ab) = (ab)^2 = abab = aabb = a^2b^2 = f(a)f(b).$$

Now, since f is a homorphism, clearly $H = \{x^2 : x \in H\}$ is the image but the kernel is where $f(x) = e$ and this is precisely when $f(x) = x^2 = e$, thus these are elements of order 2 and the identity element. The kernel is thus exacly K. Then, by the first isomorphism theorem we have that

$$G/K \cong H.$$

□

5.

Proof. Suppose $f : G \to H$ is a homomorphism of finite groups. Then, we claim the image $\{f(g) : g \in G\}$ is a subgroup of H. Suppose $a \in Im(f)$, then there exists $x \in G$ such that $f(x) = a$, and since $f(x^{-1}) = f(x)^{-1}$ for a homomorphism, then $a^{-1} = f(x)^{-1} = f(x^{-1})$ and since $x^{-1} \in G$,

$f(x^{-1}) = a^{-1} \in Im(f)$. Next, if $a, b \in Im(f)$ we have that there exists $x_1, x_2 \in G$ such that $f(x_1) = a$ and $f(x_2) = b$, so since f is a homomorphism

$$ab = f(x_1)f(x_2) = f(x_1x_2)$$

and $x_1x_2 \in G$, so $f(x_1x_2) = ab \in Im(f)$. Therefore, $Im(f) \leq H$ is a subgroup of H and by Lagrange's theorem $|Im(f)|$ divides $|H|$. By the First Isomorphism theorem

$$G/ker(f) \cong Im(f)$$

and we know that by Lagrange

$$\frac{|G|}{|ker(f)|} = [G : ker(f)]$$

but $[G : ker(f)]$ is the number of right cosets of $ker(f)$ which is exactly the order of $G/ker(f)$ which is isomorphic to $Im(f)$. Thus,

$$|G| = [G : ker(f)]|ker(f)| = |G/ker(f)||ker(f)| = |Im(f)||ker(f)|$$

implying $|Im(f)|$ divides $|G|$. k □

6.

a.

Proof. To prove that N is a normal subgroup take $x \in NK$ and $n \in N$. Because $x \in NK$, $x = n_1k_1$ and $x^{-1} = (n_1k_1)^{-1}$. Because N is normal in G, $n_1k_1 = k_1n_2$ for some $n_2 \in N$. Then, using the normality of N in G we have $k_1N = Nk_1$, so

$$xN == n_1k_1N = k_1n_2N = k_1N = Nk_1 = Nn_1k_1 = Nx.$$

□

b.

Proof. We begin by showing f is a homomorphism, so take $k_1k_2 \in K$. Note that N is normal in NK, so $Nk_1 = k_1N$. Therfore,

$$f(k_1k_2) = Nk_1k_2 = NNk_1k_2 = Nk_1Nk_2 = f(k_1)f(k_2).$$

To show the homomorphism is surjective, suppose we have $Na \in NK/N$ for $a = n_1k_1 \in NK$ and $n_1 \in N$, $k_1 \in K$. Then,

$$Na = Nn_1k_1 = Nk_1$$

so $\exists k_1 \in K$ such that

$$f(k_1) = Nk_1 = Nn_1k_1 = Na$$

and thus the homomorphism is surjective. If $f(k) = N$, then $f(k) = Nk = N$ which implies $k \in N$. Thus, the kernel must be $N \cap K$, as $k \in K$ must also be in N for $f(k) = N$. □

c.

Proof. We have met all the criteria to apply the First Isomorphism theorem and state

$$K/(N \cap K) \cong NK/N.$$

□