Math 121 Homework 7

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2.11.3.

The proof is very similar to the proof that every vector space has a basis. Let S be the family of all linearly independent subsets of V that contain A, and with the usual set inclusion, S becomes partially ordered. Let τ be a totally ordered subset of S. Define $B = \cup\{T : T \in \tau\}$. Let $v_1, \ldots, v_m \in B$ and let $a_1, \ldots, a_m \in F$ the field satisfy $a_1v_1 + \cdots + a_mv_m = 0$. For $1 \le j \le m$ take $T_j \in \tau$ such that $v_j \in T_j$. Since T_j's are totally ordered, and finite $\exists T_l$ such that $T_j \subseteq T_l$ for $1 \le j \le m$ which implies $v_j \ni T_l$. Since T_l is linearly independent $a_1 = \cdots = a_m = 0$. We can gather that B is an upper bound for S, and a maximal element C of S exists using Zorn's Lemma. By definition of S, we have $A \subseteq C$. To show C is a basis for the vector space, note that it is linearly independent. If $v \in C$, then v is in the span of C. Hence, we will now consider the case where $v \notin C$. Since C is the maximal linearly independent set that contains A, $C \cup \{v\}$ is not linearly independent. Thus, $\exists v_1, \ldots, v_n$ in $C \cup \{v\}$ and scalars $a_1, \ldots, a_n \in F$ such that $a_1v_1 + \cdots + a_nv_n = 0$ and there exists some $a_k \ne 0$ for $1 \le k \le n$. We may assume each $a_k \ne 0$. Since C is linearly independent, one of the v_k must be v. Without loss of generality, suppose $v_1 = v$. Then

$$v = \left(-\frac{a_2}{a_1}v_2\right) + \cdots + \left(-\frac{a_n}{a_1}v_n\right)$$

expresses v as the linear combination of of elements in C which implies that C is a basis that contains A.

$\square$

2.12.3.

Look at the restriction π_β to $X_\beta \times \{y\}$ $\pi_\beta : X_\beta \times \{y\} \to X_\beta$. We first want to show that π_β is an open map. Let U be open. We have that $\pi_\beta(U \times \varnothing) = \pi_\beta(\pi_\beta^{-1}(U_\beta)) = U$ which is open or we have $\pi_\beta(U \times \{y\}) = U$ which is open. Therefore, we have that π_β is an open map. Because we know that $\pi_\beta^{-1}(U) = U$ or $\pi_\beta^{-1}(U) = U \times \{y\}$ both of which are open, we know that π_β is continuous as the preimage of an open set is open. For $(x_\beta, y), (x'_\beta, y) \in X_\beta \times \{y\}$ suppose

$\pi_\beta(x_\beta, y) = \pi_\beta(x'_\beta, y)$. Then, we would have that $x_\beta = x'_\beta$ and hence it must be that $(x_\beta, y) = (x'_\beta, y)$ which implies surjectivity. To see surjectivity is quite trivial, as for any $x_\beta \in X_\beta$ we will clearly be able to find a $(x_\beta, y) \in X_\beta \times \{y\}$ such that $\pi_\beta(x_\beta, y) = x_\beta$. Lastly, $\pi_\beta^{-1} : X_\beta \to X_\beta \times \{y\}$ is clearly continuous as the preimage of any open U in X_β is still just U and will be open in $X_\beta \times \{y\}$. Hence, we have a homeomorphism.

□

2.12.4.

Suppose X_α is Hausdorff for all $\alpha \in A$ an index set. Let $X = \prod_{\alpha \in A} X_\alpha$. Let $x, y \in X$ such that $x \neq y$. Therefore, $\exists \beta \in A$ such that $x_\beta \neq y_\beta$. Since X_β is Hausdorff, $\exists U, V \in X_\beta$ open and disjoint sets such that $x_\beta \in U$ and $y_\beta \in V$. Consequently, we get $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are open and disjoint sets of the product space with $x \in \pi_\beta^{-1}(U)$ and $y \in \pi_\beta^{-1}(V)$. We conclude that X is Hausdorff.

□

2.12.7.

Let $E \subseteq X$ be connected. Then, $\pi_\alpha(E) = E_\alpha$ is connected and $E_\alpha \subseteq X_\alpha$ by the fact that if a topological space is connected, then its image under a continuous function is connected. We must have that $E_\alpha \subseteq F_\alpha \subseteq X$ where F_α is the connected component of X_α that contains E_α. Therefore, $E \subseteq \prod F_\alpha \subseteq X$ and thus the connected components are of the form $\prod F_\alpha$ where each F_α is a connected component of X_α.

□

2.12.8.

Let X_α be path connected for all $\alpha \in A$. Let $x, y \in X = \prod X_\alpha$. We know that for all $\alpha \in A$ $\exists \gamma_\alpha : [0,1] \to X_\alpha$ continuous such that $\gamma_\alpha(0) = x_\alpha$ and $\gamma_\alpha(1) = y_\alpha$. Let $\gamma : [0,1] \to X$ be continuous. Because γ is continuous, we know have previously shown that implies $\pi_\alpha \circ \gamma$ is continuous because both are continuous functions. If we set $(\gamma(t))_\alpha$ to $\gamma_\alpha(t)$ we can then conclude that $\gamma(0) = x$ and $\gamma(1) = y$.

□

2.12.9

Let S_α be a nonempty set with $\alpha \in A$ an index set. Let X_α be obtained from S_α by adjoining one point p_α. Let X_α be endowed with the cofinite topology including $\varnothing$ and $\{p_\alpha\}$. We have previously proven that any topological space with the cofinite topology is compact. Thus, X_α is compact. By Tychonoff's Theorem, $\prod X_\alpha$ is compact. Now, look at the subsets $\pi_\alpha^{-1}(S_\alpha) \subseteq \prod X_\alpha$. Note that each S_α is closed, as $X \backslash S_\alpha = \{p_\alpha\}$ which is open. Since π_α^{-1} is continuous, we know that $\pi_\alpha^{-1}(S_\alpha)$ is closed in $\prod X_\alpha$. Then, since each S_α is nonempty, $\bigcup\limits_{i=1}^{n} \left(\pi_\alpha^{-1}(S_\alpha)\right)_i \neq \varnothing$ as $\prod X_\alpha$ is compact. Thus, $\exists x \in \bigcup\limits_{i=1}^{n} \left(\pi_\alpha^{-1}(S_\alpha)\right)_i$ and each component of that element must be one element of S_α, $x_\alpha \in S_\alpha$. This is a rephrasing of the Axiom of Choice. $\square$

2.12.11.

a.

Let $x \in \prod X_\alpha$. Then, $x_\alpha \in U_\alpha$ some open subset of X_α. Hence, $x \in \prod U_\alpha$ an open subset of β. Hence, every $x \in \prod X_\alpha$ is in some element of β. For $1 \leq i \leq n$ look at a finite subset of the products $\{(\prod U_\alpha)_i\}_i \subseteq \beta$ for $1 \leq i \leq n$. Then, the intersection $\bigcap\limits_{i=1}^{n} (\prod U_\alpha)_i$ is the product of $\bigcap\limits_{i=1}^{n} U_{\alpha_i}$. The finite intersection of open sets here is of the same form as the original open sets, so we have that $\bigcap\limits_{i=1}^{n} (\prod U_\alpha)_i \in \beta$. Thus, β is closed under intersection, and we have met an equivalent definition for β being a base.

$\square$

b.

Suppose X_α has the discrete topology for all $\alpha \in A$. Then every subset $U_\alpha \subseteq X_\alpha$ is open. Thus, every $U = \prod\limits_{\alpha \in A} U_\alpha$ is open.

$\square$

c.

Let X_α have the discrete topology and consist of two points $\{0, 1\}$. Therefore, by the previous part of this problem, $\prod X_\alpha$ is discrete. Let A be infinite. Then, for every open cover, there will be no finite subcover.

$\square$

d.

Suppose X_α is Hausdorff for all $\alpha \in A$. Let $x, y \in \prod X_\alpha$ such that $x \neq y$. If $x \neq y$, then $\exists \beta \in A$ such that $x_\beta \neq y_\beta$. Since X_β is Hausdorff $\exists U_\beta, V_\beta \subseteq X_\beta$ open and disjoint such that $x_\beta \in U_\beta$ and $y_\beta \in V_\beta$. Therefore, $x \in \prod U_\alpha$ and $y \in \prod V_\alpha$ which are disjoint, so $\prod X_\alpha$ is Hausdorff.
Let each X_α be regular. Let $E \subset \prod X_\alpha$ be closed and $x \in \prod X_\alpha \backslash E$. Then, for E_α and x_α $\exists U_\alpha, V_\alpha \subseteq X_\alpha$ open and disjoint such that $E_\alpha \subset U_\alpha$ and $x_\alpha \subset V_\alpha$. Thus, $E = \prod E_\alpha \subset \prod U_\alpha$ open and $x \subset \prod V_\alpha$ with $\prod U_\alpha \cap \prod V_\alpha = \varnothing$ open. Thus, we have that it is regular.

It is not true that it is normal as the example for the half-open interval topology on $\mathbb{R}$ that $\mathbb{R} \times \mathbb{R}$ is not normal, but $\mathbb{R}$ is.

$\square$