Math 110B Homework 5

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1.

Proof. Note, that in cycle notation

$$\sigma = (138)(27)(4965).$$

The least common multiple of the lengths of these disjoint cycles is 12. Therefore, since the order of a permutation is the least common of the lengths of disjoint of the disjoint cycles, we have $|\sigma| = 12$. Since the cycles are disjoint

$$\sigma^1 000 = (138)^{1000}(27)^{1000}(4965)^{1000}$$

and since $2|1000$ and $4|1000$

$$(27)^{1000} = e, \text{ and } (4965)^{1000} = e$$

and since

$$(138)^{1000} = (138)^{999}(138)$$

and $3|999$ we have

$$(138)^{999} = e.$$

Thus,

$$\sigma = (138).$$

This is a 3−cycle, so σ^{1000} has order 3. □

2.

Proof. For k odd, suppose our k−cycle is $\sigma = (a_1, \cdots, a_k)$. Let

$$\tau = (a_1 a_{\frac{k-1}{2}+2} a_2 a_{\frac{k-1}{2}+3} \cdots a_{\frac{k-1}{2}} a_k a_{\frac{k-1}{2}+1})$$

and see that

$$\tau^2 = (a_1 a_{\frac{k-1}{2}+2} a_2 a_{\frac{k-1}{2}+3} \cdots a_{\frac{k-1}{2}} a_k a_{\frac{k-1}{2}+1})(a_1 a_{\frac{k-1}{2}+2} a_2 a_{\frac{k-1}{2}+3} \cdots a_{\frac{k-1}{2}} a_k a_{\frac{k-1}{2}+1}) = (a_1 a_2 \cdots a_k)$$

and so the claim is proven. □

3.

Proof. Let $\tau = (a_1 a_2 \cdots a_k)$ and $\sigma \in S_n$. Take $t \in [n]$, then since these are functions compositions we read right to left, so first we perform $\sigma^{-1}(t)$. If $\sigma^{-1}(t) = a_i$ for some $i \in [k]$, then $\tau(\sigma^{-1}(t)) = a_{i+1}$, otherwise $\tau(\sigma^{-1}(t)) = \sigma^{-1}(t)$. Then, if $\tau(\sigma^{-1}(t)) = a_{i+1}$, then $\sigma(\tau\sigma^{-1}(t)) = \sigma(a_{i+1})$, or $\sigma(\tau(\sigma^{-1}(t))) = \sigma(\sigma^{-1}(t)) = t$ and since this is the scenario where $t \notin (a_1 a_2 \cdots a_k)$ this is the result we want. If we say that $\sigma(a_k)$ is mapped to $\sigma(a_1)$ then we are done. $\square$

4.

Proof. Let $\sigma \in S_n$. We know that σ is the product of transpositions, so we only need to show that all transpositions can be generated. Let (i,j) be a transposition in S_n. Then, $(i,j) = (1,i)(1,j)(1,i)$. The transpositions $(1,i),(1,j),(1,i)$ are all of the desired form and thus we are done. $\square$

5.

Proof. We will begin by proving N is a subgroup. $(1)^{-1} = (1) \in N$, $((12)(34))^{-1} = (12)^{-1}(34)^{-1} = (12)(34) \in N$, and similarly $((13)(24))^{-1} = (13)(24) \in N$, $((14)(23))^{-1} = (14)(23) \in N$. Hence, every element in N has an inverse. The identity element multiplied with an other element in N is clearly also in N by observation. We now will test all of the other elements in N:

$$(12)(34)(12)(34) = (1),\ \ (12)(34)(13)(24) = (14)(23)\ \ (12)(34)(14)(23) = (13)(24)$$

$$(13)(24)(13)(24) = (1)\ \ (13)(24)(12)(34) = (14)(23)\ \ (13)(24)(14)(23) = (12)(34)$$

$$(14)(23)(14)(23) = (1)\ \ (14)(23)(12)(34) = (13)(24)\ \ (14)(23)(13)(24) = (12)(34).$$

Thus, N is closed under multiplication and is thus a subgroup. Now, we have actually done a lot of the heavy lifting showing the group is normal calculating all of the products, so

$$(1)N = \{(1)(1),(1)(12)(34),(1)(13)(24),(1)(14)(23)\} = \{(1)(1),(12)(34)(1),(13)(24)(1),(14)(23)(1)\} = N(1)$$

$$\begin{aligned}(12)(34)N &= \{(12)(34)(1),(12)(34)(12)(34),(12)(34)(13)(24),(12)(34)(14)(23)\} = \{(12)(34),(1),(14)(23),(13)(24)\} \\ &= \{(1)(12)(34),(12)(34)(12)(34),(13)(24)(12)(34),(14)(23)(12)(34)\} = N(12)(34) \\ (13)(24)N &= \{(13)(24)(1),(13)(24)(12)(34),(13)(24)(13)(24),(13)(24)(14)(23)\} = \{(13)(24),(14)(23),(1),(12)(34)\} \\ &= \{(1)(13)(24),(12)(34)(13)(24),(13)(24)(13)(24),(14)(23)(13)(24)\} = N(13)(24) \\ (14)(23)N &= \{(14)(23)(1),(14)(23)(12)(34),(14)(23)(13)(24),(14)(23)(14)(23)\} = \{(14)(23),(13)(24),(12)(34),(1)\} \\ &= \{(1)(14)(23),(12)(34)(14)(23),(13)(24)(14)(23),(14)(23)(14)(23)\} = N(14)(23).\end{aligned}$$

Thus the subgroup is normal so there is a nontrivial normal subgroup of A_4 and thus A_4 is not simple. $\square$