A NOTE ON CONSERVATIVITY CLASSES IN LINEAR LOGICS

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The conservative result was presented in a small seminar in a educational program for doctoral and master students. This small informal note is published just in case somebody could be interested in it. This note can be modified without notice.

1. Preliminaries

Definition 1.1 (conservativity)**.** *Let $\mathcal{Q}$ and $\mathcal{K}$ be classes of formulas. Then $\mathcal{Q}$ and $\mathcal{K}$ have the conservativity of logic* S *over logic* S′ *if and only if* $\vdash_{\mathrm{S}} \Gamma \Rightarrow A$ *implies* $\vdash_{\mathrm{S}'} \Gamma \Rightarrow A$ *for any* $\Gamma \subseteq \mathcal{Q}$ *and any* $A \in \mathcal{K}$*.*

We use Gentzen systems CLL and ILZ [1] as a classical and an intuitionistic linear logic system, respectively. See the appendix. We follow Troelstra's notations, also.

Notation 1.2. *From now on, we identify* $\sim A :\equiv A \multimap \mathbf{0}$ *in* CLL *and* ILZ*.*

Theorem 1.3 ([1, theorem 3.14.])**.** CLL *and* ILZ *enjoy cut elimination.*

Definition 1.4 (Positive and negative context [1, Definition 3.9.])**.** *We define positive context ($\mathcal{P}$) and negative context ($\mathcal{N}$) by simultaneous induction as follows:*

- $\mathcal{P} = * \mid A \sqcap \mathcal{P} \mid \mathcal{P} \sqcap A \mid A \sqcup \mathcal{P} \mid \mathcal{P} \sqcup A \mid A \star \mathcal{P} \mid \mathcal{P} \star A \mid A + \mathcal{P} \mid \mathcal{P} + A \mid A \multimap \mathcal{P} \mid \mathcal{N} \multimap A \mid \forall x\mathcal{P} \mid \exists x\mathcal{P}, |!\mathcal{P} \mid ?\mathcal{P};$
- $\mathcal{N} = A \sqcap \mathcal{N} \mid \mathcal{N} \sqcap A \mid A \sqcup \mathcal{N} \mid \mathcal{N} \sqcup A \mid A \star \mathcal{N} \mid \mathcal{N} \star A \mid A + \mathcal{N} \mid \mathcal{N} + A \mid A \multimap \mathcal{N} \mid \mathcal{P} \multimap A \mid \forall x\mathcal{N} \mid \exists x\mathcal{N} \mid !\mathcal{N} \mid ?\mathcal{N}$

where $*$ and Astand for a placeholder and an arbitrary formula, respectively.

2. Conservative results

Definition 2.1 (Leivant (1985) [2, Definition 3.17.])**.** *We define simultaneously classes $\mathcal{S}_i$ (i-spreading), $\mathcal{W}_i$ (i-wiping) and $\mathcal{I}_i$ (i-isolating) of formulas as follows:*

- $\perp, P, \rho, S \wedge S', S \vee S', \forall xS, \exists xS, I \to S \in \mathcal{S}_i,$
- $\perp, \rho, W \wedge W', \forall xW, S \to W \in \mathcal{W}_i,$
- $\perp, P, W, I \wedge I', I \vee I', \exists xI, S \to I \in \mathcal{I}_i.$

where P is any atomic, $S, S' \in \mathcal{S}_i$, $W, W' \in \mathcal{W}_i$, $I, I' \in \mathcal{I}_i$, and ρ varies over placeholder symbols.

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Theorem 2.2. *If $\Gamma \subseteq \mathcal{S}_i$ and $A \in \mathcal{W}_i$, then $\vdash_{\mathrm{C}} \Gamma \Rightarrow A$ implies $\vdash_{\mathrm{I}} \Gamma \Rightarrow A$.*

The above theorem is established in formula schemata, languages for predicate logic plus placeholder symbols. On the other hand, we will present a conservative result in simple predicate logic. Extending our result using schemata is future work. Leivant's result can be shown using the Gödel-Gentzen-negative translation. Here, we introduce the following context-sensitive translation. Translations $^+$ and $^-$ are designed to be applied to formulas occurring in positive and negative contexts, respectively.

Definition 2.3 (± translation)**.** *Translations $^+$ and $^-$ are defined inductively by*

- $P^{\pm} :\equiv \sim\sim P$ *for P atomic;*
- $\bot^+ :\equiv \sim\sim \bot$, $\bot^- \equiv \bot$;
- $\top^+ :\equiv \top$, $\top^- \equiv \sim\sim \top$;
- $\mathbf{0}^{\pm} :\equiv \mathbf{0}$;
- $\mathbf{1}^{\pm} :\equiv \sim\sim \mathbf{1}$;
- $(A \sqcap B)^{\pm} :\equiv A^{\pm} \sqcap B^{\pm}$;
- $(A \star B)^+ :\equiv \sim\sim (A^+ \star B^+)$, $(A \star B)^- :\equiv A^- \star B^-$;
- $(A \sqcup B)^{\pm} :\equiv \sim (\sim A^{\pm} \sqcap \sim B^{\pm})$;
- $(A + B)^{\pm} :\equiv \sim (\sim A^{\pm} \star \sim B^{\pm})$;
- $(A \multimap B)^+ :\equiv A^- \multimap B^+$, $(A \multimap B)^- :\equiv A^+ \multimap B^-$;
- $(\forall x A)^{\pm} :\equiv \forall x A^{\pm}$;
- $(\exists x A)^{\pm} :\equiv \sim \exists x \sim A^{\pm}$;
- $(!A)^+ :\equiv \sim\sim !A^+$, $(!A)^- :\equiv !A^-$;
- $(?A)^{\pm} :\equiv \sim ! \sim A^{\pm}$.

Lemma 2.4. $\vdash_{\mathrm{ILZ}} \Rightarrow \sim\sim A^+ \multimap A^+$.

Proof. By induction on the complexity of A. □

Theorem 2.5 (soundness of the translation)**.** $\vdash_{\mathrm{CLL}} \Gamma \Rightarrow A$ *implies* $\vdash_{\mathrm{ILZ}} \Gamma^- \Rightarrow A^+$.

Proof. By induction on the height of the proof of $\vdash_{\mathrm{CLL}} \Gamma \Rightarrow A$. □

Definition 2.6. *We define simultaneously classes $\mathcal{S}_1$, $\mathcal{S}_2$ (il-spreading), $\mathcal{W}$ (il-wiping) and $\mathcal{I}$ (il-isolating) of formulas as follows:*

- $\bot, \top, \mathbf{0}, \mathbf{1}, P, S_1 \sqcap S_1', S_1 \star S_1', S_1 \sqcup S_1', S_1 + S_1', \forall x S_1, \exists x S_1, !S_1, ?S_1 \in \mathcal{S}_1$,
- $\top, \mathbf{0}, \mathbf{1}, P, S_2 \sqcap S_2', S_2 \star S_2', S_2 \sqcup S_2', S_2 + S_2', \forall x S_2, \exists x S_2, ?S_2, I \multimap S_2 \in \mathcal{S}_2$,
- $\top, \mathbf{0}, W \sqcap W', \forall x W, !W, S_1 \multimap W, S_2 \multimap W \in \mathcal{W}$,
- $\bot, \top, \mathbf{0}, \mathbf{1}, P, W, I \sqcap I', I \sqcup I', \exists x I, !I \in \mathcal{I}$.

where P is any atomic, $S_1, S_1' \in \mathcal{S}_1$, $S_2, S_2' \in \mathcal{S}_2$, $W, W' \in \mathcal{W}$, and $I, I' \in \mathcal{I}$.

Lemma 2.7. *If $A \in S_2$, then* $\vdash_{\mathrm{ILZ}} \Rightarrow \sim\sim A^- \multimap A^-$.

Proof. Trivial. □

Proposition 2.8. *The following properties hold:*

- *If* $A \in \mathcal{S}_1$*, then* $\vdash_{\mathrm{ILZ}} \Rightarrow A \multimap A^-$.
- *If* $A \in \mathcal{S}_2$*, then* $\vdash_{\mathrm{ILZ}} \Rightarrow A \multimap A^-$.
- *If* $A \in \mathcal{W}$*, then* $\vdash_{\mathrm{ILZ}} \Rightarrow A^+ \multimap A$.
- *If* $A \in \mathcal{I}$*, then* $\vdash_{\mathrm{ILZ}} A^+ \multimap \sim\sim A$.

Proof. By simultaneous induction on $\mathcal{S}_1, \mathcal{S}_2, \mathcal{W}$ and $\mathcal{I}$. □

Corollary 2.9. *If* $\Gamma \subseteq \mathcal{S}_1 \cup \mathcal{S}_2$ *and* $A \in \mathcal{W}$*, then* $\vdash_{\mathrm{CLL}} \Gamma \Rightarrow A$ *implies* $\vdash_{\mathrm{ILZ}} \Gamma \Rightarrow A$.

References

1. A. S. Troelstra. *Lectures on Linear Logic*. CSLI, 1991.
2. Anne S. Troelstra and Dirk van Dalen. *Constructivism in Mathematics*, volume 1. Elsevier, 1988.

3. Acknowledgements

The author thanks some people, especially my ex-supervisor since the idea of the context-sensitive translation occurred in a dissolution with him. However, the author refrains from naming them until careful proofreading is done.

4. Appendix

Definition 4.1 (classical linear logic CLL [1, §2.2.]).

$(Ax)\ A \Rightarrow A$

$(Cut)\ \dfrac{\Gamma \Rightarrow A, \Delta \qquad \Gamma' A \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$

$(L\sqcap)\ \dfrac{\Gamma, A_i \Rightarrow \Delta}{\Gamma, A_0 \sqcap A_1, \Rightarrow \Delta}$

$(R\sqcap)\ \dfrac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \sqcap B, \Delta}$

$(L\star)\ \dfrac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \star B, \Rightarrow \Delta}$

$(R\star)\ \dfrac{\Gamma \Rightarrow A, \Delta \qquad \Gamma' \Rightarrow B, \Delta'}{\Gamma, \Gamma' \Rightarrow A \star B, \Delta, \Delta'}$

$(L\sqcup)\ \dfrac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \sqcup B \Rightarrow \Delta}$

$(R\sqcup)\ \dfrac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \sqcup B} \quad \dfrac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \sqcup B}$

$(L+)\ \dfrac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta'}{\Gamma, A + B \Rightarrow \Delta, \Delta'}$

$(R+)\ \dfrac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A + B}$

$(L\multimap)\ \dfrac{\Gamma \Rightarrow A, \Delta \qquad \Gamma', B \Rightarrow \Delta'}{\Gamma, \Gamma', A \multimap B \Rightarrow \Delta, \Delta'}$

$(R\multimap)\ \dfrac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \multimap B, \Delta}$

$(L\forall)\ \dfrac{\Gamma, A[x/t] \Rightarrow \Delta}{\Gamma, \forall x A \Rightarrow \Delta}$

$(R\forall)\ \dfrac{\Gamma \Rightarrow \Delta, A[x/y]}{\Gamma \Rightarrow \Delta, \forall x A}$

$(L\exists)\ \dfrac{\Gamma, A[x/y] \Rightarrow \Delta}{\Gamma, \exists x A \Rightarrow \Delta}$

$(R\exists)\ \dfrac{\Gamma \Rightarrow \Delta, A[x/t]}{\Gamma \Rightarrow \Delta, \exists x A}$

$(L\mathbf{1})\ \dfrac{\Gamma \Rightarrow \Delta}{\Gamma, \mathbf{1} \Rightarrow \Delta}$

$\Rightarrow \mathbf{1}$

no $(L\top)$

$(R\top)\ \Gamma \Rightarrow \top, \Delta$

$(L\mathbf{0})\ \mathbf{0} \Rightarrow$

$(R\mathbf{0})\ \dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \mathbf{0}}$

$(L\bot)\ \Gamma, \bot \Rightarrow \Delta$

no $(R\bot)$

$(W!)\ \dfrac{\Gamma \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta}$

$(W?)\ \dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow ?A, \Delta}$

$(L!)\ \dfrac{\Gamma, A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta}$

$(L?)\ \dfrac{!\Gamma, A \Rightarrow ?\Delta}{!\Gamma, ?A \Rightarrow ?\Delta}$

$(R!)\ \dfrac{!\Gamma \Rightarrow A.?\Delta}{!\Gamma \Rightarrow !A, ?\Delta}$

$(R?)\ \dfrac{\Gamma \Rightarrow A, \Delta}{\Gamma, \Rightarrow ?A, \Delta}$

$(C!)\ \dfrac{\Gamma, !A, !A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta}$

$(C?)\ \dfrac{\Gamma \Rightarrow ?A, ?A, \Delta}{\Gamma \Rightarrow ?A, \Delta}$

Definition 4.2 (intuitionistic linear logic ILL [1, §2.5.]).

$(Ax)\ A \Rightarrow A$ $\qquad$ $(Cut)\ \dfrac{\Gamma \Rightarrow A \qquad \Gamma', A \Rightarrow C}{\Gamma, \Gamma' \Rightarrow C}$

$(L\sqcap)\ \dfrac{\Gamma, A_i \Rightarrow C}{\Gamma, A_0 \sqcap A_1, \Rightarrow C}$ $\qquad$ $(R\sqcap)\ \dfrac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \sqcap B}$

$(L\star)\ \dfrac{\Gamma, A, B \Rightarrow C}{\Gamma, A \star B, \Rightarrow C}$ $\qquad$ $(R\star)\ \dfrac{\Gamma \Rightarrow A \qquad \Gamma' \Rightarrow B}{\Gamma, \Gamma' \Rightarrow A \star B}$

$(L\sqcup)\ \dfrac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \sqcup B \Rightarrow C}$ $\qquad$ $(R\sqcup)\ \dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \sqcup B} \quad \dfrac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \sqcup B}$

$(L+)\ \dfrac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A + B \Rightarrow C}$ $\qquad$ *no* $(L+)$

$(L\multimap)\ \dfrac{\Gamma \Rightarrow A \qquad \Gamma', B \Rightarrow C}{\Gamma, \Gamma', A \multimap B \Rightarrow C}$ $\qquad$ $(R\multimap)\ \dfrac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \multimap B}$

$(L\forall)\ \dfrac{\Gamma, A[x/t] \Rightarrow C}{\Gamma, \forall x A \Rightarrow C}$ $\qquad$ $(R\forall)\ \dfrac{\Gamma \Rightarrow A[x/y]}{\Gamma \Rightarrow \forall x A}$

$(L\exists)\ \dfrac{\Gamma, A[x/y] \Rightarrow C}{\Gamma, \exists x A \Rightarrow C}$ $\qquad$ $(R\exists)\ \dfrac{\Gamma \Rightarrow [x/t]}{\Gamma \Rightarrow \exists x A}$

$(L\mathbf{1})\ \dfrac{\Gamma \Rightarrow C}{\Gamma, \mathbf{1} \Rightarrow C}$ $\qquad$ $\Rightarrow \mathbf{1}$

no $(L\top)$ $\qquad$ $(R\top)\ \Gamma \Rightarrow \top$

$(L\mathbf{0})\ \mathbf{0} \Rightarrow$ $\qquad$ $(L\mathbf{0})\ \dfrac{\Gamma \Rightarrow}{\Gamma \Rightarrow \mathbf{0}}$

$(L\bot)\ \Gamma, \bot \Rightarrow C$ $\qquad$ *no* $(R\bot)$

$(W!)\ \dfrac{\Gamma \Rightarrow C}{\Gamma, !A \Rightarrow C}$ $\qquad$ $(W?)\ \dfrac{\Gamma \Rightarrow}{\Gamma \Rightarrow ?A}$

$(L!)\ \dfrac{\Gamma, A \Rightarrow C}{\Gamma, !A \Rightarrow C}$ $\qquad$ $(L?)\ \dfrac{!\Gamma, A \Rightarrow ?C}{!\Gamma, ?A \Rightarrow ?C}$

$(R!)\ \dfrac{!\Gamma \Rightarrow A}{!\Gamma \Rightarrow !A}$ $\qquad$ $(R?)\ \dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow ?A}$

$(C!)\ \dfrac{\Gamma, !A, !A \Rightarrow C}{\Gamma, !A \Rightarrow C}$, $\qquad$ *no* $(C?)$

In $L\exists$ and $R\forall$, variable y is not free in the conclusion .

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