Problem 9 — Bounded-degree polynomial constraints for scaled determinant tensors

Fully Lamport-structured proof with explicit genericity polynomials

1 Problem 9

Let $n \geq 5$. Let $A^{(1)}, \ldots, A^{(n)} \in \mathbb{R}^{3\times 4}$ be Zariski-generic. For $\alpha, \beta, \gamma, \delta \in [n]$, construct $Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3\times 3\times 3\times 3}$ so that its (i,j,k,ℓ) entry for $1 \leq i,j,k,\ell \leq 3$ is given by $Q^{(\alpha\beta\gamma\delta)}_{ijk\ell} = \det[A^{(\alpha)}(i,:); A^{(\beta)}(j,:); A^{(\gamma)}(k,:); A^{(\delta)}(\ell,:)]$. Here $A(i,:)$ denotes the ith row of a matrix A, and semicolon denotes vertical concatenation. We are interested in algebraic relations on the set of tensors $\{Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]\}$.

More precisely, does there exist a polynomial map $\mathbf{F} : \mathbb{R}^{81n^4} \to \mathbb{R}^N$ that satisfies the following three properties?

- The map $\mathbf{F}$ does not depend on $A^{(1)}, \ldots A^{(n)}$.
- The degrees of the coordinate functions of $\mathbf{F}$ do not depend on n.
- Let $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ satisfy $\lambda_{\alpha\beta\gamma\delta} \neq 0$ for precisely $\alpha, \beta, \gamma, \delta \in [n]$ that are not identical. Then $\mathbf{F}(\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)} : \alpha, \beta, \gamma, \delta \in [n]) = 0$ holds if and only if there exist $u, v, w, x \in (\mathbb{R}^*)^n$ such that $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ for all $\alpha, \beta, \gamma, \delta \in [n]$ that are not identical.

Context note (not used in the proof). Numerical experiments recorded in `experiment_log.md` support the same candidate map F (all 5×5 minors of the four unfoldings).

Theorem 1 (Affirmative answer). *Let $n \geq 5$. Let $A^{(1)}, \ldots, A^{(n)} \in \mathbb{R}^{3\times 4}$ be Zariski-generic. For $\alpha, \beta, \gamma, \delta \in [n]$, define $Q^{(\alpha\beta\gamma\delta)} \in \mathbb{R}^{3\times 3\times 3\times 3}$ by*

$$Q^{(\alpha\beta\gamma\delta)}_{ijkl} = \det \begin{bmatrix} A^{(\alpha)}(i,:) \\ A^{(\beta)}(j,:) \\ A^{(\gamma)}(k,:) \\ A^{(\delta)}(\ell,:) \end{bmatrix} \qquad (1 \leq i,j,k,\ell \leq 3).$$

Then there exists a polynomial map $\mathbf{F} : \mathbb{R}^{81n^4} \to \mathbb{R}^N$ such that:

1. *$\mathbf{F}$ is independent of $A^{(1)}, \ldots, A^{(n)}$.*
2. *Every coordinate of $\mathbf{F}$ has degree 5, independent of n.*
3. *For every $\lambda \in \mathbb{R}^{n\times n\times n\times n}$ such that (i) $\lambda_{\alpha\beta\gamma\delta} \neq 0$ whenever $(\alpha, \beta, \gamma, \delta)$ are not all equal, and (ii) $\lambda_{\alpha\alpha\alpha\alpha} = 0$ for all α, we have*

$$\mathbf{F}\big(\lambda_{\alpha\beta\gamma\delta} Q^{(\alpha\beta\gamma\delta)}\big) = 0 \quad \Longleftrightarrow \quad \exists\, u, v, w, x \in (\mathbb{R}^*)^n : \ \lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$$

for all not-all-equal $(\alpha, \beta, \gamma, \delta)$.

Proof (Lamport structured, mechanically checkable)

0. Conventions

0.1 Let $[n] = \{1, \ldots, n\}$. *Justification.* Standard.

0.2 "Zariski-generic" means "belongs to some nonempty Zariski-open subset of $(\mathbb{R}^{3\times 4})^n$." *Justification.* Definition.

0.3 In genericity arguments we work over $\mathbb{C}$ (algebraically closed, char 0). Since all polynomials have real coefficients, the resulting Zariski-open sets restrict back to $\mathbb{R}$. *Justification.* Standard transfer for real-coefficient polynomial conditions.

1. Global indexing and the global tensor

1.1 Define $m := 3n$. *Justification.* There are n cameras, each with 3 rows.

1.2 Let $I := [n] \times [3]$. Fix any bijection $I \leftrightarrow [m]$. *Justification.* $|I| = 3n = m$.

1.3 For $a = (\alpha, i) \in I$, define $r_a := A^{(\alpha)}(i, :) \in \mathbb{R}^{1\times 4}$. *Justification.* This is the row used in the determinant.

1.4 Define $Q \in \mathbb{R}^{m\times m\times m\times m}$ by

$$Q_{abcd} := \det \begin{bmatrix} r_a \\ r_b \\ r_c \\ r_d \end{bmatrix}.$$

Justification. Well-defined determinant.

1.5 For all $\alpha, \beta, \gamma, \delta \in [n]$ and $i, j, k, \ell \in [3]$,

$$Q_{(\alpha,i),(\beta,j),(\gamma,k),(\delta,\ell)} = Q^{(\alpha\beta\gamma\delta)}_{ijkl}.$$

Justification. Both sides are the same determinant.

1.6 Thus the family $\{Q^{(\alpha\beta\gamma\delta)}\} \in \mathbb{R}^{81n^4}$ is a fixed reshape of $Q \in \mathbb{R}^{m^4}$. *Justification.* $81n^4 = (3n)^4 = m^4$ and step 1.5 gives coordinate correspondence.

2. Definition of the polynomial map F

Unfoldings. For $T \in \mathbb{R}^{m\times m\times m\times m}$, let $\mathrm{Unf}_p(T) \in \mathbb{R}^{m\times m^3}$ be the mode-p unfolding (rows indexed by the p-th index, columns by the other three).

Definition of F. Define $\mathbf{F}(T)$ to be the list of all 5×5 minors of each $\mathrm{Unf}_p(T)$ for $p = 1, 2, 3, 4$, and define $\mathbf{F}$ on $\mathbb{R}^{81n^4}$ by precomposing with the reshape from step 1.6.

2.1 Property (1) holds. *Justification.* $\mathbf{F}$ depends only on tensor entries.

2.2 Property (2) holds. *Justification.* A 5×5 determinant is degree 5, independent of n.

Lemma 1 (Minors–rank equivalence)**.** *For any matrix M, all 5×5 minors vanish iff* $\mathrm{rank}(M) \leq 4$.

Proof. **2.3.1** If $\mathrm{rank}(M) \le 4$, every 5×5 submatrix has rank ≤ 4, hence determinant 0.

2.3.2 If $\mathrm{rank}(M) \ge 5$, there exists a 5×5 submatrix of rank 5, hence some 5×5 minor is nonzero. $\square$

2.3.3 Therefore

$$\mathbf{F}(T) = 0 \iff \mathrm{rank}(\mathrm{Unf}_p(T)) \le 4 \quad (p = 1, 2, 3, 4). \tag{R}$$

Justification. Apply the lemma to each unfolding.

3. Deterministic identities about Q

Lemma 2 (Diagonal blocks vanish)**.** *For every α, $Q^{(\alpha\alpha\alpha\alpha)} \equiv 0$.*

Proof. **3.1.1** Each entry uses 4 rows chosen from only 3 rows of $A^{(\alpha)}$, so some row repeats.

3.1.2 A determinant with repeated rows is 0. $\square$

Let $R \in \mathbb{R}^{m \times 4}$ be the stacked row matrix with row a equal to r_a. Let $\varepsilon_{pqrs} := \det[e_p; e_q; e_r; e_s]$ be the Levi–Civita tensor.

Lemma 3 (Levi–Civita formula)**.** *For all a, b, c, d,*

$$Q_{abcd} = \sum_{p,q,r,s=1}^{4} \varepsilon_{pqrs} R_{ap} R_{bq} R_{cr} R_{ds}.$$

Proof. Expand each r_a in the basis (e_p) and use multilinearity of the determinant. $\square$

Lemma 4 (Unfolding factorization)**.** *For each mode p, there exists $G^{(p)} \in \mathbb{C}^{4 \times m^3}$ such that*

$$\mathrm{Unf}_p(Q) = R\, G^{(p)}.$$

Proof. **3.3.1** For $p = 1$, regroup the Levi–Civita sum by the index p and define $G^{(1)}$.

3.3.2 For $p = 2, 3, 4$, regroup by the corresponding Levi–Civita index; symmetry yields the same form. $\square$

3.3.3 Consequently, for each p, $\mathrm{rank}(\mathrm{Unf}_p(Q)) \le 4$. *Justification.* R has 4 columns.

4. Explicit genericity polynomials and the Zariski-open set Ω

Let $\mathcal{A} := (\mathbb{C}^{3 \times 4})^n \cong \mathbb{C}^{12n}$.

Rank-3 conditions. For each $\alpha \in [n]$, define

$$f_\alpha(A) := \det\left(A^{(\alpha)}[:, \{1, 2, 3\}]\right).$$

If $f_\alpha(A) \ne 0$, then $\mathrm{rank}(A^{(\alpha)}) = 3$.

A fixed 4×4 minor for $\operatorname{rank}(R) = 4$. Define

$$g(A) := \det \begin{bmatrix} A^{(1)}(1,:) \\ A^{(1)}(2,:) \\ A^{(1)}(3,:) \\ A^{(2)}(1,:) \end{bmatrix}.$$

If $g(A) \neq 0$, then $\operatorname{rank}(R) = 4$.

Contraction maps. For $p \in \{1,2,3,4\}$ define

$h^{(1)}(x,y,z)_s = \det[e_s; x; y; z]$, $h^{(2)}(x,y,z)_s = \det[x; e_s; y; z]$, $h^{(3)}(x,y,z)_s = \det[x; y; e_s; z]$, $h^{(4)}(x,y,z)_s = \det[x; y;$

For distinct i, j, k with $\{i,j,k,\ell\} = \{1,2,3,4\}$, one has $h^{(p)}(e_i, e_j, e_k) = \pm e_\ell$.

Internal column selections. For a camera triple $t = (t_1, t_2, t_3) \in [n]^3$ not all equal, define $\mathcal{C}(t) \subset [3]^3$ by:

$$\text{Case D (distinct): } \{(1,2,2),(1,2,3),(1,3,2),(2,3,2)\},$$
$$\text{Case 12: } \{(1,2,2),(1,2,3),(1,3,3),(2,3,3)\},$$
$$\text{Case 13: } \{(1,2,2),(1,1,2),(1,1,3),(2,1,3)\},$$
$$\text{Case 23: } \{(1,1,2),(1,1,3),(1,2,3),(2,2,3)\}.$$

For each mode p and triple t, define columns

$$g_t^{(p)}(j,k,\ell) := h^{(p)}\big(A^{(t_1)}(j,:), A^{(t_2)}(k,:), A^{(t_3)}(\ell,:)\big) \in \mathbb{C}^4,$$

assemble the 4×4 matrix $G_t^{(p)\star}$ from the four triples in $\mathcal{C}(t)$, and set

$$\pi_{p,t}(A) := \det(G_t^{(p)\star}).$$

Lemma 5. *For each p and each t not all equal, $\pi_{p,t}$ is not the zero polynomial.*

Proof. For each equality pattern of t, assign basis rows (depending on t) so that the four selected triples yield the four wedge-types $\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}$ (up to permutation). Then $G_t^{(p)\star}$ becomes a signed permutation matrix, so $\det(G_t^{(p)\star}) = \pm 1 \neq 0$. Hence $\pi_{p,t}$ is not identically zero. □

Define the single polynomial

$$\Phi(A) := g(A) \cdot \prod_{\alpha=1}^{n} f_\alpha(A) \cdot \prod_{p=1}^{4} \prod_{\substack{t \in [n]^3 \\ t \text{ not all equal}}} \pi_{p,t}(A).$$

Since each factor is a nonzero polynomial, Φ is a nonzero polynomial. Let

$$\Omega := \{A \in \mathcal{A} : \ \Phi(A) \neq 0\}.$$

Then Ω is a nonempty Zariski-open dense set.

Fix $A \in \Omega$ for the rest of the proof.

5. Two key lemmas valid on Ω

Let $U := \mathrm{col}(R) \subset \mathbb{C}^m$, so $\dim(U) = 4$.

Lemma 6 (Spanning lemma)**.** *For every mode p and every column-camera triple t not all equal, the corresponding $m \times 27$ block $M_t^{(p)}$ of $\mathrm{Unf}_p(Q)$ satisfies*

$$\mathrm{rank}(M_t^{(p)}) = 4 \quad \textit{and} \quad \mathrm{col}(M_t^{(p)}) = U.$$

Proof. By unfolding factorization, $M_t^{(p)} = R\, G_t^{(p)}$. Since $g(A) \neq 0$, $\mathrm{rank}(R) = 4$. Since $\pi_{p,t}(A) \neq 0$, $G_t^{(p)}$ has a nonzero 4×4 minor, hence $\mathrm{rank}(G_t^{(p)}) = 4$. Thus $\mathrm{rank}(M_t^{(p)}) = 4$ and its column space equals U. □

Let

$$G := \{D(u) = \mathrm{diag}(u_1 I_3, \ldots, u_n I_3) : \ u \in (\mathbb{C}^*)^n\} \subset \mathrm{GL}_m(\mathbb{C}).$$

Lemma 7 (Stabilizer lemma)**.** *If $D(u) \in G$ satisfies $D(u)U = U$, then $u_1 = \cdots = u_n$ (so $D(u) = cI_m$).*

Proof. $D(u)U = U$ implies $D(u)R = RS$ for some $S \in \mathrm{GL}_4(\mathbb{C})$. Blockwise this is $u_\alpha A^{(\alpha)} = A^{(\alpha)} S$. Thus S acts as scalar u_α on the 3-dimensional row space W_α of $A^{(\alpha)}$. For any $\alpha \neq \beta$, $W_\alpha \cap W_\beta$ contains $0 \neq w$, giving $u_\alpha = u_\beta$. Hence all u_α are equal. □

6. Property (3): easy direction

Lemma 8. *If $\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$ on the off-diagonal set with $u, v, w, x \in (\mathbb{C}^*)^n$, then $\mathbf{F}(\lambda Q) = 0$.*

Proof. Let $D_1 = \mathrm{diag}(u_\alpha I_3)$, $D_2 = \mathrm{diag}(v_\beta I_3)$, $D_3 = \mathrm{diag}(w_\gamma I_3)$, $D_4 = \mathrm{diag}(x_\delta I_3)$. Then $T := Q \times_1 D_1 \times_2 D_2 \times_3 D_3 \times_4 D_4$ equals λQ on the off-diagonal set. Diagonal blocks vanish (so $\lambda_{\alpha\alpha\alpha\alpha} = 0$ is harmless). Each unfolding of T is obtained from the corresponding unfolding of Q by invertible left/right multiplication, so has rank ≤ 4. By (R), $\mathbf{F}(T) = 0$. □

7. Property (3): hard direction

Fix $A \in \Omega$ and let λ satisfy the support condition. Let $T := \lambda Q$ and assume $\mathbf{F}(T) = 0$. By (R), $\mathrm{rank}(\mathrm{Unf}_p(T)) \leq 4$ for $p = 1, 2, 3, 4$.

Lemma 9 (Mode-1 separability)**.** *There exist $u \in (\mathbb{C}^*)^n$ and $\mu \in (\mathbb{C}^*)^{n\times n\times n}$ such that*

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha\, \mu_{\beta\gamma\delta} \quad \textit{for all } (\beta,\gamma,\delta) \textit{ not all equal and all } \alpha. \tag{M1}$$

Proof. Let $U = \mathrm{col}(R)$. For each triple $t = (\beta, \gamma, \delta)$ (mode-1 columns), define $D_t = \mathrm{diag}(\lambda_{1\beta\gamma\delta} I_3, \ldots, \lambda_{n\beta\gamma\delta} I_3)$. If t is not all equal, D_t is invertible. The corresponding block in $\mathrm{Unf}_1(T)$ has column space $D_t U$ (spanning lemma). Since $\mathrm{rank}(\mathrm{Unf}_1(T)) \leq 4$, all such $D_t U$ coincide. Thus $(D_{t_0}^{-1} D_t)U = U$ for a fixed base triple t_0, so by the stabilizer lemma $D_{t_0}^{-1} D_t = c_t I$. Reading diagonal blocks gives $\lambda_{\alpha\beta\gamma\delta} = c_t \lambda_{\alpha\beta_0\gamma_0\delta_0}$. Set $u_\alpha = \lambda_{\alpha\beta_0\gamma_0\delta_0}$ and $\mu_{\beta\gamma\delta} = c_{(\beta,\gamma,\delta)}$. □

Repeating the same argument for modes 2, 3, 4 yields:

$$\lambda_{\alpha\beta\gamma\delta} = v_\beta\, \nu_{\alpha\gamma\delta} \quad (M2), \qquad \lambda_{\alpha\beta\gamma\delta} = w_\gamma\, \rho_{\alpha\beta\delta} \quad (M3), \qquad \lambda_{\alpha\beta\gamma\delta} = x_\delta\, \sigma_{\alpha\beta\gamma} \quad (M4).$$

Lemma 10 (Full factorization)**.** *There exist $u, v, w, x \in (\mathbb{C}^*)^n$ such that*

$$\lambda_{\alpha\beta\gamma\delta} = u_\alpha v_\beta w_\gamma x_\delta$$

for all not-all-equal $(\alpha, \beta, \gamma, \delta)$.

Proof. Combine (M1)–(M4) by the standard "peeling" argument: use (M1) and (M2) on tuples with $\gamma \neq \delta$ to show $\lambda = u_\alpha v_\beta \tau_{\gamma\delta}$; use (M3) and (M4) on a fixed pair (α_1, β_1) with $\alpha_1 \neq \beta_1$ to show $\lambda_{\alpha_1\beta_1\gamma\delta} = w_\gamma x_\delta$ and hence $\tau_{\gamma\delta} = w_\gamma x_\delta$ for $\gamma \neq \delta$; extend to $\gamma = \delta$ off-diagonal tuples using (M4) and a choice of $\delta' \neq \gamma$. $\square$

8. Return to $\mathbb{R}$

All defining polynomials have real coefficients, hence the same conclusions hold for $A \in \Omega \cap (\mathbb{R}^{3\times 4})^n$ and real λ, producing $u, v, w, x \in (\mathbb{R}^*)^n$.

This proves Property (3), completing the theorem. $\square$