Exercises in Introduction to Mathematical Statistics (Ch. 11)

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Note

- Not all Solution.s are provided: exercises that are too simple or not very important to me are skipped.
- Texts in red are just attentions to me. Please ignore them.

11 Bayesian Statistics

Note: I solved some of the problems only in 11.1.

11.1. Bayesian Procedures

11.1.1. Let Y have a binomial distribution in which $n = 20$ and $p = \theta$. The prior probabilities on θ are $P(\theta = 0.3) = 2/3$ and $P(\theta = 0.5) = 1/3$. If $y = 9$, what are the posterior probabilities for $\theta = 0.3$ and $\theta = 0.5$?

Solution.

The model is

$$\begin{aligned} Y|\theta &\sim \text{iid Binom}(20, y) \\ \Theta &\sim h(\theta), \end{aligned}$$

where

$$\begin{aligned} f(y|\theta) &= \binom{20}{y}\theta^y(1-\theta)^{20-y}, \\ h(\theta) &= \begin{cases} 2/3 & \theta = 0.3 \\ 1/3 & \theta = 0.5. \end{cases} \end{aligned}$$

Note that in this case the sample size $n = 1$, or the likelihood function equals the pdf of Y. Hence, the conditional probability of θ given $y = 9$ is

$$g(\theta|y = 9) = \frac{L(y = 9|\theta)h(\theta)}{g(y = 9)} = \frac{f(y = 9|\theta)h(\theta)}{f(y = 9|\theta = 0.3)h(0.3) + f(y = 9|\theta = 0.5)h(0.5)}.$$

Since

$$\begin{aligned} f(y = 9|\theta = 0.3)h(0.3) &= \binom{20}{9}0.3^9(0.7)^{11}(2/3) \\ f(y = 9|\theta = 0.5)h(0.5) &= \binom{20}{9}(0.5)^{20}(1/3), \end{aligned}$$

the posterior probabilities for $\theta = 0.3$ and $\theta = 0.5$ is

$$g(\theta = 0.3|y = 9) = \frac{\binom{20}{9}0.3^9(0.7)^{11}(2/3)}{\binom{20}{9}0.3^9(0.7)^{11}(2/3) + \binom{20}{9}(0.5)^{20}(1/3)} = 0.449,$$
$$g(\theta = 0.5|y = 9) = 1 - g(\theta = 0.3|y = 9) = 0.551.$$

11.1.2. Let $X_1, X_2, ..., X_n$ be a random sample from a distribution that is $b(1, \theta)$. Let the prior of Θ be a beta one with parameters α and β. Show that the posterior pdf $k(\theta|x_1, x_2, ..., x_n)$ is exactly the same as $k(\theta|y)$ given in Example 11.1.2.

Solution.

The model is

$$\begin{aligned}\mathbf{X}|\theta &\sim L(\mathbf{x}|\theta) = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}\\ \Theta &\sim \text{Beta}(\alpha, \beta).\end{aligned}$$

Hence, the posterior pdf is given by

$$\begin{aligned}k(\theta|\mathbf{x}) &\propto L(\mathbf{x}|\theta)h(\theta)\\ &= \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}(1/\text{Beta}(\alpha,\beta))\theta^{\alpha-1}(1-\theta)^{\beta-1}\\ &\propto \theta^{\alpha+\sum x_i - 1}(1-\theta)^{\beta+n-\sum x_i - 1},\end{aligned}$$

meaning $\Theta|\mathbf{x} \sim \text{Beta}\left(\alpha + \sum x_i, \beta + n - \sum x_i\right)$. Thus, $\Theta|\mathbf{x}$ equals $\Theta|y$ given in Example 11.1.2, where Y is the sufficient statistic $Y = \sum X_i$ for θ.

11.1.4. Let $X_1, X_2, ..., X_n$ denote a random sample from a Poisson distribution with mean θ, $0 < \theta < \infty$. Let $Y = \sum_1^n X_i$. Use the loss function $L[\theta, \delta(y)] = [\theta - \delta(y)]^2$. Let θ be an observed value of the random variable θ. If θ has the prior pdf $h(\theta) = \theta^{\alpha-1}e^{-\theta/\beta}/\Gamma(\alpha)\beta^\alpha$, for $0 < \theta < \infty$, zero elsewhere, where $\alpha > 0$, $\beta > 0$ are known numbers, find the Bayes solution $\delta(y)$ for a point estimate for θ.

Solution.

The model is

$$\begin{aligned}\mathbf{X}|\theta &\sim L(\mathbf{x}|\theta)\\ \Theta &\sim \Gamma(\alpha, \beta)\\ \Theta|\mathbf{x} &\sim g(\theta|\mathbf{x}).\end{aligned}$$

Since we know that Y is sufficient for θ, the above model is equivalent to

$$\begin{aligned}Y|\theta &\sim \text{Poisson}(n\theta)\\ \Theta &\sim \Gamma(\alpha, \beta)\\ \Theta|y &\sim g(\theta|y),\end{aligned}$$

where

$$\begin{aligned}g(\theta|y) &\propto f(y|\theta)h(\theta)\\ &= \left[\frac{e^{-n\theta}(n\theta)^y}{y!}\right]\left[\frac{\theta^{\alpha-1}e^{-\theta/\beta}}{\Gamma(\alpha)\beta^\alpha}\right]\\ &\propto \theta^{y+\alpha-1}e^{-\theta(n+1/\beta)},\end{aligned}$$

indicating $\Theta|y \sim \Gamma(y + \alpha - 1, 1/(n + 1/\beta))$. Hence,

$$\begin{aligned}\delta(y) = E(\Theta|y) = \frac{y+\alpha}{n+1/\beta} &= \frac{\beta(y+\alpha)}{n\beta+1}\\ &= \frac{n\beta}{n\beta+1}\frac{y}{n} + \frac{1}{n\beta+1}\alpha\beta.\end{aligned}$$

Indeed, the estimate is the weighted average of the MLE of θ and the prior mean.

11.1.5. Let Y_n be the nth order statistic of a random sample of size n from a distribution with pdf $f(x|\theta) = 1/\theta$, $0 < x < \theta$, zero elsewhere. Take the loss function to be $L[\theta, \delta(y)] = [\theta - \delta(y_n)]^2$. Let θ be an observed value of the random variable Θ, which has the prior pdf $h(\theta) = \beta\alpha^\beta/\theta^{\beta+1}$, $\alpha < \theta < \infty$, zero elsewhere, with $\alpha > 0$,$\beta > 0$. Find the Bayes solution $\delta(y_n)$ for a point estimate of θ.

Solution.

The model is

$$\begin{aligned}
\mathbf{X}|\theta &\sim L(\mathbf{x}|\theta) \\
\Theta &\sim h(\theta) \\
\Theta|\mathbf{x} &\sim g(\theta|\mathbf{x}),\ 0 < x_i < \theta
\end{aligned}$$

Since we know that Y_n is sufficient for θ, the above model is equivalent to

$$\begin{aligned}
Y_n|\theta &\sim f_{Y_n}(y_n) \\
\Theta &\sim h(\theta) \\
\Theta|y_n &\sim g(\theta|y_n),\ y_n < \theta.
\end{aligned}$$

By the previous example, we know that the pdf of Y_n is

$$f_{Y_n}(y_n) = \frac{ny^{n-1}}{\theta^n},\ y_n < \theta.$$

Thus, the posterior pdf is

$$\begin{aligned}
g(\theta|y) &\propto f_{Y_n}(y_n)h(\theta) \\
&= \frac{ny^{n-1}}{\theta^n}\frac{\beta\alpha^\beta}{\theta^{\beta+1}} \\
&\propto \theta^{-(n+\beta+1)}.
\end{aligned}$$

Therefore,

$$\delta(y_n) = E(\Theta|y_n) = \int_{y_n}^{\infty} \theta^{-(n+\beta)} d\theta = \left[-\frac{\theta^{-(n+\beta)+1}}{n+\beta-1}\right]_{y_n}^{\infty} = \frac{y_n^{-n-\beta+1}}{n+\beta-1}.$$