Exercises in Introduction to Mathematical Statistics (Ch. 2)

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Note

- Not all solutions are provided: exercises that are too simple or not very important to me are skipped.
- Texts in red are just attentions to me. Please ignore them.

2 Multivariate Distributions

2.1 Distributions of Two Random Variables

2.1.1. Let $f(x_1, x_2) = 4x_1x_2$, $0 < x_1 < 1$, $0 < x_2 < 1$, zero elsewhere, be the pdf of X_1 and X_2. Find $P(0 < X_1 < \frac{1}{2}, \frac{1}{4} < X_2 < 1)$, $P(X_1 = X_2)$, $P(X_1 < X_2)$, and $P(X_1 \leq X_2)$.

Solution.

$$
\begin{aligned}
&P\left(0 < X_1 < \frac{1}{2}, \frac{1}{4} < X_2 < 1\right) = \int_{1/4}^{1}\int_{0}^{1/2} 4x_1x_2dx_1dx_2 = \cdots = \frac{15}{64}\\
&P(X_1 = X_2) = 0 \quad \text{since the support is a segment not area}\\
&P(X_1 < X_2) = \int_0^1\int_0^{x_2} 4x_1x_2dx_1dx_2 = \int_0^1 2x_1^2x_2|_{x_1=0}^{x_1=x_2}dx_1dx_2 = \int_0^1 2x_2^3dx_2 = \frac{1}{2}.\\
&P(X_1 \leq X_2) = P(X_1 < X_2) + P(X_1 = X_2) = P(X_1 < X_2) = \frac{1}{2}.
\end{aligned}
$$

2.1.2. Let $A_1 = \{(x, y) : x \leq 2, y \leq 4\}$, $A_2 = \{(x, y) : x \leq 2, y \leq 1\}$, $A_3 = \{(x, y) : x \leq 0, y \leq 4\}$, and $A_4 = \{(x, y) : x \leq 0, y \leq 1\}$ be subsets of the space $\mathcal{A}$ of two random variables X and Y, which is the entire two-dimensional plane. If $P(A_1) = \frac{7}{8}$, $P(A_2) = \frac{4}{8}$, $P(A_3) = \frac{3}{8}$, and $P(A_4) = \frac{2}{8}$, find $P(A_5)$, where $A_5 = \{(x, y) : 0 < x \leq 2, 1 < y \leq 4\}$.

Solution. $P(A_5) = P(A_1) - P(A_2) - P(A_3) + P(A_4) = \frac{2}{8}$.

2.1.3. Let $F(x, y)$ be the distribution function of X and Y. For all real constants $a < b, c < d$, show that

$$P(a < X \leq b, c < Y \leq d) = F(b, d) - F(b, c) - F(a, d) + F(a, c).$$

Solution.

$$
\begin{aligned}
P(a < X \leq b, c < Y \leq d) &= P(X \leq b, c < Y \leq d) - P(X \leq a, c < Y \leq d)\\
&= P(X \leq b, Y \leq d) - P(X \leq b, Y \leq c) - P(X \leq a, Y \leq d) + P(X \leq a, Y \leq c)\\
&= F(b, d) - F(b, c) - F(a, d) + F(a, c).
\end{aligned}
$$

2.1.7. Let $f(x, y) = e^{-x-y}$, $0 < x < \infty$, $0 < y < \infty$, zero elsewhere, be the pdf of X and Y. Then if $Z = X + Y$, compute $P(Z \leq 0)$, $P(Z \leq 6)$, and, more generally, $P(Z \leq z)$, for $0 < z < \infty$. What is the pdf of Z.

Solution.

Compute the general probability:

$$
\begin{aligned}
F(z) &= P(Z \le z) = P(X+Y \le z) = P(Y \le -X+z) \\
&= \int_0^z \int_0^{z-x} e^{-x-y} dydx = \int_0^z (e^{-x} - e^{-z}) dx = 1 - e^{-z} - ze^{-z}.
\end{aligned}
$$

Hence, $P(Z \le 0) = 0$, $P(Z \le 6) = 1 - 7e^{-6}$, and $f(z) = F'(z) = ze^{-z}$, $0 < z < \infty$, zero elsewhere.

2.1.8. Let X and Y have the pdf $f(x, y) = 1$, $0 < x < 1$, $0 < y < 1$, zero elsewhere. Find the cdf and pdf of the product $Z = XY$.

Solution.

If $z \le 0$, then $F(z) = P(Z \le z) = 0$ because $Z > 0$.

$$
F(z) = P(Z \le z) = P(Y \le z/X) = \int_0^z \int_0^1 dydx + \int_z^1 \int_0^{z/x} dydx = z - z\log z, \quad 0 < z < 1,
$$

and one $z \ge 1$. Hence, the pdf pf Z is

$$
f_Z(z) = F'(z) = -\log z, \quad 0 < z < 1,
$$

zero elsewhere.

2.1.11. Let X_1 and X_2 have the joint pdf $f(x_1, x_2) = 15x_1^2 x_2$, $0 < x_1 < x_2 < 1$, zero elsewhere. Find the marginal pdfs and compute $P(X_1 + X_2 \le 1)$.

Solution.

$$
\begin{aligned}
f_{X_1}(x_1) &= \int_{x_1}^1 15x_1^2 x_2 dx_2 = \frac{15x_1^2(1-x_1^2)}{2}, \quad 0 < x_1 < 1, \\
f_{X_2}(x_2) &= \int_0^{x_2} 15x_1^2 x_2 dx_1 = 5x_2^4, \quad 0 < x_2 < 1, \\
P(X_1 + X_2 \le 1) &= 15 \int_0^{1/2} x_1^2 \left(\int_{x_1}^{1-x_1} x_2 dx_2 \right) dx_1 = \cdots = \frac{5}{64}.
\end{aligned}
$$

2.1.13. Let X_1, X_2 be two random variables with the joint pmf $p(x_1, x_2) = (x_1 + x_2)/12$, for $x_1 = 1, 2$, $x_2 = 1, 2$, zero elsewhere. Compute $E(X_1)$, $E(X_1^2)$, $E(X_2)$, $E(X_2^2)$, and $E(X_1X_2)$. Is $E(X_1X_2) = E(X_1)E(X_2)$? Find $E(2X_1 - 6X_2^2 + 7X_1X_2)$.

Solution.

First, find the marginal pdfs:

$$
p_{X_1}(x_1) = \sum_{x_2=1}^{2} \frac{x_1 + x_2}{12} = \frac{x_1+1}{12} + \frac{x_1+2}{12} = \frac{2x_1+3}{12}, \quad p_{X_2}(x_2) = \frac{2x_2+3}{12}.
$$

Hence

$$
\begin{aligned}
E(X_1) &= \sum_{x_1=1}^{2} x_1 p(x_1) = p_{X_1}(1) + 2p_{X_1}(2) = \frac{5}{12} + \frac{14}{12} = \frac{19}{12}, \\
E(X_1^2) &= p_{X_1}(1) + 2^2 p_{X_1}(2) = \frac{33}{12}, \\
E(X_2) &= E(X_1) = \frac{19}{12}, \quad E(X_2^2) = E(X_1^2) = \frac{33}{12}.
\end{aligned}
$$

Also, use the joint mgf to obtain

$$E(X_1X_2) = \sum_{x_1x_2} x_1x_2p(x_1,x_2) = p(1,1) + 2p(2,1) + 2p(1,2) + 4p(2,2) = \frac{5}{2} \neq E(X_1)E(X_2).$$

Therefore,

$$E(2X_1 - 6X_2^2 + 7X_1X_2) = 2\frac{19}{12} - 6\frac{33}{12} + 7\frac{5}{2} = \frac{25}{6}.$$

2.1.15. Let X_1, X_2 be two random variables with joint pmf $p(x_1,x_2) = (1/2)^{x_1+x_2}$, for $1 \leq x_i < \infty$, $i = 1,2$, where X_1 and X_2 are integers, zero elsewhere. Determine the joint mgf of X_1, X_2. Show that $M(t_1,t_2) = M(t_1,0)M(0,t_2)$.

Solution.

$$\begin{aligned}
p(x_1) &= \sum_{x_2=1}^{\infty} (1/2)^{x_1+x_2} = \frac{(1/2)^{x_1+1}}{1-1/2} = (1/2)^{x_1}, \quad p(x_1) = (1/2)^{x_2} \\
M_{X_1}(t) &= \sum_{x_1=1}^{\infty} (e^t/2)^{x_1} = \frac{e^t/2}{1-e^t/2} = \frac{e^t}{2-e^t} = M_{X_2}(t), \quad t < \log 2, \\
M(t_1,t_2) &= \sum_{x_1=1}^{\infty}\sum_{x_2=1}^{\infty} e^{t_1x_1+t_2x_2}(1/2)^{x_1+x_2} = \sum_{x_1=1}^{\infty}(e^{t_1}/2)^{x_1}\sum_{x_2=1}^{\infty}(e^{t_2}/2)^{x_2} \\
&= M_{X_1}(t_1)M_{X_2}(t_2) = M(t_1,0)M(0,t_2).
\end{aligned}$$

2.2 Transformations: Bivariate Random Variables

2.2.1. If $p(x_1,x_2) = (\frac{2}{3})^{x_1+x_2}(\frac{1}{3})^{2-x_1-x_2}$, $(x_1,x_2) = (0,0),(0,1),(1,0),(1,1)$, zero elsewhere, is the joint pmf of X_1 and X_2, find the joint pmf of $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$.

Solution.

The support of (Y_1,Y_2) is $(y_1,y_2) = (0,0),(-1,1),(1,1),(0,2)$. Since the one-to-one inverse functions are $x_1 = (y_1+y_2)/2$ and $x_2 = (y_2-y_1)/2$,

$$p_{Y_1,Y_2}(y_1,y_2) = p\left(\frac{y_1+y_2}{2}, \frac{y_2-y_1}{2}\right) = \left(\frac{2}{3}\right)^{y_1}\left(\frac{1}{3}\right)^{2-y_1},$$

zero outside the support.

2.2.5. Let X_1 and X_2 be continuous random variables with the joint pdf $f_{X_1,X_2}(x_1,x_2)$, $-\infty < x_i < \infty$, $i = 1,2$. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_2$.

(a) Find the joint pdf f_{Y_1,Y_2}.

Solution.

The inverse functions are $x_1 = y_1 - y_2$, $x_2 = y_2$ and then the Jacobian $J = 1$. Hence

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(y_1-y_2,y_2)|J| = f_{X_1,X_2}(y_1-y_2,y_2).$$

(b) Show that

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(y_1-y_2,y_2)dy_2, \qquad (2.2.5)$$

which is sometimes called the **convolution formula**.

Solution.

The support is $-\infty < y_1 - y_2 < \infty,\ -\infty < y_2 < \infty$, i.e., $-\infty < y_i < \infty,\ i = 1,2$, which gives (2.2.5).

2.2.6. Suppose X_1 and X_2 have the joint pdf $f(x_1, x_2) = e^{-(x_1+x_2)}$, $0 < x_i < \infty$, $i = 1, 2$, zero elsewhere.

(a) Use formula (2.2.5) to find the pdf of $Y_1 = X_1 + X_2$.

Solution.

Since the support of (Y_1, Y_2) is $0 < y_1 - y_2 < \infty,\ 0 < y_2 < \infty \ \Rightarrow\ 0 < y_2 < y_1 < \infty$,

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(y_1 - y_2, y_2)dy_2 = \int_0^{y_1} e^{-y_1}dy_2 = y_1 e^{-y_1}, \quad y_1 > 0.$$

(b) Find the mgf of Y_1

Solution.

$$M(t) = \int_0^{\infty} y_1 e^{-(1-t)y_1}dy_1 = \Gamma(2)\left(\frac{1}{1-t}\right)^2 = \frac{1}{(1-t)^2}, \quad t < 1.$$

2.2.7. Use the formula (2.2.5) to find the pdf of $Y_1 = X_1 + X_2$, where X_1 and X_2 have the joint pdf $f_{X_1,X_2}(x_1, x_2) = 2e^{-(x_1+x_2)}$, $0 < x_1 < x_2 < \infty$, zero elsewhere.

Solution.

Since the support of Y_1 and Y_2 is $0 < y_1 - y_2 < y_2$, $0 < y_2 < \infty \ \Rightarrow\ 0 < y_1/2 < y_2 < y_1 < \infty$,

$$f_{Y_1}(y_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(y_1 - y_2, y_2)dy_2 = \int_{y_1/2}^{y_1} 2e^{-y_1}dy_2 = y_1 e^{-y_1}, \quad y_1 > 0,$$

which means $Y \sim \text{Exp}(1)$.

2.2.8. Suppose X_1 and X_2 have the joint pdf

$$f(x_1, x_2) = \begin{cases} e^{-x_1}e^{-x_2} & x_1 > 0, x_2 > 0 \\ 0 & \text{elsewhere} \end{cases}.$$

For constants $w_1 > 0$ and $w_2 > 0$, let $W = w_1 X_1 + w_2 X_2$.

(a) Show that the pdf pf W is

$$f(x_1, x_2) = \begin{cases} \frac{1}{w_1 - w_2}(e^{-w/w_1} - e^{-w/w_2}) & w > 0 \\ 0 & \text{elsewhere} \end{cases}.$$

Solution.

Let $Z = w_1 X_1 - w_2 X_2$. This is one-to-one transformation so that we have

$$x_1 = \frac{w+z}{2w_1}, \quad x_2 = \frac{w-z}{2w_2}.$$

Then the Jacobian is given by

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial w} & \frac{\partial x_1}{\partial z} \\ \frac{\partial x_2}{\partial w} & \frac{\partial x_2}{\partial z} \end{vmatrix} = \begin{vmatrix} 1/2w_1 & 1/2w_1 \\ 1/2w_2 & -1/2w_2 \end{vmatrix} = -\frac{1}{2w_1 w_2}.$$

Hence the joint pdf of W and Z is

$$f_{W,Z}(w, z) = f\left(\frac{w+z}{2w_1}, \frac{w-z}{2w_2}\right)|J| = e^{-\frac{w+z}{2w_1}} e^{-\frac{w-z}{2w_2}} \frac{1}{2w_1 w_2} = \frac{1}{2w_1 w_2} e^{-\frac{w_1+w_2}{2w_1 w_2}w} e^{\frac{w_1 - w_2}{2w_1 w_2}z}.$$

The support is

$$\frac{w+z}{2w_1} > 0, \quad \frac{w-z}{2w_2} > 0 \quad \Rightarrow \quad w > 0, \quad -w < z < w.$$

Hence the marginal pdf of W is

$$\begin{aligned}
f_W(w) &= \frac{1}{2w_1w_2} e^{-\frac{w_1+w_2}{2w_1w_2}w} \int_{-w}^{w} e^{\frac{w_1-w_2}{2w_1w_2}z} dz \\
&= \frac{1}{w_1-w_2} e^{-\frac{w_1+w_2}{2w_1w_2}w} \left[e^{\frac{w_1-w_2}{2w_1w_2}z} \right]_{-w}^{w} \\
&= \frac{1}{w_1-w_2} e^{-\frac{w_1+w_2}{2w_1w_2}w} \left(e^{\frac{w_1-w_2}{2w_1w_2}w} - e^{-\frac{w_1-w_2}{2w_1w_2}w} \right) \\
&= \frac{1}{w_1-w_2} (e^{-w/w_1} - e^{-w/w_2}), \quad w > 0.
\end{aligned}$$

(b) Verify that $f_W(w) > 0$ for $w > 0$.

Solution.

If $w_1 > w_2$, then $w_1 - w_2 > 0$, $e^{-w/w_1} - e^{-w/w_2} > 0$ because $g(x) = e^{-a/x}$ is increasing for $a > 0$. If $w_1 < w_2$, then $w_1 - w_2 < 0$, $e^{-w/w_1} - e^{-w/w_2} < 0$. Hence, $f_W(w) > 0$ for $w > 0$.

(c) Note that the pdf $f_W(w)$ has an indeterminate form when $w_1 = w_2$. Rewrite $f_W(w)$ using h defined as $w_1 - w_2 = h$. Then use l'Hˆopital's rule to show that when $w_1 = w_2$, the pdf is given by $f_W(w) = (w/w_1^2)\exp\{-w/w_1\}$ for $w > 0$ and zero elsewhere.

Solution.

When $w_1 = w_2$, or equivalently $h \to 0$,

$$\begin{aligned}
\lim_{h\to 0} f_W(w) &= \lim_{h\to 0} \frac{\left[e^{-w/w_1} - e^{-w/(w_1-h)}\right]}{h} \\
&= \lim_{h\to 0} \frac{\frac{d}{dh}\left[e^{-w/w_1} - e^{-w/(w_1-h)}\right]}{dh/dh} \\
&= \lim_{h\to 0} \frac{\left[0 + \{w/(w_1-h)^2\}e^{-w/(w_1-h)}\right]}{1} \\
&= w/w_1^2 e^{-w/w_1}.
\end{aligned}$$

2.3 Conditional Distributions and Expectations

2.3.5. Let X_1 and X_2 be two random variables such that the conditional distributions and means exist. Show that:

(a) $E(X_1 + X_2|X_2) = E(X_1|X_2) + X_2$.

Solution.

Consider $X_2 = x_2$ (a fixed number) first.

$$E(X_1 + X_2|X_2 = x_2) = E(X_1|X_2 = x_2) + x_2 \;\Rightarrow\; E(X_1 + X_2|X_2) = E(X_1|X_2) + X_2.$$

(b) $E(u(X_2)|X_2) = u(X_2)$.

Solution. $E(u(X_2)|X_2 = x_2) = E(u(x_2)) = u(x_2) \;\Rightarrow\; E(u(X_2)|X_2) = u(X_2)$.

2.3.6. Let the joint pdf of X and Y be given by

$$f(x,y) = \begin{cases} \frac{2}{(1+x+y)^3} & 0 < x < \infty,\; 0 < x < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Compute the marginal pdf of X and the conditional pdf of Y, given $X = x$.

Solution.

$$f(x) = \int_0^\infty \frac{2}{(1+x+y)^3} dy = \left[-\frac{1}{(1+x+y)^2}\right]_0^\infty = \frac{1}{(1+x)^2} \quad 0 < x < \infty,$$
$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{2(1+x)^2}{(1+x+y)^3} \quad 0 < x < \infty,\ 0 < x < \infty,$$

zero elsewhere.

(b) For a fixed $X = x$, compute $E(1 + x + Y|x)$ and use the result to compute $E(Y|x)$.

Solution.

$$E(1+x+Y|x) = \int_0^\infty (1+x+y)\frac{2(1+x)^2}{(1+x+y)^3} dy = \int_0^\infty \frac{2(1+x)^2}{(1+x+y)^2} dy = \left[\frac{-2(1+x)^2}{(1+x+y)}\right]_0^\infty = 2(1+x).$$

Since $E(1 + x + Y|x) = 1 + x + E(Y|x)$, $E(Y|x) = 1 + x$.

2.3.7. Suppose X_1 and X_2 are discrete random variables which have the joint pmf $p(x_1, x_2) = (3x_1 + x_2)/24$, $(x_1, x_2) = (1, 1), (1, 2), (2, 1), (2, 2)$, zero elsewhere. Find the conditional mean $E(X_2|x_1)$, when $x_1 = 1$.

Solution.

$$E(X_2|x_1 = 1) = \sum_{x_2 \in (1,2)} x_2 p(1, x_2) = p(1, 1) + 2p(2, 1) = \frac{4}{24} + 2\frac{5}{24} = \frac{7}{12}.$$

2.3.8. Let X and Y have the joint pdf $f(x, y) = 2\exp\{-(x + y)\}$, $0 < x < y < \infty$, zero elsewhere. Find the conditional mean $E(Y|x)$ of Y, given $X = x$.

Solution.

$$f(x) = \int_x^\infty 2\exp\{-(x+y)\}dy = 2e^{-2x} \ \Rightarrow\ f_{2|1}(y|x) = \frac{f(x,y)}{f(x)} = e^{x-y} \quad 0 < x < y < \infty.$$

Hence,

$$E(Y|x) = \int_x^\infty y e^{x-y} dy = \int_0^\infty (x+t)e^{-t} dt = x + 1,\ x > 0.$$

2.3.10. Let X_1 and X_2 have the joint pmf $p(x1, x2)$ described as follows:

(x_1, x_2)	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(2,0)$	$(2,1)$
$p(x_1, x_2)$	$\frac{1}{18}$	$\frac{3}{18}$	$\frac{4}{18}$	$\frac{3}{18}$	$\frac{6}{18}$	$\frac{1}{18}$

and $p(x_1, x_2)$ is equal to zero elsewhere. Find the two marginal probability mass functions and the two conditional means.
Hint: Write the probabilities in a rectangular array.

Solution.

$$p(x_1) = \begin{cases} \frac{11}{18} & x_2 = 0 \\ \frac{7}{18} & x_2 = 1 \end{cases}, \quad p(x_2) = \begin{cases} \frac{4}{18} & x_1 = 0 \\ \frac{7}{18} & x_1 = 1 \\ \frac{7}{18} & x_1 = 2 \end{cases},$$
$$E(X_1|X_2 = x_2) = \begin{cases} \frac{16}{18} & x_2 = 0 \\ \frac{5}{18} & x_2 = 1 \end{cases}, \quad E(X_2|X_1 = x_1) = \begin{cases} \frac{3}{18} & x_1 = 0 \\ \frac{3}{18} & x_1 = 1 \\ \frac{1}{18} & x_1 = 2 \end{cases}.$$

2.3.11. Let us choose at random a point from the interval $(0,1)$ and let the random variable X_1 be equal to the number that corresponds to that point. Then choose a point at random from the interval $(0,x_1)$, where x_1 is the experimental value of X_1; and let the random variable X_2 be equal to the number that corresponds to this point.

(a) Make assumptions about the marginal pdf $f_1(x_1)$ and the conditional pdf $f_{2|1}(x_2|x_1)$.

Solution.

Assume that $X_1 \sim U(0,1)$ and $X_2|X_1 = x_1 \sim U(0,x_2)$:

$$f(x_1) = I(0 < x_1 < 1), \quad f(x_2|x_1) = \frac{1}{x_1} I(0 < x_2 < x_1).$$

(b) Compute $P(X_1 + X_2 \geq 1)$.

Solution.

By (a), $f_{1,2}(x_1,x_2) = f(x_2|x_1)f(x_1) = 1/x_1$, $0 < x_2 < x_1 < 1$. Hence,

$$P(X_1 + X_2 \geq 1) = P(X_2 \geq 1 - X_1) = \int_{1/2}^{1} \int_{1-x_1}^{x_1} \frac{1}{x_1} dx_2 dx_1 = \int_{1/2}^{1} \left(2 - \frac{1}{x_1}\right) dx_1 = 1 - \log 2.$$

(c) Find the conditional mean $E(X_1|x_2)$

Solution.

Find $f(x_2)$ to get $f(x_1|x_2)$.

$$f(x_2) = \int_{x_2}^{1} \frac{1}{x_1} dx_1 = -\log x_2,\ 0 < x_2 < 1 \ \Rightarrow \ f(x_1|x_2) = \frac{f(x_1,x_2)}{f(x_2)} = -\frac{1}{x_1 \log x_2},\ \ 0 < x_2 < x_1 < 1.$$

Hence,

$$E(X_1|X_2 = x_2) = \int_{x_2}^{1} -\frac{1}{\log x_2} dx_1 = \frac{1 - x_2}{\log(1/x_2)},\ \ 0 < x_2 < 1.$$

2.3.12. Let $f(x)$ and $F(x)$ denote, respectively, the pdf and the cdf of the random variable X. The conditional pdf of X, given $X > x_0$, x_0 a fixed number, is defined by $f(x|X > x_0) = f(x)/[1 - F(x_0)]$, $x_0 < x$, zero elsewhere. This kind of conditional pdf finds application in a problem of time until death, given survival until time x_0.

(a) Show that $f(x|X > x_0)$ is a pdf.

Solution.

Since $f(x) > 0$ and $0 < F(x) < 1$, $f(x|X > x_0) = f(x)/[1 - F(x_0)] > 0$. Also,

$$\int_{x_0}^{\infty} f(x|X > x_0) dx = \int_{x_0}^{\infty} \frac{f(x)}{[1 - F(x_0)]} dx = \frac{1}{[1 - F(x_0)]} [F(x)]_{x_0}^{\infty} = 1 \quad \text{since } F(\infty) = 1.$$

(b) Let $f(x) = e^{-x}$, $0 < x < \infty$, and zero elsewhere. Compute $P(X > 2|X > 1)$.

Solution.

Since $F(x) = 1 - e^{-x}, x > 0$, $f(x|X > 1) = f(x)/[1 - F(1)] = e^{-x+1}$. Hence,

$$P(X > 2|X > 1) = \int_{2}^{\infty} f(x|X > 1) dx = \int_{2}^{\infty} e^{-x+1} dx = [-e^{-x+1}]_2^{\infty} = e^{-1}.$$

2.4 Independent Random Variables

2.4.1. Show that the random variables X_1 and X_2 with joint pdf

$$f(x_1, x_2) = \begin{cases} 12x_1x_2(1-x_2) & 0 < x_1 < 1,\ 0 < x_2 < 1 \\ 0 & \text{elsewhere} \end{cases}$$

are independent.

Solution.

The support is rectangular (a product space). And $f(x_1, x_2)$ can be written as a product of a nonnegative function of x_1 and a nonnegative function of x_2 : $f(x_1, x_2) \equiv g(x_1)h(x_2)$, where $g(x_1) = 12x_1 I(0 < x_1 < 1)$ and $h(x_2) = x_2(1 - x_2)I(0 < x_2 < 1)$. Thus, X_1 and X_2 are independent.

Another solution is $f(x_1, x_2) = f(x_1)f(x_2)$, where $f(x_1) = 2x_1$ and $f(x_2) = 6x_2(1 - x_2)$ are marginal pdfs of X_1 and X_2.

2.4.2. If the random variables X_1 and X_2 have the joint pdf $f(x_1, x_2) = 2e^{-x_1-x_2}$, $0 < x_1 < x_2$, $0 < x_2 < \infty$, zero elsewhere, show that X_1 and X_2 are dependent.

Solution.

Although the joint pdf can be expressed by a product of two nonnegative functions of x_1 and x_2, respectively, $0 < x_1 < x_2 < \infty$ is not a product space, which implies that X_1 and X_2 are dependent.

2.4.3. Let $p(x_1, x_2) = \frac{1}{16}$, $x_1 = 1, 2, 3, 4$, and $x_2 = 1, 2, 3, 4$, zero elsewhere, be the joint pmf of X_1 and X_2. Show that X_1 and X_2 are independent.

Solution.

The marginal pdfs of X_1 and X_2 are $p(x_1) = p(x_2) = 1/4$. So $p(x_1, x_2) = p(x_1)p(x_2)$ and the space is rectangular, which gives us X_1 and X_2 are independent.

2.4.4. Find $P(0 < X_1 < \frac{1}{3}\,, 0 < X_2 < \frac{1}{3})$ if the random variables X_1 and X_2 have the joint pdf $f(x_1, x_2) = 4x_1(1 - x_2)$, $0 < x_1 < 1$, $0 < x_2 < 1$, zero elsewhere.

Solution.

Since $f(x_1) = 2x_1,\ 0 < x_1 < 1$ and $f(x_2) = 2(1 - x_2),\ 0 < x_2 < 1$ and X_1 and X_2 are independent,

$$\begin{aligned} P\left(0 < X_1 < \frac{1}{3}, 0 < X_2 < \frac{1}{3}\right) &= P\left(0 < X_1 < \frac{1}{3}\right) P\left(0 < X_2 < \frac{1}{3}\right) \\ &= \left(\int_0^{1/3} 2x_1 dx_1\right)\left(\int_0^{1/3} 2(1-x_2)dx_2\right) \\ &= \left(\frac{1}{9}\right)\left(\frac{5}{9}\right) = \frac{5}{81}. \end{aligned}$$

2.4.5. Find the probability of the union of the events $a < X_1 < b$, $-\infty < X_2 < \infty$, and $-\infty < X_1 < \infty$, $c < X_2 < d$ if X_1 and X_2 are two independent variables with $P(a < X_1 < b) = \frac{2}{3}$ and $P(c < X_2 < d) = \frac{5}{8}$.

Solution.

$$\begin{aligned} &P(\{a < X_1 < b, \infty < X_2 < \infty\} \cup \{-\infty < X_1 < \infty,\ c < X_2 < d\}) \\ &= P(\{a < X_1 < b\} \cup \{c < X_2 < d\}) \\ &= P(a < X_1 < b) + P(c < X_2 < d) - P(\{a < X_1 < b\} \cap \{c < X_2 < d\}) \\ &= P(a < X_1 < b) + P(c < X_2 < d) - P(a < X_1 < b)P(c < X_2 < d) \\ &= \frac{2}{3} + \frac{5}{8} - \frac{2}{3}\left(\frac{5}{8}\right) = \frac{7}{8}. \end{aligned}$$

2.4.8. Let X and Y have the joint pdf $f(x,y) = 3x$, $0 < y < x < 1$, zero elsewhere. Are X and Y independent? If not, find $E(X|y)$.

Solution.

X and Y are not independent because the support $0 < y < x < 1$ is not rectangular (not a product space). So find $f(y)$ first: $f(y) = \int_y^1 3x dx = 3(1-y^2)/2$, $0 < y < 1$, zero elsewhere. Hence

$$E(X|y) = \int_{-\infty}^{\infty} x \frac{f(x,y)}{f(y)} dx = \int_y^1 \frac{2x^2}{(1-y^2)} dx = \frac{2(1-y^3)}{3(1-y^2)} = \frac{2(1+y+y^2)}{3(1+y)}, \ 0 < y < 1.$$

2.4.10. Let X and Y be random variables with the space consisting of the four points $(0,0)$, $(1,1)$, $(1,0)$, $(1,-1)$. Assign positive probabilities to these four points so that the correlation coefficient is equal to zero. Are X and Y independent?

Solution.

Assume the uniform distribution as shown below:

x_1, x_2	-1	0	1	$p_{X_1}(x_1)$
0	0	1/4	0	1/4
1	1/4	1/4	1/4	3/4
$p_{X_2}(x_2)$	1/4	1/2	1/4	

Then, correlation coefficient $\rho = 0$ because

$$E(X) = 3/4, \quad E(Y) = 0, \quad E(XY) = -1/4 + 1/4 = 0 \Rightarrow E(XY) - E(X)E(Y) = 0.$$

However, $P(X_1 = X_2 = 1) = 1/4 \neq 3/16 = p_{X_1}(1)p_{X_2}(1)$, meaning that X and Y are *not* independent.

2.4.11. Two line segments, each of length two units, are placed along the x-axis. The midpoint of the first is between $x = 0$ and $x = 14$ and that of the second is between $x = 6$ and $x = 20$. Assuming independence and uniform distributions for these midpoints, find the probability that the line segments overlap.

Solution.

Since $X_1 \sim U(0,14)$ and $X_2 \sim U(6,20)$, the joint pdf of X_1 and X_2 is $f(x_1,x_2) = 1/14^2$. The desired probability is

$$P(X_1 \geq X_2) = \int_6^{14} \int_6^{x_1} \frac{1}{14^2} dx_2 dx_1 = \frac{(x_1-6)^2}{2(14^2)}\Big|_6^{14} = \frac{8}{49}.$$

2.4.12. Cast a fair die and let $X = 0$ if 1, 2, or 3 spots appear, let $X = 1$ if 4 or 5 spots appear, and let $X = 2$ if 6 spots appear. Do this two independent times, obtaining X_1 and X_2. Calculate $P(|X_1 - X_2| = 1)$.

Solution.

$|X_1 - X_2| = 1$ when $(X_1, X_2) = (0,1), (1,0), (1,2), (2,1)$ with probabilities of 1/6, 1/6, 1/18, and 1/18, respectively. Hence the desired probability is $2(1/6 + 1/18) = 4/9$.

2.4.13. For X_1 and X_2 in Example 2.4.6, show that the mgf of $Y = X_1 + X_2$ is $e^{2t}/(2-e^t)^2$, $t < \log 2$, and then compute the mean and variance of Y.

Solution.

Let $t = t_1 = t_2$ then

$$M_Y(t) = M_{X_1,X_2}(t,t) = \left(\frac{e^t}{2-e^t}\right)^2 = \frac{e^{2t}}{(2-e^t)^2}, \quad t < \log 2.$$

Let $\psi(t) = \log M_Y(t) = 2t - 2\log(2 - e^t)$. Then

$$E(Y) = \psi'(0) = 2 + \frac{2e^t}{2 - e^t}\Big|_{t=0} = 4,$$
$$\mathrm{Var}(Y) = \psi''(0) = \frac{4e^t}{(2 - e^t)^2}\Big|_{t=0} = 4.$$

2.5. The Correlation Coefficient

2.5.1. Let the random variables X and Y have the joint pmf

(a) $p(x, y) = \frac{1}{3}$, $(x, y) = (0, 0), (1, 1), (2, 2)$, zero elsewhere.

(b) $p(x, y) = \frac{1}{3}$, $(x, y) = (0, 2), (1, 1), (2, 0)$, zero elsewhere.

(c) $p(x, y) = \frac{1}{3}$, $(x, y) = (0, 0), (1, 1), (2, 0)$, zero elsewhere.

In each case compute the correlation coefficient of X and Y.

Solution.

For (a) and (b), the scatter plots clearly show that $\rho = 1$ and $\rho = -1$, respectively.

For (c), since $E(X) = 1, E(Y) = \frac{1}{3}$, and $E(XY) = \frac{1}{3}$, $\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0$. Thus, $\rho = 0$.

2.5.3. Let $f(x, y) = 2$, $0 < x < y$, $0 < y < 1$, zero elsewhere, be the joint pdf of X and Y. Show that the conditional means are, respectively, $(1 + x)/2$, $0 < x < 1$, and $y/2$, $0 < y < 1$. Show that the correlation coefficient of X and Y is $\rho = \frac{1}{2}$.

Solution.

Find the marginal pdfs of X and Y first.

$$f(x) = \int_x^1 2dy = 2(1 - x),\ 0 < x < 1, \quad f(y) = \int_0^y 2dx = 2y,\ 0 < y < 1.$$

Hence,

$$E(Y|X = x) = \int_{-\infty}^{\infty} y f(y|x)dy = \int_{-\infty}^{\infty} y\frac{f(x, y)}{f(x)}dy = \int_x^1 \frac{y}{1 - x}dy = \frac{1 + x}{2}, \quad 0 < x < 1,$$
$$E(X|Y = y) = \int_{-\infty}^{\infty} x f(x|y)dy = \int_{-\infty}^{\infty} x\frac{f(x, y)}{f(y)}dy = \int_0^y \frac{x}{y}dy = \frac{y}{2}, \quad 0 < y < 1.$$

2.5.4. Show that the variance of the conditional distribution of Y, given $X = x$, in Exercise 2.5.3, is $(1 - x)^2/12$, $0 < x < 1$, and that the variance of the conditional distribution of X, given $Y = y$, is $y^2/12$, $0 < y < 1$.

Solution.

$$E(Y^2|X = x) = \int_x^1 \frac{y^2}{1 - x}dy = \frac{1 + x + x^2}{3}, \quad 0 < x < 1,$$
$$E(X^2|Y = y) = \int_0^y \frac{x^2}{y}dy = \frac{y^2}{3}, \quad 0 < y < 1.$$

Hence,

$$\mathrm{Var}(Y|X = x) = E(Y^2|X = x) - [E(Y|X = x)]^2 = \frac{1 + x + x^2}{3} - \frac{(1 + x)^2}{4} = \frac{(1 - x)^2}{12}, \quad 0 < x < 1,$$
$$\mathrm{Var}(X|Y = y) = E(X^2|Y = y) - [E(X|Y = y)]^2 = \frac{y^2}{3} - \frac{y^2}{4} = \frac{y^2}{12}, \quad 0 < y < 1.$$

2.5.5. Verify the results of equations (2.5.11) of this section.

Solution. See Exercise 2.5.8 because using $\psi(t_1, t_2)$ is easier to compute them.

2.5.6. Let X and Y have the joint pdf $f(x, y) = 1$, $-x < y < x$, $0 < x < 1$, zero elsewhere. Show that, on the set of positive probability density, the graph of $E(Y|x)$ is a straight line, whereas that of $E(X|y)$ is not a straight line.

Solution. Find the marginal pdfs of X and Y first.

$$f(x) = \int_{-x}^{x} dy = 2x,\ 0 < x < 1, \quad f(y) = \begin{cases} \int_y^1 dx = 1 - y & 0 < y < 1 \\ \int_0^1 dx = 1 & -1 < y \le 0 \end{cases}.$$

Hence,

$$E(Y|x) = \int_{-\infty}^{\infty} y f(y|x) dy = \int_{-\infty}^{\infty} y \frac{f(x,y)}{f(x)} dy = \int_{-x}^{x} \frac{y}{2x} dy = 0, \quad 0 < x < 1,$$
$$E(X|y) = \int_{-\infty}^{\infty} x f(x|y) dy = \int_{-\infty}^{\infty} x \frac{f(x,y)}{f(y)} dy = \begin{cases} \int_y^1 \frac{x}{1-y} dy = \frac{1+y}{2} & 0 < y < 1 \\ \int_0^1 x dy = \frac{1}{2} & -1 < y \le 0, \end{cases}$$

which means that the graph of $E(Y|x)$ is a straight line, whereas that of $E(X|y)$ is not a straight line.

2.5.8. Let $\psi(t_1, t_2) = \log M(t_1, t_2)$, where $M(t_1, t_2)$ is the mgf of X and Y. Show that

$$\frac{\partial \psi(0,0)}{\partial t_i},\ \frac{\partial^2 \psi(0,0)}{\partial t_i^2},\ i = 1, 2,$$

and

$$\frac{\partial^2 \psi(0,0)}{\partial t_1 t_2}$$

yield the means, the variances, and the covariance of the two random variables. Use this result to find the means, the variances, and the covariance of X and Y of Example 2.5.6.

Solution.

Note that $M(0, 0) = E(1) = 1$. When $i = 1$,

$$\frac{\partial \psi(0,0)}{\partial t_1} = \frac{\partial M(0,0)/\partial t_1}{M(0,0)} = \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f(x,y) dy dx = \int_{-\infty}^{\infty} x f(x) dx = E(X),$$
$$\frac{\partial^2 \psi(0,0)}{\partial t_1^2} = \frac{M(0,0)\partial^2 M(0,0)/\partial t_1^2 - [\partial M(0,0)/\partial t_1]^2}{M(0,0)^2} = E(X^2) - [E(X)]^2 = \mathrm{Var}(X).$$

Same for $i = 2$. And

$$\begin{aligned} \frac{\partial^2 \psi(0,0)}{\partial t_1 t_2} &= \frac{\partial}{\partial t_2} \frac{\partial M(0,0)/\partial t_1}{M(0,0)} \\ &= \frac{[\partial^2 M(0,0)/\partial t_1 t_2] M(0,0) - [\partial M(0,0)/\partial t_1][\partial M(0,0)/\partial t_2]}{M(0,0)^2} \\ &= E(XY) - E(X)E(Y) = \mathrm{Cov}(X, Y). \end{aligned}$$

Hence, for Example 2.5.6,

$$\begin{aligned} \psi(t_1, t_2) &= \log M(t_1, t_2) = -\log(1 - t_1 - t_2) - \log(1 - t_2), \\ \frac{\partial \psi(t_1,t_2)}{\partial t_1} &= \frac{1}{1 - t_1 - t_2}, \quad \frac{\partial \psi(t_1,t_2)}{\partial t_2} = \frac{1}{1 - t_1 - t_2} + \frac{1}{1 - t_2} \\ \frac{\partial^2 \psi(t_1,t_2)}{\partial t_1^2} &= \frac{1}{(1 - t_1 - t_2)^2}, \quad \frac{\partial^2 \psi(t_1,t_2)}{\partial t_2^2} = \frac{1}{(1 - t_1 - t_2)^2} + \frac{1}{(1 - t_2)^2} \\ \frac{\partial^2 \psi(t_1,t_2)}{\partial t_1 t_2} &= \frac{1}{(1 - t_1 - t_2)^2}. \end{aligned}$$

Therefore,

$$\mu_1 = E(X) = \frac{\partial\psi(0,0)}{\partial t_1} = 1, \quad \mu_2 = E(Y) = \frac{\partial\psi(0,0)}{\partial t_2} = 2$$
$$\sigma_1^2 = \mathrm{Var}(X) = \frac{\partial^2\psi(0,0)}{\partial t_1^2} = 1, \quad \sigma_2^2 = \mathrm{Var}(Y) = \frac{\partial^2\psi(0,0)}{\partial t_2^2} = 2$$
$$E[(X-\mu_1)(Y-\mu_2)] = \mathrm{Cov}(X,Y) = \frac{\partial^2\psi(0,0)}{\partial t_1 t_2} = 1.$$

2.5.9. Let X and Y have the joint pmf $p(x,y) = \frac{1}{7}$, $(0,0),(1,0),(0,1),(1,1),(2,1),(1,2),(2,2)$, zero elsewhere. Find the correlation coefficient ρ.

Solution.

$$E(X) = E(Y) = \frac{1+1+2+1+2}{7} = 1, \quad E(X^2) = E(Y^2) = \frac{1+1+4+1+4}{7} = \frac{11}{7}$$
$$\Rightarrow \sigma_X^2 = \sigma_Y^2 = \frac{11}{7} - 1 = \frac{4}{7}, \quad E(XY) = \frac{1+2+2+4}{7} = \frac{9}{7}.$$

Hence,

$$\rho = \frac{E(XY) - E(X)E(Y)}{\sigma_X\sigma_Y} = \frac{2/7}{4/7} = \frac{1}{2}.$$

2.5.11. Let $\sigma_1^2 = \sigma_2^2 = \sigma^2$ be the common variance of X_1 and X_2 and let ρ be the correlation coefficient of X_1 and X_2. Show for $k > 0$ that

$$P[|(X_1-\mu_1) + (X_2-\mu_2)| \geq k\sigma] \leq \frac{2(1+\rho)}{k^2}.$$

Solution.

$$\begin{aligned}
P[|(X_1-\mu_1) + (X_2-\mu_2)| \geq k\sigma] &= P[|(X_1-\mu_1) + (X_2-\mu_2)|^2 \geq k^2\sigma^2] \\
&= P[(X_1-\mu_1)^2 + (X_2-\mu_2)^2 + 2(X_1-\mu_1)(X_2-\mu_2) \geq k^2\sigma^2] \\
&\leq P[(X_1-\mu_1)^2 \geq k^2\sigma^2] + P[(X_2-\mu_2)^2 \geq k^2\sigma^2] \\
&\quad + P[2(X_1-\mu_1)(X_2-\mu_2) \geq k^2\sigma^2] \\
&= P(|X_1-\mu_1| \geq k\sigma) + P(|X_2-\mu_2| \geq k\sigma) \\
&\quad + P[2(X_1-\mu_1)(X_2-\mu_2) \geq k^2\sigma^2] \\
&\leq \frac{1}{k^2} + \frac{1}{k^2} + \frac{2E(X_1-\mu_1)(X_2-\mu_2)}{k^2\sigma^2} \\
&= \frac{2(1+\rho)}{k^2} \quad \text{since } \frac{E(X_1-\mu_1)(X_2-\mu_2)}{\sigma^2} = \rho.
\end{aligned}$$

2.6. Extension to Several Random Variables

2.6.1. Let X, Y, Z have joint pdf $f(x,y,z) = 2(x+y+z)/3$, $0 < x < 1$, $0 < y < 1$, $0 < z < 1$, zero elsewhere.

(a) Find the marginal probability density functions of X, Y, and Z.

Solution.

$$f_X(x) = \int_0^1\int_0^1 \frac{2(x+y+z)}{3}dzdy = \cdots = \frac{2(x+1)}{3}.$$

Similarly,

$$f_Y(y) = \frac{2(y+1)}{3}, \quad f_Z(z) = \frac{2(z+1)}{3}.$$

(b) Compute $P(0 < X < \frac{1}{2}, 0 < Y < \frac{1}{2}, 0 < Z < \frac{1}{2})$ and $P(0 < X < \frac{1}{2}) = P(0 < Y < \frac{1}{2}) = P(0 < Z < \frac{1}{2})$.

Solution. Skipped. We can solve part (c) without computing them.

(c) Are X, Y, and Z independent?

Solution. No; $f(x, y, x) \neq f(x)f(y)f(z)$ although the support is a product space.

(d) Compute $E(X^2YZ + 3XY^4Z^2)$.

Solution. Skipped.

(e) Determine the cdf of X, Y, and Z.

Solution.

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x \frac{2(t+1)}{3} dt = \frac{(x+1)^2 - 1}{3} = \frac{x^2+2x}{3} & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}.$$

Similarly,

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ \frac{y^2+2y}{3} & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}, \quad F_Z(z) = \begin{cases} 0 & z \leq 0 \\ \frac{z^2+2z}{3} & 0 < z < 1 \\ 1 & z \geq 1 \end{cases}.$$

(f) Find the conditional distribution of X and Y, given $Z = z$, and evaluate $E(X + Y|z)$.

Solution.

$$f(x, y|z) = \frac{f(x, y, z)}{f(z)} = \frac{x + y + z}{z + 1}, \; 0 < x < 1, \; 0 < y < 1.$$

Hence,

$$\begin{aligned}
E(X + Y|z) &= \int_0^1 \int_0^1 (x + y)\frac{x + y + z}{z + 1} dydx \\
&= \int_0^1 \int_0^1 \frac{(x + y)^2 + z(x + y)}{z + 1} dydx \\
&= \frac{1}{z + 1} \int_0^1 \left[\frac{(x + y)^3}{3} + \frac{z(x + y)^2}{2}\right]_{y=0}^{y=1} dx \\
&= \frac{1}{z + 1} \int_0^1 \left[\frac{(x + 1)^3}{3} + \frac{z(x + 1)^2}{2} - \frac{x^3}{3} - \frac{zx^2}{2}\right] dx \\
&= \frac{1}{z + 1} \left[\frac{(x + 1)^4}{12} + \frac{z(x + 1)^3}{6} - \frac{x^4}{12} - \frac{zx^3}{6}\right]_0^1 \\
&= \frac{z + 7/6}{z + 1} = \frac{6z + 7}{6(z + 1)}, \; 0 < z < 1.
\end{aligned}$$

(g) Determine the conditional distribution of X, given $Y = y$ and $Z = z$, and compute $E(X|y, z)$.

Solution.

$$f(y, z) = \int_0^1 \frac{2(x + y + z)}{3} dx = \frac{2y + 2z + 1}{3}$$
$$f(x|y, z) = \frac{f(x, y, z)}{f(y, z)} = \frac{2(x + y + z)}{2y + 2z + 1}.$$

Hence,

$$E(X|y,z) = \int_0^1 x\frac{2(x+y+z)}{2y+2z+1}dx = \int_0^1 \frac{2x^2+2x(y+z)}{2y+2z+1} = \cdots = \frac{3y+3z+2}{3(2y+2z+1)}, \quad 0<y,z<1.$$

2.6.2. Let $f(x_1,x_2,x_3) = \exp[-(x_1+x_2+x_3)]$, $0<x_1<\infty$, $0<x_2<\infty$, $0<x_3<\infty$, zero elsewhere, be the joint pdf of X_1, X_2, X_3.

(a) Compute $P(X_1<X_2<X_3)$ and $P(X_1=X_2<X_3)$.

Solution.

$$\begin{aligned}
P(X_1<X_2<X_3) &= \int_0^\infty \int_0^{x_3} \int_0^{x_2} e^{-x_1-x_2-x_3} dx_1dx_2dx_3 \\
&= \int_0^\infty \int_0^{x_3} [e^{-x_2-x_3} - e^{-2x_2-x_3}] dx_2dx_3 \\
&= \int_0^\infty [(e^{-x_3} - e^{-2x_3}) - (e^{-x_3}/2 - e^{-3x_3}/2)] dx_3 \\
&= (1-1/2)-(1/2-1/6) = \frac{1}{6}, \\
P(X_1=X_2<X_3) &= \int_0^\infty \int_0^{x_3} \int_{x_2}^{x_2} e^{-x_1-x_2-x_3} dx_1dx_2dx_3 = 0.
\end{aligned}$$

(b) Determine the joint mgf of X_1, X_2, and X_3. Are these random variables independent?

Solution.

$$\begin{aligned}
M(t_1,t_2,t_3) &= \int_0^\infty \int_0^\infty \int_0^\infty e^{-(1-t_1)x_1} e^{-(1-t_2)x_2} e^{-(1-t_3)x_3} dx_1dx_2dx_3 \\
&= \int_0^\infty e^{-(1-t_1)x_1} dx_1 \int_0^\infty e^{-(1-t_2)x_2} dx_2 \int_0^\infty e^{-(1-t_3)x_3} dx_3 \\
&= \frac{1}{(1-t_1)(1-t_2)(1-t_3)}, \; t_1<1, t_2<1, t_3<1 \\
&= M_{X_1}(t_1)M_{X_2}(t_2)M_{X_3}(t_3),
\end{aligned}$$

which clearly shows that these three random varialbes are independent.

2.6.7. Prove Corollary 2.6.1: Suppose $X_1, X_2, ..., X_n$ are iid random variables with the common mgf $M(t)$, for $-h<t<h$, where $h>0$. Let $T=\sum_{i=1}^n X_i$. Then T has the mgf given by

$$M_T(t) = [M(t)]^n, \quad -h<t<h.$$

Solution.

$$\begin{aligned}
M_T(t) = E\left[e^{\sum_{i=1}^n X_i t}\right] &= \prod_{i=1}^n E(e^{X_i t}) \quad (X_i's \text{ are independent}) \\
&= [E(e^{Xt})]^n \quad (X_i's \text{ are identical}) \\
&= [M_X(t)]^n.
\end{aligned}$$

2.6.9. Let X_1, X_2, X_3 be iid with common pdf $f(x) = \exp(-x)$, $0<x<\infty$, zero elsewhere. Evaluate:

(a) $P(X_1<X_2|X_1<2X_2)$.

Solution.

$$P(X_1<X_2|X_1<2X_2) = \frac{P(X_1<X_2, X_1<2X_2)}{P(X_1<2X_2)} = \frac{P(X_1<X_2)}{P(X_1<2X_2)}.$$

For the numerator,

$$P(X_1 < X_2) = \int_0^\infty \int_{x_1}^\infty e^{-x_1-x_2} dx_2 dx_1 = \int_0^\infty e^{-2x_1} dx_2 = \frac{1}{2}.$$

For the denominator,

$$P(X_1 < 2X_2) = \int_0^\infty \int_{x_1/2}^\infty e^{-x_1-x_2} dx_2 dx_1 = \int_0^\infty e^{-3x_1/2} dx_2 = \frac{2}{3}.$$

Thus, $P(X_1 < X_2 | X_1 < 2X_2) = \frac{1/2}{2/3} = \frac{3}{4}$.

(b) $P(X_1 < X_2 < X_3 | X_3 < 1)$.

Solution.

$$P(X_1 < X_2 < X_3 | X_3 < 1) = \frac{P(X_1 < X_2 < X_3 < 1)}{P(X_3 < 1)}.$$

For the numerator,

$$\begin{aligned}
P(X_1 < X_2 < X_3 < 1) &= \int_0^1 \int_0^{x_3} \int_0^{x_2} e^{-x_1-x_2-x_3} dx_1 dx_2 dx_3 \\
&= \int_0^1 \int_0^{x_3} [e^{-x_2-x_3} - e^{-2x_2-x_3}] dx_2 dx_3 \\
&= \int_0^1 [(e^{-x_3} - e^{-2x_3}) - (e^{-x_3}/2 - e^{-3x_3}/2)] dx_3 \\
&= \int_0^1 [e^{-x_3}/2 - e^{-2x_3} + e^{-3x_3}/2)] dx_3 \\
&= \frac{1-e^{-1}}{2} - \frac{1-e^{-2}}{2} + \frac{1-e^{-3}}{6} \\
&= \frac{1 - 3e^{-1} + 3e^{-2} - e^{-3}}{6}
\end{aligned}$$

For the denominator,

$$P(X_3 < 1) = \int_0^1 e^{-x_3} dx_3 = 1 - e^{-1}.$$

Hence

$$P(X_1 < X_2 < X_3 | X_3 < 1) = \frac{P(X_1 < X_2 < X_3 < 1)}{P(X_3 < 1)} = \frac{1 - 3e^{-1} + 3e^{-2} - e^{-3}}{6(1-e^{-1})} \approx 0.0666.$$

2.7. Transformations for Several Random Variables

Skipped because of a just extension from two random variables.

2.8. Linear Combinations of Random Variables

2.8.3. Let X_1 and X_2 be two independent random variables so that the variances of X_1 and X_2 are $\sigma_1^2 = k$ and $\sigma_2^2 = 2$, respectively. Given that the variance of $Y = 3X_2 - X_1$ is 25, find k.

Solution.

$$\begin{aligned}
\text{Var}(Y) &= 3^2 \text{Var}(X_2) + \text{Var}(X_1) \quad X_1, X_2 \text{ are independent} \\
&= 9\sigma_2^2 + \sigma_1^2 = 18 + k.
\end{aligned}$$

Hence, $\text{Var}(Y) = 25 \Rightarrow k = 7$.

2.8.6. Determine the mean and variance of the sample mean X=$5^{-1}\sum_{i=1}^{5} X_i$, where $X_1, \ldots, X_5$ is a random sample from a distribution having pdf $f(x) = 4x^3$, $0 < x < 1$, zero elsewhere.

Solution.

$$E(X) = \int_0^1 x(4x^3)dx = \frac{4}{5}, \quad E(X^2) = \int_0^1 x^2(4x^3)dx = \frac{2}{3} \Rightarrow \text{Var}(X) = \frac{2}{75}.$$

Hence,

$$E(\bar{X}) = E(X) = \frac{4}{5} = 0.8, \quad \text{Var}(\bar{X}) = \frac{\text{Var}(X)}{5} = \frac{2}{375} \approx 0.00533.$$

2.8.7. Let X and Y be random variables with $\mu_1 = 1$, $\mu_2 = 4$, $\sigma_1^2 = 4$, $\sigma_2^2 = 6$, $\rho = \frac{1}{2}$. Find the mean and variance of the random variable $Z = 3X - 2Y$.

Solution.

$$\begin{aligned} E(Z) &= 3E(X) - 2E(Y) = 3\mu_1 - 2\mu_2 = -5 \\ \text{Var}(Z) &= 3^2\text{Var}(X) + 2^2\text{Var}(Y) - 12\text{Cov}(X, Y) \\ &= 9\sigma_1^2 + 4\sigma_2^2 - 12\rho\sigma_1\sigma_2 \\ &= 60 - 12\sqrt{6} \approx 30.6. \end{aligned}$$

2.8.8. Let X and Y be independent random variables with means μ_1, μ_2 and variances σ_1^2, σ_2^2. Determine the correlation coefficient of X and $Z = X - Y$ in terms of $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$.

Solution. Since X and Y are independent,

$$\begin{aligned} \text{Var}(Z) &= \text{Var}(X) + \text{Var}(Y) = \sigma_1^2 + \sigma_2^2, \\ \text{Cov}(X, Z) &= \text{Cov}(X, X - Y) = \text{Var}(X) - \text{Cov}(X, Y) = \sigma_1^2. \end{aligned}$$

Hence, the correlation coefficient is

$$\rho = \frac{\text{Cov}(X, Z)}{\sqrt{\text{Var}(X)\text{Var}(Z)}} = \frac{\sigma_1^2}{\sqrt{\sigma_1^2(\sigma_1^2 + \sigma_2^2)}} = \frac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2}}.$$

2.8.10. Determine the correlation coefficient of the random variables X and Y if $\text{var}(X) = 4, \text{var}(Y) = 2$, and $\text{var}(X + 2Y) = 15$.

Solution.

$$15 = \text{Var}(X + 2Y) = \text{Var}(X) + 4\text{Var}(Y) + 4\text{Cov}(X, Y) = 4 + 4(2) + 4\rho\sqrt{4}\sqrt{2} = 12 + 8\sqrt{2}\rho.$$

Hence, $\rho = 3/(8\sqrt{2}) \approx 0.265$.

2.8.11. Let X and Y be random variables with means μ_1, μ_2; variances σ_1^2, σ_2^2; and correlation coefficient ρ. Show that the correlation coefficient of $W = aX + b$, $a > 0$, and $Z = cY + d$, $c > 0$, is ρ.

Solution.

$$\text{Var}(W) = a^2\text{Var}(X) = a^2\sigma_1^2, \quad \text{Var}(Z) = c^2\text{Var}(Y) = c^2\sigma_2^2, \quad \text{Cov}(W, Z) = ac\text{Cov}(X, Y) = ac\rho\sigma_1\sigma_2.$$

Hence, $\text{Corr}(W, Z) = \text{Cov}(W, Z)/(\sqrt{\text{Var}(W)\text{Var}(Z)}) = \rho$ because $a > 0$ and $c > 0$.

2.8.13. Let X_1 and X_2 be independent random variables with nonzero variances. Find the correlation coefficient of $Y = X_1X_2$ and X_1 in terms of the means and variances of X_1 and X_2.

Solution.

Let μ_1, μ_2 and σ_1^2, σ_2^2 denote the means and the variances of X_1 and X_2, respectively. Since the two r.v.s. are independent,

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{Var}(X_1X_2)\\
&= E(X_1^2X_2^2) - E(X_1X_2)^2\\
&= E(X_1^2)E(X_2^2) - E(X_1)^2E(X_2)^2\\
&= (\mu_1^2+\sigma_1^2)(\mu_2^2+\sigma_2^2) - \mu_1^2\mu_2^2\\
&= \mu_1^2\sigma_2^2 + \sigma_1^2\mu_2^2 + \sigma_1^2\sigma_2^2,\\
\mathrm{Cov}(Y, X_1) &= \mathrm{Cov}(X_1X_2, X_1)\\
&= E(X_1^2X_2) - E(X_1X_2)E(X_1)\\
&= E(X_1^2)E(X_2) - E(X_1)^2E(X_2)\\
&= (\mu_1^2+\sigma_1^2)\mu_2 - \mu_1^2\mu_2\\
&= \sigma_1^2\mu_2
\end{aligned}$$

Hence,

$$\rho = \frac{\mathrm{Cov}(Y, X_1)}{\sqrt{\mathrm{Var}(Y)\mathrm{Var}(X_1)}} = \frac{\sigma_1^2\mu_2}{\sqrt{\mu_1^2\sigma_2^2+\sigma_1^2\mu_2^2+\sigma_1^2\sigma_2^2}(\sigma_1)} = \frac{\sigma_1\mu_2}{\sqrt{\mu_1^2\sigma_2^2+\sigma_1^2\mu_2^2+\sigma_1^2\sigma_2^2}}.$$

2.8.15. Let X_1, X_2, and X_3 be random variables with equal variances but with correlation coefficients $\rho_{12} = 0.3$, $\rho_{13} = 0.5$, and $\rho_{23} = 0.2$. Find the correlation coefficient of the linear functions $Y = X_1 + X_2$ and $Z = X_2 + X_3$.

Solution.

Let σ^2 denote the variance of X_1, X_2, and X_3. Then

$$\begin{aligned}
\mathrm{Var}(Y) &= \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2\mathrm{Cov}(X_1, X_2) = 2\sigma^2(1+\rho_{12}) = 2.6\sigma^2,\\
\mathrm{Var}(Z) &= \mathrm{Var}(X_2) + \mathrm{Var}(X_3) + 2\mathrm{Cov}(X_2, X_3) = 2\sigma^2(1+\rho_{23}) = 2.4\sigma^2,\\
\mathrm{Cov}(Y, Z) &= \mathrm{Cov}(X_1+X_2, X_2+X_3) = \sigma^2(\rho_{12}+\rho_{13}+1+\rho_{23}) = 2\sigma^2.
\end{aligned}$$

Therefore, the correlation coefficient, ρ, is

$$\rho = \frac{\mathrm{Cov}(Y, Z)}{\sqrt{\mathrm{Var}(Y)\mathrm{Var}(Z)}} = \frac{2\sigma^2}{\sqrt{2.6(2.4)}\sigma^2} \approx 0.801.$$

2.8.17. Let X and Y have the parameters $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$, and ρ. Show that the correlation coefficient of X and $[Y - \rho(\sigma_2/\sigma_1)X]$ is zero.

Solution.

$$\mathrm{Cov}(X, Y - \rho(\sigma_2/\sigma_1)X) = \mathrm{Cov}(X, Y) - \rho(\sigma_2/\sigma_1)\mathrm{Var}(X) = \rho\sigma_1\sigma_2 - \rho(\sigma_2/\sigma_1)\sigma_1^2 = 0.$$