Exercises in Introduction to Mathematical Statistics (Ch. 8)

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Note

- Not all Solution.s are provided: exercises that are too simple or not very important to me are skipped.
- Texts in red are just attentions to me. Please ignore them.

8 Optimal Tests of Hypotheses

Note that I use the reverise definition:

$$\frac{L(\theta''; \mathbf{x})}{L(\theta'; \mathbf{x})} \geq k$$

because I learned this in a class.

8.1. Most Powerful Tests

Let $k > 0$ and $K > 0$ in this section.

8.1.1. In Example 8.1.2 of this section, let the simple hypotheses read $H_0 : \theta = \theta' = 0$ and $H_1 : \theta = \theta'' = -1$. Show that the best test of H_0 against H_1 may be carried out by use of the statistic $\overline{X}$, and that if $n = 25$ and $\alpha = 0.05$, the power of the test is 0.9996 when H_1 is true.

Solution.

$$\frac{L(-1)}{L(0)} = \exp\left[-\sum x_i - \frac{n}{2}\right] \geq k \ \Rightarrow \ \sum_1^n x_i \leq -\log k - \frac{n}{2} = k' \ \Rightarrow \ \overline{x} \leq c.$$

Hence, the best critical region is

$$C = \{\mathbf{x} : \overline{x} \leq c\}.$$

When $n = 25$ and $\alpha = 0.05$, since $\overline{X} \sim N(0, 1/25)$ under H_0,

$$0.05 = P_{H_0}\left(\overline{X} \leq c\right) = P_{H_0}\left(5\overline{X} \leq 5c\right) = \Phi(5c) \ \Rightarrow \ 5c = -1.645 \ \Rightarrow \ c = -0.329.$$

On the other hand, since $\overline{X} \sim N(-1, 1/25)$ under H_1,

$$1 - \beta = P_{H_1}\left(\overline{X} \leq 0.329\right) P_{H_1}\left(5(\overline{X} + 1) \leq 3.355\right) = \Phi(3.355) = 0.9996.$$

8.1.2. Let the random variable X have the pdf $f(x; \theta) = (1/\theta)e^{-x/\theta}$, $0 < x < \infty$, zero elsewhere. Consider the simple hypothesis $H_0 : \theta = \theta' = 2$ and the alternative hypothesis $H_1 : \theta = \theta'' = 4$. Let X_1, X_2 denote a random sample of size 2 from this distribution. Show that the best test of H_0 against H_1 may be carried out by use of the statistic $X_1 + X_2$.

Solution.

$$\frac{L(4)}{L(2)} \geq k \;\Rightarrow\; x_1 + x_2 \geq c = \chi^2_{4,1-\alpha}.$$

because $X_1 + X_2 \sim \Gamma(2,2) = \chi^2(4)$ under $H_0 : \theta = 2$.

8.1.3. Repeat Exercise 8.1.2 when $H_1 : \theta = \theta'' = 6$. Generalize this for every $\theta'' > 2$.

Solution.

Show only the general case: $\theta'' > 2$.

$$\frac{L(\theta'')}{L(2)} \geq k \;\Rightarrow\; (\theta'' - 2)(x_1 + x_2) \geq k' \;\Rightarrow\; x_1 + x_2 \geq c = \chi^2_{4,1-\alpha}$$

under H_0, which is consistent with the precious exercise.

8.1.4. Let $X_1, X_2, ..., X_{10}$ be a random sample of size 10 from a normal distribution $N(0, \sigma^2)$. Find a best critical region of size $\alpha = 0.05$ for testing $H_0 : \sigma^2 = 1$ against $H_1 : \sigma^2 = 2$. Is this a best critical region of size 0.05 for testing $H_0 : \sigma^2 = 1$ against $H_1 : \sigma^2 = 4$? against $H_1 : \sigma^2 = \sigma_1^2 > 1$?

Solution.

Show only the general case: $\sigma^2 = \sigma_1^2 > 1$.

$$\frac{L(\sigma_1^2)}{L(1)} \geq k \;\Rightarrow\; (\sigma_1^2 - 1)\sum_1^{10} x_i^2 \geq k' \;\Rightarrow\; \sum_1^{10} x_i^2 \geq c = \chi^2_{10,\;0.95} = 18.30$$

because $X_i \sim N(0,1) \Rightarrow \sum_1^{10} X_i^2 \sim \chi^2_{10}$ under H_0.

8.1.5. If $X_1, X_2, ..., X_n$ is a random sample from a distribution having pdf of the form $f(x;\theta) = \theta x^{\theta-1}$, $0 < x < 1$, zero elsewhere, show that a best critical region for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$ is $C = \{(x_1, x_2, ..., x_n) : c \leq \prod_{i=1}^n x_i\}$.

Solution.

$$\frac{L(2)}{L(1)} = 2^n \prod_1^n x_i \geq k \;\Rightarrow\; \prod_1^n x_i \geq c,$$

where $c = 2^{-n}k$.

8.1.6. Let $X_1, X_2, ..., X_{10}$ be a random sample from a distribution that is $N(\theta 1, \theta 2)$. Find a best test of the simple hypothesis $H_0 : \theta_1 = \theta_1' = 0, \theta_2 = \theta_2' = 1$ against the alternative simple hypothesis $H_1 : \theta_1 = \theta_1'' = 1, \theta_2 = \theta_2'' = 4$.

Solution.

$$\frac{L(1,4)}{L(0,1)} \geq k \;\Rightarrow\; \sum_1^{10}(3x_i - 1)(x_i + 1) \geq c \text{ or } \sum_1^{10}(3x_i^2 + 2x_i) \geq c'.$$

8.1.8. If $X_1, X_2, ..., X_n$ is a random sample from a beta distribution with parameters $\alpha = \beta = \theta > 0$, find a best critical region for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$.

Solution.

$$\frac{L(2)}{L(1)} = K \prod_1^n x_i(1 - x_i) \geq k \;\Rightarrow\; \prod_1^n x_i(1 - x_i) \geq c,$$

where $K = B(1,1)/B(2,2)$ that we do not have to compute.

8.1.9. Let $X_1, X_2, ..., X_n$ be iid with pmf $f(x;p) = p^x(1-p)^{1-x}$, $x = 0, 1$, zero elsewhere. Show that $C = \{(x_1, ..., x_n) : \sum_1^n x_i \leq c\}$ is a best critical region for testing $H_0 : p = \frac{1}{2}$ against $H_1 : p = \frac{1}{3}$. Use the Central Limit Theorem to find n and c so that approximately $P_{H_0}(\sum_1^n X_i \leq c) = 0.10$ and $P_{H_1}(\sum_1^n X_i \leq c) = 0.80$.

Solution.

$$\frac{L(1/3)}{L(1/2)} = \frac{(1/3)^{\sum x_i}(2/3)^{n-\sum x_i}}{(2/3)^{\sum x_i}(1/3)^{n-\sum x_i}} = 2^n 4^{-\sum x_i} \geq k \;\Rightarrow\; \sum_1^n x_i \leq c.$$

Since $\sum_1^n X_i \sim b(n,p)$, using the CLT to obtain $\sum_1^n X_i \overset{D}{\sim} N(np, np(1-p))$. We can find n and c by solving

$$\frac{c - n/2}{\sqrt{n/4}} = -1.28, \qquad \frac{c - n/3}{\sqrt{2n/9}} = 0.84.$$

In fact, $n = 38.6 \approx 39$ and $c = 15.34 \approx 15$.

8.1.10. Let $X_1, X_2, ..., X_{10}$ denote a random sample of size 10 from a Poisson distribution with mean θ. Show that the critical region C defined by $\sum_1^{10} x_i \geq 3$ is a best critical region for testing $H_0 : \theta = 0.1$ against $H_1 : \theta = 0.5$. Determine, for this test, the significance level α and the power at $\theta = 0.5$. Use the R function `ppois`.

Solution.

$$\frac{L(0.5)}{L(0.1)} = e^{-4} 5^{\sum_1^{10} x_i} \geq k \;\Rightarrow\; \sum_1^{10} x_i \geq c.$$

Given that $c = 3$ and $\sum_1^{10} X_i \sim \text{Poisson}(10\theta)$,

$$\alpha = P_{H_0}\left(\sum_1^{10} X_i \geq 3\right) = \texttt{1 - ppois(2,1)} = 0.08$$

$$1 - \beta = P_{H_1}\left(\sum_1^{10} X_i \geq 3\right) = \texttt{1 - ppois(2,5)} = 0.875.$$

8.2 Uniformly Most Powerful Tests

8.2.1. Let X have the pmf $f(x;\theta) = \theta^x(1-\theta)^{1-x}$, $x = 0, 1$, zero elsewhere. We test the simple hypothesis $H_0 : \theta = \frac{1}{4}$ against the alternative composite hypothesis $H_1 : \theta < \frac{1}{4}$ by taking a random sample of size 10 and rejecting $H_0 : \theta = \frac{1}{4}$ if and only if the observed values $x_1, x_2, ..., x_{10}$ of the sample observations are such that $\sum_1^{10} x_i \leq 1$. Find the power function $\gamma(\theta), 0 < \theta \leq \frac{1}{4}$, of this test.

Solution.

Use NP theorem to obtain. Let $\theta' < \frac{1}{4}$.

$$\frac{L(\theta')}{L(1/4)} = \cdots = K\left(\frac{3\theta'}{1-\theta'}\right)^{\sum_1^{10} x_i} \geq k \;\Rightarrow\; \sum_1^{10} x_i \log\left(\frac{3\theta'}{1-\theta'}\right) \geq \log k - \log K$$

$$\Rightarrow\; \sum_1^{10} x_i \leq c \quad \text{since } 0 < \frac{3\theta'}{1-\theta'} < 1 \;\left(\theta' < \frac{1}{4}\right).$$

Let $\Omega = \{\theta \leq \frac{1}{4}\}$. Then $Y = \sum_1^{10} x_i \sim b(10, \theta)$ under Ω, when $c = 1$,

$$r(\theta) = P_\Omega(Y \leq 1) = P_\Omega(Y = 0) + P_\Omega(Y = 1) = (1-\theta)^{10} + 10\theta(1-\theta)^9 = (1-\theta)^9(1+9\theta).$$

8.2.2. Let X have a pdf of the form $f(x;\theta) = 1/\theta$, $0 < x < \theta$, zero elsewhere. Let $Y_1 < Y_2 < Y_3 < Y_4$ denote the order statistics of a random sample of size 4 from this distribution. Let the observed value of Y_4 be y_4.

We reject $H_0 : \theta = 1$ and accept $H_1 : \theta \neq 1$ if either $y_4 \leq \frac{1}{2}$ or $y_4 > 1$. Find the power function $\gamma(\theta)$, $0 < \theta$, of the test.

Solution.

By the previous exercise, we have

$$F_{Y_4}(y_4) = \frac{y_4^4}{\theta^4},\ 0 < y_4 < \theta.$$

Hence,

$$\gamma(\theta) = P\left(Y_4 \leq \frac{1}{2}\right) + P\left(Y_4 > 1\right) = F_{Y_4}(1/2) + (1 - F_{Y_4}(1)) = 1 - \frac{15}{16\theta^4},\ \theta > 0.$$

8.2.6. If, in Example 8.2.2 of this section, $H_0 : \theta = \theta'$, where θ' is a fixed positive number, and $H_A : \theta \neq \theta'$, show that there is no uniformly most powerful test for testing H_0 against H_1

Solution.

If $\theta'' > \theta$, then we want to use a critical region of the from $\sum x_i^2 > c$. If $\theta'' < \theta$, the critical region is like $\sum x_i^2 < c$. That is, we cannot find one test that will be best for each type of alternative.

8.2.7. Let $X_1, X_2, ..., X_{25}$ denote a random sample of size 25 from a normal distribution $N(\theta, 100)$. Find a uniformly most powerful critical region of size $\alpha = 0.10$ for testing $H_0 : \theta = 75$ against $H_1 : \theta > 75$.

Solution.

Let $\theta' > 75$. Use NP theorem:

$$\begin{aligned} \frac{L(\theta')}{L(75)} &= \exp\left[\sum (x_i - 75)^2/200 - \sum (x_i - \theta')^2/200\right] \\ &= \exp\left[(\theta' - 75)\sum (2x_i - 75 - \theta')/200\right] \geq k. \end{aligned}$$

Hence,

$$\begin{aligned} &(\theta' - 75)\sum (2x_i - 75 - \theta')/200 \geq \log k \\ &\Rightarrow \sum (2x_i - 75 - \theta') \geq 200 \log k/(\theta' - 75) \\ &\Rightarrow \sum 2x_i \geq 200 \log k/(\theta' - 75) + n(75 + \theta') \\ &\Rightarrow \sum x_i \geq 100 \log k/(\theta' - 75) + n(75 + \theta')/2 = k' \\ &\Rightarrow \overline{x} \geq c \end{aligned}$$

for every $\theta' > 75$. Hence $C = \{\mathbf{x} : \overline{x} \geq c\}$ is the UMP critical region. Furthermore, since $\overline{X} \sim N(75, 4)$,

$$\overline{X} \geq c \ \Rightarrow\ \frac{\overline{X} - 75}{2} \geq \frac{c - 75}{2} \ \Rightarrow\ \frac{c - 75}{2} = z_{0.90} = 1.28 \ \Rightarrow\ c = 75 + 2.56 = 77.56.$$

That is H_0 is rejected if $\overline{x} \geq 77.56$.

8.2.12. Let X have the pdf $f(x; \theta) = \theta^x (1 - \theta)^{1-x}$, $x = 0, 1$, zero elsewhere. We test $H_0 : \theta = \frac{1}{2}$ against $H_1 : \theta < \frac{1}{2}$ by taking a random sample $X_1, X_2, ..., X_5$ of size $n = 5$ and rejecting H_0 if $Y = \sum_1^n X_i$ is observed to be less than or equal to a constant c.

(a) Show that this is a uniformly most powerful test.

Solution.

Use NP theorem to obtain. Let $\theta' < \frac{1}{2}$.

$$\frac{L(\theta')}{L(1/2)} = \cdots = K\left(\frac{\theta'}{1-\theta'}\right)^{\sum_1^5 x_i} \geq k \;\Rightarrow\; \sum_1^5 x_i \log\left(\frac{\theta'}{1-\theta'}\right) \geq \log k - \log K$$

$$\Rightarrow \sum_1^5 x_i \leq c \quad \text{since } 0 < \frac{\theta'}{1-\theta'} < 1 \; \left(\theta' < \frac{1}{2}\right).$$

(b) Find the significance level when $c = 1$.

Solution.

Since $Y = \sum_1^5 x_i \sim b(5, 1/2)$ under H_0,

$$P_{\theta=1/2}(Y \leq 1) = P_{\theta=1/2}(Y = 0) + P_{\theta=1}(Y = 0) = \frac{1}{32} + \frac{5}{32} = \frac{6}{32}.$$

(c) Find the significance level when $c = 0$.

Solution.

$$P_{\theta=1/2}(Y \leq 0) = P_{\theta=1/2}(Y = 0) = \frac{1}{32}.$$

(d) By using a randomized test, as discussed in Example 4.6.4, modify the tests given in parts (b) and (c) to find a test with significance level $\alpha = \frac{2}{32}$.

Solution.

If $y = 0$, the test rejects H_0. If $y = 1$, then the test rejects H_0 with probability p, where

$$\frac{1}{36} + \frac{5}{36}p = \frac{2}{32} \;\Rightarrow\; p = \frac{1}{5}.$$

8.2.13. Let $X_1, ..., X_n$ denote a random sample from a gamma-type distribution with $\alpha = 2$ and $\beta = \theta$. Let $H_0 : \theta = 1$ and $H_1 : \theta > 1$.

(a) Show that there exists a uniformly most powerful test for H_0 against H_1, determine the statistic Y upon which the test may be based, and indicate the nature of the best critical region.

Solution.

Use NP theorem to obtain. Let $\theta' > 1$.

$$\frac{L(\theta')}{L(1)} = \cdots = K \exp\left[\frac{\theta'-1}{\theta'}\sum x_i\right] \geq k \;\Rightarrow\; \frac{\theta'-1}{\theta'}\sum_1^n x_i \geq \log k - \log K$$

$$\Rightarrow \sum_1^n x_i \geq c$$

for every $\theta' > 1$. Hence, the UMP critical region $C = \{\mathbf{x} : \sum_1^n x_i \geq c\}$ defines a UMP test. If $Y = \sum_1^n X_i$, then $Y \sim \Gamma(2n, 1)$ or $2Y \sim \Gamma(2n, 2) = \chi^2(4n)$ under H_0.

(b) Find the pdf of the statistic Y in part (a). If we want a significance level of 0.05, write an equation that can be used to determine the critical region. Let $\gamma(\theta)$, $\theta \geq 1$, be the power function of the test. Express the power function as an integral.

Solution.

$$f_Y(y) = \frac{1}{\Gamma(2n)} x^{2n-1} e^{-x}, \;\; 0 < x < \infty.$$

Hence,

$$0.05 = P_{H_0}\left(Y \geq c\right) = \int_c^\infty \frac{1}{\Gamma(2n)} x^{2n-1} e^{-x} dx.$$

Or, $2c = \chi^2_{4n,0.95} \Rightarrow c = \chi^2_{4n,0.95}/2$. Finally, the power function is given by

$$\gamma(\theta) = P_\Omega\left(Y \geq c\right) = \int_c^\infty \frac{1}{\Gamma(2n)\theta^{2n}} x^{2n-1} e^{-x/\theta} dx.$$

8.3. Likelihood Ratio Tests

8.3.2. Verify Equations (8.3.2) of Example 8.3.1 of this section.

Solution.

Since

$$\ell(\omega) = -\frac{n+m}{2}\log(2\pi\theta_3) - \frac{1}{2\theta_3}\left[\sum_1^n (x_i-\theta_1)^2 + \sum_1^m (y_i-\theta_1)^2\right],$$

the derivative with respect to θ_1 and θ_3 are, respectively,

$$\begin{aligned}
\frac{\partial \ell(\omega)}{\partial \theta_1} &= \frac{1}{\theta_3}\left[\sum_1^n (x_i-\theta_1) + \sum_1^m (y_i-\theta_1)\right] = \frac{1}{\theta_3}\left[\sum_1^n x_i + \sum_1^m y_i - (n+m)\theta_1\right], \\
\frac{\partial \ell(\omega)}{\partial \theta_3} &= -\frac{n+m}{2\theta_3} + \frac{1}{2\theta_3^2}\left[\sum_1^n (x_i-\theta_1)^2 + \sum_1^m (y_i-\theta_1)^2\right].
\end{aligned}$$

u and w are, indeed, solutions for θ_1 and θ_3 that satisfy that these derivatives are zero.

8.3.3. Verify Equations (8.3.3) of Example 8.3.1 of this section.

Solution.

Since

$$\ell(\Omega) = -\frac{n+m}{2}\log(2\pi\theta_3) - \frac{1}{2\theta_3}\left[\sum_1^n (x_i-\theta_1)^2 + \sum_1^m (y_i-\theta_2)^2\right],$$

the derivative with respect to θ_1, θ_2, and θ_3 are, respectively,

$$\begin{aligned}
\frac{\partial \ell(\Omega)}{\partial \theta_1} &= \frac{1}{\theta_3}\left[\sum_1^n (x_i-\theta_1)\right] = \frac{1}{\theta_3}\left[\sum_1^n x_i - n\theta_1\right], \\
\frac{\partial \ell(\Omega)}{\partial \theta_2} &= \frac{1}{\theta_3}\left[\sum_1^n (y_i-\theta_2)\right] = \frac{1}{\theta_3}\left[\sum_1^m x_i - m\theta_2\right], \\
\frac{\partial \ell(\Omega)}{\partial \theta_3} &= -\frac{n+m}{2\theta_3} + \frac{1}{2\theta_3^2}\left[\sum_1^n (x_i-\theta_1)^2 + \sum_1^m (y_i-\theta_2)^2\right].
\end{aligned}$$

u_1, u_2, and w' are, indeed, the solutions for θ_1, θ_2, and θ_3 that satisfy that these derivatives are zero.

8.3.4. Let $X_1, ..., X_n$ and $Y_1, ..., Y_m$ follow the location model

$$\begin{aligned}
X_i &= \theta_1 + Z_i, \quad i = 1, ...n \\
Y_i &= \theta_2 + Z_{n+i}, \quad i = 1, ...m,
\end{aligned}$$

where $Z_1, ..., Z_{n+m}$ are iid random variables with common pdf $f(z)$. Assume that $E(Z_i) = 0$ and $\text{Var}(Zi) = \theta_3 < \infty$.

(a) Show that $E(X_i) = \theta_1$, $E(Y_i) = \theta_2$, and $\text{Var}(X_i) = \text{Var}(Y_i) = \theta_3$.

Solution.

$$E(X_i) = E(\theta_1) + E(Z_i) = \theta_1, \quad E(Y_i) = E(\theta_2) + E(Z_{n+i}) = \theta_2,$$
$$\text{Var}(X_i) = \text{Var}(Z_i) = \theta_3, \quad \text{Var}(Y_i) = \text{Var}(Z_{n+i}) = \theta_3.$$

because all parameters are fixed.

(b) Consider the hypotheses of Example 8.3.1, i.e.,

$$H_0 : \theta_1 = \theta_2 \text{ versus } H_1 : \theta_1 \neq \theta_2.$$

Show that under H_0, the test statistic T given in expression (8.3.4) has a limiting $N(0,1)$ distribution.

Solution.

Since we know that the T has a t-distribution with $n+m-2$ degrees of freedom, it converges to the standard normal as $n, m \to \infty$.

(c) Using part (b), determine the corresponding large sample test (decision rule) of H_0 versus H_1. (This shows that the test in Example 8.3.1 is asymptotically correct.)

Solution.

The decision rule is $\alpha = P(|T| \geq z_{\alpha/2})$. If, for instance, $\alpha = 0.05$, then $z_{\alpha/2} = 1.96$, which is not far from $c = \texttt{qt(0.975, 14)} = 2.1448$ as shown on page 491 of the textbook as an example.

8.3.7. Show that the likelihood ratio principle leads to the same test when testing a simple hypothesis H_0 against an alternative simple hypothesis H_1, as that given by the Neyman–Pearson theorem. Note that there are only two points in Ω.

Solution.

When $H_0 : \theta = \theta_0$ against $H_0 : \theta = \theta_1$, $\Omega = \{\theta', \theta''\}$, the likelihood ratio is

$$\Lambda = \frac{L(\widehat{\theta})}{L(\theta')} = \begin{cases} \frac{L(\theta'')}{L(\theta')} & L(\theta'') \geq L(\theta') \\ 1 & L(\theta'') < L(\theta') \end{cases}.$$

If $\Lambda = 1$, H_0 is not rejected; otherwise, $\Lambda \geq k$ is the same critical region by The Neyman–Pearson theorem.

8.3.9. Let $X_1, X_2, ..., X_n$ be iid $N(\theta_1, \theta_2)$. Show that the likelihood ratio principle for testing $H_0 : \theta_2 = \theta_2'$ specified, and θ_1 unspecified, against $H_1 : \theta_2 \neq \theta_2'$, θ_1 unspecified, leads to a test that rejects when $\sum_1^n (x_i - \overline{x})^2 \leq c_1$ or $\sum_1^n (x_i - \overline{x})^2 \geq c_2$, where $c_1 < c_2$ are selected appropriately.

Solution.

The LRT statistic is

$$\Lambda = \frac{L(\widehat{\theta}_1, \widehat{\theta}_2)}{L(\widehat{\theta}_{10}, \theta_2')}.$$

On the whole space Ω, the mles of θ_1 and θ_2 are, respectively,

$$\widehat{\theta}_1 = \overline{X}, \quad \widehat{\theta}_2 = n^{-1} \sum_1^n (X_i - \widehat{\theta}_1) = n^{-1} \sum_1^n (X_i - \overline{X}),$$

while, under H_0, $\widehat{\theta}_{10} = \overline{X}$. Hence, let $w = \sum_1^n (x_i - \overline{x})/\theta_2'$,

$$\Lambda = \frac{L(\widehat{\theta}_1, \widehat{\theta}_2)}{L(\widehat{\theta}_{10}, \theta_2')} = \frac{(2\pi\widehat{\theta}_2)^{-n/2} e^{-\sum(x_i - \overline{x})/2\widehat{\theta}_2}}{(2\pi\theta_2')^{-n/2} e^{-\sum(x_i - \overline{x})/2\theta_2'}} = \left(\frac{n}{ew}\right)^{n/2} e^{w/2} = Kg(w),$$

where $g(w) = w^{-n/2}e^{w/2}$. Consider $\log g(w) = -n/2 \log w + w/2$.

$$[\log g(w)]' = -\frac{n}{2w} + \frac{1}{2} \;\Rightarrow\; [\log g(n)]' = 0$$
$$[\log g(w)]'' = \frac{n}{2w^2} > 0$$

indicates that $g(w)$ is convex with a minimum at $w = n$. Thus,

$$\Lambda \geq k \;\Rightarrow\; w \leq k_1,\; w \geq k_2 \;\Rightarrow\; \sum_1^n (x_i - \overline{x})^2 \leq c_1,\; \sum_1^n (x_i - \overline{x})^2 \geq c_2.$$

8.3.12. Let $Y_1 < Y_2 < \cdots < Y_5$ be the order statistics of a random sample of size $n = 5$ from a distribution with pdf $f(x;\theta) = \frac{1}{2}e^{-|x-\theta|}$, $-\infty < x < \infty$, for all real θ. Find the likelihood ratio test Λ for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$.

Solution.

We know that the mle of θ is $\widehat{\theta} = Y_3$ under Ω. Hence, the LRT statistic is

$$\Lambda = \frac{L(\widehat{\theta})}{L(\theta_0)} = \exp\left[\sum_1^5 |x_i - \theta_0| - \sum_1^5 |x_i - y_3|\right]$$

Since $\sum_1^5 |x_i - \theta_0| = |y_3 - \theta_0| + \sum_1^5 |x_i - y_3|$, $\Lambda = |y_3 - \theta_0| \geq c$ is a critical region.

8.3.13. A random sample $X_1, X_2, ..., X_n$ arises from a distribution given by

$$H_0 : f(x;\theta) = \frac{1}{\theta},\; 0 < x < \theta,\; \text{zero elsewhere},$$

or

$$H_1 : f(x;\theta) = \frac{1}{\theta}e^{-x/\theta},\; 0 < x < \infty,\; \text{zero elsewhere}.$$

Determine the likelihood ratio (Λ) test associated with the test of H_0 against H_1.

Solution.

The mle of θ is $\widehat{\theta}_0 = Y_n$ under H_0, while the mle of θ is $\widehat{\theta} = \overline{X}$ or Y_n under Ω. If $\widehat{\theta} = Y_n$, we do not reject H_0. If $\widehat{\theta} = \overline{X}$. then the LRT statistic is

$$\Lambda = \frac{L_\Omega(\widehat{\theta})}{L_{H_0}(\widehat{\theta}_0)} = \frac{L_\Omega(\overline{X})}{L_{H_0}(Y_n)} = \frac{(1/\overline{x})^n e^{-n}}{(1/y_n)^n} \geq k \;\Rightarrow\; \left(\frac{y_n}{\overline{x}}\right)^n \geq k' \;\Rightarrow\; \frac{y_n}{\overline{x}} \geq c$$

because $\overline{x}$ and y_n are both positive.

8.3.14. Consider a random sample $X_1, X_2, ..., X_n$ from a distribution with pdf $f(x;\theta) = \theta(1-x)^{\theta-1}$, $0 < x < 1$, zero elsewhere, where $\theta > 0$.

(a) Find the form of the uniformly most powerful test of $H_0 : \theta = 1$ against $H_1 : \theta > 1$.

Solution.

Suppose $H_1 : \theta = \theta' > 1$ and use the NP theorem.

$$\frac{L(\theta')}{L(1)} = \theta'^n \prod (1-x_i)^{\theta'-1} = \theta'^n \left[\prod (1-x_i)\right]^{\theta'-1} \geq k \;\Rightarrow\; \prod (1-x_i) \geq c$$

for every $\theta' > 1$. Hence, $C = \{\mathbf{x} : \prod (1-x_i) \geq c\}$ is the UMP critical region, setting the UMP test.

(b) What is the likelihood ratio Λ for testing $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$?

Solution.

By the previous exercise, we obtain the mle:

$$\widehat{\theta} = -n/\log \prod (1 - x_i).$$

Hence, the LRT statistic is

$$\Lambda = \frac{L(\widehat{\theta})}{L(1)} = \widehat{\theta}^n \left[\prod (1 - x_i)\right]^{\widehat{\theta}-1}.$$