Exercises in Introduction to Mathematical Statistics (Ch. 1)

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Note

- Not all solutions are provided: exercises that are too simple or not very important to me are skipped.
- Texts in red are just attentions to me. Please ignore them.

1 Probability and Distribution

1.2 Sets

1.2.6. Show that the following sequences of sets, $\{C_k\}$, are nondecreasing, (1.2.16), then find $\lim_{k\to\infty} C_k$.

(a) $C_k = \{x : 1/k \le x \le 3 - 1/k\}$, $k = 1, 2, 3, \ldots$.

Solution.

By (1.2.16),

$$\lim_{k\to\infty} C_k = \bigcup_{k=1}^{\infty} C_k = \{x : 0 < x < 3\}.$$

(b) $C_k = \{(x, y) : 1/k \le x^2 + y^2 \le 4 - 1/k\}$, $k = 1, 2, 3, \ldots$.

Solution.

$$\lim_{k\to\infty} C_k = \bigcup_{k=1}^{\infty} C_k = \{(x, y) : 0 < x^2 + y^2 < 4\}.$$

1.2.7. Show that the following sequences of sets, $\{C_k\}$, are nonincreasing, (1.2.17), then find $\lim_{k\to\infty} C_k$.

(a) $C_k = \{x : 2 - 1/k < x \le 2\}$, $k = 1, 2, 3, \ldots$.

Solution.

By (1.2.17),

$$\lim_{k\to\infty} C_k = \bigcap_{k=1}^{\infty} C_k = \{x : x = 2\}.$$

because the lower limit converges to 2 as $k \to \infty$, which is the upper limit.

(b) $C_k = \{x : 2 < x \le 2 + 1/k\}$, $k = 1, 2, 3, \ldots$.

Solution.

$$\lim_{k\to\infty} C_k = \bigcap_{k=1}^{\infty} C_k = \phi$$

because the upper limit attains 2 as $k \to \infty$, which is less than the lower limit.

(c) $C_k = \{(x,y) : 0 \le x^2 + y^2 \le 1/k\}$, $k = 1, 2, 3, ...$

Solution.

$$\lim_{k\to\infty} C_k = \bigcap_{k=1}^{\infty} C_k = \{(x,y) : x^2 + y^2 = 0\} = \{(x,y) : x = y = 0\}.$$

because the upper limit attains 0 as $k \to \infty$, which is the lower limit.

1.2.10. For every two-dimensional set C contained in R^2 for which the integral exists, let $Q(C) = \int\int_C (x^2 + y^2)dxdy$. If $C_1 = \{(x,y) : -1 \le x \le 1, -1 \le y \le 1\}$, $C_2 = \{(x,y) : -1 \le x = y \le 1\}$, and $C_3 = \{(x,y) : x^2 + y^2 \le 1\}$, find $Q(C_1), Q(C_2)$, and $Q(C_3)$.

Solution.

$$Q(C_1) = \int_{-1}^{1}\int_{-1}^{1}(x^2+y^2)dxdy = \int_{-1}^{1}\left(\frac{x^3}{3} + y^2x\Big|_{-1}^{1}\right)dy = \int_{-1}^{1}\left(\frac{2}{3} + 2y^2\right)dy = \frac{2y}{3} + \frac{2y^3}{3}\Big|_{-1}^{1} = \frac{3}{8},$$

$$Q(C_2) = \int_{-1}^{1}\int_{y}^{y}(x^2+y^2)dxdy = 0 \quad \text{since } C_2 \text{ has no area, just segment,}$$

$$Q(C_3) = \int\int_{x^2+y^2\le 1}(x^2+y^2)dxdy = \int_0^1\int_0^{2\pi} r^2(rd\theta dr) = \frac{\pi}{2}.$$

1.2.14. To join a certain club, a person must be either a statistician or a mathematician or both. Of the 25 members in this club, 19 are statisticians and 16 are mathematicians. How many persons in the club are both a statistician and a mathematician?

Solution.

Let $\mathcal{C} = \{\text{statistician, mathematician}\}$, $C_1 = \{\text{statistician}\}$, and $C_2 = \{\text{mathematician}\}$ Then

$$Q(\mathcal{C}) = Q(C_1) + Q(C_2) - Q(C_1 \cap C_2) \Rightarrow 25 = 19 + 16 - Q(C_1 \cap C_2)$$

Hence, the number of persons who are both a statistician and a mathematician is $Q(C_1 \cap C_2) = 10$.

1.3 The Probability Set Function

1.3.2. A random experiment consists of drawing a card from an ordinary deck of 52 playing cards. Let the probability set function P assign a probability of $\frac{1}{52}$ to each of the 52 possible outcomes. Let C_1 denote the collection of the 13 hearts and let C_2 denote the collection of the 4 kings. Compute $P(C_1)$, $P(C_2)$, $P(C_1 \cap C_2)$, and $P(C_1 \cup C_2)$.

Solution.

$$P(C_1) = \frac{\#(C_1)}{\#(\mathcal{C})} = \frac{13}{52} = \frac{1}{4}, \quad P(C_2) = \frac{\#(C_2)}{\#(\mathcal{C})} = \frac{4}{52} = \frac{1}{13},$$

$$P(C_1 \cap C_2) = \frac{\#(\text{King of Heart})}{\#(\mathcal{C})} = \frac{1}{52}, \quad P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2) = \frac{16}{52} = \frac{4}{13}.$$

1.3.4. If the sample space is $\mathcal{C} = C_1 \cup C_2$ and if $P(C_1) = 0.8$ and $P(C_2) = 0.5$, find $P(C_1 \cap C_2)$.

Solution.

$$1 = P(\mathcal{C}) = P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2) = 1.3 - P(C_1 \cap C_2).$$

Hence $P(C_1 \cap C_2) = 0.3$.

1.3.6. If the sample space is $\mathcal{C} = \{c : -\infty < c < \infty\}$ and if $C \subset \mathcal{C}$ is a set for which the integral $\int_C e^{-|x|}dx$ exists, show that this set function is not a probability set function. What constant do we multiply the integrand by to make it a probability set function?

Solution.

$$P(\mathcal{C}) = \int_0^\infty e^{-x}dx + \int_{-\infty}^0 e^x dx = -e^{-x}\big|_0^\infty + e^x\big|_{-\infty}^0 = 1 + 1 = 2,$$

indicating that we need to multiply it by 1/2.

1.3.8. Let C_1, C_2, and C_3 be three mutually disjoint subsets of the sample space $\mathcal{C}$. Find $P[(C_1 \cup C_2) \cap C_3]$ and $P(C_1^c \cup C_2^c)$.

Solution.

$$\begin{aligned} P[(C_1 \cup C_2) \cap C_3] &= P[(C_1 \cap C_3) \cup (C_2 \cap C_3)] = P(\phi \cup \phi) = P(\phi) = 0 \\ P(C_1^c \cup C_2^c) &= P[(C_1 \cap C_2)^c] = P[\phi^c] = P(\mathcal{C}) = 1. \end{aligned}$$

1.3.11. A person has purchased 10 of 1000 tickets sold in a certain raffle. To determine the five prize winners, five tickets are to be drawn at random and without replacement. Compute the probability that this person wins at least one prize.

Solution.

$$1 - P(\text{person does not win any price}) = 1 - \binom{990}{5} \Big/ \binom{1000}{5}$$

1.3.22. Consider the events C_1, C_2, C_3.

(a) Suppose C_1, C_2, C_3 are mutually exclusive events. If $P(C_i) = p_i,\ i = 1, 2, 3$, what is the restriction on the sum $p_1 + p_2 + p_3$?

Solution. $0 \le p_1 + p_2 + p_3 \le 1$.

(b) In the notation of part (a), if $p_1 = 4/10, p_2 = 3/10$, and $p_3 = 5/10$, are C_1, C_2, C_3 mutually exclusive?

Solution. No because $p_1 + p_2 + p_3 > 1$ in part (b).

1.4 Conditional Probability and Independence

1.4.1. If $P(A_1) > 0$ and if $A_2, A_3, A_4, ...$ are mutually disjoint sets, show that

$$P(A_2 \cup A_3 \cup \cdots | A_1) = P(A_2|A_1) + P(A_3|A_1) + \cdots .$$

Solution.

$$\begin{aligned} P(A_2 \cup A_3 \cup \cdots | A_1) &= \frac{P(A_2 \cup A_3 \cup \cdots)}{P(A_1)} \quad \text{by definition of the conditional probability} \\ &= \frac{P(A_2) + P(A_3) + \cdots}{P(A_1)} \quad \text{since } A_2, A_3, \ldots \text{ are mutually disjoint} \\ &= P(A_2|A_1) + P(A_3|A_1) + \cdots \quad \text{by definition of the conditional probability.} \end{aligned}$$

1.4.2. Assume that $P(A_1 \cap A_2 \cap A_3) > 0$. Prove that

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)P(A_4|A_1 \cap A_2 \cap A_3).$$

Solution.

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3 \cap A_4) &= P(A_4|A_1 \cap A_2 \cap A_3)P(A_1 \cap A_2 \cap A_3) \\ &= P(A_4|A_1 \cap A_2 \cap A_3)P(A_3|A_1 \cap A_2)P(A_1 \cap A_2) \\ &= P(A_4|A_1 \cap A_2 \cap A_3)P(A_3|A_1 \cap A_2)P(A_2|A_1)P(A_1). \end{aligned}$$

1.4.4. From a well-shuffled deck of ordinary playing cards, four cards are turned over one at a time without replacement. What is the probability that the Ss and red cards alternate?

Solution.

$$P(S)P(R|S)P(S|S\cap R)P(R|S\cap R\cap S)+P(R)P(S|R)P(R|R\cap S)P(S|R\cap S\cap R)$$
$$=\frac{13}{52}\frac{26}{51}\frac{12}{50}\frac{25}{49}\times 2=0.031.$$

1.4.9. Bowl I contains six red chips and four blue chips. Five of these 10 chips are selected at random and without replacement and put in bowl II, which was originally empty. One chip is then drawn at random from bowl II. Given that this chip is blue, find the conditional probability that two red chips and three blue chips are transferred from bowl I to bowl II.

Solution.

$$P(\text{2 reds and 3 blues in I}|\text{blue in II})=\frac{P(\text{2 reds and 3 blues in I}\cap\text{blue in II})}{P(\text{blue in II})}$$
$$=\frac{\binom{6}{2}\binom{4}{3}/\binom{10}{5}\times\frac{3}{5}}{4/10}=\frac{5}{14}.$$

1.4.12. Let C_1 and C_2 be independent events with $P(C_1)=0.6$ and $P(C_2)=0.3$. Compute (a) $P(C_1\cap C_2)$, (b) $P(C_1\cup C_2)$, and $P(C_1\cup C_2^c)$.

Solution.

$$\begin{aligned}
(a)\ P(C_1\cap C_2)&=P(C_1)P(C_2)=0.18.\\
(b)\ P(C_1\cup C_2)&=P(C_1)+P(C_2)-P(C_1\cap C_2)=0.72.\\
(c)\ P(C_1\cup C_2^c)&=P(C_1)+P(C_2^c)-P(C_1\cap C_2^c)\\
&=P(C_1)+P(C_2^c)-P(C_1)P(C_2^c)\\
&=0.6+(1-0.3)-(0.6)(1-0.3)\\
&=0.88.
\end{aligned}$$

1.4.20. A person answers each of two multiple choice questions at random. If there are four possible choices on each question, what is the conditional probability that both answers are correct given that at least one is correct?

Solution.

$$\frac{P(\text{both correct})}{P(\text{at lease one correct})}=\frac{P(\text{both correct})}{1-P(\text{both incorrect})}=\frac{(1/4)^2}{1-(3/4)^2}=\frac{1}{7}.$$

1.4.28. A bowl contains 10 chips numbered $1,2,\ldots,10$, respectively. Five chips are drawn at random, one at a time, and without replacement. What is the probability that two even-numbered chips are drawn and they occur on even-numbered draws?

Solution.

$$\begin{aligned}
&P(\{\text{two even-numbered chips}\}\cap\{\text{two even-numbered draws}\})\\
&=P(\text{two even-numbered chips})P(\text{two even-numbered draws}|\text{two even-numbered chips})\\
&=\frac{\binom{5}{2}}{\binom{10}{5}}\times 1=\frac{5}{126}\approx 0.040.
\end{aligned}$$

1.4.32. Hunters A and B shoot at a target; the probabilities of hitting the target are p_1 and p_2, respectively. Assuming independence, can p_1 and p_2 be selected so that

$$P(\text{zero hits})=P(\text{one hit})=P(\text{two hits})?$$

Solution.

$$\begin{aligned}
&P(\text{zero hits}) = P(\text{one hit}) = P(\text{two hits})\\
&\Rightarrow (1-p_1)(1-p_2) = p_1(1-p_2) + (1-p_1)p_2 = p_1p_2\\
&\Rightarrow 3p_1 - 3p_1 + 1 = 0.
\end{aligned}$$

However, the last equation cannot hold since $3p_1 - 3p_1 + 1 = 3(p_1 - 1/2)^2 + 1/4 > 0$. Thus, the answer is No.

1.5 Random Variables

1.5.2. For each of the following, find the constant c so that $p(x)$ satisfies the condition of being a pmf of one random variable X.

(a) $p(x) = c(\frac{2}{3})^x$, $x = 1, 2, 3, \ldots,$, zero elsewhere.

Solution.

$$\sum_{x=-\infty}^{\infty} p(x) = \sum_{x=1}^{\infty} c\left(\frac{2}{3}\right)^x = c\,\frac{2/3}{1-2/3} = 2c \quad \Rightarrow \quad c = \frac{1}{2}.$$

(b) $p(x) = cx$ $x = 1, 2, 3, 4, 5, 6$, zero elsewhere.

Solution.

$$\sum_{x=-\infty}^{\infty} p(x) = \sum_{x=1}^{6} cx = \frac{c(6)(7)}{2} = 21c \quad \Rightarrow \quad c = \frac{1}{21}.$$

1.5.3. Let $p_X(x) = x/15$, $x = 1, 2, 3, 4, 5$, zero elsewhere, be the pmf of X. Find $P(X = 1 \text{ or } 2), P(\frac{1}{2} < X < \frac{5}{2})$, and $P(1 \le X \le 2)$.

Solution.

$$P(X = 1 \text{ or } 2) = P\left(\frac{1}{2} < X < \frac{5}{2}\right) = P(1 \le X \le 2) = P(X=1) + P(X=2) = \frac{1}{15} + \frac{2}{15} = \frac{1}{5}.$$

1.5.4. Let $p_X(x)$ be the pmf of a random variable X. Find the cdf $F(x)$ of X (and sketch its graph along with that of $p_X(x)$) if:

(a) $p_X(x) = 1,\ x = 0$, zero elsewhere.

Solution.

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 & x \ge 0 \end{cases}$$

(b) $p_X(x) = \frac{1}{3},\ x = -1, 0, 1$, zero elsewhere.

Solution.

$$F(x) = \begin{cases} 0 & x < -1 \\ 1/3 & -1 \le x < 0 \\ 2/3 & 0 \le x < 1 \\ 1 & x \ge 1. \end{cases}$$

(c) $p_X(x) = x/15,\ x = 1,2,3,4,5$, zero elsewhere.

Solution.

$$F(x) = \begin{cases} 0 & x < 1 \\ 1/15 & 1 \le x < 2 \\ 3/15 & 2 \le x < 3 \\ 6/15 & 3 \le x < 4 \\ 10/15 & 4 \le x < 5 \\ 1 & x \ge 5. \end{cases}$$

1.5.5. Let us select five cards at random and without replacement from an ordinary deck of playing cards.

(a) Find the pmf of X, the number of hearts in the five cards.

Solution.

$$f_X(x) = \frac{\binom{13}{x}\binom{39}{5-x}}{\binom{52}{5}}, \quad x = 0,1,2,3,4,5,$$

zero elsewhere.

(b) Determine $P(X \le 1)$.

Solution.

$$P(X \le 1) = P(X = 0) + P(X = 1) = f_X(0) + f_X(1) = \frac{\binom{39}{5} + \binom{13}{1}\binom{39}{4}}{\binom{52}{5}}.$$

1.5.6. Let the probability set function of the random variable X be $P_X(D) = \int_D f(x)dx$, where $f(x) = 2x/9$, for $x \in \mathcal{D} = \{x : 0 < x < 3\}$. Define the events $D_1 = \{x : 0 < x < 1\}$ and $D_2 = \{x : 2 < x < 3\}$. Compute $P_X(D_1), P_X(D_2)$, and $P_X(D_1 \cup D_2)$.

Solution.

$$\begin{aligned} P_X(D_1) &= \int_0^1 2x/9dx = [x^2/9]_0^1 = \frac{1}{9} \\ P_X(D_2) &= \int_2^3 2x/9dx = [x^2/9]_2^3 = \frac{5}{9} \\ P_X(D_1 \cup D_2) &= P_X(D_1) + P_X(D_2) = \frac{2}{3} \quad \text{since } D_1 \cap D_2 = \phi. \end{aligned}$$

1.5.7. Let the space of the random variable X be $\mathcal{D} = \{x : 0 < x < 1\}$. If $D_1 = \{x : 0 < x < \frac{1}{2}\}$ and $D_2 = \{x : \frac{1}{2} \le x < 1\}$, find $P_X(D_2)$ if $P_X(D_1) = \frac{1}{4}$.

Solution.

Since $\mathcal{D} = D_1 \cup D_2$ and $D_1 \cap D_2 = \phi$,

$$1 = P(\mathcal{D}) = P(D_1 \cup D_2) = P(D_1) + P(D_2) = \frac{1}{4} + P(D_2) \ \Rightarrow \ P(D_2) = \frac{3}{4}.$$

1.6 Discrete Random Variables

1.6.2. Let a bowl contain 10 chips of the same size and shape. One and only one of these chips is red. Continue to draw chips from the bowl, one at a time and at random and without replacement, until the red chip is drawn.

(a) Find the pmf of X, the number of trials needed to draw the red chip.

Solution.

Note that the order is important in this case.

$$p_X(x) = \frac{{}_9P_{x-1} \cdot 1}{{}_{10}P_x} = \frac{9!/(9-x+1)!}{10!/(10-x)!} = \frac{1}{10}, \quad x = 1, 2, \ldots 10.$$

(b) Compute $P(X \leq 4)$.

Solution.

$$P(X \leq 4) = \sum_{x=1}^{4} p_X(x) = \frac{4}{10}.$$

1.6.3. Cast a die a number of independent times until a six appears on the up side of the die.

(a) Find the pmf $p(x)$ of X , the number of casts needed to obtain that first six.

Solution.

$$p(x) = \frac{1}{6}\left(\frac{5}{6}\right)^{x-1}, \quad x = 1, 2, \ldots.$$

(b) Show that $\sum_{x=1}^{\infty} p(x) = 1$.

Solution.

$$\sum_{x=1}^{\infty} p(x) = \frac{1}{6}\sum_{x=1}^{\infty}\left(\frac{5}{6}\right)^{x-1} = \frac{1}{6}\frac{1}{1-5/6} = 1.$$

(c) Determine $P(X = 1, 3, 5, 7, \ldots)$.

Solution.

$$P(X = 1, 3, 5, 7, \ldots) = \sum_{k=1}^{\infty} p(2k-1) = \frac{1}{6}\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^{2(k-1)} = \frac{1}{6}\sum_{k=1}^{\infty}\left(\frac{25}{36}\right)^{k-1} = \frac{1}{6}\frac{1}{1-25/36} = \frac{6}{11}.$$

(d) Find the cdf $F(x) = P(X \leq x)$.

Solution.

$$F(x) = \sum_{l=1}^{x} p(x) = \frac{1}{6}\sum_{l=1}^{x}\left(\frac{5}{6}\right)^{l-1} = \frac{1}{6}\frac{1-(5/6)^x}{1-5/6} = 1 - \left(\frac{5}{6}\right)^x.$$

1.6.7. Let X have a pmf $p(x) = \frac{1}{3}$, $x = 1, 2, 3$, zero elsewhere. Find the pmf of $Y = 2X + 1$.

Solution.

Since $g(x) = 2x + 1$ is one-to-one transformation,

$$p_Y(y) = P(Y = y) = P(2X + 1 = y) = P\left(X = \frac{y-1}{2}\right) = p_X\left(\frac{y-1}{2}\right) = \frac{1}{3}, \quad y = 3, 5, 7,$$

zero elsewhere.

1.6.8. Let X have the pmf $p(x) = (\frac{1}{2})^x$, $x = 1, 2, 3, \ldots$, zero elsewhere. Find the pmf of $Y = X^3$.

Solution.

This transformation is also one-to-one and $x = g^{-1}(y) = \sqrt[3]{y}$. Hence,

$$p_Y(y) = p_X(\sqrt[3]{y}) = \left(\frac{1}{2}\right)^{\sqrt[3]{y}}, \quad y = 1, 8, 27, \ldots,$$

zero elsewhere.

1.6.9. Let X have the pmf $p(x) = 1/3, \; x = -1, 0, 1$. Find the pmf of $Y = X^2$.

Solution.

This is not one-to-one transformation; $Y = 0$ only if $X = 0$ but $Y = 1$ if $X = 1$ or $X = -1$. Thus,

$$p_Y(y) = p_X(\pm\sqrt{y}) = \begin{cases} p_X(0) = 1/3 & y = 0 \\ p_X(1) + p_X(-1) = 2/3 & y = 1, \end{cases}$$

zero elsewhere.

1.6.10. Let X have the pmf

$$p(x) = \left(\frac{1}{2}\right)^{|x|}, \quad x = -1, -2, -3, \ldots.$$

Find the pmf of $Y = X^4$.

Solution.

The transformation $g(x) = x^4$ maps $\mathcal{D}_X = \{x : -1, -2, -3, \ldots\}$ onto $\mathcal{D}_Y = \{y : 1, 2^4, 3^4, \ldots\}$. That is, we have the single-valued inverse function $x = g^{-1}(y) = -\sqrt[4]{y}$ (not $\sqrt[4]{y}$). Hence,

$$p_Y(y) = p_X(-\sqrt[4]{y}) = \left(\frac{1}{2}\right)^{|-\sqrt[4]{y}|} = \left(\frac{1}{2}\right)^{\sqrt[4]{y}}, \quad y = 1, 2^4, 3^4, \ldots,$$

zero elsewhere.

1.7 Continuous Random Variables

1.7.3. Let the subsets $C_1 = \{\frac{1}{4} < x < \frac{1}{2}\}$ and $C_2 = \{\frac{1}{2} \le x < 1\}$ of the space $\mathcal{C} = \{x : 0 < x < 1\}$ of the random variable X be such that $P_X(C_1) = \frac{1}{8}$ and $P_X(C_2) = \frac{1}{2}$. Find $P_X(C_1 \cup C_2)$, $P_X(C_1^c)$, and $P_X(C_1^c \cap C_2^c)$.

Solution.

Since $C_1 \cap C_2 = \phi$

$$\begin{aligned} P_X(C_1 \cup C_2) &= P_X(C_1) + P_X(C_2) = \frac{5}{8} \\ P_X(C_1^c) &= 1 - P(C_1) = \frac{7}{8} \\ P_X(C_1^c \cap C_2^c) &= P_X[(C_1 \cup C_2)^c] = 1 - P_X(C_1 \cup C_2) = \frac{3}{8}. \end{aligned}$$

1.7.4. Given $\int_C [1/\pi(1+x^2)]dx$, where $C \subset \mathcal{C} = \{x : -\infty < x < \infty\}$. Show that the integral could serve as a probability set function of a random variable X whose space is $\mathcal{C}$.

Solution.

First, $1/\pi(1+x^2) > 0$ for all $x \in C$.

Next, let $x = \tan\theta$, $dx = 1/\cos^2\theta d\theta$ and the support transforms to $-\pi/2 < \theta < \pi/2$. Thus,

$$\int_{-\infty}^{\infty} \left[\frac{1}{\pi(1+x^2)}\right] dx = \int_{-\pi/2}^{\pi/2} \left[\frac{1}{\pi(1+\tan^2\theta)}\right] \frac{1}{\cos^2\theta} d\theta = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} d\theta = 1.$$

1.7.6. For each of the following pdfs of X, find $P(|X| < 1)$ and $P(X^2 < 9)$.

(a) $f(x) = x^2/18,\ -3 < x < 3$, zero elsewhere.

Solution.

$$\begin{aligned} P(|X| < 1) = P(-1 < X < 1) &= \int_{-1}^{1} x^2/18 dx = [x^3/54]_{-1}^{1} = \frac{1}{27} \\ P(X^2 < 9) = P(-3 < x < 3) &= 1 \quad \text{since } f(x) \text{ is a pdf .} \end{aligned}$$

(b) $f(x) = (x+2)/18,\ -2 < x < 4$, zero elsewhere.

Solution.

$$\begin{aligned} P(-1 < X < 1) &= \int_{-1}^{1} (x+2)/18 dx = [(x+2)^2/36]_{-1}^{1} = \frac{2}{9} \\ P(-3 < x < 3) &= P(-2 < x < 3) = [(x+2)^2/36]_{-2}^{3} = \frac{25}{36}. \end{aligned}$$

1.7.7. Let $f(x) = 1/x^2,\ 1 < x < \infty$, zero elsewhere, be the pdf of X. If $C_1 = \{x : 1 < x < 2\}$ and $C_2 = \{x : 4 < x < 5\}$, find $P_X(C_1 \cup C_2)$ and $P_X(C_1 \cap C_2)$.

Solution.

Since $C_1 \cap C_2 = \phi$

$$\begin{aligned} &P_X(C_1 \cup C_2) = P_X(C_1) + P_X(C_2) = \int_1^2 1/x^2 dx + \int_4^5 1/x^2 dx = [-1/x]_1^2 + [-1/x]_4^5 = \frac{11}{20} \\ &P_X(C_1 \cap C_2) = 0. \end{aligned}$$

1.7.14. Let X have the pdf $f(x) = 2x$, $0 < x < 1$, zero elsewhere. Compute the probability that X is at least $\frac{3}{4}$ given that X is at least $\frac{1}{2}$.

Solution.

$$P(X \geq 3/4 | X \geq 1/2) = \frac{P(X \geq 3/4, X \geq 1/2)}{P(X \geq 1/2)} = \frac{P(X \geq 3/4)}{P(X \geq 1/2)} = \frac{\int_{3/4}^{1} 2x dx}{\int_0 1/2^1 2x dx} = \frac{[x^2]_{3/4}^{1}}{[x^2]_{1/2}^{1}} = \frac{7}{12}.$$

1.7.17. Divide a line segment into two parts by selecting a point at random. Find the probability that the length of the larger segment is at least three times the length of the shorter segment. Assume a uniform distribution.

Solution.

Let $X \sim U(0,1)$, which is the length of one of the segments. Then the desired possibility is

$$P\left(\frac{1-X}{X} \geq 3 \text{ or } \frac{X}{1-X} \geq 3\right) = P(X \leq 1/4 \text{ or } X \geq 3/4) = F_X(1/4) + 1 - F_X(3/4) = \frac{1}{2}$$

because the cdf $F_X(x) = x$, $0 < x < 1$.

1.7.22. Let X have the pdf $f(x) = x^2/9$, $0 < x < 3$, zero elsewhere. Find the pdf of $Y = X^3$.

Solution.

This transformation is one-to-one. Since $x = y^{1/3}$ and $dx/dy = (1/3)y^{-2/3}$,

$$f_Y(y) = f_X(y^{1/3})|dx/dy| = \frac{y^{2/3}}{9}\left(\frac{y^{-2/3}}{3}\right) = \frac{1}{27}, \quad 0 < y < 27,$$

zero elsewhere.

(Another solution)

$$F_Y(y) = P(Y \le y) = P(X \le y^{1/3}) = F_X(y^{1/3}) = \frac{y}{27} \;\Rightarrow\; f_Y(y) = F'_Y(y) = \frac{1}{27}.$$

1.7.23. If the pdf of X is $f(x) = 2xe^{-x^2}$, $0 < x < \infty$, zero elsewhere, determine the pdf of $Y = X^2$.

Solution.

Since $x > 0$, the inverse function is $x = g^{-1}(y) = y^{1/2}$ and then $dx/dy = y^{-1/2}/2$. Hence

$$f_Y(y) = f_X(y^{1/2})|dx/dy| = 2y^{1/2}e^{-y}y^{-1/2}/2 = e^{-y}, \quad 0 < y < \infty,$$

zero elsewhere. That is $Y \sim \text{Exp}(1) = \text{Gamma}(1,1)$.

1.7.25. Let X have the pdf $f(x) = 4x^3$, $0 < x < 1$, zero elsewhere. Find the cdf and the pdf of $Y = -\ln X^4$.

Solution.

This transformation is one-to-one. The inverse function is $x = e^{-y/4}$ and then $dx/dy = -(1/4)e^{-y/4}$. Thus,

$$f_Y(y) = f_X(e^{-y/4})|dx/dy| = 4(e^{-y/4})^3(1/4)e^{-y/4} = e^{-y}, \quad 0 < y < \infty,$$

zero elsewhere. The cdf of Y is

$$F_Y(y) = \begin{cases} 0 & y \le 0 \\ \int_0^y e^{-t}dt = 1 - e^{-y} & y > 0. \end{cases}$$

1.8. Expectation of a Random Variable

1.8.4. Suppose that $p(x) = \frac{1}{5}$, $x = 1, 2, 3, 4, 5$, zero elsewhere, is the pmf of the discrete-type random variable X. Compute $E(X)$ and $E(X^2)$. Use these two results to find $E[(X+2)^2]$ by writing $(X+2)^2 = X^2+4X+4$.

Solution.

Since $E(X) = \frac{1}{5}\frac{5(6)}{2} = 3$ and $E(X^2) = \frac{1}{5}\frac{5(6)(11)}{6} = 11$, $E(X^2+4X+4) = E(X^2)+4E(X)+4 = 27$.

1.8.6. Let the pmf $p(x)$ be positive at $x = -1, 0, 1$ and zero elsewhere.

(a) If $P(0) = \frac{1}{4}$, find $E(X^2)$.

Solution. $E(X^2) = p(-1) + p(1) = 1 - p(0) = \frac{3}{4}$.

(b) If $P(0) = \frac{1}{4}$ and if $E(X) = \frac{1}{4}$, determine $p(-1)$ and $p(1)$.

Solution.

$$\begin{aligned} P(0) = \frac{1}{4} &\Rightarrow p(-1) + p(1) = \frac{3}{4} \\ E(X) = \frac{1}{4} &\Rightarrow -p(-1) + p(1) = \frac{1}{4}, \end{aligned}$$

which gives $p(-1) = \frac{1}{4}$ and $p(1) = \frac{1}{2}$.

1.8.7. Let X have the pdf $f(x) = 3x^2$, $0 < x < 1$, zero elsewhere. Consider a random rectangle whose sides are X and $(1-X)$. Determine the expected value of the area of the rectangle.

Solution.

$$E(X(1-X)) = E(X - X^2) = \int_0^1 3(x^3 - x^4)dx = 3[x^4/4 - x^5/5]_0^1 = \frac{3}{20}.$$

1.8.8. A bowl contains 10 chips, of which 8 are marked \$2 each and 2 are marked \$5 each. Let a person choose, at random and without replacement, three chips from this bowl. If the person is to receive the sum of the resulting amounts, find his expectation.

Solution.

Let X denote the number of chips marked \$2, then the pmf is

$$p_X(x) = \frac{\binom{8}{x}\binom{2}{3-x}}{\binom{10}{3}}, \quad x = 1, 2, 3,$$

meaning that that X has a hypergeometric distribution. Then

$$E(X) = \frac{1(8)(1) + 2(28)(2) + 3(56)(1))}{\binom{10}{3}} = \frac{288}{120} = \frac{12}{5} = 2.4.$$

Since the sum of the resulting amount is $2X + 5(3 - X) = 15 - 3X$, his expectation is

$$E(15 - 3X) = 15 - 3E(X) = 15 - 3(2.4) = 7.8.$$

1.8.9. Let $f(x) = 2x$, $0 < x < 1$, zero elsewhere, be the pdf of X.

(a) Compute $E(1/X)$.

Solution. $E(1/X) = \int_0^1 2dx = 2$.

(b) Find the cdf and the pdf of $Y = 1/X$.

Solution.

Since $F_X(x) = x^2$, $0 < x < 1$,

$$F_Y(y) = P(Y \le y) = P(X \ge 1/y) = 1 - F_X(1/y) = 1 - \frac{1}{y^2}, \quad f_Y(y) = \frac{2}{y^3}, \quad y > 1.$$

(c) Compute $E(Y)$ and compare this result with the answer obtained in part (a).

Solution.

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y) = \int_1^{\infty} \frac{2}{y^2} = [-2/y]_1^{\infty} = 2,$$

which is equal to part (a).

1.8.11. Let X have a Cauchy distribution which has the pdf

$$f(x) = \frac{1}{\pi}\frac{1}{x^2 + 1}, \quad -\infty < x < \infty.$$

Then X is symmetrically distributed about 0 (why?). Why isn't $E(X) = 0$?

Solution.

$$f(-x) = \frac{1}{\pi}\frac{1}{(-x)^2 + 1} = f(x),$$

which implies that X is symmetric around 0. Also, $E(X)$ does not exist because

$$E(|X|) = \int_{-\infty}^{\infty} |x| f(x) dx = 2\int_0^{\infty} x f(x) dx = \frac{1}{\pi}\int_0^{\infty} \frac{2x}{x^2 + 1} dx = \frac{1}{\pi}[\log(x^2 + 1)]_0^{\infty} = \infty.$$

1.8.12. Let X have the pdf $f(x) = 3x^2$, $0 < x < 1$, zero elsewhere.

(a) Compute $E(X^3)$

Solution.

$$E(X^3) = \int_0^1 3x^5 dx = [x^6/2]_0^1 = \frac{1}{2}.$$

(b) Show that $Y = X^3$ has a uniform(0, 1) distribution.

Solution.

The support of Y is $y \in (0,1)$ and the inverse function is $x = y^{1/3}$ and $dx/dy = (1/3)y^{-2/3}$. Hence,

$$f_Y(y) = f_X(y^{1/3})|dx/dy| = 3y^{2/3}(1/3)y^{-2/3} = 1,$$

which means that $Y \sim U(0,1)$.

(c) Compute $E(Y)$ and compare this result with the answer obtained in part (a).

Solution.

$$E(Y) = \int_0^1 y f_Y(y) dy = [y^2/2]_0^1 = \frac{1}{2},$$

which is consistent with part (a).

1.9. Some Special Expectation

1.9.1. Find the mean and variance, if they exist, of each of the following distributions.

(a) $p(x) = \frac{3!}{x!(3-x)!}(\frac{1}{2})^3$, $x = 0, 1, 2, 3$, zero elsewhere.

Solution. Since $X \sim \text{Binomial}(3, 1/2)$, $E(X) = np = 3/2$ and $\text{Var}(X) = np(1-p) = 3/4$.

(b) $f(x) = 6x(1-x)$, $0 < x < 1$, zero elsewhere.

Solution.

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = 6\int_0^1 (x^2 - x^3) dx = 6[x^3/3 - x^4/4]_0^1 = \frac{1}{2}$$

$$E(X^2) = 6\int_0^1 (x^3 - x^4) dx = \frac{3}{10} \Rightarrow \text{Var}(X) = E(X^2) = [E(X)]^2 = \frac{1}{20}.$$

(c) $f(x) = 2/x^3$, $1 < x < \infty$, zero elsewhere.

Solution.

$$E(X) = \int_1^{\infty} 2/x^2 dx = [-2/x]_1^{\infty} = 2$$

However, the variance does not exist since

$$E(X^2) = \int_1^{\infty} 2/x dx = [2\log x]_1^{\infty} = \infty.$$

1.9.2. Let $p(x) = (\frac{1}{2})^x$, $x = 1, 2, 3, ...$, zero elsewhere, be the pmf of the random variable X. Find the mgf, the mean, and the variance of X.

$$M(t) = E(e^{tX}) = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x = \frac{e^t/2}{1 - e^t/2} = \frac{e^t}{2 - e^t}, \quad t < \log 2.$$

Here, let $\psi(t) = \log M(t) = t - \log(2 - e^t)$. Then,

$$E(X) = \psi'(0) = 1 + \frac{e^t}{2 - e^t}\Big|_{t=0} = 2,$$
$$\mathrm{Var}(X) = \psi''(0) = \frac{2e^t}{(2 - e^t)^2}\Big|_{t=0} = 2.$$

1.9.4. If the variance of the random variable X exists, show that

$$E(X^2) \geq [E(X)^2].$$

Solution. $E(X^2) - [E(X)^2] = \mathrm{Var}(X) \geq 0$.

1.9.5. Let a random variable X of the continuous type have a pdf $f(x)$ whose graph is symmetric with respect to $x = c$. If the mean value of X exists, show that $E(X) = c$.

Solution.

$$\begin{aligned}
E(X - c) &= \int_{-\infty}^{\infty} (x - c) f(x) dx \\
&= \int_{-\infty}^{c} (x - c) f(x) dx + \int_{c}^{\infty} (x - c) f(x) dx \\
&= \int_{0}^{\infty} (-y) f(c - y) dy + \int_{0}^{\infty} z f(c + z) dz \quad (y = c - x,\ z = x - c) \\
&= \int_{0}^{\infty} z[f(c + z) - f(c - z)] dz \\
&= 0 \quad \text{since } f(c + z) = f(c - z), \forall z.
\end{aligned}$$

Since c is constant, $E(X) = c$.

1.9.8. Let X be a random variable such that $E[(X - b)^2]$ exists for all real b. Show that $E[(X - b)^2]$ is a minimum when $b = E(X)$.

Solution.

$$h(b) = E[(X - b)^2] = b^2 - 2E(X)b + [E(X)^2], \quad h'(b) = 2b - 2E(X).$$

Solving $h'(b)$ gets $b = E(X)$. Since $h(b)$ is convex, $h(b)$ minimizes at $b = E(X)$.

1.9.10. Let X denote a random variable for which $E[(X - a)^2]$ exists. Give an example of a distribution of a discrete type such that this expectation is zero. Such a distribution is called a **degenerate** distribution.

Solution.

Since $(X - a)^2 \geq 0$, $E[(X - a)^2] = 0 \Rightarrow X = a$. That means that $P(X = a) = 1$ or X is degenerate.

1.9.16. Let the random variable X have pmf

$$p(x) = \begin{cases} p & x = -1, 1 \\ 1 - 2p & x = 0 \\ 0 & \text{elsewhere,} \end{cases}$$

where $0 < p < \frac{1}{2}$. Find the measure of kurtosis as a function of p. Determine its value when $p = \frac{1}{3}$, $p = \frac{1}{5}$, $p = \frac{1}{10}$, and $p = \frac{1}{100}$. Note that the kurtosis increases as p decreases.

Solution.

$$\mu = E(X) = \sum_{x=-1}^{1} x p(x) = (-1)p(-1) + p(1) = -p + p = 0$$

$$\sigma^2 = E(X^2) = \sum_{x=-1}^{1} x^2 p(x) = (-1)^2 p(-1) + p(1) = p + p = 2p.$$

Hence, the kurtosis is

$$\kappa(p) = \frac{E(X^4)}{\sigma^4} = \frac{(-1)^4 p(-1) + p(1)}{(2p)^2} = \frac{1}{2p},$$

which gives

$$\kappa(1/3) = \frac{3}{2}, \quad \kappa(1/5) = \frac{5}{2}, \quad \kappa(1/10) = 5, \quad \kappa(1/100) = 50.$$

1.9.17. Let $\psi(t) = \log M(t)$, where $M(t)$ is the mgf of a distribution. Prove that $\psi'(0) = \mu$ and $\psi''(0) = \sigma^2$. The function $\psi(t)$ is called the **cumulant generating function**.

Solution.

$$\begin{aligned}
\psi'(0) &= \frac{M'(0)}{M(0)} = \frac{E(X)}{1} = \mu, \\
\psi''(0) &= \frac{M''(0)M(0) - [M'(0)]^2}{[M(0)]^2} = \frac{E(X^2) - [E(X)]^2}{1} = \sigma^2.
\end{aligned}$$

1.9.21. Let X be a random variable of the continuous type with pdf $f(x)$, which is positive provided $0 < x < b < \infty$, and is equal to zero elsewhere. Show that

$$E(X) = \int_0^b [1 - F(x)]dx,$$

where $F(x)$ is the cdf of X.

Solution.

$$E(X) = \int_0^b xf(x)dx = \int_0^b \left(\int_0^x dt\right) f(x)dx = \int_0^b \int_t^\infty f(x)dxdt = \int_0^b P(X > t)dt = \int_0^b [1 - F(x)]dx.$$

1.9.22. Let X be a random variable of the discrete type with pmf $p(x)$ that is positive on the nonnegative integers and is equal to zero elsewhere. Show that

$$E(X) = \sum_{x=0}^\infty [1 - F(x)],$$

where $F(x)$ is the cdf of X.

Solution.

$$\begin{aligned}
E(X) = \sum_{x=0}^\infty xp(x) &= P(1) + 2p(2) + 3p(3) + 4p(4) + \cdots \\
&= p(1) + p(2) + p(3) + p(4) + \cdots \\
&\quad + p(2) + p(3) + p(4) + \cdots \\
&\quad + p(3) + p(4) + \cdots \\
&\quad + p(4) + \cdots \\
&= \sum_{x=0}^\infty P(X > x) = \sum_{x=0}^\infty [1 - F(x)].
\end{aligned}$$

1.9.24. Let X have the cdf $F(x)$ that is a mixture of the continuous and discrete types, namely

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x+1}{4} & 0 \le x < 1 \\ 1 & 1 \le x. \end{cases}$$

Determine reasonable definitions of $\mu = E(X)$ and $\sigma^2 = \text{Var}(X)$ and compute each.

Solution.

The pdf of X is written by

$$f(x) = \begin{cases} \frac{1}{4} & x = 0 \\ \frac{1}{4} & 0 < x < 1 \\ \frac{1}{2} & x = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$E(X) = 0[f(0)] + \int_0^1 \frac{x}{4} dx + 1f(1) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$$
$$E(X^2) = \int_0^1 \frac{x^2}{4} dx + 1f(1) = \frac{7}{12} \Rightarrow \sigma^2 = \frac{7}{12} - \left(\frac{5}{8}\right) = \frac{37}{192}.$$

1.9.25. Consider k continuous-type distributions with the following characteristics: pdf $f_i(x)$, mean μ_i, and variance σ_i^2, $i = 1, 2, \ldots, k$. If $c_i \geq 0$, $i = 1, 2, \ldots, k$, and $c_1 + c_2 + \cdots + c_k = 1$, show that the mean and the variance of the distribution having pdf $c_1 f_1(x) + \cdots + c_k f_k(x)$ are $\mu = \sum_{i=1}^k c_i \mu_i$ and $\sigma^2 = \sum_{i=1}^k c_i[\sigma_i^2 + (\mu_i - \mu)^2]$, respectively.

Solution.

$$\mu = E(X) = \int_{-\infty}^{\infty} x \sum_{i=1}^k c_i f_i(x) dx = \sum_{i=1}^k c_i \int_{-\infty}^{\infty} x f_i(x) dx = \sum_{i=1}^k c_i \mu_i,$$
$$\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} (x-\mu)^2 \sum_{i=1}^k c_i f_i(x) dx = \sum_{i=1}^k c_i \int_{-\infty}^{\infty} (x-\mu)^2 f_i(x) dx \\
&= \sum_{i=1}^k c_i \int_{-\infty}^{\infty} [(x-\mu_i) + (\mu_i - \mu)]^2 f_i(x) dx \\
&= \sum_{i=1}^k c_i \left[\int_{-\infty}^{\infty} (x-\mu_i)^2 f_i(x) dx + 2(\mu_i - \mu) \int_{-\infty}^{\infty} (x-\mu_i) f_i(x) dx + (\mu_i - \mu)^2 \int_{-\infty}^{\infty} f_i(x) dx\right] \\
&= \sum_{i=1}^k c_i \left[\sigma_i^2 + (\mu_i - \mu)^2\right] \quad \text{since} \int_{-\infty}^{\infty} (x-\mu_i) f_i(x) dx = E(X - \mu_i) = E(X) - \mu_i = 0.
\end{aligned}$$

1.9.26. Let X be a random variable with a pdf $f(x)$ and mgf $M(t)$. Suppose f is symmetric about 0; i.e., $f(-x) = f(x)$. Show that $M(-t) = M(t)$.

Solution.

$$M(-t) = E(e^{-tX}) = \int_{-\infty}^{\infty} e^{-tx} f(x) dx = \int_{\infty}^{-\infty} e^{tu} f(-u) d(-u) = \int_{-\infty}^{\infty} e^{tu} f(u) du = M(t) \quad (u = -x).$$

1.9.27. Let X have the exponential pdf, $f(x) = \beta^{-1} \exp\{-x/\beta\}$, $0 < x < \infty$, zero elsewhere. Find the mgf, the mean, and the variance of X

$$M(t) = \int_0^{\infty} \frac{1}{\beta} \exp\{-(1/\beta - t)x\} dx = \frac{1}{\beta}\left[-\frac{\exp\{-(1/\beta - t)x\}}{1/\beta - t}\right]_0^{\infty} = \frac{1}{\beta}\frac{1}{1/\beta - t} = \frac{1}{1-\beta t}$$

provided $1/\beta - t > 0 \Rightarrow t < 1/\beta$. Then

$$\psi(t) = -\log(1 - \beta t), \quad \psi'(t) = \frac{\beta}{1-\beta t} \quad \psi''(t) = \frac{\beta^2}{(1-\beta t)^2}$$
$$\Rightarrow E(X) = \psi'(0) = \beta \quad \text{Var}(X) = \psi''(0) = \beta^2.$$

1.10. Important Inequalities

1.10.1. Let X be a random variable with mean μ and let $E[(X-\mu)^{2k}]$ exist. Show, with $d>0$, that $P(|X-\mu|\geq d)\leq E[(X-\mu)^{2k}]/d^{2k}$. This is essentially Chebyshev's inequality when $k=1$. The fact that this holds for all $k=1,2,3,...$, when those $(2k)$th moments exist, usually provides a much smaller upper bound for $P(|X-\mu|\geq d)$ than does Chebyshev's result.

Solution. For Markov's inequality

$$P(g(X)\geq a)\leq\frac{E[g(X)]}{a},$$

Let $g(X)=(X-\mu)^{2k}$ and $a=d^{2k}$, then

$$P[(X-\mu)^{2k}\geq d^{2k}]=P(|X-\mu|\geq d)\leq\frac{E[(X-\mu)^{2k}]}{d^{2k}}.$$

1.10.2. Let X be a random variable such that $P(X\leq 0)=0$ and let $\mu=E(X)$ exist. Show that $P(X\geq 2\mu)\leq\frac{1}{2}$.

Solution.

For Markov's inequality, let $g(X)=X$ and $a=2\mu$, then

$$P(X\geq 2\mu)\leq\frac{E(X)}{2\mu}=\frac{1}{2}.$$

1.10.3. If X is a random variable such that $E(X)=3$ and $E(X^2)=13$, use Chebyshev's inequality to determine a lower bound for the probability $P(-2<X<8)$.

Solution.

We have $\text{Var}(X)=E(X^2)-[E(X)]^2=4$.

$$P(-2<X<8)=P(|X-3|<5)=1-P(|X-3|\geq 5)\geq 1-\frac{\text{Var}(X)}{5^2}=\frac{21}{25}=0.84.$$

1.10.4. Suppose X has a Laplace distribution with pdf (1.9.20). Show that the mean and variance of X are 0 and 2, respectively. Using Chebyshev's inequality determine the upper bound for $P(|X|\geq 5)$ and then compare it with the exact probability.

Solution.

$$E(X)=\int_{-\infty}^{\infty}\frac{x}{2}e^{-|x|}dx=\int_0^{\infty}\frac{x}{2}e^{-x}dx+\int_{-\infty}^{0}\frac{x}{2}e^{x}dx=\int_0^{\infty}\frac{x}{2}e^{-x}dx+\int_0^{\infty}\frac{-u}{2}e^{-u}du=0\quad(u=-x)$$

$$\text{Var}(X)=E(X^2)=\int_{-\infty}^{\infty}\frac{x^2}{2}e^{-|x|}dx=\int_0^{\infty}\frac{x^2}{2}e^{-x}dx+\int_{-\infty}^{0}\frac{x^2}{2}e^{x}dx=\int_0^{\infty}x^2e^{-x}dx=\Gamma(3)=2.$$

Hence, using Chebyshev's inequality, we obtain

$$P(|X|\geq 5)\leq\frac{\text{Var}(X)}{25}=0.08.$$

On the other hand, the exact probability is

$$P(|X|\geq 5)=2\int_5^{\infty}\frac{1}{2}e^{-x}dx=e^{-5}=0.0067,$$

which does not attain the upper bound.

1.10.5. Let X be a random variable with mgf $M(t)$, $-h<t<h$. Prove that

$$P(X\geq a)\leq e^{-at}M(t),\quad 0<t<h,$$

and that

$$P(X \leq a) \leq e^{-at}M(t), \quad -h < t < 0.$$

Solution.

Let $g(x) = e^{tx}$ and $a = e^{ta}$, then Markov's inequality gives

$$P(e^{tX} \geq e^{ta}) \leq \frac{E[e^{tX}]}{e^{ta}} = e^{-at}M(t).$$

Here, if $0 < t < h$ then $e^{tX} \geq e^{ta} \Rightarrow X \geq a$; $X \leq a$ if $-h < t < 0$, which are the desired results.

1.10.6. The mgf of X exists for all real values of t and is given by

$$M(t) = \frac{e^t - e^{-t}}{2t}, \quad t \neq 0, \quad M(0) = 1.$$

Use the results of the preceding exercise to show that $P(X \geq 1) = 0$ and $P(X \leq -1) = 0$. Note that here h is infinite.

Solution.

$$\begin{aligned}
P(X \geq 1) \leq e^{-t}M(t) &= \frac{1 - e^{-2t}}{2t}, \quad 0 < \forall t < \infty \\
&\to 0 \quad \text{as } t \to \infty, \\
P(X \leq -1) \leq e^{t}M(t) &= \frac{e^{2t} - 1}{2t}, \quad -\infty < \forall t < 0 \\
&\to 0 \quad \text{as } t \to -\infty.
\end{aligned}$$