Solutions in Introduction to Mathematical Statistics

8th edition by Hogg, McKean, and Craig
https://minerva.it.manchester.ac.uk/~saralees/statbook2.pdf

EXERCISES

1.2.8

For every one-dimensional set C, define the function $Q(C) = \sum_C f(x)$, where $f(x) = (\frac{2}{3})(\frac{1}{3})^x, x = 0, 1, 2, \ldots,$ zero elsewhere. If $C_1 = x : x = 0, 1, 2, 3$ and $C_2 = x : x = 0, 1, 2, \ldots$, find $Q(C1)$ and $Q(C2)$.

Solution:

(a) $C_k = \{x : \frac{1}{k} \leq x \leq 3 - \frac{1}{k}\}, C_{k+1} = \{x : \frac{1}{k+1} \leq x \leq 3 - \frac{1}{k+1}\}$, so $C_k \subset C_{k+1}$.

$$\lim_{k\to\infty} C_k = \bigcup_{k=1}^{\infty} C_k = \{x : 0 < x < 3\}.$$

(b) $C_k = \{(x, y) : \frac{1}{k} \leq x^2 + y^2 \leq 4 - \frac{1}{k}\}, C_{k+1} = \{(x, y) : \frac{1}{k+1} \leq x^2 + y^2 \leq 4 - \frac{1}{k+1}\}$, so $C_k \subset C_{k+1}$.

$$\lim_{k\to\infty} C_k = \bigcup_{k=1}^{\infty} C_k = \{(x, y) : 0 < x^2 + y^2 < 4\}.$$

1.2.9

For every one-dimensional set C for which the integral exists, let $Q(C) = \int_C f(x)dx$, where $f(x) = 6x(1 - x), 0 < x < 1$, zero elsewhere; otherwise, let $Q(C)$ be undefined. If $C_1 = \{x : \frac{1}{4} < x < \frac{3}{4}\}, C_2 = \{\frac{1}{2}\}$, and $C_3 = \{x : 0 < x < 10\}$, find $Q(C_1), Q(C_2)$, and $Q(C_3)$.

Solution:

(a) $C_k = \{x : 2 - \frac{1}{k} < x \leq 2\}, C_{k+1} = \{x : 2 - \frac{1}{k+1} < x \leq 2\}$, so $C_k \supset C_{k+1}$.

$$\lim_{k\to\infty} C_k = \bigcap_{k=1}^{\infty} C_k = \{x : 2 < x \leq 2\} = \{x : x = 2\}.$$

(b) $C_k = \{x : 2 < x \leq 2 + \frac{1}{k}\}, C_{k+1} = \{x : 2 < x \leq 2 + \frac{1}{k+1}\}$, so $C_k \supset C_{k+1}$.

$$\lim_{k\to\infty} C_k = \bigcap_{k=1}^{\infty} C_k = \emptyset.$$

(c) $C_k = \{(x, y) : 0 \leq x^2 + y^2 \leq \frac{1}{k}\}, C_{k+1} = \{(x, y) : 0 \leq x^2 + y^2 \leq \frac{1}{k+1}\}$, so $C_k \supset C_{k+1}$.

$$\lim_{k\to\infty} C_k = \bigcap_{k=1}^{\infty} C_k = \{(x,y) : x^2 + y^2 = 0\} = \{(x,y) : x = y = 0\}.$$

1.3.6

If the sample space is $\mathcal{C} = \{c : -\infty < c < \infty\}$ and if $C \subset \mathcal{C}$ is a set for which the integral $\int_C e^{-|x|}dx$ exists, show that this set function is not a probability set function. What constant do we multiply the integrand by to make it a probability set function?

Solution:

$$\int_C e^{-|x|} = \int_{-\infty}^{\infty} e^{-|x|} = \int_{-\infty}^{0} e^{x} + \int_{0}^{\infty} e^{-x} = 2,$$

which means that this set function is not a probability set function and the constant is $\frac{1}{2}$.

1.10.5

Let X be a random variable with mgf $M(t), -h < t < h$. Prove that

$$P(X \ge a) \le^{-at} M_X(t), \quad 0 < t < h,$$

and that

$$P(X \le a) \le^{-at} M_X(t), \quad -h < t < 0.$$

Solution:

If $0 < t < h$,

$$P(X \ge a) \overset{(1)}{=} P(e^{tX} \ge e^{ta}) \overset{(2)}{\le} \frac{E(e^{tX})}{e^{at}} = e^{-at}M_X(t) \quad \text{for } \forall a > 0,$$

(1) since e^{tx} is increasing for x;
(2) by replacing $X \to e^{tX}(> 0), a \to e^{at}(> 0)$ in Markov's inequality.

If $-h < t < 0$,

$$P(X \le a) \overset{(3)}{=} P(e^{tX} \ge e^{ta}) \overset{(4)}{\le} \frac{E(e^{tX})}{e^{at}} = e^{-at}M_X(t) \quad \text{for } \forall a > 0,$$

(3) since e^{tx} is decreasing for x;
(4) by replacing $X \to e^{tX}(> 0), a \to e^{at}(> 0)$ in Markov's inequality.

3.3.15

Let X have a Poisson distribution with parameter m. If m is an experimental value of a random variable having a gamma distribution with $\alpha = 2$ and $\beta = 1$, compute $P(X = 0, 1, 2)$.

Solution:

$$
\begin{aligned}
f_{X,M}(x,m) &= f_{X|M}(x|m)f_M(m) = \frac{e^{-m}m^x}{x!}\frac{me^{-m}}{\Gamma(2)} = \frac{e^{-2m}m^{x+1}}{x!} \\
f_X(x) &= \int_0^\infty f_{X,M}(x,m)dm = \frac{1}{x!}\int_0^\infty \left(\frac{t}{2}\right)^{x+1}\frac{e^{-t}}{2}dt \;\; (2m = t) \\
&= \frac{1}{x!2^{x+2}}\int_0^\infty t^{x+1}e^{-t}dt = \frac{\Gamma(x+2)}{x!2^{x+2}}
\end{aligned}
$$

Thus,

$$
P(X=0) = \frac{\Gamma(2)}{0!2^2} = \frac{1}{4}, P(X=1) = \frac{\Gamma(3)}{1!2^3} = \frac{1}{4}, P(X=2) = \frac{\Gamma(4)}{2!2^4} = \frac{3}{16}
$$
$$
\Rightarrow P(X=0,1,2) = \frac{11}{16}.
$$

3.3.24

Let X_1, X_2 be two independent random variables having gamma distributions with parameters $\alpha_1 = 3, \beta_1 = 3$ and α$\alpha_2 = 5, \beta_2 = 1$, respectively.
(a) Find the mgf of $Y = 2X_1 + 6X_2$.
(b) What is the distribution of Y ?

Solution:

$$
\begin{aligned}
M_Y(t) &= M_{X_1,X_2}(2t, 6t) = M_{X_1}(2t)M_{X_2}(6t) \\
&= (1-\beta_1(2t))^{-\alpha_1}(1-\beta_2(6t))^{-\alpha_2} = (1-6t)^{-3}(1-6t)^{-5} = (1-6t)^{-8}
\end{aligned}
$$

provided that $t < 1/6$. Thus $Y \sim \Gamma(8, 6)$.

3.4.29.

Let X_1 and X_2 be independent with normal distributions N(6, 1) and N(7, 1), respectively. Find $P(X_1 > X_2)$.

Solution:

Let $Y = X_1 - X_2$ then since X_1 and X_2 are independent, mgf of Y is

$$
M_Y(t) = M_{X_1}(t)M_{X_2}(-t) = \exp(6t + t^2/2)\exp(-7t + t^2/2) = \exp(-t + t^2)
$$

indicating that $Y \sim N(-1, 2)$. Hence

$$
\begin{aligned}
P(X_1 > X_2) = P(Y > 0) &= 1 - P(Y < 0) \\
&= 1 - P\Big(\frac{Y+1}{\sqrt{2}} < \frac{1}{\sqrt{2}}\Big) \\
&= 1 - \Phi(0.7071) = 0.240.
\end{aligned}
$$

3.4.30.

Compute $P(X_1 + 2X_2 - 2X_3 > 7)$ if X_1, X_2, X_3 are iid with common distribution $N(1, 4)$.

Solution:

Let $Y = X_1 + 2X_2 - 2X_3$ then since X_1, X_2, and X_3 are independent, mgf of Y is

$$M_Y(t) = M_{X_1}(t)M_{X_2}(2t)M_{X_3}(-2t) = \exp(t + 2t^2)\exp(2t + 8t^2)\exp(-2t + 8t^2) = \exp(t + 18t^2)$$

which means that $Y \sim N(1, 36)$. Thus

$$\begin{aligned} P(Y > 7) &= 1 - P(Y < 7) \\ &= 1 - P\Big(\frac{Y-1}{\sqrt{18}} < \frac{6}{6}\Big) \\ &= 1 - \Phi(1) = 0.159. \end{aligned}$$