Mathematical Proof 1: Random Search

Prove that Random Search needs with probability $1 - e^{-\Omega(n)}$ at least a budget of $2^{\frac{n}{2}}$ fitness evaluations to reach an optimal search point for the function F1.

F1: OneMax

$$\mathrm{OM} : \{0,1\}^n \to [0..n], x \mapsto \sum_{i=1}^{n} x_i.$$

The problem has a very smooth and non-deceptive fitness landscape. Due to the well-known coupon collector effect, it is relatively easy to make progress when the function values are small, and the probability of obtaining an improving move decreases considerably with increasing function value.

1) The OneMax function counts the number of ones in a bitstring of length n. Additionally, the search space is $\{0,\ 1\}^n$ which has size 2^n, and the unique optimum is the all-ones string $x^* = 1^n$ with fitness n.

2) Let X denote the number of times the optimum x^* is sampled in t independent uniform trials from $\{0,\ 1\}^n$. Then X is a non-negative random variable with,

$$\boldsymbol{X \sim Bin(t, p)}, \qquad \boldsymbol{p} = \frac{1}{2^n} = \boldsymbol{2^{-n}}, \qquad \boldsymbol{E[X] = t \cdot p}$$

3) By Markov's inequality, for any non-negative random variable X and any $s > 0$,

$$\boldsymbol{Pr(X \geq s \cdot E[X]) \leq \frac{1}{s}}$$

To bound the probability such that $X \geq 1$, we set $s = \frac{1}{E[X]} > 0$ in Markov's inequality (which is valid since $E[X] > 0$ whenever $t > 0$). This yields,

$$Pr\left(X \geq \frac{1}{E[X]} \cdot E[X]\right) \leq \frac{1}{\frac{1}{E[X]}}$$

$$\therefore Pr(X \geq 1) \leq E[X]$$

$$\therefore \boldsymbol{Pr(X \geq 1) \leq t \cdot p}$$

Thus, the probability that Random Search reaches an optimum search value within t evaluations is at most tp.

4) Now setting $t = 2^{\frac{n}{2}}$, and since $p = 2^{-\mathrm{n}}$,

$$\boldsymbol{t \cdot p} = 2^{\frac{n}{2}} \cdot 2^{-\mathrm{n}} = \mathbf{2^{-\frac{n}{2}}}$$

This can be written as,

$$2^{-\frac{\mathrm{n}}{2}} = e^{-\left(\frac{n}{2}\right)(\ln 2)} = \boldsymbol{e^{-\Omega(n)}}$$

5) Therefore, the probability that Random Search finds the optimal search value within $2^{\frac{n}{2}}$ evaluations is at most,

$$Pr\left(find\ optimum\ within\ 2^{\frac{n}{2}}\right) \leq e^{-\Omega(n)}$$

Thus, the probability that Random Search does not find the optimal search value within $2^{\frac{n}{2}}$ evaluations is,

$$Pr\left(no\ optimum\ within\ 2^{\frac{n}{2}}\right) \geq 1 - Prob\left(find\ optimum\ within\ 2^{\frac{n}{2}}\right)$$

$$\therefore\ \boldsymbol{Pr\left(no\ optimum\ within\ 2^{\frac{n}{2}}\right) \geq 1 - e^{-\Omega(n)}}$$

So, with probability at least $1 - e^{-\Omega(n)}$, Random Search requires at least $2^{\frac{n}{2}}$ evaluations to reach an optimal search point for the function F1.