No.2　弾性基礎式の導出

20D4104016H　森智紀

2022 年 11 月 7 日

1　課題 1：x,y,z 座標系で、応力を変位で表せ。

3 次元のフックの法則は以下のようになる。

$$
\begin{cases}
\sigma_x = \dfrac{E}{(1-2\nu)(1+\nu)}\left\{(1-\nu)\epsilon_x + \nu(\epsilon_y + \epsilon_z)\right\} \\
\sigma_y = \dfrac{E}{(1-2\nu)(1+\nu)}\left\{(1-\nu)\epsilon_y + \nu(\epsilon_z + \epsilon_x)\right\} \\
\sigma_z = \dfrac{E}{(1-2\nu)(1+\nu)}\left\{(1-\nu)\epsilon_z + \nu(\epsilon_x + \epsilon_y)\right\}
\end{cases}
\tag{1}
$$

ここで、ひずみを変位で表すと、

$$\epsilon_x = \frac{\partial u}{\partial x} \tag{2}$$

$$\epsilon_y = \frac{\partial v}{\partial y} \tag{3}$$

$$\epsilon_z = \frac{\partial w}{\partial z} \tag{4}$$

であるから (1) 式に代入すると

$$
\begin{cases}
\sigma_x = \dfrac{E}{(1-2\nu)(1+\nu)}\left\{(1-\nu)\dfrac{\partial u}{\partial x} + \nu(\dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z})\right\} \\
\sigma_y = \dfrac{E}{(1-2\nu)(1+\nu)}\left\{(1-\nu)\dfrac{\partial v}{\partial y} + \nu(\dfrac{\partial w}{\partial z} + \dfrac{\partial u}{\partial x})\right\} \\
\sigma_z = \dfrac{E}{(1-2\nu)(1+\nu)}\left\{(1-\nu)\dfrac{\partial w}{\partial z} + \nu(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y})\right\}
\end{cases}
\tag{5}
$$

を得る。

さらに、せん断応力 τ_{xy},τ_{yz},τ_{zx} を表す。せん断ひずみ γ_{xy},γ_{yz},γ_{zx} は、

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{6}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \tag{7}$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{8}$$

と表せるので、

$$\tau_{xy} = G\gamma_{xy} = G\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{9}$$

$$\tau_{yz} = G\gamma_{yz} = G\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \tag{10}$$

$$\tau_{zx} = G\gamma_{zx} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{11}$$

を得る。

ここで、応力の釣り合い式を立てる。x 方向のみを考えると、

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\right)dydz + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\right)dxdz + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\right)dxdy - \sigma_x dydz - \tau_{yx}dxdz - \tau_{zx}dydx + Xdxdydz = 0 \tag{12}$$

となる。これを変形すると、

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = -X \tag{13}$$

を得る。y, z 方向も同様に考えると、以下のようになる。

$$\left\{\begin{array}{l} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} = -X \\ \dfrac{\partial \tau_{yx}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} = -Y \\ \dfrac{\partial \tau_{yz}}{\partial x} + \dfrac{\partial \tau_{xz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} = -Z \end{array}\right. \tag{14}$$

2　x,y,z 座標系で変位に関する微分方程式を導け

式 (13) に式 (5),(10),(11) を代入することにより求める。

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = -X \tag{15}$$

$$\frac{E}{(1-2\nu)(1+\nu)} \left\{ (1-\nu)\frac{\partial u}{\partial x} + \nu(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) \right\} + G \left\{ \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \right\} \tag{16}$$

$$\frac{E(1-\nu)}{(1-2\nu)(1+\nu)}\frac{\partial^2 u}{\partial x^2} + \frac{E}{(1-2\nu)(1+\nu)}\nu\frac{\partial}{\partial x}(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) + G\left\{ \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial}{\partial x}(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) \right\} = -X \tag{17}$$

ここで、$E = 2G(1+\nu)$ より、以下のように変形できる。

$$\frac{2G(1-\nu)}{(1-2\nu)}\frac{\partial^2 u}{\partial x^2} + \frac{2G\nu}{(1-2\nu)}\frac{\partial}{\partial x}(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) + G\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + G\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})\right) = -X \tag{18}$$

$$\frac{2(1-\nu)}{(1-2\nu)}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{2\nu}{1-2\nu}\frac{\partial}{\partial x}(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) + \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = -\frac{X}{G} \tag{19}$$

$$\frac{1}{(1-2\nu)}\frac{\partial^2 u}{\partial x^2} + \nabla^2 u + \frac{1}{1-2\nu}\frac{\partial}{\partial x}(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = -\frac{X}{G} \tag{20}$$

$$(1-2\nu)\nabla^2 u + \frac{\partial}{\partial x}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = -\frac{1-2\nu}{G}X \tag{21}$$

ここで、体積弾性率 e は $e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ と表せるので、式 (20) は

$$(1-2\nu)\nabla^2 u + \frac{\partial e}{\partial x} = -\frac{1-2\nu}{G}X \tag{22}$$

と表せる。y, z 方向も同様に考えると、以下のようになる。

$$\begin{cases} (1-2\nu)\nabla^2 u + \frac{\partial e}{\partial x} = -\frac{1-2\nu}{G}X \\ (1-2\nu)\nabla^2 v + \frac{\partial e}{\partial y} = -\frac{1-2\nu}{G}Y \\ (1-2\nu)\nabla^2 w + \frac{\partial e}{\partial z} = -\frac{1-2\nu}{G}Z \end{cases} \tag{23}$$