Q1: $A \in \mathbb{R}^{m\times m}$ UNITARY AND UPPER TRIANGULAR. SHOW IT USING INDUCTION.

BASE CASE $m=1$. $A=[a]$ is INVERTIBLE $a\neq 0$.

m : A_m is DIAGONAL UNITARY

$m+1$:

$$A = \left[\begin{array}{c|c} A_m & w \\ \hline 0 \cdots 0 & a \end{array}\right]$$

column j (a_j), column a_{m+1}.

By orthonormality of the columns

$a_{m+1}\cdot a_j = 0$, for $j<m+1$

$\Rightarrow w\cdot a_j = w_j a_{jj} = 0$

$\Rightarrow w_j = 0$ since A is unitary (and thus $a_{jj}\neq 0$).

ALSO SINCE $\|Ax\|=\|x\|$ $\forall x$ $\|Ae_{m+1}\|=1 \Rightarrow$

$|a|=1$. $\left(Ae_{m+1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ a \end{bmatrix} \right)$

Alternatively we can look at $A^TA=I$, form $A^TA \Rightarrow$

$$\begin{bmatrix} A^T & 0 \\ w^T & a \end{bmatrix}\begin{bmatrix} A & w \\ 0 & a \end{bmatrix} = \begin{bmatrix} A^TA & A^Tw \\ w^TA & a\cdot a \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$\Rightarrow a=1$ and $A^Tw=0$. THE LATTER NEEDS JUSTIFICATION.

Since $w\neq 0$, THE ONLY WAY FOR $A^Tw=0$ IS TO HAVE A RANK DEFICIENT. $\rightarrow$ CONTRADICTION. Thus $A^Tw=0 \Rightarrow \underline{w=0}$ $\square$.
(A is UNITARY)

QUESTION 2

: Let $\|\ \|$ be a vector-induced norm. (ANY NORM NOT ONLY THE 2-NORM). Let $A\in\mathbb{R}^{n\times n}$; SHOW THAT $(I-A)$ is invertible. We SHOW THIS BY CONTRADICTION.

Assume that $I-A$ is not INVERTIBLE. Then IT HAS A NON TRIVIAL NULL SPACE, I.E., $\exists x$:

non trivial nullspace.

WITHOUT LOSS OF GENERALITY ASSUME $\|x\|=1$.

$(I-A)x = 0$, $x\neq 0 \Rightarrow Ax=x. \Rightarrow \|Ax\|=\|x\|$.

SINCE $\|A\| = \max_{\|x\|=1}\|Ax\|$; $\|Ax\|=\|x\|$ IMPLIES THAT

$\|A\|\geq 1$. THIS IS A CONTRADICTION SINCE WE ASSUMED $\|A\|<1$.

Q3

$$A = U\Sigma V^T \quad ; \quad \underset{U}{\underbrace{4\times 2}} \; \underset{\Sigma}{\underbrace{2\times 2}} \; \underset{V^T}{\underbrace{2\times 3}} \quad ; \quad A \in \mathbb{R}^{4\times 3}$$

A IS RANK DEFICIENT. RANGE (A) : span (u_i)

Row (A) : SPAN (v_i) $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right\}$.

Null (A) : span $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; a vector perpendicular to v_i

$$\|A\|_2 = \sigma_{max} = 1$$

$$\text{cond}_2(A) = \frac{\sigma_1}{\sigma_1} = 1.$$

Q4

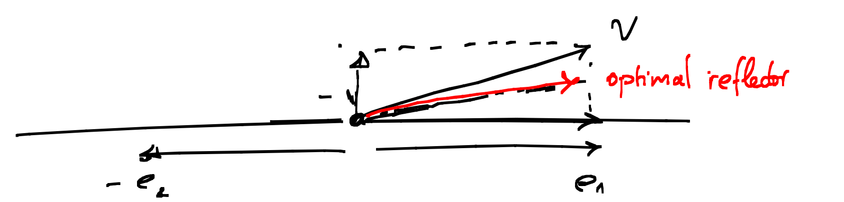
: H is UNITARY. $H^{-1} = H^T$

H is symmetric $H^T = H$;

$$\Rightarrow \quad Hv = w \Rightarrow w = H^{-1} v = Hv.$$

Q5

$$v = \begin{bmatrix} 1 \\ \epsilon_M \end{bmatrix}$$



$r_1 = v - \|v\| e_1$; $r_1 = \frac{r_1}{\|r_1\|_2}$; $\|v\| = \sqrt{1+\epsilon_M^2} = 1$; $r_1 = \begin{bmatrix} 0 \\ \epsilon_M \end{bmatrix} / \epsilon_M = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ✗

This will not align v to e_1

$r_2 = v + \|v\| e_1$; $r_2 = \frac{r_2}{\|r_2\|_2}$

$$r_2 = \begin{bmatrix} 1 \\ \epsilon_M \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ \epsilon_M \end{bmatrix} \; ; \quad r_2 = \begin{bmatrix} 2 \\ \epsilon_M \end{bmatrix} \Big/ \sqrt{4+\epsilon_M^2} = \begin{bmatrix} 1 \\ \epsilon_M/2 \end{bmatrix} \; ✓$$

Q6

MANY ALGORITHMS SUGGESTED. ASSUME $\|v\| = 1$.

E.g. Projector to Null (v) $P: (I - vv^T)$; P is not ORTHONORMAL ; USE QR WITH HOUSEHOLDER.

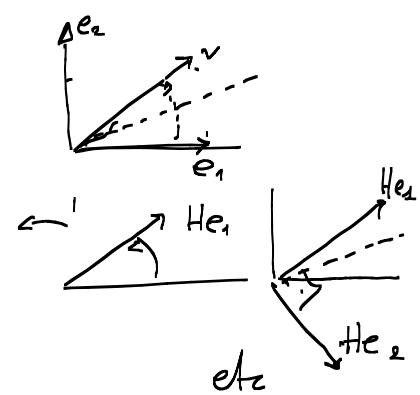
BUT PIVOTING IS NEEDED TO PICK THE EXACT COLUMNS.

(e.g. $v = e_1$; $I - e_1 e_1^T - \begin{bmatrix} 0 & 0 & 0 \\ \vdots & 1 & 0 \\ \vdots & \vdots & 1 \end{bmatrix}$.)

THIS ALGORITHM IS $O(n^3)$. BUT THERE IS AN $O(1)$ ALGO THAT DOESN'T REQUIRE PIVOTING AND IT IS STABLE.

- Use HOUSEHOLDER TRANSFORMATIONS TO ALIGN THE CANONICAL BASIS WITH E1.

  - Let $k = \max_k \{v_k\}$; PICK THE CANONICAL DIRECTION v POINTS TO.
  - WITHOUT LOSS OF GENERALITY ASSUME $k = 1$.
  - Find HOUSEHOLDER : $He_1 = v$.



For $i = 2 \ldots, n$

$$q_i = He_i. \qquad O(n^2) \text{ WORK.}$$

- THEN a) $q_i \cdot v = 0 \quad \forall i \geq 2$

  b) $q_i \cdot q_j = \delta_{ij}$

$$\text{a)}: \quad He_i \cdot He_1 = e_i \cdot H^* H \, e_1 = e_i \cdot e_1 = 0$$

$$\text{b)}: \quad He_i \cdot He_j = e_i \cdot e_j = \delta_{ij}$$

COMPLEXITY : $q_i = \underbrace{e_i - 2u}_{O(n) \text{ work}} \underbrace{(u^* e_i)}_{O(1)}$

$i = 2 \ldots n \rightarrow O(n^2)$ work

THE ALGORITHM IS NUMERICALLY STABLE.

$\boxed{Q7}$

- $\| [f(x)] - f(y) \| = O(\epsilon_M) \|f\|$ WITH $\|x - y\| = O(\epsilon_M) \|x\|$.

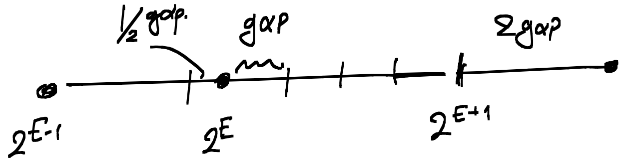
THEN $\| [f(x)] - f(x) \| = \| [f(x)] - f(y) + f(y) - f(x) \| \leq O(\epsilon_M) \|f\| + \| f(x) - f(y) \|$

$$\frac{\| [f(x)] - f(x) \|}{\|f\|} \leq O(\epsilon_M) \frac{\|f\|}{\|f\|} + \frac{\| f(x) - f(y) \|}{\|f\|} \leq (1 + \kappa_f) \, O(\epsilon_M)$$

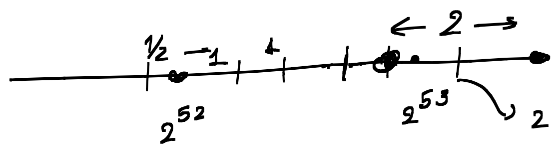
$\blacksquare$

Q8

FLOATING POINT NUMBERS $\qquad gap = 2^E \epsilon_M$.



Let $\epsilon = 52$. Then the gap is $2^E \epsilon_M = 2^{52} 2^{-52} = 2^0 = 1$.



$2^{53}+1$ is not part of the floating point numbers.

Q9

A IS UNDERDETERMINED AND FULL RANK $\qquad m \; \underbrace{\qquad}_{n} \qquad \underline{\underline{m < n}}$.

$[Q,R] = qr(A^T) = \underset{Q}{m \times m} \; R \qquad Q^T = \qquad ; \; QQ^T = \qquad = n$

↳ not invertible.

$\Rightarrow R^T Q^T x = b$; R^T is invertible $\Rightarrow Q^T x = R^{-T} b$.

[some of you inverted QQ^T and $A^T A$]

Since x SHOULD BE IN THE ROWSPACE (A) LET

$x = Qy$;

Then $Q^T Q y = R^{-T} b \quad ; \; Q^T Q = I_{m \times m}$

$\Rightarrow \quad y = R^{-T} b \quad \Rightarrow \quad \boxed{x = Q R^{-T} b}$

THIS SOLUTION IS EQUIVALENT TO SVD BY EQUIVALENCE TO THE PSEUDOINVERSE OF A^T OR BY THE UNIQUENESS OF THE MAP: ROWSPACE (A) ⟶ RANGE (A).

(ALSO YOU CAN TAKE THE SVD OF R AND GET THE SVD OF A BY UNIQUNESS OF THE DECOMPOSITION).

NOTICE THAT WE CAN STILL REQUIRE $A^T r = 0$ BUT $A^T A$ IS NOT INVERTIBLE. BUT AA^T IS.

$\boxed{Q10}$

THE RESIDUAL $r = Ax - b$ IS GIVEN BY

$$r = \begin{bmatrix} A & b \end{bmatrix}\begin{bmatrix} x \\ -1 \end{bmatrix} = \begin{bmatrix} Q & q \end{bmatrix}\begin{bmatrix} R & w \\ 0 & \beta \end{bmatrix}\begin{bmatrix} x \\ -1 \end{bmatrix} = \begin{bmatrix} Q & q \end{bmatrix}\begin{bmatrix} Rx - w \\ -\beta \end{bmatrix}$$

$$\Rightarrow \quad r = Q(Rx - w) - q\beta$$

↳ notice that $b = \underbrace{Qw}_{\text{Range}(A)} + \underbrace{q\beta}_{\mathbb{C}^m \setminus \text{Range}(A)}$, and $q \perp Q$.

In the least squares $x = \underset{x}{\arg\min} \|r\|_2^2$

$$\|r\|_2 = \|Q(Rx - w) - \|q\beta\|_2^2$$; SINCE Q and q are orthogonal to each other

$$= \|Q(Rx - w)\|_2^2 + |\beta|^2$$

since β^2 does not depend on x,

$$\underset{x}{\arg\min} \|r\|_2^2 = \underset{x}{\arg\min} \|Q(Rx \cdot w)\|_2^2 .$$

Notice that $\|Q(Rx - w)\|_2^2 \geq 0$ since it is a norm.

Notice that $x = R^{-1}w$ sets the norm to zero. thus $x = R^{-1}w$ is a minimizer.

- This least squares solution is stable by the stability of computing the R factor using Gram-Schmidt.

- Notice we only use the fact that R is square invertible. We are not using orthogonality of Q. But we construct w so that Qw is the projection of b to the range of A, so we can solve the problem uniquely.