The University of Texas at Austin
Department of Electrical and Computer Engineering

EE381K: Large Scale Optimization — Fall 2015

PROBLEM SET THREE

Constantine Caramanis Due: Tuesday, September 22, 2015.

Reading Assignments

1. (?) Reading: Boyd & Vandenberghe: Chapters 9.1 - 9.5.

Matlab and Computational Assignments. Please provide a printout of the Matlab code you wrote to generate the solutions to the problems below.

1. Consider the non-quadratic problem given in Eq. (9.20) in B & V. Implement five flavors of gradient descent algorithms, and provide the convergence plots for all five.
   (a) Standard gradient descent with backtracking.
   (b) Two kinds of Steepest Descent, using the two matrices suggested in the book:
$$P_1 = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}.$$
   (c) Cyclic coordinate descent, and greedy coordinate descent, as defined in class.

Written Problems

1. **Coordinate Descent**
   (a) Give an example that shows that coordinate descent may not find the optimum of a convex function. That is, provide a simple function f and a point x such that coordinate descent starting from x will *not* get to the global minimum of f.
   (b) Let $f(x, y) = x^2 + y^2 + 3xy$, where x, y are scalars. Note that f is not convex. Would coordinate descent with exact line search always converge to a stationary point ?

2. **Condition Number**. We saw in class that a fixed step size is able to guarantee linear convergence. The choice of step size we gave in class, however, depended on the function f. Show that it is not possible to choose a fixed step size t, that gives convergence for any strongly convex function. That is, for any fixed step size t, show that there exists (by finding one!) a smooth (twice continuously-differentiable) strongly convex function with bounded Hessian, such that a fixed-stepsize gradient algorithm starting from some point x_0, does not converge to the optimal solution.

3. **Decreasing Stepsize.**[1] The previous problem shows that no constant step-size works for every strongly convex function. Consider now, a decreasing step size. Thus, at time k, you use step size $t_k \geq 0$. Show that if this sequence of step sizes satisfies:
$$\lim_k t_k = 0, \qquad \sum_{k=0}^{\infty} t_k = \infty,$$

[1] This problem borrowed from Nati Srebro.

then gradient descent converges to the global optimal solution. Hint: Recall that strong convexity implies lower and upper bounds on the Hessian. Each of these bounds in turn gives lower and upper bounds on the value of $f(y)$ with respect to $f(x)$. Use one of these two show that for k large enough,

$$f(x_{k+1}) \leq f(x_k) - \frac{1}{2} t_k \|\nabla f(x_k)\|_2^2.$$

Use the other inequality to get (lower) bound on $\|\nabla f(x_k)\|$ in terms of the optimality gap. Then put these together to conclude that gradient descent must converge.

4. **Convex functions**

   (a) If f_i are convex functions, show that $f(x) := \sup_i f_i(x)$ is also convex.

   (b) Show that the largest eigenvalue of a matrix is a convex function of the matrix (i.e. $\lambda_{\max}(M)$ is a convex function of M). Is the same true for the eigenvalue of largest magnitude ?

   (c) Consider a weighted graph with edge weight vector w. Fix two nodes a and b. The *weighted shortest path* from a to b is the path whose sum of edge weights is the minimum, among all paths with one endpoint at a and another at b. Let $f(w)$ be the weight of this path. Show that f is a concave function of w.

5. **Convex functions: Jensen's Inequality**. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any function. It's epigraph is defined as the set:

$$\text{epi}(f) = \{(x, y) \in \mathbf{R}^{n+1} \, : \, y \geq f(x)\}.$$

   (a) Show that if f is convex, then epi(f) is also convex.

   (b) Prove (the finite version of) Jensen's inequality. Jensen's inequality says that if p is a distribution on $\{x_1, \ldots, x_m\}$ with weights $p_1, \ldots, p_m$, and f is any concave function, then

$$\mathbb{E}[f(X)] \leq f(\mathbb{E}(X)).$$

6. **Projection**. We have been discussing only unconstrained problems. We will soon consider constraints. One update we will consider has the following form:

$$x^{(k+1)} = \arg\min_{x \in \mathcal{X}} \left\{ \langle x, \nabla f(x^{(k)} \rangle + \frac{1}{2t_k} \|x - x^{(k)}\|_2^2 \right\}.$$

   Show that the solution is:

$$x^{(k+1)} = \text{Proj}_{\mathcal{X}}(x^{(k)} - t_k \nabla f(x^{(k)})).$$

   This is called the *Projected Gradient* algorithm.

7. **Computing Projections**. For the given convex set $\mathcal{X}$, compute the projection of a point z.

   (a) $\mathcal{X}$ is a rectangle defined by vectors L and U that satisfy $U_i \geq L_i$. Thus, $\mathcal{X} = \{x \, : \, L_i \leq x_i \leq U_i, \; i = 1, \ldots, n\}$.

   (b) $\mathcal{X} = \mathbb{R}^n_+$.

   (c) Euclidean ball: $\{x \, : \, \|x\|_2 \leq 1\}$.

   (d) 1-norm ball: $\{x \, : \, \sum_i |x_i| \leq 1\}$.

   (e) Positive semidefinite cone: $S^n_+ = \{M \in S^n \, : \, x^\top M x \geq 0, \; \forall x \in \mathbb{R}^n\}$.

   (f) Probability simplex: $\mathcal{X} = \{x \, : \, \sum_i x_i = 1, \; x_i \geq 0, \; i = 1, \ldots, n\}$.