56/100

Fall 2013, CSE 383, Numerical Linear Algebra, Midterm
Tuesday, October 31,
This is a closed-book exam.

YOUR NAME : Stephen Shannon

1. [**10 points**,] Let $A \in \mathbb{R}^{m\times m}$ be unitary and upper triangular. Using induction with base case $m = 1$, show that A is diagonal with nonzero entries in the diagonal.

A unitary
A upper triangular

Base case $k=1$ $A = [a_{11}] = [\pm 1]$

Hyp: true for $k=m-1$ then $A = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{m-1,m-1} \end{bmatrix}$ where $a_{11} \ldots a_{m-1,m-1}$ are ± 1

unitary and diagonal

$k=m$ $A = \begin{bmatrix} A^{(m-1)\times(m-1)} & \begin{matrix} a_{1m} \\ \vdots \end{matrix} \\ a_{m1} \cdots & a_{mm} \end{bmatrix}$

$\Rightarrow a_{1m} = \ldots = a_{m(m-1)} = 0$
$a_{m1} = \ldots = a_{(m-1)m} = 0$
$a_{mm} = \pm 1$
(b.c. $a_1^* a_m = a_2^* a_m = \ldots = a_{m-1}^* a_m = 0$)

2. [**5 points**,] Let $A \in \mathbb{R}^{m\times m}$ and let $\|A\|$ be a vector-induced matrix norm. Let I be the $m \times m$ identity matrix. Show that if $\|A\| < 1$ then $I - A$ is invertible.

$\|A\| = \max_{\|x\|=1} \|Ax\|$

$0 \le \|A\| < 1$

$\|I - A\| \le \|I\| + \|-A\| = \|I\| + \|A\| = 1 + \|A\| \ge 1$

$\|I\| + \|A\| \geq 1$
but $\|I-A\| \le \|I\| + \|A\|$

-4

Since $\|I-A\| \ge 1$, $I-A$ is invertible

4×3

A

$m \times r$ $\quad$ $r \times r$ $\quad$ $r \times n$

U $\quad$ Σ $\quad$ V^*

3. **[20 points,**] Let

$$A = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

truncated

(a) Give a set of orthonormal vectors that span range(A); (b) Give a set of orthonormal vectors that span rowspace(A); (c) give a set of orthonormal vectors that span null(A); (d) What is $\|A\|_2$?; (e) What is the 2-norm condition number of the linear map defined by $y = Ax : x \in \text{rowspace}(A)$?

(a) $\left\{ \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix} \right\} \left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \right\}$

(b) $\left\{ \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \right\} \left\{ \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{pmatrix} \right\}$ $\quad$ (c) $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

(d) $\|A\|_2 = 2$ (max sing. value)

(e) $\kappa_2(A) = \frac{2}{1} = 2$ $\left(\frac{\sigma_{max}}{\sigma_{min}}\right)$

4. **[5 points,**] Let $u \in \mathbb{R}^n$ such that $\|u\|_2 = 1$ be given. Let $H = I - 2uu^T$. Let $w \in \mathbb{R}^n$ be given. Give a formula for finding a vector v such that $Hv = w$. Explain.

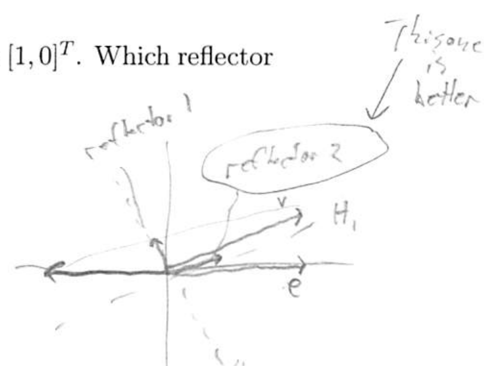
$(I - 2uu^T)v = w$

$H = (I - 2uu^T)$

$v = H^* w$

since H is unitary

5. **[5 points,**] Let $v = [1, \epsilon_M]^T$. Find the two reflectors that align v with $[1, 0]^T$. Which reflector is numerically stable? Explain.



$$\underset{\sim}{u}_1 = \frac{\underset{\sim}{v} - \underset{\sim}{e}_1}{\|v - \underset{\sim}{e}_1\|_2}$$

$$\underset{\sim}{u}_2 = \frac{\underset{\sim}{v} - -\underset{\sim}{e}_1}{\|v - -\underset{\sim}{e}_1\|_2} = \boxed{\frac{\underset{\sim}{v} + \underset{\sim}{e}_1}{\|v + \underset{\sim}{e}_1\|_2}}$$

We use the reflector that reflects further in order to avoid rounding error due to subtraction.

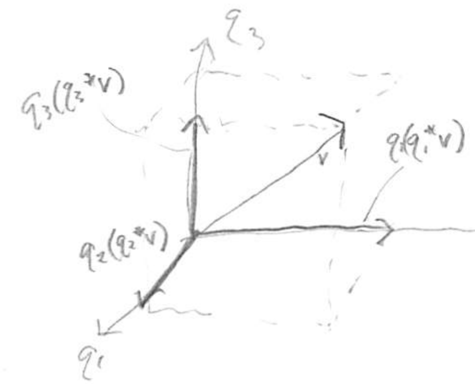
6. [**20 points**,] Let $v \in \mathbb{R}^n$. Suggest an algorithm for constructing an orthonormal basis for $\mathbb{R}^n \backslash \mathrm{span}\{v\}$. Give an estimate for the work complexity of your algorithm.

-15

$\underset{\sim}{v} = v_1 + \dots + v_n$

projects v onto q_1

$r = v - q_1 q_1^* v - q_2 q_2^* v \cdots q_n q_n^* v$



V is a single vector and so it spans 1 dimension

$\mathbb{R}^n - v$ should be of dimension $\mathbb{R}^{n-1}$ ✓

$q_n = \frac{v}{\|v\|}$

Build $n-1$ more vectors orthogonal to q_n ← how? that's the question.

Then we have $\underset{n \times n}{Q} = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$

→ This spans $\mathbb{R}^n$

Then remove q_n so we're left with $\underset{n \times n-1}{\tilde{Q}} = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_{n-1} \\ | & & | \end{bmatrix}$

which spans $\mathbb{R}^n - v$

7. [**10 points**,] Assume you performing a numerical stability analysis for a function $y = f(x)$. Let $[f(x)]$ denote the numerical evaluation of f using IEEE-754-compliant hardware. Assume that you've managed to show that for every x there exists a y such that $\|[f(x)] - f(y)\| = \mathcal{O}(\epsilon_M)$ with $\|x - y\| = \mathcal{O}(\epsilon_M)$. Using this fact, show that the algorithm for computing f numerically stable.

0

Def. numerical stability $\forall x \ \exists y : \frac{\|y-x\|}{\|x\|} = O(\varepsilon_n)$ and $\frac{\|\tilde{f}(x) - f(\tilde{x})\|}{f(\tilde{x})} = O(\varepsilon_n)$

This is definition of forward stable

$\forall x \ \exists y : \|x-y\| = O(\varepsilon_n)$ and $\|\tilde{f}(x) - f(y)\| = O(\varepsilon_n)$

8. [**5 points**,] What is the smallest positive integer that is not a double precision floating point number (i.e., cannot be represented exactly in IEEE-754)?

-5

$\boxed{3}$

$(1 = 1.00 \ldots \times 2^0)$

$(2 = 1.00 \ldots \times 2^1)$

9. [**10 points**,] Let $A \in \mathbb{R}^{m \times n}$ be a full rank matrix with $m < n$. Show how you can use the Householder QR factorization of A^T to compute the least squares solution to $Ax = b$ that is equal to $x = V\Sigma^{-1}U^T b$, the solution we obtain using the reduced SVD of A. (Without using the actual SVD of A or any other matrix.) Explain.

-10

10. [**10 points**,] Let $A \in \mathbb{A}^{m \times n}$ be a full-rank matrix with $m > n$. Let Q and R be the (reduced) QR factorization of A computed using the modified Gram-Schmidt (MGS) orthogonalization. Since Q is not orthonormal, we cannot use this factorization to solve $Ax = b$. However, consider using the MGS-factorization of the augmented matrix $[A \quad b]$

$$[A \quad b] = [Q \quad q] \begin{bmatrix} R & w \\ 0 & \beta \end{bmatrix}$$

Noticing

$$Ax - b = [A \quad b] \begin{bmatrix} x \\ -1 \end{bmatrix}$$

-10

show that this factorization can be used to compute the least squares solution of $Ax = b$ by setting $x = R^{-1}w$. Is this algorithm numerically stable?