Fall 2016, CSE/CS 383, Homework 1

Due 9:30am, Sep/8. Also submit electronic PDF (scans are ok)

Reading: Text book lectures 1-3.

1. [**10 points.** *Linear maps*] Prove that if $f : \mathbb{C}^n \to \mathbb{C}^m$ is a linear map, then $f(\sum_{i=1}^{N} \lambda_i \mathbf{x}_i) = \sum_{i=1}^{N} \lambda_i f(\mathbf{x}_i)$, where $\lambda_i \in \mathbb{C}$, $\mathbf{x}_i$ in $\mathbb{C}^n$, and N is an integer. (Hint: Use induction.)

2. [**40 points.** *Review*] Textbook exercises: 1.1, 1.3, 2.2 (Hint for 1.3: use induction.)

3. [**10 points.** *Outer product*] Let $u(\neq 0), v(\neq 0) \in \mathbb{R}^n$. Let $A = uv^*$. What is the dimension of the range space of A and A^*? What is the dimension of the null space of A?

4. [**15 points.** *Subspaces*] Let $A, B, C \in \mathbb{R}^{m\times n}$.

   (a) Let $m = 20$ and $n = 10$. Assume that A has linearly independent columns. What are the dimensions of $\text{Range}(A), \text{Range}(A^*), \text{Null}(A), \text{Null}(A^*)$?

   (b) Show that $\dim(\text{Range}(A)) + \dim(\text{Null}(A)) = n$.

   (c) Let $m = n$. Show that if B, C are invertible, then $\text{rank}(A) = \text{rank}(BAC)$.

   (Note: the statements above also are also true for complex matrices.)

5. [**10 points.** *Inner product*] An inner product on a complex vector space $\mathcal{V}$ is a mapping $\xi : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$, i.e.

   $$\mathbf{x} \in \mathcal{V},\ \mathbf{y} \in \mathcal{V} \to \xi(\mathbf{x}, \mathbf{y}) \in \mathbb{C}.$$

   An inner product satisfies three conditions:
   (1) it is linear in its first argument $\mathbf{x}$;
   (2) it is Hermitian, i.e, $\xi(\mathbf{y}, \mathbf{x}) = \overline{\xi(\mathbf{x}, \mathbf{y})}$ $\forall \mathbf{x}, \mathbf{y} \in \mathcal{V}$; and
   (3) it is positive definite, i.e., $\xi(\mathbf{x}, \mathbf{x}) > 0$, $\forall \mathbf{x} \neq 0$.

   For $\mathcal{V} = \mathbb{C}^2$ verify these properties for the inner product we defined in the class (also textbook, lecture 2):

   $$\xi(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} = \mathbf{y}^* \mathbf{x} = \sum_{i=1}^{2} \overline{y_i} x_i,$$

   where $\overline{\beta}$ is the complex conjugate of $\beta \in \mathbb{C}$.

6. [**15 points.** *Subspaces*] Let $q(\neq 0) \in \mathbb{R}^n$ and assume $\|q\|_2 \neq 1$.

   (a) Compute the rank of $I - qq^*$.

   (b) Show that the columns of $I - qq^*$ and q are linearly dependent.

   (c) Given $x \in \mathbb{R}^n$, show that $(q^* x)q$ is not orthogonal to the columns of $I - qq^*$.