LAST TIME

- LINEAR DEPENDENCE OF VECTORS
- MATRIX-VECTOR MULTPLICATION
    - OUTPUT IS LINEAR COMBINATION OF COLUMNS
    - LINEAR FUNCTON ON INPUT VECTOR
    - INNER PRODUCTS WITH ROWS.
- TODAY : ORTOGONALITY AND RANK

- ORTHOGONALITY OF TWO VECTORS.

$x, y \in \mathbb{C}^n$ ARE ORTHOGONAL IFF $(x \cdot y) = 0$

- EXAMPLE -

- ORTHONORMAL VECTORS

$x, y \in \mathbb{C}^n$ ARE ORTHONORMAL IFF $(x \cdot y) = 0$ AND $\|x\|_2 = \|y\|_2 = 1.$

- SET OF ORTHOGONAL VECTORS

$\{x_1, x_2, \cdots, x_m\} \in \mathbb{C}^n$ ARE ORTHOGONAL IFF $(x_i \cdot x_j) = \delta_{ij}$

WHERE δ_{ij} IS THE KRONECKER DELTA : $\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{otw.} \end{cases}$

THERE IS AN ANALOGOUS DEFINITION FOR ORTHONORMAL VECTORS

- Adjoint OR CONJUGATE TRANSPOSE OF A MATRIX.

- LET $A \in \mathbb{C}^{m \times n}$. THE ADJOINT A^* of A IS THE UNIQUE MATRIX THAT SATISFIES

$$(Ax \cdot y) = (x \cdot A^* y) \quad \forall\, x \in \mathbb{C}^n, y \in \mathbb{C}^m$$

- USING THIS PROPERTY WE CAN SHOW THAT $A^*(i,j) = \overline{A(j,i)}$

- EXAMPLE

$$\begin{bmatrix} 1 & -i & 0 \\ -1 & i & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ i & -i \\ 0 & 1 \end{bmatrix}$$

- IF $A \in \mathbb{R}^{m\times n}$ THEN $A^* = A^T$, THE TRANSPOSE OF A DEFINED BY $A^T(i,j) = A(j,i)$. NOTICE THAT IF A IS IN $\mathbb{C}^{m\times n}$ THEN $A^* \neq A^T$.

- GIVEN THE DEFINITION OF ADJOINT IS EASY TO PROVE BASIC PROPERTIES:

$$(A+B)^* = A^* + B^* \; ; \; (\lambda A)^* = \lambda^* A^* \; ;$$

(I) MATRIX-MATRIX MULTIPLICATION.

GIVEN $A \in \mathbb{C}^{m\times k}$; $B \in \mathbb{C}^{k\times n}$ WE DEFINE $(AB) \in \mathbb{C}^{m\times n}$ AS THE LINEAR OPERATOR THAT SATISFIES

$$(AB)x = A(Bx) \quad , \forall x \in \mathbb{C}^n.$$

- USING THIS DEFINITION ONE CAN PROVE THE FAMILIAR FORMULA

$$(AB)_{ij} = \sum_{l=1}^{k} A_{il} B_{lj} \quad ; i=1,\dots,m \; ; j=1,\dots,n$$

(II) ADJOINT OF (AB) : $(AB)^* = B^* A^*$

PF:

$$(ABx \cdot y) = (A(Bx) \cdot y) = (Bx \cdot A^* y) = (x \cdot B^*(A^* y))$$

$$= (x \cdot (B^* A^*) y) \qquad \forall x, y. \qquad \square$$

(III) INVERSE OF A SQUARE MATRIX : A^{-1} ; $A, A^{-1} \in \mathbb{C}^{m\times m}$

- A^{-1} IS THE INVERSE OF A IFF $A^{-1}(Ax) = x \; , \forall x \in \mathbb{C}^m$.
- NOT ALL $A \in \mathbb{C}^{m\times m}$ HAVE AN INVERSE.

(IV) IDENTITY MATRIX, $I \in \mathbb{R}^{m\times m}$;

$$Ix = x \; , \forall x \in \mathbb{C}^m. \qquad I_{ij} = \delta_{ij}$$

① UNITARY MATRIX $U \in \mathbb{C}^{m\times m}$.

U is UNITARY $\iff$ U IS SQUARE AND $U^*(Ux) = x, \forall x \in \mathbb{C}^m$

CONSIDER AN EXAMPLE:

$$U = \begin{bmatrix} | & | & | & & | \\ q_1 & q_2 & q_3 & \cdots & q_m \\ | & | & | & & | \end{bmatrix} \qquad U^* = \begin{bmatrix} — & q_1^* & — \\ — & q_2^* & — \\ — & q_3^* & — \\ & \vdots & \\ — & q_m^* & — \end{bmatrix}$$

THEN

$$U^*U = \begin{bmatrix} q_1^* q_1 & q_1^* q_2 & \cdots & q_1^* q_m \\ & & & \vdots \\ \cdots & & & q_m^* q_m \end{bmatrix}$$

$= I$ since $U^*(Ux) = Ix = x$. (WHAT ABOUT $UU^* = ?$)

Therefore $q_i^* q_j = (q_i \cdot q_j) = \delta_{ij}$

Therefore the columns of U are orthonormal.

- U HAS MANY OTHER IMPORTANT PROPERTIES.

THE MOST IMPORTANT IS THAT IT PRESERVES THE EUCLIDEAN NORM.

$$\|Ux\|_2 = \|x\|_2$$

Pf: $\|Ux\|_2 = \sqrt{(Ux \cdot Ux)} = \sqrt{(x \cdot U^*(Ux))} = \sqrt{x \cdot (U^*U)x} = \sqrt{(x \cdot Ix)} = \|x\|_2$

- ESSENTIALLY U REPRESENTS A CHANGE OF BASIS. WE WILL REVISIT THIS.

- Given a set of orthogonal vectors $\{q_i\}_{i=1}^{n} \in \mathbb{C}^m$ with $n < m$, and given a vector x in $\mathbb{C}^m$

we can write $x = z + r$ where

$z \in \text{span}\{q_i\}$ and $r \cdot q_i = 0, \forall i$.

that is,

NOTATION FOR SET DIFFERENCE/COMPLEMENT SET

$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$

$$r \in \mathbb{C}^m \setminus \text{span}\{q_i\}$$

we can write

$$z = \sum_{i=1}^{n} a_{(i)} q_i = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} a_{(1)} \\ \vdots \\ a_{(n)} \end{bmatrix}$$

with $a_{(i)} = x \cdot q_i$ or

$$a = \begin{bmatrix} - & q_1^* & - \\ - & q_2^* & - \\ & \vdots & \\ - & q_n^* & - \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix}$$

- Matrix-vector multiplication as a row vector

$$b = \begin{bmatrix} - & a_1^* & - \\ - & a_2^* & - \\ - & a_n^* & - \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} x \cdot a_1 \\ x \cdot a_2 \\ \vdots \\ x \cdot a_n \end{bmatrix}.$$

- Range space of $A \in \mathbb{C}^{m \times n}$
  - span of A's columns, subset of $\mathbb{C}^m$
  - $\dim(\text{Range}(A)) \leq \min(m,n)$
- Row space of A (subset of $\mathbb{C}^n$)
- Null space of A (subset of $\mathbb{C}^n$)
- Column rank of A: $\dim(\text{range}(A))$
- Row rank of A: $\dim(\text{row space}(A))$.
- A $m \times n$ matrix is full rank if $m = \dim(\text{range}(A))$
- THM: Let A in $\mathbb{C}^{m \times n}$ and $m \leq n$.
  Then, A Full rank $\Longleftrightarrow \dim(\text{Null}(A)) = 0$

  This means that if $Ax = b$ and $Ay = b$ then $x = y$. (why?)

- THM: Column rank = row rank = rank

Proof: Let $A \in \mathbb{C}^{m \times n}$

Let μ = column rank = $\dim(\text{Range}(A))$

ν = row rank = $\dim(\text{row space}(A))$

// : plan: need to show 1: $\mu \geq \nu$

2: $\nu \geq \mu$

1. Let $\{r_1, \dots, r_v\}$ be a set of linearly independent vectors in Rowspace(A).

Then $\{Ar_1, \dots, Ar_v\}$ are linearly independent.

Assume they are not. Then $\exists\, \lambda_i$ such that $\sum_i \lambda_i A r_i = 0$.

Since A is linear

$$A\Big(\underbrace{\sum_i \lambda_i r_i}_{w}\Big) = 0$$

$$\Rightarrow Aw = 0 \Rightarrow \begin{bmatrix} - & a_1^* & - \\ - & a_2^* & - \\ & \vdots & \\ - & a_m^* & - \end{bmatrix} w = 0$$

$\Rightarrow a_j \cdot w = 0$ for all j

That means w is orthogonal to the rows of A. This is impossible since by construction $r_i \in \operatorname{span}\{a_j\}_{j=1}^m$ (*)

see note at the end

Since $\{Ar_i\}$ are independent

and $\dim\left[\operatorname{span}\{Ar_i\}\right] = \vartheta$

that means that $\mu \geq \vartheta$.

$\boxed{2}$ Repeating the same argument for A^*, gives $\vartheta \leq \mu$.

Therefore $\vartheta = \mu$.

(*) to see this you can select $\{r_i\} \subseteq \{a_i\}$ a subset of the rows of A that is independent then $\alpha_i \cdot a_i \neq 0$.