Lecture 1. Matrix-Vector Multiplication

You already know the formula for matrix-vector multiplication. Nevertheless, the purpose of this first lecture is to describe a way of interpreting such products that may be less familiar. If $b = Ax$, then b *is a linear combination of the columns of* A.

Familiar Definitions

Let x be an n-dimensional column vector and let A be an $m \times n$ matrix (m rows, n columns). Then the matrix-vector product $b = Ax$ is the m-dimensional column vector defined as follows:

$$b_i = \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, \ldots, m. \tag{1.1}$$

Here b_i denotes the ith entry of b, a_{ij} denotes the i, j entry of A (ith row, jth column), and x_j denotes the jth entry of x. For simplicity, we assume in all but a few lectures of this book that quantities such as these belong to $\mathbb{C}$, the field of complex numbers. The space of m-vectors is $\mathbb{C}^m$, and the space of $m \times n$ matrices is $\mathbb{C}^{m \times n}$.

The map $x \mapsto Ax$ is *linear*, which means that, for any $x, y \in \mathbb{C}^n$ and any $\alpha \in \mathbb{C}$,

$$\begin{aligned} A(x + y) &= Ax + Ay, \\ A(\alpha x) &= \alpha Ax. \end{aligned}$$

Conversely, every linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$ can be expressed as multiplication by an $m \times n$ matrix.

A Matrix Times a Vector

Let a_j denote the jth column of A, an m-vector. Then (1.1) can be rewritten

$$b = Ax = \sum_{j=1}^{n} x_j a_j. \tag{1.2}$$

This equation can be displayed schematically as follows:

$$\left[\begin{array}{c} \\ \\ b \\ \\ \\ \end{array}\right] = \left[\begin{array}{c|c|c|c} \\ \\ a_1 & a_2 & \cdots & a_n \\ \\ \\ \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \left[\begin{array}{c} \\ \\ a_1 \\ \\ \\ \end{array}\right] + x_2 \left[\begin{array}{c} \\ \\ a_2 \\ \\ \\ \end{array}\right] + \cdots + x_n \left[\begin{array}{c} \\ \\ a_n \\ \\ \\ \end{array}\right].$$

In (1.2), b is expressed as a linear combination of the columns a_j. Nothing but a slight change of notation has occurred in going from (1.1) to (1.2). Yet thinking of Ax in terms of the form (1.2) is essential for a proper understanding of the algorithms of numerical linear algebra.

We can summarize these different descriptions of matrix-vector products in the following way. As mathematicians, we are used to viewing the formula $Ax = b$ as a statement that A acts on x to produce b. The formula (1.2), by contrast, suggests the interpretation that x acts on A to produce b.

Example 1.1. Vandermonde Matrix. Fix a sequence of numbers $\{x_1, x_2, \ldots, x_m\}$. If p and q are polynomials of degree $< n$ and α is a scalar, then $p+q$ and αp are also polynomials of degree $< n$. Moreover, the values of these polynomials at the points x_i satisfy the following linearity properties:

$$\begin{aligned} (p+q)(x_i) &= p(x_i) + q(x_i), \\ (\alpha p)(x_i) &= \alpha(p(x_i)). \end{aligned}$$

Thus the map from vectors of coefficients of polynomials p of degree $< n$ to vectors $(p(x_1), p(x_2), \ldots, p(x_m))$ of sampled polynomial values is linear. Any linear map can be expressed as multiplication by a matrix; this is an example. In fact, it is expressed by an $m \times n$ *Vandermonde matrix*

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{bmatrix}.$$

If c is the column vector of coefficients of p,

$$c = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-1} \end{bmatrix}, \qquad p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1},$$

then the product Ac gives the sampled polynomial values. That is, for each i from 1 to m, we have

$$(Ac)_i \;=\; c_0 + c_1x_i + c_2x_i^2 + \cdots + c_{n-1}x_i^{n-1} \;=\; p(x_i). \tag{1.3}$$

In this example, it is clear that the matrix-vector product Ac need not be thought of as m distinct scalar summations, each giving a different linear combination of the entries of c, as (1.1) might suggest. Instead, A can be viewed as a matrix of columns, each giving sampled values of a monomial,

$$A = \left[\begin{array}{c|c|c|c|c} \\ 1 & x & x^2 & \cdots & x^{n-1} \\ \\ \end{array}\right], \tag{1.4}$$

and the product Ac should be understood as a single vector summation in the form of (1.2) that at once gives a linear combination of these monomials,

$$Ac \;=\; c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1} \;=\; p(x). \qquad \square$$

The remainder of this lecture will review some fundamental concepts in linear algebra from the point of view of (1.2).

A Matrix Times a Matrix

For the matrix-matrix product $B = AC$, *each column of B is a linear combination of the columns of A*. To derive this fact, we begin with the usual formula for matrix products. If A is $\ell \times m$ and C is $m \times n$, then B is $\ell \times n$, with entries defined by

$$b_{ij} = \sum_{k=1}^{m} a_{ik}c_{kj}. \tag{1.5}$$

Here b_{ij}, a_{ik}, and c_{kj} are entries of B, A, and C, respectively. Written in terms of columns, the product is

$$\left[\begin{array}{c|c|c|c} \\ b_1 & b_2 & \cdots & b_n \\ \\ \end{array}\right] = \left[\begin{array}{c|c|c|c} \\ a_1 & a_2 & \cdots & a_m \\ \\ \end{array}\right] \left[\begin{array}{c|c|c|c} c_1 & c_2 & \cdots & c_n \end{array}\right],$$

and (1.5) becomes

$$b_j = Ac_j = \sum_{k=1}^{m} c_{kj} a_k. \tag{1.6}$$

Thus b_j is a linear combination of the columns a_k with coefficients c_{kj}.

Example 1.2. Outer Product. A simple example of a matrix-matrix product is the *outer product*. This is the product of an m-dimensional column vector u with an n-dimensional row vector v; the result is an $m \times n$ matrix of rank 1. The outer product can be written

$$\left[\begin{array}{c} \\ u \\ \\ \end{array}\right]\left[\begin{array}{cccc} v_1 & v_2 & \cdots & v_n \end{array}\right] = \left[\begin{array}{c|c|c|c} & & & \\ v_1 u & v_2 u & \cdots & v_n u \\ & & & \end{array}\right] = \left[\begin{array}{ccc} v_1 u_1 & \cdots & v_n u_1 \\ \vdots & & \vdots \\ v_1 u_m & \cdots & v_n u_m \end{array}\right].$$

The columns are all multiples of the same vector u, and similarly, the rows are all multiples of the same vector v. □

Example 1.3. As a second illustration, consider $B = AR$, where R is the upper-triangular $n \times n$ matrix with entries $r_{ij} = 1$ for $i \leq j$ and $r_{ij} = 0$ for $i > j$. This product can be written

$$\left[\begin{array}{c|c|c} & & \\ b_1 & \cdots & b_n \\ & & \end{array}\right] = \left[\begin{array}{c|c|c} & & \\ a_1 & \cdots & a_n \\ & & \end{array}\right]\left[\begin{array}{ccc} 1 & \cdots & 1 \\ & \ddots & \vdots \\ & & 1 \end{array}\right].$$

The column formula (1.6) now gives

$$b_j = Ar_j = \sum_{k=1}^{j} a_k. \tag{1.7}$$

That is, the jth column of B is the sum of the first j columns of A. The matrix R is a discrete analogue of an indefinite integral operator. □

Range and Nullspace

The *range* of a matrix A, written range(A), is the set of vectors that can be expressed as Ax for some x. The formula (1.2) leads naturally to the following characterization of range(A).

Theorem 1.1. range(A) *is the space spanned by the columns of* A.

Proof. By (1.2), any Ax is a linear combination of the columns of A. Conversely, any vector y in the space spanned by the columns of A can be written as a linear combination of the columns, $y = \sum_{j=1}^{n} x_j a_j$. Forming a vector x out of the coefficients x_j, we have $y = Ax$, and thus y is in the range of A. $\square$

In view of Theorem 1.1, the range of a matrix A is also called the *column space* of A.

The *nullspace* of $A \in \mathbb{C}^{m\times n}$, written null$(A)$, is the set of vectors x that satisfy $Ax = 0$, where 0 is the 0-vector in $\mathbb{C}^m$. The entries of each vector $x \in$ null(A) give the coefficients of an expansion of zero as a linear combination of columns of A: $0 = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$.

Rank

The *column rank* of a matrix is the dimension of its column space. Similarly, the *row rank* of a matrix is the dimension of the space spanned by its rows. Row rank always equals column rank (among other proofs, this is a corollary of the singular value decomposition, discussed in Lectures 4 and 5), so we refer to this number simply as the *rank* of a matrix.

An $m \times n$ matrix of *full rank* is one that has the maximal possible rank (the lesser of m and n). This means that a matrix of full rank with $m \geq n$ must have n linearly independent columns. Such a matrix can also be characterized by the property that the map it defines is one-to-one.

Theorem 1.2. *A matrix $A \in \mathbb{C}^{m\times n}$ with $m \geq n$ has full rank if and only if it maps no two distinct vectors to the same vector.*

Proof. ($\Longrightarrow$) If A is of full rank, its columns are linearly independent, so they form a basis for range(A). This means that every $b \in$ range(A) has a unique linear expansion in terms of the columns of A, and therefore, by (1.2), every $b \in$ range(A) has a unique x such that $b = Ax$. ($\Longleftarrow$) Conversely, if A is not of full rank, its columns a_j are dependent, and there is a nontrivial linear combination such that $\sum_{j=1}^{n} c_j a_j = 0$. The nonzero vector c formed from the coefficients c_j satisfies $Ac = 0$. But then A maps distinct vectors to the same vector since, for any x, $Ax = A(x + c)$. $\square$

Inverse

A *nonsingular* or *invertible* matrix is a square matrix of full rank. Note that the m columns of a nonsingular $m \times m$ matrix A form a basis for the whole space $\mathbb{C}^m$. Therefore, we can uniquely express any vector as a linear combination of them. In particular, the canonical unit vector with 1 in the jth entry and zeros elsewhere, written e_j, can be expanded:

$$e_j = \sum_{i=1}^{m} z_{ij} a_i. \tag{1.8}$$

Let Z be the matrix with entries z_{ij}, and let z_j denote the jth column of Z. Then (1.8) can be written $e_j = Az_j$. This equation has the form (1.6); it can be written again, most concisely, as

$$\left[\begin{array}{c|c|c} \\ e_1 & \cdots & e_m \\ \\ \end{array}\right] = I = AZ,$$

where I is the $m \times m$ matrix known as the *identity*. The matrix Z is the *inverse* of A. Any square nonsingular matrix A has a unique inverse, written A^{-1}, that satisfies $AA^{-1} = A^{-1}A = I$.

The following theorem records a number of equivalent conditions that hold when a square matrix is nonsingular. These conditions appear in linear algebra texts, and we shall not give a proof here. Concerning (f), see Lecture 5.

Theorem 1.3. *For $A \in \mathbb{C}^{m\times m}$, the following conditions are equivalent:*

(a) *A has an inverse A^{-1},*
(b) $\operatorname{rank}(A) = m$,
(c) $\operatorname{range}(A) = \mathbb{C}^m$,
(d) $\operatorname{null}(A) = \{0\}$,
(e) 0 *is not an eigenvalue of A,*
(f) 0 *is not a singular value of A,*
(g) $\det(A) \neq 0$.

Concerning (g), we mention that the determinant, though a convenient notion theoretically, rarely finds a useful role in numerical algorithms.

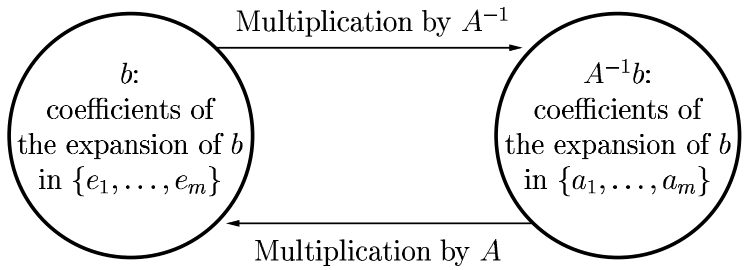
A Matrix Inverse Times a Vector

When writing the product $x = A^{-1}b$, it is important not to let the inverse-matrix notation obscure what is really going on! Rather than thinking of x as the result of applying A^{-1} to b, we should understand it as the unique vector that satisfies the equation $Ax = b$. By (1.2), this means that x is the vector of coefficients of the unique linear expansion of b in the basis of columns of A.

This point cannot be emphasized too much, so we repeat:

$A^{-1}b$ is the vector of coefficients of the expansion of b
in the basis of columns of A.

Multiplication by A^{-1} is a *change of basis* operation:



In this description we are being casual with terminology, using "b" in one instance to denote an m-tuple of numbers, and in another, as a point in an abstract vector space. The reader should think about these matters until he or she is comfortable with the distinction.

A Note on$m$ and n

Throughout numerical linear algebra, it is customary to take a rectangular matrix to have dimensions $m \times n$. We follow this convention in this book.

What if the matrix is square? The usual convention is to give it dimensions $n \times n$, but in this book we shall generally take the other choice, $m \times m$. Many of our algorithms require us to look at rectangular submatrices formed by taking a subset of the columns of a square matrix. If the submatrix is to be $m \times n$, the original matrix had better be $m \times m$.

Exercises

1.1. Let B be a 4×4 matrix to which we apply the following operations:

1. double column 1,
2. halve row 3,
3. add row 3 to row 1,
4. interchange columns 1 and 4,
5. subtract row 2 from each of the other rows,
6. replace column 4 by column 3,
7. delete column 1 (so that the column dimension is reduced by 1).

(a) Write the result as a product of eight matrices.

(b) Write it again as a product ABC (same B) of three matrices.

1.2. Suppose masses m_1, m_2, m_3, m_4 are located at positions x_1, x_2, x_3, x_4 in a line and connected by springs with spring constants k_{12}, k_{23}, k_{34} whose natural lengths of extension are $\ell_{12}, \ell_{23}, \ell_{34}$. Let f_1, f_2, f_3, f_4 denote the rightward forces on the masses, e.g., $f_1 = k_{12}(x_2 - x_1 - \ell_{12})$.

(a) Write the 4×4 matrix equation relating the column vectors f and x. Let K denote the matrix in this equation.

(b) What are the dimensions of the entries of K in the physics sense (e.g., mass times time, distance divided by mass, etc.)?

(c) What are the dimensions of $\det(K)$, again in the physics sense?

(d) Suppose K is given numerical values based on the units meters, kilograms, and seconds. Now the system is rewritten with a matrix K' based on centimeters, grams, and seconds. What is the relationship of K' to K? What is the relationship of $\det(K')$ to $\det(K)$?

1.3. Generalizing Example 1.3, we say that a square or rectangular matrix R with entries r_{ij} is *upper-triangular* if $r_{ij} = 0$ for $i > j$. By considering what space is spanned by the first n columns of R and using (1.8), show that if R is a nonsingular $m \times m$ upper-triangular matrix, then R^{-1} is also upper-triangular. (The analogous result also holds for lower-triangular matrices.)

1.4. Let $f_1, \ldots, f_8$ be a set of functions defined on the interval $[1, 8]$ with the property that for any numbers $d_1, \ldots, d_8$, there exists a set of coefficients $c_1, \ldots, c_8$ such that

$$\sum_{j=1}^{8} c_j f_j(i) = d_i, \qquad i = 1, \ldots, 8.$$

(a) Show by appealing to the theorems of this lecture that $d_1, \ldots, d_8$ determine $c_1, \ldots, c_8$ uniquely.

(b) Let A be the 8×8 matrix representing the linear mapping from data $d_1, \ldots, d_8$ to coefficients $c_1, \ldots, c_8$. What is the i, j entry of A^{-1}?

Lecture 2. Orthogonal Vectors and Matrices

Since the 1960s, many of the best algorithms of numerical linear algebra have been based in one way or another on orthogonality. In this lecture we present the ingredients: orthogonal vectors and orthogonal (unitary) matrices.

Adjoint

The *complex conjugate* of a scalar z, written $\overline{z}$ or z^*, is obtained by negating its imaginary part. For real z, $\overline{z} = z$.

The *hermitian conjugate* or *adjoint* of an $m \times n$ matrix A, written A^*, is the $n \times m$ matrix whose i, j entry is the complex conjugate of the j, i entry of A. For example,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \qquad \Longrightarrow \qquad A^* = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{21} & \overline{a}_{31} \\ \overline{a}_{12} & \overline{a}_{22} & \overline{a}_{32} \end{bmatrix}.$$

If $A = A^*$, A is *hermitian*. By definition, a hermitian matrix must be square. For real A, the adjoint simply interchanges the rows and columns of A. In this case, the adjoint is also known as the *transpose*, and is written A^T. If a real matrix is hermitian, that is, $A = A^T$, then it is also said to be *symmetric*.

Most textbooks of numerical linear algebra assume that the matrices under discussion are real and thus principally use ^{T} instead of *. Since most of the ideas to be dealt with are not intrinsically restricted to the reals, however, we have followed the other course. Thus, for example, in this book a row vector will usually be denoted by, say, a^* rather than a^T. The reader who prefers to imagine that all quantities are real and that * is a synonym for ^{T} will rarely get into trouble.

Inner Product

The *inner product* of two column vectors $x, y \in \mathbb{C}^m$ is the product of the adjoint of x by y:

$$x^*y = \sum_{i=1}^{m} \overline{x}_i y_i. \tag{2.1}$$

The Euclidean length of x may be written $\|x\|$ (vector norms such as this are discussed systematically in the next lecture), and can be defined as the square root of the inner product of x with itself:

$$\|x\| = \sqrt{x^*x} = \left(\sum_{i=1}^{m} |x_i|^2 \right)^{1/2}. \tag{2.2}$$

The cosine of the angle α between x and y can also be expressed in terms of the inner product:

$$\cos\alpha = \frac{x^*y}{\|x\|\,\|y\|}. \tag{2.3}$$

At various points of this book, as here, we mention geometric interpretations of algebraic formulas. For these geometric interpretations, the reader should think of the vectors as real rather than complex, although usually the interpretations can be carried over in one way or another to the complex case too.

The inner product is *bilinear*, which means that it is linear in each vector separately:

$$\begin{aligned}
(x_1 + x_2)^* y &= x_1^* y + x_2^* y, \\
x^*(y_1 + y_2) &= x^* y_1 + x^* y_2, \\
(\alpha x)^*(\beta y) &= \overline{\alpha}\beta x^* y.
\end{aligned}$$

We shall also frequently use the easily proved property that for any matrices or vectors A and B of compatible dimensions,

$$(AB)^* = B^*A^*. \tag{2.4}$$

This is analogous to the equally important formula for products of invertible square matrices,

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{2.5}$$

The notation A^{-*} is a shorthand for $(A^*)^{-1}$ or $(A^{-1})^*$; these two are equal, as can be verified by applying (2.4) with $B = A^{-1}$.

Orthogonal Vectors

A pair of vectors x and y are *orthogonal* if $x^*y = 0$. If x and y are real, this means they lie at right angles to each other in $\mathbb{R}^m$. Two sets of vectors X and Y are orthogonal (also stated "X is orthogonal to Y") if every $x \in X$ is orthogonal to every $y \in Y$.

A set of nonzero vectors S is *orthogonal* if its elements are pairwise orthogonal, i.e., if for $x, y \in S$, $x \neq y \Rightarrow x^*y = 0$. A set of vectors is *orthonormal* if it is orthogonal and, in addition, every $x \in S$ has $\|x\| = 1$.

Theorem 2.1. *The vectors in an orthogonal set S are linearly independent.*

Proof. If the vectors in S are not independent, then some $v_k \in S$ can be expressed as a linear combination of other members $v_1, \ldots, v_n \in S$,

$$v_k = \sum_{\substack{i=1 \\ i \neq k}}^{n} c_i v_i.$$

Since $v_k \neq 0$, $v_k^* v_k = \|v_k\|^2 > 0$. Using the bilinearity of inner products and the orthogonality of S, we calculate

$$v_k^* v_k = \sum_{\substack{i=1 \\ i \neq k}}^{n} c_i v_k^* v_i = 0,$$

which contradicts the assumption that the vectors in S are nonzero. □

As a corollary of Theorem 2.1 it follows that if an orthogonal set $S \subseteq \mathbb{C}^m$ contains m vectors, then it is a basis for $\mathbb{C}^m$.

Components of a Vector

The most important idea to draw from the concepts of inner products and orthogonality is this: inner products can be used to decompose arbitrary vectors into orthogonal components.

For example, suppose that $\{q_1, q_2, \ldots, q_n\}$ is an orthonormal set, and let v be an arbitrary vector. The quantity $q_j^* v$ is a scalar. Utilizing these scalars as coordinates in an expansion, we find that the vector

$$r = v - (q_1^* v) q_1 - (q_2^* v) q_2 - \cdots - (q_n^* v) q_n \tag{2.6}$$

is orthogonal to $\{q_1, q_2, \ldots, q_n\}$. This can be verified by computing $q_i^* r$:

$$q_i^* r \;=\; q_i^* v - (q_1^* v)(q_i^* q_1) - \cdots - (q_n^* v)(q_i^* q_n).$$

This sum collapses, since $q_i^* q_j = 0$ for $i \neq j$:

$$q_i^* r \;=\; q_i^* v - (q_i^* v)(q_i^* q_i) \;=\; 0.$$

Thus we see that v can be decomposed into $n+1$ orthogonal components:

$$v = r + \sum_{i=1}^{n} (q_i^* v) q_i = r + \sum_{i=1}^{n} (q_i q_i^*) v. \tag{2.7}$$

In this decomposition, r is the part of v orthogonal to the set of vectors $\{q_1, q_2, \ldots, q_n\}$, or, equivalently, to the subspace spanned by this set of vectors, and $(q_i^* v) q_i$ is the part of v in the direction of q_i.

If $\{q_i\}$ is a basis for $\mathbb{C}^m$, then n must be equal to m and r must be the zero vector, so v is completely decomposed into m orthogonal components in the directions of the q_i:

$$v = \sum_{i=1}^{m} (q_i^* v) q_i = \sum_{i=1}^{m} (q_i q_i^*) v. \tag{2.8}$$

In both (2.7) and (2.8) we have written the formula in two different ways, once with $(q_i^* v) q_i$ and again with $(q_i q_i^*) v$. These expressions are equal, but they have different interpretations. In the first case, we view v as a sum of coefficients $q_i^* v$ times vectors q_i. In the second, we view v as a sum of orthogonal projections of v onto the various directions q_i. The ith projection operation is achieved by the very special rank-one matrix $q_i q_i^*$. We shall discuss this and other projection processes in Lecture 6.

Unitary Matrices

A square matrix $Q \in \mathbb{C}^{m \times m}$ is *unitary* (in the real case, we also say *orthogonal*) if $Q^* = Q^{-1}$, i.e, if $Q^* Q = I$. In terms of the columns of Q, this product can be written

$$\left[\begin{array}{c} \quad q_1^* \quad \\ \hline q_2^* \\ \hline \vdots \\ \hline q_m^* \end{array}\right] \left[\begin{array}{c|c|c|c} & & & \\ q_1 & q_2 & \cdots & q_m \\ & & & \end{array}\right] = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

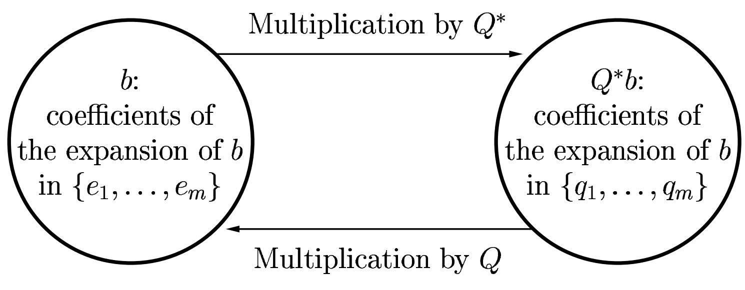
In other words, $q_i^* q_j = \delta_{ij}$, and the columns of a unitary matrix Q form an orthonormal basis of $\mathbb{C}^m$. The symbol δ_{ij} is the *Kronecker delta*, equal to 1 if $i = j$ and 0 if $i \neq j$.

Multiplication by a Unitary Matrix

In the last lecture we discussed the interpretation of matrix-vector products Ax and $A^{-1}b$. If A is a unitary matrix Q, these products become Qx and $Q^* b$, and the same interpretations are of course still valid. As before, Qx is the linear combination of the columns of Q with coefficients x. Conversely,

Q^*b is the vector of coefficients of the expansion of b
in the basis of columns of Q.

Schematically, the situation looks like this:



These processes of multiplication by a unitary matrix or its adjoint preserve geometric structure in the Euclidean sense, because inner products are preserved. That is, for unitary Q,

$$(Qx)^*(Qy) = x^*y, \tag{2.9}$$

as is readily verified by (2.4). The invariance of inner products means that angles between vectors are preserved, and so are their lengths:

$$\|Qx\| = \|x\|. \tag{2.10}$$

In the real case, multiplication by an orthogonal matrix Q corresponds to a rigid rotation (if $\det Q = 1$) or reflection (if $\det Q = -1$) of the vector space.

Exercises

2.1. Show that if a matrix A is both triangular and unitary, then it is diagonal.

2.2. The Pythagorean theorem asserts that for a set of n orthogonal vectors $\{x_i\}$,

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n} \|x_i\|^2.$$

(a) Prove this in the case $n = 2$ by an explicit computation of $\|x_1 + x_2\|^2$.
(b) Show that this computation also establishes the general case, by induction.

2.3. Let $A \in \mathbb{C}^{m\times m}$ be hermitian. An eigenvector of A is a nonzero vector $x \in \mathbb{C}^m$ such that $Ax = \lambda x$ for some $\lambda \in \mathbb{C}$, the corresponding eigenvalue.
(a) Prove that all eigenvalues of A are real.

(b) Prove that if x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal.

2.4. What can be said about the eigenvalues of a unitary matrix?

2.5. Let $S \in \mathbb{C}^{m\times m}$ be *skew-hermitian*, i.e., $S^* = -S$.

(a) Show by using Exercise 2.3 that the eigenvalues of S are pure imaginary.

(b) Show that $I - S$ is nonsingular.

(c) Show that the matrix $Q = (I-S)^{-1}(I+S)$, known as the *Cayley transform* of S, is unitary. (This is a matrix analogue of a linear fractional transformation $(1+s)/(1-s)$, which maps the left half of the complex s-plane conformally onto the unit disk.)

2.6. If u and v are m-vectors, the matrix $A = I + uv^*$ is known as a *rank-one perturbation of the identity*. Show that if A is nonsingular, then its inverse has the form $A^{-1} = I + \alpha uv^*$ for some scalar α, and give an expression for α. For what u and v is A singular? If it is singular, what is $\mathrm{null}(A)$?

2.7. A *Hadamard matrix* is a matrix whose entries are all ± 1 and whose transpose is equal to its inverse times a constant factor. It is known that if A is a Hadamard matrix of dimension $m > 2$, then m is a multiple of 4. It is not known, however, whether there is a Hadamard matrix for every such m, though examples have been found for all cases $m \le 424$.

Show that the following recursive description provides a Hadamard matrix of each dimension $m = 2^k$, $k = 0, 1, 2, \ldots$:

$$H_0 = \begin{bmatrix} 1 \end{bmatrix}, \qquad H_{k+1} = \begin{bmatrix} H_k & H_k \\ H_k & -H_k \end{bmatrix}.$$