Fall 2016 CSE/CS 383 HW#3 Po-Cheng Pan (UT EID: pp22828)

1. Let $A \in \mathbb{R}^{m\times n}$ be full rank and let $\{q_j\}_{j=1}^{n}$ be the orthonormal vectors from classical Gram-Schmidt orthogonalization of the columns of A

Let $P_j = q_j q_j^*$

$$W = \left[\prod_{j=1}^{i-1}(I-P_j)\right] A(:,i)\ ,\quad q_i = \frac{W}{\|W\|_2}\quad , i=1,\cdots,m$$

From the textbook, we know the classical Gram-Schmidt as below:

$$q_1 = \frac{A(:,1)}{\|A(:,1)\|_2}\quad q_2 = \frac{A(:,2) - q_1^*\cdot A(:,1)\, q_1}{\|A(:,2) - q_1^* A(:,1)\, q_1\|_2}\quad \cdots\quad q_n = \frac{A(:,n) - \sum_{i=1}^{n-1} q_i^* A(:,i) q_i}{\|A(:,n) - \sum_{i=1}^{n-1} q_i^* A(:,i) q_i\|_2}$$

& From (8.1) & (8.3)

q_i can be also expressed as $q_1 = \frac{A(:,1)}{\|A(:,1)\|_2}\quad q_2 = \frac{P_2' A(:,2)}{\|P_2' A(:,2)\|_2}\quad \cdots\quad q_n = \frac{P_n' A(:,n)}{\|P_n' A(:,n)\|_2}$ where $P_j' = I - Q_{j-1} Q_{j-1}^*$

$Q_{j-1} = [q_1 | q_2 | \cdots | q_{j-1}]$

Then consider modified Gram-Schmidt algorithm, $P_j = q_j q_j^*$

$$i=1:\quad \prod_{j=1}^{0}(I-P_j) = I \Rightarrow \frac{W_1}{\|W_1\|_2} = \frac{A(:,1)}{\|A(:,1)\|_2}$$

$$i=2:\quad \prod_{j=1}^{1}(I-P_j) = I - P_1 = I - q_1 q_1^* = I - Q_1 Q_1^* = P_2' \Rightarrow \frac{W_2}{\|W_2\|_2} = \frac{P_2' A(:,2)}{\|P_2' A(:,2)\|_2} = q_2$$

We suppose when $i=k$, $\prod_{j=1}^{k-1}(I-P_j) = P_k' \Rightarrow \frac{W_k}{\|W_k\|_2} = q_k$

Consider $i=k+1$

$$\begin{aligned}
\prod_{j=1}^{k}(I-P_j) &= \prod_{j=1}^{k-1}(I-P_j)\cdot(I-P_k) \\
&= P_k'\cdot(I - q_k q_k^*) \\
&= (I - Q_{k-1}Q_{k-1}^*)(I - q_k q_k^*) \\
&= I - Q_{k-1}Q_{k-1}^* - q_k q_k^* + Q_{k-1}Q_{k-1}^* q_k q_k^*
\end{aligned}$$

($\because q_1, q_2, \cdots, q_k$ all are orthonormal $\therefore Q_{k-1}Q_{k-1}^* q_k q_k^* = 0$)

$$\begin{aligned}
&= I - Q_{k-1}Q_{k-1}^* - q_k q_k^* \\
&= I - (Q_{k-1}Q_{k-1}^* + q_k q_k^*)
\end{aligned}$$

($\because Q_{k-1}Q_{k-1}^*$ & $q_k q_k^*$ both are $\mathbb{R}^{n\times n}$ $\therefore$ We can combine q_k into Q_{k-1} as $Q_k = [q_1 | q_2 | \cdots | q_k]$ s.t. $Q_{k-1}Q_{k-1}^* + q_k q_k^* = Q_k Q_k^*$)

$$\begin{aligned}
&= I - Q_k Q_k^* \\
&= P_{k+1}' \Rightarrow \frac{W_{k+1}}{\|W_{k+1}\|_2} = q_{k+1}
\end{aligned}$$

By induction, we prove $q_i = \frac{W}{\|W\|_2}$, $i=1,\cdots,m$. Hence, both classical and modified algorithm are mathematically equivalent.