LAST TIME

- ORTHOGONALITY, MATRIX MULTIPLICATION, ADJOINT, INVERSE (RIGHT INVERSE, LEFT INVERSE)
- $A \in \mathbb{C}^{m \times n}$
    - $\text{RANGE}(A) \subset \mathbb{C}^m \overset{\text{DEF}}{=} \{ y \in \mathbb{C}^m : y = Ax \text{ for some } x \in \mathbb{C}^n \}$ COLUMN SPACE] span of columns

Column rank
    - $\dim(\text{RANGE}(A)) \leq \min(m,n)$
    - $\text{RANGE}(A^*) \subset \mathbb{C}^n \overset{\text{DEF}}{=} \{ x \in \mathbb{C}^n : x = A^* y \text{ for some } y \in \mathbb{C}^m \}$ ROW SPACE

span of rows

Row Rank
    - $\dim(\text{RANGE}(A^*)) \leq \min(m,n)$
- ROW RANK = COLUMN RANK $\leq \min(m,n)$
- if RANK $= \min(m,n)$ we say matrix is full rank
- $\text{Null}(A) \overset{\text{DEF}}{=} \{ x \in \mathbb{C}^n : Ax = 0 \}$

WE SAY THAT A HAS A NON-TRIVIAL NULL SPACE if $\dim\{\text{Null}(A)\} > 0$: This is equivalent to $\exists\, x : \|x\|_2 = 1$ & $Ax = 0$.

NOTE: $\text{Null}(A)$, $\text{RANGE}(A)$, $\text{RANGE}(A^*)$ ARE PROPER SUBSETS of $\mathbb{C}^n, \mathbb{C}^m, \mathbb{C}^n$ RESPECTIVELY. (I.E, closed under ADDITION & SCALAR MULTIPLICATION)

EXAMPLE

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} :$$

$\text{Range}(A) =: \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$; column rank $= 2$

$\text{Range}(A^*) =: \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$; $\dim = 2$. since VECTORS AR LINEARLY DEPENDENT

$\text{Null}(A) = \{\emptyset\}$; $\dim(\text{Null}(A)) = 0$

$$A = \begin{bmatrix} 1 & 0 & \gamma_1 & \gamma_2 \\ 0 & 1 & \beta_1 & \beta_2 \\ 0 & 1 & \beta_1 & \beta_2 \\ 1 & 0 & \gamma_1 & \gamma_2 \end{bmatrix} :$$

3rd and 4th column: $\gamma_1 a_1 + \beta_1 a_2$, $\gamma_2 a_1 + \beta_2 a_2$; a_1, a_2 ARE THE FIRST TWO COLUMNS OF A.

$\text{RANGE}(A) = \text{span}\{a_1, a_2\}$; rank $= 2$

$$\text{RANGE}(A^*) \to \begin{bmatrix} 1 \\ 0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \beta_1 \\ \beta_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ \beta_1 \\ \beta_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} \Rightarrow \text{span}\Big\{ \underbrace{\begin{matrix} 1 \\ 0 \\ \gamma_1 \\ \gamma_2 \end{matrix}}_{v_1} \; \underbrace{\begin{matrix} 0 \\ 1 \\ \beta_1 \\ \beta_2 \end{matrix}}_{v_2} \Big\} \quad \dim = 2.$$

$\text{Null}(A)$: Consider the case of $\gamma_i = \beta_i = 0$

WE can see that $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ span the Null space so $\dim(\text{Null}(A)) = 2$).

Since dim of rowspace of $A = 2$, $\exists$ 2-D subspace of $\mathbb{C}^4$ $\mathbb{N}$ so that every vector in $\mathbb{N}$ is orthogonal to rows of A. Thus the dimension of Null space is 2.

Example: Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times p}$. Then,

① • $\mathrm{Range}(AB) \subseteq \mathrm{Range}(A)$. If B has lin-ind rows then we have equality

② • $\mathrm{Null}(B) \subseteq \mathrm{Null}(AB)$. If A has lin-ind columns we have equality

Pf:

① $b \in \mathrm{Range}(AB) \Rightarrow \exists x: b = ABx$. $y := Bx$. $\Rightarrow b = Ay \Rightarrow b \in \mathrm{Range}(A)$

Thus $\mathrm{Range}(AB) \subset \mathrm{Range}(A)$

Since row rank = column rank

Let $b \in \mathrm{Range}(A) \Rightarrow \exists y \in \mathbb{C}^n$: $b = Ay$.

B full row rank $\Leftrightarrow \dim(\mathrm{Range}(B^*)) = n \Rightarrow \mathrm{Range}(B) = \mathbb{C}^n$

Thus, $\forall y \in \mathbb{C}^n$, $\exists x \in \mathbb{C}^p$: $x = By$. Thus $b = ABx \in \mathrm{Range}(AB)$

Thus, $\mathrm{Range}(A) \subset \mathrm{Range}(AB)$.

$\rightarrow$ Equality.

② If $x \in \mathrm{Null}(B) \Rightarrow Bx = 0 \Rightarrow ABx = 0 \Rightarrow x \in \mathrm{Null}(AB) \Rightarrow$ $\mathrm{Null}(B) \subset \mathrm{Null}(AB)$.

A has full column rank $\Leftrightarrow \mathrm{Null}(A) = \{0\}$. $ABx = 0 \Rightarrow Bx = 0$

$\Rightarrow \mathrm{Null}(AB) \subset \mathrm{Null}(B)$.

$\rightarrow$ Equality..

More concisely: $\mathrm{Rank}(AB) = \mathrm{Rank}(A)$ if B is non-singular (for square A, B).
$\mathrm{Null}(AB) = \mathrm{Null}(B)$ if A is non-singular

• GIVEN SUBSPACE of A VECTOR SPACE CONSTRUCT DECOMPOSITION OF A VECTOR TO SUBSPACE COMPONENT AND COMPONENT ORTHOGONAL TO SUBSPACE

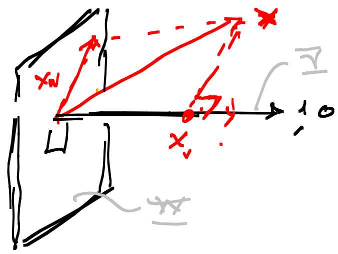
SPACE: $\mathbb{C}^n$

$\mathbb{V}$: SUBSPACE, $\operatorname{span}\left\{\underbrace{\begin{bmatrix}1\\ \vdots\end{bmatrix}}_{e_1}\right\}$

$\|e_1\|_2 = 1$

$x = x_V + x_W, \ \forall x$

$\mathbb{C}^n = \mathbb{V} + \mathbb{W}$



Subspaces: closed under vector addition and scalar multiplication

$x \in \mathbb{V} + \mathbb{W} \Leftrightarrow \exists v \in \mathbb{V}, w \in \mathbb{W} : x = v + w$

$\oplus$: subspace addition.

we also want $\mathbb{V} \cap \mathbb{W} = \{0\}$: Subspace intersection

$x \in \mathbb{V} \cap \mathbb{W} \Leftrightarrow x \in \mathbb{V}$ and $x \in \mathbb{W}$

• CONSTRUCTION: $x_V = (e_1^T x) e_1 = e_1 (e_1^T x)$

$$x_W = x - x_V = x - e_1(e_1^T x) = x - (e_1 e_1^T) x = (I - e_1 e_1^T) x$$

• PROJECTIONS / UNITARY PROJECTIONS

WE SAW THAT: $\{q_i\}_{i=1}^n \, ; \, q_i \in \mathbb{C}^m$

and q_i are orthonormal $(\Rightarrow n \leq m)$.

GIVEN $x = \sum x(i) e_i$ WE CAN WRITE

$$x_q = \underbrace{Q Q^* x}$$

$$\begin{bmatrix} q_i^* x \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ \alpha_i \\ \vdots \end{bmatrix}$$

$$\sum \alpha_i q_i$$

$$\Rightarrow \quad x_q = \sum_{i=1}^{n} (q_i q_i^*) \, x \, .$$

$[x_q = x \text{ iff } n = m]$,

$(q_i q_i^*)$ ⟶ rank-1 matrix, outerproduct

Ex.: $Q \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$.

verify: $q_1 \cdot q_1 = \left(-\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} + \frac{2}{4} = 1$.

$q_1 \cdot q_2 = -\frac{2}{4} + \frac{2}{4} = 0$.

Thus q_i are orthonormal.

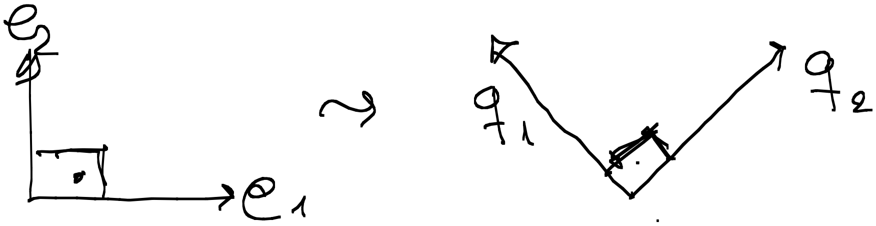
let $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2e_1 + 1e_2$.

$$Q^* x = \begin{bmatrix} -\sqrt{2} + \sqrt{2}/2 \\ \sqrt{2} + \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ 3\sqrt{2}/2 \end{bmatrix} = a$$

$$Qa = -\sqrt{2}/2\, q_1 + 3\sqrt{2}/2\, q_2$$

$$= \begin{bmatrix} 2/4 \\ -2/4 \end{bmatrix} + \begin{bmatrix} 3 \cdot 2/4 \\ 3\ 2/4 \end{bmatrix} = \begin{bmatrix} 1/2 + 3/2 \\ -1/2 + 3/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Therefore, we say that a unitary transformation is a change of coordinates

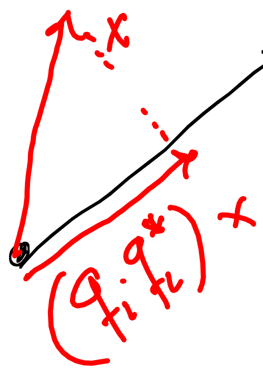


The matrices $q_i q_i^*$ are called 1-rank projections.

Geometrix interpretation

$$(q_i q_i^*)x = q_i \underbrace{(q_i^* x)}_{\text{coordinate of } x \text{ in } q_i}$$

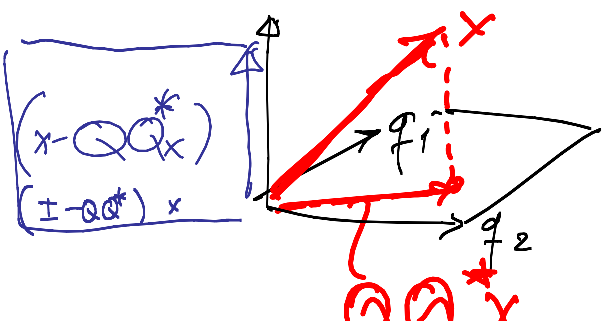
vector in direction of q_i



In general, if we have several vectors q_i, the matrix QQ^* is called a <u>projection</u>, as it projects x to a hyperplane $(\text{span}\{q_i\})$

Notice that

$$QQ^* = \sum_{i=1} q_i q_i^*$$



MATRIX INVERSE

Let A be a square complex matrix
(i.e. $A \in \mathbb{C}^{m \times m}$)

WE SAY THAT A HAS AN INVERSE
IF: $\forall b$, $\exists$ unique x such that

$b = Ax$.

GIVEN b, WE WRITE $x = A^{-1} b$

[EX. show that A^{-1} is a linear operator].

THM: A is invertible $\Leftrightarrow$
RANGE $(A) = m$ (or A is full rank) or $\mathrm{Null}(A) = \{0\}$

Proof:

$\Rightarrow$: Suppose RANGE $(A) < m$
Then: $\exists\, b_i \in \mathbb{C}$ such that

$\sum_i b_i a_i = 0$; where a_i are the
columns of A AND $\sum |b_i| > 0$.

Let $z = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$; then $Az = 0$, $\|z\|_2 \neq 0$
(I.E. A HAS NON TRIVIAL NULL SPACE)

Let $b = Ax$ for some $x \in \mathbb{C}^m$

Then $b = A(x+z)$. So for the same b we have 2 vectors that are mapped to b. This violates uniquness

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$\Leftarrow$ If A is full rank, a_i form a basis in $\mathbb{C}^m$.

By definition, every $b \in \mathbb{C}^m$ can be written as $b = \sum x_i a_i$ $\boxed{\text{uniquely}}$ or $b = Ax$. Therefore A is invertible $\blacksquare$

WE DID NOT PROVE THIS BUT IT IS A DIRECT CONSEQUENCE OF THE DEFN.

A invertible $\Longleftrightarrow$

null space $\{A\} = \{0\}$

$\det(A) \neq 0$.

no zero eigenvalues or singular values.

det, eigenvalues & singular values have not been covered yet. Just a reminder.

Note that inversion can be seen again as change of coordinates.

Let $b = Ax$

Then $b = \sum_i e_i\, b(i)$

Columns of A

$b = \sum_i a_i\, x(i)$

To compute x_i, we "invert" A.

[Remark: By "invert" in this class we mean solving $Ax=b$ by an efficient algorithm.]

DEF: IDENTITY MATRIX.

$I: \; Ix = x \quad \forall x \in \mathbb{C}^m$.

[Ex: Show that $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \vdots & 1 & 0 \\ 0 & 0 & \vdots & 1 \end{bmatrix}$.

a diagonal matrix.

$$A A^{-1} = A^{-1} A = I$$

Since: $(AA^{-1})b = A(A^{-1}b) = Ax = b, \forall b$

$$(A^{-1}A)x = A^{-1}(b) = x, \forall x.$$

SUPPOSE $Q \in \mathbb{C}^{m}$ is unitary.

Then $QQ^{*}x = x \quad \forall x \Rightarrow$

$$QQ^{*} = I. \Rightarrow \boxed{Q^{-1} = Q^{*}} \;!$$

TO INVERT AN ORTHOGONAL MATRIX JUST TAKE THEIR CONJUGATE TRANSPOSE.

ANOTHER CLASS OF MATRICES THAT ARE EASY TO INVERT:

DIAGONAL MATRICES.

VECTOR NORMS $\|x\|$

- GENERAL PROPERTIES

$$\|x\| : \mathbb{C}^m \to [0,\infty)$$

$\|x\| = 0 \iff x = 0$

$\|x+y\| \leq \|x\| + \|y\|$ (TRIANGLE INEQU)

$\|a x\| = |a| \, \|x\|, \quad a \in \mathbb{C}$

- COMMONLY USED NORMS

  - p-norm: $\|x\|_p = \left[\sum_{i=1}^{m} |x(i)|^p \right]^{1/p}, \quad p \geq 1$

Eg
$\|e_j\|_p = 1, p \geq 1$
$\|\mathbb{1}\|_1 = n$
$\infty = 1$
$p = n^{1/p}$

$p = \infty$ $\quad \|x\|_\infty = \max_i |x(i)|$ (infinity norm)

$p = 2$ $\quad \|x\|_2 = \sqrt{x^* x}$

$p = 0$ $\quad$: # NO ZERO ELEMENTS IN x. (NOT A NORM)

  - MATRIX WEIGHTED NORM

LET W WITH RANK m. $\quad \|Wx\|_p = \|x\|_W$ for ANY p.norm

- IMPORTANT PROPERTIES

  HOLDER AND CAUCHY-SCHWARZ

$|x^* y| \leq \|x\|_2 \, \|y\|_2$

$|x^* y| \leq \|x\|_1 \, \|y\|_\infty$

BOTH SHARP INEQUALITIES

  - Ex. USING THEM: $|\sum_i x_i| \leq n \|x\|_\infty$
    $\leq \sqrt{n} \, \|x\|_2$

  - CONVEXITY, DIFFERENTIABILITY $(1 < p < \infty)$, ROTATIONAL INVARIANCE $(p=2)$

MATRIX NORMS

- SAME GENERIC PROPERTIES WITH VECTORS
- FROBENIUS NORM: $\|A\|_F = \sqrt{tr(A^* A)}$ $\quad$ tr: TRACE $= \sum_i (\cdot)_{ii}$
- (VECTOR) INDUCED MATRIX NORMS

$$\|A\|_p = \max_{\|x\|_p = 1} \|Ax\|_p \quad \text{for ANY VECTOR NORM.}$$

IMMEDIATE PROPERTIES: $\|Ax\|_p \leq \|A\|_p \|x\|_p$

$\|AB\|_p \leq \|A\|_p \|B\|_p$

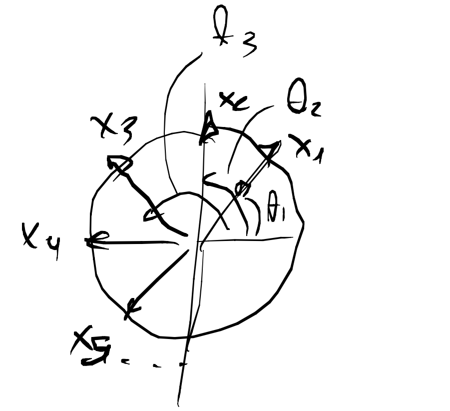
$\max_j \|A(:,j)\|_1$

$\|A\|_1 = \max_j \sum_i |A(i,j)|$ $\quad$ max one-norm of columns

$\|A\|_\infty = \max_i \sum_j |A(i,j)|$ $\quad$ max one-norm of rows

$\max_i \|A(i,:)\|_1$

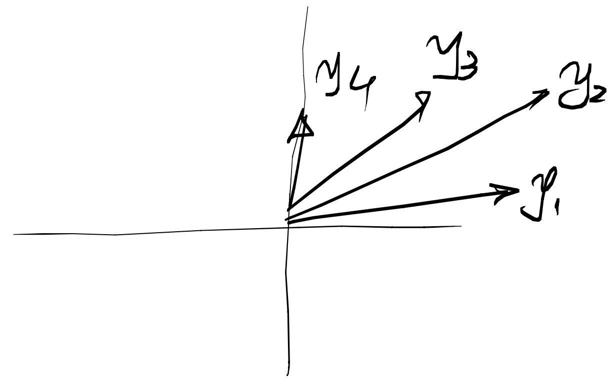
- Approximating $\|A\|$, $\mathbb{R}^2$.



- Take unit sphere,

let $\theta_i = \frac{2\pi}{N} i$, $i = 1, \dots, N$.

let $\underset{\sim}{x}_i = \begin{Bmatrix} \cos\theta_i \\ \sin\theta_i \end{Bmatrix}$



let $\underset{\sim}{y}_i = A \underset{\sim}{x}_i$

$$\|A\| = \max_i \|y_i\|.$$

- Approximating $\|A\|$ in $\mathbb{C}^m$; anorm=0

```
loop a = rand(m,1)*2 -1     ] // choose a random
     b = rand(m,1)*2 - 1    ] // vector
     x = a+bi
     x = x/||x||                // normalize vector
     anorm = max(anorm, ||Ax||). // Compute
                                 //norm of A
end. // when anorm doesn't change, terminate
```

SINGULAR VALUE DECOMPOSITION

LET $A \in \mathbb{C}^{m \times n}$ WITH $r \stackrel{\text{DEF}}{=} \text{RANK}(A)$.

THERE EXIST $U \in \mathbb{C}^{m \times m}$, $V \in \mathbb{C}^{n \times n}$, $\Sigma \in \mathbb{C}^{m \times n}$

(NON-UNIQUE) (NON-UNIQUE) UNIQUE IF WE SPECIFY ORDERING.

SUCH THAT:

- $A = U \Sigma V^*$
- U : ORTHONORMAL.
- Σ : DIAGONAL ; $\Sigma(i,i) \geq 0$; $\Sigma(i,j) = 0$

  $1 \leq i \leq \min(m,n)$

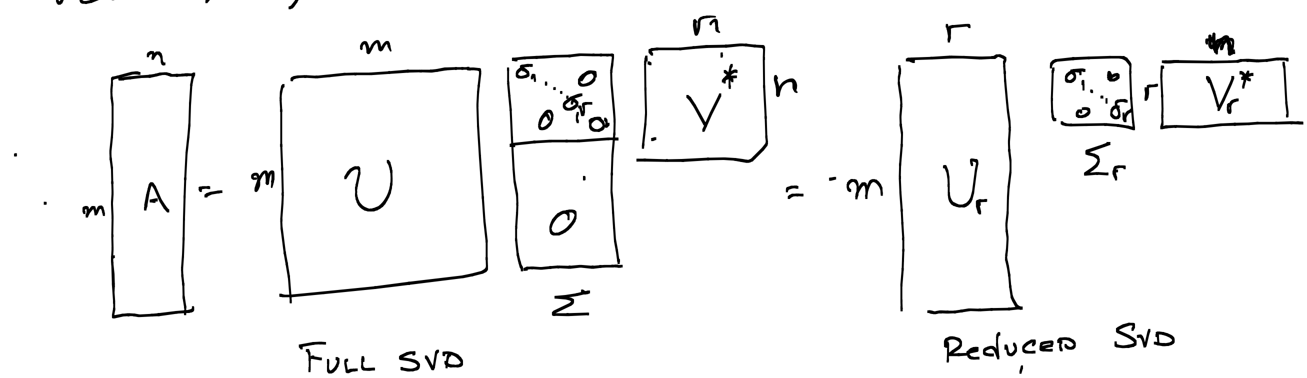
  $\sigma_i = \Sigma(i,i)$: SINGULAR VALUES.

  $\sigma_i \geq \sigma_{i+1} \geq 0$ (ORDERED IN DECREASING ORDER)

  $\forall\, i \geq r$; $\sigma_{r+1} = 0$
- V : ORTHONORMAL.

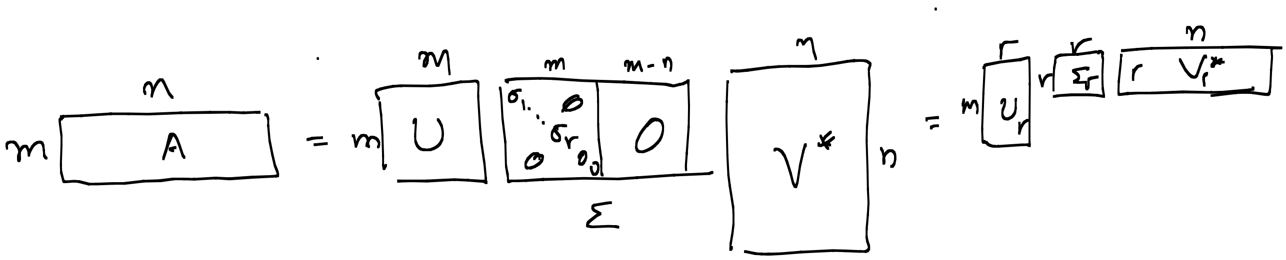
(i.e. $UU^* = I$, $VV^* = I$)

LET $m \geq n$, $r \leq n$



FULL SVD REDUCED SVD

LET $m < n$, $r \leq m$



INTEPRENTATIONS

TO APPLY A to x

(1) Rroject x to the span $\{\underset{\sim}{v_i}\}_{i=1}^{r}$

$$w = V^* x = \begin{bmatrix} v_1^* x \\ \vdots \\ v_r^* x \end{bmatrix}$$

(2) Scale $w(i)$ to $\sigma_i\, w(i)$

$$s = \Sigma w \quad ; \quad \boxed{\sigma_i = \Sigma_{ii}}$$

singular values of A

(3) $y = Us = \sum_{i=1}^{r} u_i s(i)$

$(v_i^* x)\sigma_i$

Any matrix A can be written as a sum of of rank-1 matrices

$$A = \sum_{i=1}^{r} \sigma_i (u_i v_i^*)$$

; r can be SUBSTITUTED BY min(m,n) BUT ANY σ_i WITH $i>r$ WILL BE ZERO.

- $\text{Range}(A) = \text{span}\{u_i\}_{i=1}^{r}$

  $\{u_i\}_{i=1}^{r}$: an orthonormal basis of the range of A.

- $\text{Range}(A^*) = \text{Range}(V \Sigma U^*)$

  $\{v_i\}_{i=1}^{r}$: orthonormal basis of $\text{Range}\{A^*\}$.

- Let $A \in \mathbb{C}^{m \times m}$

  Solve $Ax = b$

If A is invertible, it is full rank

Thus, $r = m$ $\quad A = {}^{m}\overset{m}{\boxed{U}} \; \boxed{\begin{matrix} \Sigma & 0 \\ 0 & \end{matrix}} \; \boxed{V^*}$

- $\text{Null}(A)$ is orthogonal to $\text{span}\{\underset{\sim}{v}_i\}$.

$$Ax = b \Rightarrow U\Sigma V^* x = b \Rightarrow$$

$$U^*(U\Sigma V^* x) = U^* b \Rightarrow$$

$$\Sigma V^* x = U^* b.$$ ["Due to orthonormality of U]

Assume $x = Vz$ (Since V is a basis)

then $\Sigma V^* V z = \Sigma z$

thus $z = \Sigma^{-1}(U^* b)$ and

$$\boxed{x = V\Sigma^{-1}U^* b}$$

i.e. $A^{-1} = V\Sigma^{-1}U^*$

Why all these goodies?

Because: All matrices are diagonal with appropriate change of coordinates.