LAST TIME: SVD DECOMPOSITION ; EXISTENCE/UNIQU.

ANOTHER IMPORTANT PROPERTY OF SVD (LOW RANK APPROXIMAT.)

LET $r = \operatorname{rank}(A)$, $A \in \mathbb{C}^{m\times n}$, $r \leq \min(m,n)$

LET $\nu < r$. THEN $\operatorname*{argmin}_{B,\, \operatorname{ran}(B)=\nu} \|A - B\|_2 = \sum_{i=1}^{\nu} \sigma_i u_i v_i^* = A_\nu$

and $\|A - A_\nu\|_2 = \sigma_{\nu+1}$.

PROJECTION OPERATORS

- DEFINITION: (1) $P \in \mathbb{C}^{m\times m}$ (SQUARE)

  (2) $P^2 = P$.

- P PROJECTOR $\iff$ $I-P$ PROJECTOR & RANGE$(I-P)$ = NULL(P); NULL$(I-P)$ = RANGE(P); RANGE$(P) \perp$ NUL(P).

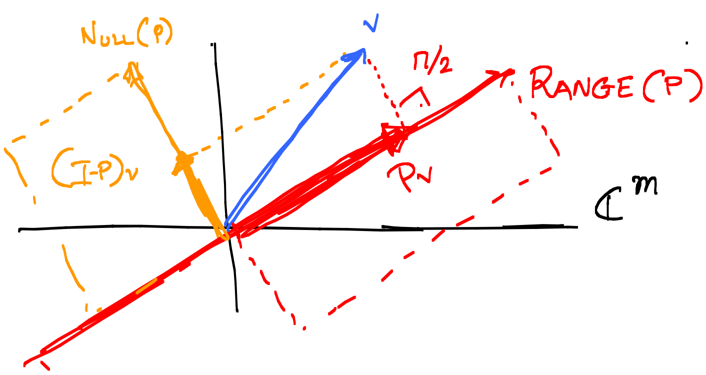
  $I-P$: COMPLEMENTARY PROJECTOR

- DEFINITION: A PROJECTOR IS ORTHOGONAL IF $P = P^*$

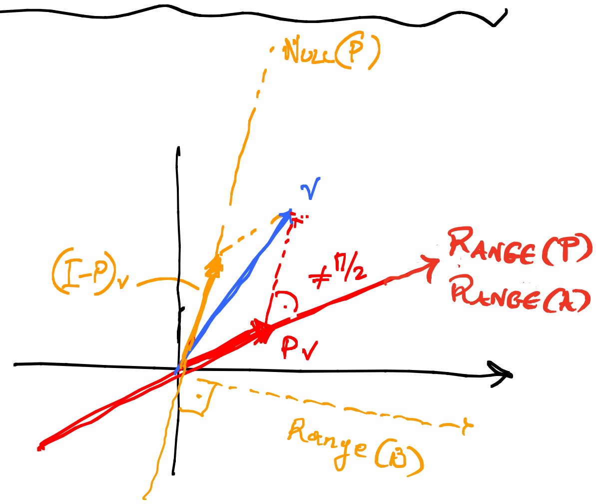
  - $\{q_i\}_{i=1}^{n}$ ORTHONORMAL VECTORS: $P = QQ^*$; $Q = \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix}$
  - $\{a_i\}_{i=1}^{n}$ ANY LIN-IND VECTORS $P = A(A^*A)^{-1}A^*$, $A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$

- OBLIQUE PROJECTION: A spans RANGE(P), B spans $\perp$ Null(P) $\quad P = A(B^TA)^{-1}B^T$

GEOMETRIC INTERPRETATION



ORTHOGONAL PROJECTOR



OBLIQUE PROJECTOR

- ORTHONORMAL VECTORS $\{q_i\}$; $i=1,\dots,n$, $q_i \in \mathbb{C}^m$ $\quad n \le m$

$$P = QQ^* = \sum_{i=1}^{n} q_i (q_i^*)$$

E.X. $q = \frac{1}{\sqrt{6}}\begin{Bmatrix} 1 \\ 2 \\ -1 \end{Bmatrix}$; $P = \frac{1}{6}\begin{Bmatrix} 1 \\ 2 \\ -1 \end{Bmatrix}\{1 \;\; 2 \;\; -1\} = \frac{1}{6}\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}$

$(I-P) = \frac{1}{6}\begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}$. you can check $(I-P)q = 0$.

- COMPLEMENTARY PROJECTOR

$\forall v$; $Pv = 0 \Rightarrow (I-P)v = v \Rightarrow v \in \text{RANGE}(I-P) \Rightarrow \text{NULL}(P) \subseteq \text{Range}(I-P)$

$(I-P)v = v - Pv$. $P(v-Pv) = Pv - Pv = 0 \Rightarrow v - Pv \in \text{NULL}(P) \Rightarrow \text{RANGE}(I-P) \subset \text{Null}(P)$

$\Rightarrow \text{RANGE}(I-P) = \text{NULL}(P)$.

P DECOMPOSES $\mathbb{C}^m$ TO 2 SUBSPACES RANGE(P) & RANGE$(I-P)$

NOTATION TO INDICATE THAT THE TWO SUBSPACES ARE ORTHOGONAL

- ORTHOGONAL PROJECTION : $\text{NULL}(P) \perp \text{RANGE}(P)$

$(I-P)^* P = 0 \Rightarrow P - P^*P = 0 \Rightarrow P^*P = P \quad = PP \quad \Rightarrow (P^* - P)P = 0$

- ARBITRARY LIN-IND BASIS

LET $\{a_j\}_{j=1}^{n}$, $a_j \in \mathbb{C}^m$ WE WANT P ORTHOGONAL ONTO RANGE(A)

$A = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdot & a_n \\ | & | & & | \end{bmatrix}$ input vector

LET $y = Pv$. SINCE $y \in \text{RANGE}(A)$ $\exists\, x : y = Ax$

$(v-y) \in \text{RANGE}(I-P)$

BY ORTHOGONALITY OF $(I-P)$ TO P : $A^*(v-y) = 0$

$\Rightarrow A^*(v - Ax) = 0 \Rightarrow A^*v = A^*Ax$

IF A IS FULL RANK, THEN A^*A IS INVERTIBLE (SYD)

$\Rightarrow x = (A^*A)^{-1}A^*v$

$\Rightarrow Pv = y = Ax = A(A^*A)^{-1}A^*v \rightarrow \underline{P = A(A^*A)^{-1}A^*}$.

EASY TO CHECK THAT $\begin{cases} P^2 = P & \text{(PROJECTION)} \\ P^* = P & \text{(ORTHOGONALITY)} \end{cases}$

QR FACTORIZATION

Given $\{a_j\}_{j=1}^{n}$, $a_j \in \mathbb{R}^m$ WE WANT TO CONSTRUCT NEW SET OF VECTORS

$$\{q_j\}_{j=1}^{n}$$

SUCH THAT (1) : $q_i^T q_j = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{otw} \end{cases}$

(2) : $q_j \in \text{span}\{a_1, \dots, a_j\}$; $j = 1, \dots, n$

THE PROCESS OF FINDING q_j IS CALLED ORTHOGONALIZATION OF a_j

$\boxed{j=1}$ $\|q_1\|_2 = 1$ & $q_1 \in \text{span}\{a_1\}$

$$\Rightarrow \boxed{q_1 = \frac{a_1}{\|a_1\|_2}}$$

$\boxed{j=2}$ $\|q_2\|_2 = 1$ & $q_1^T q_2 = 0$ & $q_2 \in \text{span}\{a_1, a_2\}$

LET $w = a_2 + \lambda_1 q_1$; NOTICE $w \in \text{span}\{a_1, a_2\}$

$$q_1^T w = 0 \Rightarrow a_2^T q_1 + \lambda_1 q_1^T q_1 = 0 \Rightarrow \lambda_1 = -a_2^T q_1 \Rightarrow$$

$$w = (I - q_1 q_1^T) a_2$$

But we also want $\|q_2\|_2 = 1 \Rightarrow q_2 = \frac{w}{\|w\|_2}$

WITH THIS CONSTRUCTION ALL CONDITIONS ARE SATISFIED FOR $j=2$

$\boxed{j=3}$

LET $w = a_3 + \lambda_1 q_1 + \lambda_2 q_2$

- $w^T q_1 = 0 \Rightarrow \lambda_1 = -q_1^T a_3$
- $w^T q_2 = 0 \Rightarrow \lambda_2 = -q_2^T a_3$

$$\Rightarrow w = \left(I - q_1 q_1^T - q_2 q_2^T\right) a_3$$

SETTING $q_3 = \dfrac{w}{\|w\|_2}$ WE SATISFY ALL CONDITIONS

$\boxed{\text{FOR } j}$

$$w = a_j + \sum_{i=1}^{j-1} \lambda_i q_i \ , \quad \lambda_i = -q_i^T a_j$$

$$\Rightarrow w = a_j - \sum_{i=1}^{j-1} q_i \left(q_i^T a_j\right) = \left(I - \sum_{i=1}^{j-1} q_i q_i^T\right) a_j$$

$$q_j = \frac{w}{\|w\|_2}$$

THIS PROCEDURE IS THE <u>GRAM-SCHMIDT ORTHOGONALIZATION</u>

MORE PRECISELY, THE CLASSICAL GRAM-SCHMIDT.

Eg. $a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$; $a_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$; $q_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \dfrac{1}{\sqrt{2}}$

$$w = \left(\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \boxed{\begin{matrix} 1 & 1 & 0 & 0 \end{matrix}} \boxed{\begin{matrix} 0 \\ 1 \\ 1 \\ -1 \end{matrix}}\right) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow q_2 = w / \|w\|_2 \ ; \ \|w\|_2 = \sqrt{\tfrac{5}{2}}$$