# LECTURE 1 SUMMARY

- Syllabus
- COURSE SUMMARY:
  - WHAT: Solving $Ax=b$, Spectral decompositions
  - WHY: SCIENCE, ENGINEERING, DATA
- NOTATION
  - Scalar quantities ($\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$) Greek lower case
    real complex integer
    exception: $m, n, i, j, k, \ell \in \mathbb{N}$
  - Vectors: lower case roman $(x, y, w, u \ldots)$

    All vectors will be column vectors.

    $v^*$ will indicate a row vector whose elements are the complex conjugate of $v$.

complex conjugate $v(1) = a + ib$
$\bar{v}(1) = a - ib$

$$v = \begin{bmatrix} v(1) \\ v(2) \\ v(3) \end{bmatrix}; \quad v^* = \begin{bmatrix} \bar{v}(1) & \bar{v}(2) & \bar{v}(3) \end{bmatrix};$$

$v(i)$ indicates the ith element of the vector.

Occasionally, to denote a set of vectors, I will use an undertilde. e.g. $\underset{\sim}{v}_1, \underset{\sim}{v}_2$ indicate vectors.

Matrices: uppercase roman: $A, B, C, \ldots$
& GREEK: $\Lambda, \Xi$

ABSTRACT VECTOR spaces: $\mathcal{V}, \mathcal{W}, \mathcal{X}$

Addition: Associativity, Commutativity, Identity, Negative

Multiplication with scalar: Distributivity, Identity, Associativity

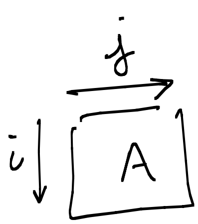
Matrices as a set of column vectors

e.g.

$$A = \begin{bmatrix} | & | & | & | \\ a_1 & a_2 & a_3 & a_4 \\ | & | & | & | \end{bmatrix}$$

Matrices as row vectors

$$A = \begin{bmatrix} — & a_1^* & — \\ — & a_2^* & — \\ & \vdots & \\ — & a_n^* & — \end{bmatrix}.$$




Matrix entries

indexes rows

indexes columns

- elements $A_{ij}$, $A(i,j)$
- columns $A_{:i}$ ; $A(:,i)$
- rows $A_{i:}$ ; $A(i:)$

- Linear combinations of vectors

$\{v_1, \dots, v_n\}$ ; n vectors ; $v_i \in \mathbb{C}^m$ (or $\mathbb{R}^m$)

$\{\lambda_1, \dots, \lambda_n\}$ in scalars ; $\lambda_i \in \mathbb{C}$ (or $\mathbb{R}$)

$w = \sum_{i=1}^{n} \lambda_i v_i$ is a linear combination of $\{v_i\}$

- SPAN of $\{v_i\}_{i=1}^{n}$ ; $v_i \in \mathbb{C}^m$

all possible linear combinations of $\{v_i\}$

- LINEARLY INDEPENDENT SETS OF VECTORS : $\nexists \{\lambda_i\}_{i=1}^{n} : \sum_i |\lambda_i| > 0$ AND $\sum_i \lambda_i v_i = 0$

THEN $\{v_i\}$ ARE LIN-IND.

- LINEARLY DEPENDENT SETS OF VECTORS: WE SAY $\{v_i\}_{i=1}^{n}$ ARE LIN-DEP IFF

$\exists \{\lambda_i\}_{i=1}^{n} : \sum_i |\lambda_i| > 0$ AND $\sum_i \lambda_i v_i = 0$ ; $\lambda_i \in \mathbb{C}$ ; $v_i \in \mathbb{C}^n$.

- Basis of a vector space

Let $\mathbb{V}$ be A VECTOR SPASE. Let $\{v_i\}_{i=1}^{n} \in \mathbb{V}$.

It is a BASIS IF
(1) $\mathbb{V} = \text{span}\{v_i\}$
(2) $\{v_i\}$ IS lin-INDEP.

- Dimension of a vector space

$\dim(\mathbb{V})$ = # OF VECTORS IN A BASIS.

- Subspace of a vector space

- Inner product $x, y \in \mathbb{C}^n$

$(x \cdot y) = x^* y = \sum_{j=1}^{n} \overline{x_j}\, y_j$

PROPERTIES

$(x \cdot y) = \overline{(y \cdot x)}$

$\alpha (x \cdot y) = (\alpha x \cdot y)$

$(x + y, z) = (x, z) + (y, z)$

$(x, x) > 0$, $\forall x \neq 0$

- Euclidean norm of a vector

  $\|x\|_2 = \sqrt{(x \cdot x)}$

- Angle between two vectors

  $\cos a = (x \cdot y) / \|x\|_2 \cdot \|y\|_2$

---

functions (or maps) of vectors

Linear maps

Let $f : \mathcal{V} \to \mathcal{W}$

(that is $v \in \mathcal{V}$; $f(v) \in \mathcal{W}$)

$f$ is linear iff (if and only if)

$$f(\lambda_1 v_1 + \lambda_2 v_2) =$$

$$\lambda_1 f(v_1) + \lambda_2 f(v_2)$$

$$\forall \lambda_1, \lambda_2 \in \mathbb{C} ; v_1, v_2 \in \mathcal{V}$$

---

Given a linear map $f : \mathbb{C}^n \to \mathbb{C}^m$, we can represent it as a matrix

$$Ax = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = f(x)$$

$$= \sum_i x_i a_i \quad , \text{where } a_i = f(e_i)$$

and $\underset{\sim}{e_i}$ is the canonical Basis in $\mathbb{C}^n$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} ; \; e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} ; \cdots ; \; e_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}.$$

That is, a applying a matrix to a vector gives another vector, which is a linear combination of the columns of the matrix.

---

ALTERNATIVELY, WE CAN VIEW A MATRIX VECTOR MULTIPLICATION AS A SET OF INNER PRODUCTS.

GIVEN $a_i \in \mathbb{C}^n$ and $x \in \mathbb{C}^n$ WITH $i = 1, \dots, m$

WE WISH TO COMPUTE $y_i = (a_i \cdot x) \quad i = 1, \dots, m$

THIS OPERATION CAN BE WRITTEN AS

$$\begin{Bmatrix} | \\ y \\ | \end{Bmatrix} = \begin{bmatrix} — a_1^* — \\ — a_2^* — \\ \vdots \\ — a_m^* — \end{bmatrix} \begin{Bmatrix} \\ x \\ \\ \end{Bmatrix}$$

so that

$$y_i = \sum_{j=1}^{n} \underbrace{\boxed{\bar{a}_i(j)}}_{A_{ij}} x_j = \sum_{j=1}^{n} A_{ij} x_j$$

which is again the classical matrix VECTOR MULTIPLICATION FORMULA.

- THESE SETS OF INNER PRODUCTS CAN BE VIEWED AS A WAY TO DEFINE A PARTICULAR MAP, SO THIS DEFINITION DOES NOT CONTRADICT THE MOST GENERAL ONE THAT VIEWS A MATRIX-VECTOR MULTIPLICATION AS A LINEAR FUNCTION.