LECTURE

LAST TIME

- GRAM - SCHIMDT

BASIC ALGORITHM , $A \in \mathbb{C}^{m\times n}$ $A(:,j) \sim a_j$; full Rank

$$q_1 = \frac{a_1}{\|a_1\|_2} \quad r_{11}$$

$$q_2 = a_2 - q_1(\underbrace{q_1^* a_2}_{r_{12}}) \; ; \; q_2 = \frac{q_2}{\|q_2\|_2} \quad r_{22}$$

$$q_3 = a_3 - q_1(\underbrace{q_1^* a_3}_{r_{13}}) - q_2(\underbrace{q_2^* a_3}_{r_{23}}) \; , \; q_3 = \frac{q_3}{\|q_3\|_2}$$

$$\vdots$$

$$q_n = a_n - \sum_{j=1}^{n-1} q_j(\underbrace{q_j^* a_n}_{r_{jn}}) \; ; \; q_n = \frac{q_n}{\|q_n\|_2} \quad r_{nn}$$

MATLAB
```
q=A(:,1) ; q=q/norm(q)
Q=[q]; Im=eye(m) ;
for j=2:n
    q =(Im-QQ*)*A(:,j)
    q = q/norm(q);
    Q = [Q,q];
end
```
ALGO 1

WE CAN VERIFY THAT $A=QR$

THE EXPESSION FOR q_j is

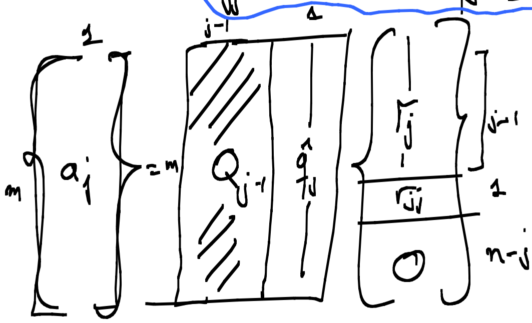
$$q_j = (I - Q_{j-1}Q_{j-1}^*)a_j \quad ; \quad Q_{j-1} = [q_1 \cdots q_{j-1}]$$

$$\hat{q}_j = q_j / \|q_j\|_2$$

NOTICE THAT $r_j \in \mathbb{C}^{j-1}$

IF WE DEFINE $\boxed{r_j = Q_{j-1}^* a_j}$, THEN $q_j = a_j - Q_{j-1} r_j$ $\rightarrow$

AND $\boxed{r_{jj} \in \mathbb{R} = \|q_j\|_2}$ $\Big\}$ $a_j = q_j + Q_{j-1} r_j = \hat{q}_j r_{jj} + Q_{j-1} r_j$



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TESTING ALGORITHM.

SUPPOSE WE IMPLEMENTED ALGORITHM 1.
HOW DO WE TEST IT?

(1) SELECT R AND Q

SET $A = Q*R;$

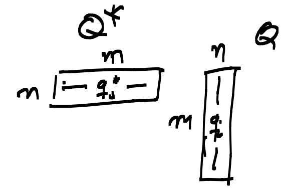
RUN ALGORITHM AND CHECK IF WE GET BACK Q & R.

EXPENSIVE, A LOT OF EXTRA CALCULATIONS

(2) CHECK PROPERTIES OF Q

(A) ORTHONORMAL

(B) SPANS RANGE OF A.



• (A) : $\|Q^*Q - I_m\|_2$ SHOULD BE ZERO $\Leftarrow (Q^*Q)_{ij} = q_i^* q_j = \delta_{ij}$

• (B) : $z = \text{randn}(n,1)$; $Q^*(Az) \neq 0$

$z = \text{randn}(m,1)$; $A^*(I - QQ^*)z = 0$

CHECK THIS FOR VARIUS A. BUT HOW MANY?

- check rand(m,n); randn(m,n); EVERYTHING WORK WELL
- RUN CODE INTO PRACTICE: FAILURE. Eg. : $A = [\,\vdots\;\vdots\,] + 1e{-}2 * \text{rand}(3,2)$

STABILITY OF NUMERICAL COMPUTATIONS.

- THREE MAIN SOURCES OF ERRORS

  (1) DATA UNCERTAINTY

  Imperfect knowledge
  Sensor errors, incomplete information
  APPLICATON DEPENDENT
  ALWAYS PRESENT

  (2) TRUNCATION

  DISCRETIZATION OF ODEs, PDEs
  INEXACT ALGORITHMS

  MAJOR TASK OF NUMERICAL ANALYSIS

  (3) ROUNDING ERRORS

  FINITE PRECISION ARITHMETIC.
  FLOATING POINT NUMBERS.
  QUITE TRICKY AND TIME CONSUMING.

- WHAT IS THE EFFECT OF THESE ERRORS IN THE COMPUTATION?

SUPPOSE WE WANT TO COMPUTE $y = f(x)$

A BASIC QUESTION IS THE FOLLOWING:

GIVEN SOME ACCURACY IN THE INPUT x WHAT IS THE EXPECTED ACCURACY IN y?

EXAMPLE:

$f(x) = 9^x$, $x = 50$

Suppose a perturb x to $x = 50.5$

then $9^{50.5} = 3 f(50)$

Is this ok? depends on the application.

Input relative perturbation: $\frac{50.5}{50} = 1\%$

Output relative perturbation: $\frac{3f(50)}{f(50)} = 300\%$.

Example 2

Solve $\begin{bmatrix} 1/3 & 1/3 \\ 1/3 & 0.3 \end{bmatrix} \{x\} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -27 \\ 30 \end{bmatrix}$

INPUT PERTURBATION SIZE $|1/3 - 0.3|$ ← change this to 1/3 and no solutions exist. → OUTPUT PERTURBATION UNBOUNDED.

CONDITION NUMBER OF A PROBLEM

TO FORMALIZE THE ANALYSIS ABOVE WE INTRODUCE THE CONDITION NUMBER OF A FUNCTION (AND AS WE WILL SEE LATER, OF A MATRIX.)

- well-conditioned → small perturbations in input result in small perturbations in output
- ill conditioned → small perturbations in input result in large perturbations in output
- ILL POSED → SMALL PERTURBATIONS IN INPUT RESULT IN UNBOUNDED PERTURBATIONS IN OUTPUT.

CONDITION NUMBER OF A FUNCTION f

let $f: \mathbb{C}^n \to \mathbb{C}^m$ (COULD BE NONLINEAR).

LET δx BE A PERTURBATION IN x.

THEN $\delta f = f(x+\delta x) - f(x)$

ABSOLUTE CONDITION NUMBER AT x:

$$\lim_{|\delta x| \to 0} \sup_{\delta x} \frac{\|\delta f\|}{\|\delta x\|}$$

(NOTE THAT IT DEPENDS ON x)

IF f IS DIFFERENTIABLE, THIS IS JUST A MAGNITUDE OF THE DERIVATIVE OF f.

$\|\delta f\|$, $\|\delta x\|$

THE NORMS ARE CHOSEN TO BE CONSISTENT.

TYPICALLY WE'RE MORE INTERESTED IN RELATIVE CHANGES

RELATIVE CONDITION NUMBER

OR JUST CONDITION NUMBER

$$K(x) = \frac{\text{relative output perturbation at } x}{\text{relative input perturbation at } x}$$

$$K(x) = \lim_{\|\delta x\| \to 0} \sup_{\|\delta x\|} \frac{\|\delta f\| / \|f(x)\|}{\|\delta x\| / \|x\|}$$

IF f IS DIFFERENTIABLE, $J := \frac{\partial f}{\partial x}$

$$K(x) = \frac{\|J(x)\| \, \|x\|}{\|f(x)\|}$$

<u>Example</u>: $f(x) = ax$; $a \in \mathbb{R}$

$K(x) = 1$. well conditioned.

Example $\quad f(x) = \underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{J} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 - x_2$

$$\|J\|_\infty = 2$$

$$\kappa(x) = \frac{\|J\|_\infty \|x\|_\infty}{|x_1 - x_2|} = \frac{2\max(|x_1|, |x_2|)}{|x_1 - x_2|}$$

problematic if $|x_1 - x_2| \ll \|x\|_\infty$

$0 < a, \delta \ll 1$

e.g.: $x_1 = 1 + a$; $x_2 = 1$; $\delta x_2 = \delta$, $\delta x_1 = 0$; $\delta f = a - \delta$, $\delta x = \delta$; $f = a$, $\|x\| \approx 2$ $\Big]$ $\kappa(x) = \dfrac{2(a-\delta)}{a\delta}$, $a = 1E{-9}$, $\delta = 1E{-3}$, $\kappa(x) = ?$

Condition number of matrix-vector multiplication

Let $A \in \mathbb{C}^{m \times m}$, A full rank ; $f(x) = Ax$

We consider perturbations on x. $\delta f: A\delta x$

$$\boxed{K(x)} = \lim_{\delta x \to 0} \sup_{\|\delta x\|} \frac{\|A\delta x\|}{\|Ax\|} \Big/ \frac{\|\delta x\|}{\|x\|}$$

These terms don't depend on δx.

$$= \frac{\|x\|}{\|Ax\|} \underbrace{\lim_{\delta x \to 0} \sup_{\|\delta x\|} \frac{\|A\delta x\|}{\|\delta x\|}}_{\text{Norm of } A \text{ by definition.}} = \boxed{\frac{\|x\|}{\|Ax\|} \|A\|}$$

Notice that this quantity depends on x.

We generilize this to a quantity that ~~does not~~ depend on x. We call it **the condition number of a matrix**

$$\text{cond}(A) \overset{\text{DEF}}{:=} \max_{x} K(x)$$

Since A is invertible, $\exists\, y \in \mathbb{C}^m : x = A^{-1}y$

$$K(x) = \frac{\|x\| \|A\|}{\|Ax\|} = \frac{\|A^{-1}y\| \ \|A\|}{\|AA^{-1}y\|} \leq \frac{\|A^{-1}\| \|A\| \|y\|}{\|y\|}$$

$\rightarrow K(x) \leq \|A^{-1}\| \|A\|$, $\forall x$

Also can show (exercise) : $\exists\, x : K(x) = \|A\| \|A^{-1}\|$.

$$\Rightarrow \boxed{\text{cond}(A) = \|A\| \|A^{-1}\|}, \ A \in \mathbb{C}^{m \times m}, \text{ full rank.}$$

EXAMPLE : $\text{cond}(A^{-1}) = \|A\| \|A^{-1}\| = \text{cond}(A)$.

CONDITION NUMBER OF A SQUARE UNITARY MATRIX U IN THE $\| \|_2$

$$U^{-1} = U^{*} ; \quad \|U\|_2 = \|U^{*}\|_2 = 1.$$

$$\kappa_2(U) = 1.$$

CONDITION NUMBER USING THE SVD

$$A = U \Sigma V^{*} \Rightarrow \|A\| \leq \|U\| \|V\| \|\Sigma\|$$

SINCE U, V ARE UNITARY,

$$\|U\|_2 = \|V\|_2 = 1$$

SO IF WE USE THE 2-norm,

$$\|A\|_2 = \sigma_{max} = \sigma_1$$

SIMILARLY

$$\|A^{-1}\|_2 = \|\Sigma^{-1}\|_2 = 1/\sigma_{min}$$

THEREFORE $\boxed{K_2(A) = \dfrac{\sigma_{max}}{\sigma_{min}}}$

2-norm of a square invertible matrix

page 105., THM 12.2

$Ax = b$; perturb δA

Again $K = \|A\| \, \|A^{-1}\|$.

REMARK

THE CONDITION NUMBER OF f HAS NOTHING TO DO ON HOW WE COMPUTE IT;

IT IS INHERENT TO THE STRUCTURE OF f. (i.e. NORM OF THE DERIVATIVE)