Fall 2016, CSE/CS 383 HW 1

Name: Po-Cheng Pan
UT EID: pp 22828

1、 $\because f$ is a linear map

$\therefore$ ① $f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(v_1) + \lambda_2 f(v_2)$, $\forall\ v_1, v_2 \in \mathbb{C}^n$, $\forall\ \lambda_1, \lambda_2 \in \mathbb{C}$

② $f(\lambda_1 v_1) = \lambda_1 f(v_1)$, $\forall\ v_1 \in \mathbb{C}^n$, $\lambda_1 \in \mathbb{C}$

For $N=1$:

$f(\lambda_1 x_1) = \lambda_1 f(x_1)$ is true because f is a linear map.

For $N=2$:

$$f\left(\sum_{i=1}^{2} \lambda_i x_i\right) = f(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 f(x_1) + \lambda_2 f(x_2) = \sum_{i=1}^{2} \lambda_i f(x_i)$$

Because f is a linear map

For $N=k$, we suppose that $f\left(\sum_{i=1}^{k} \lambda_i x_i\right) = \sum_{i=1}^{k} \lambda_i f(x_i)$, where $\lambda_i \in \mathbb{C}$, x_i in $\mathbb{C}^n$ and k is an integer.

For $N=k+1$,

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) = f\left(\sum_{i=1}^{k} \lambda_i x_i + \lambda_{k+1} x_k\right) \overset{\text{Because } f \text{ is a linear map}}{=} f\left(\sum_{i=1}^{k} \lambda_i x_i\right) + \lambda_{k+1} f(x_k)$$

$$\overset{\text{Because the supposition above}}{=} \sum_{i=1}^{k} \lambda_i f(x_i) + \lambda_{k+1} f(x_k)$$

$$= \sum_{i=1}^{k+1} \lambda_i f(x_i)$$

Hence, by induction, we prove that if $f: \mathbb{C}^n \to \mathbb{C}^m$ is a linear map, then $f\left(\sum_{i=1}^{N} \lambda_i x_i\right) = \sum_{i=1}^{N} \lambda_i f(x_i)$ where $\lambda_i \in \mathbb{C}$, x_i in $\mathbb{C}^n$ and N is an integer.

2、

Textbook exercise 1.1:

B is a 4×4 matrix $= \begin{bmatrix} - r_1 - \\ - r_2 - \\ - r_3 - \\ - r_4 - \end{bmatrix} = \begin{bmatrix} | & | & | & | \\ c_1 & c_2 & c_3 & c_4 \\ | & | & | & | \end{bmatrix}$

(a)

① double column 1

We can build a 4×4 matrix $M_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

s.t. $B \cdot M_1 = \begin{bmatrix} | & | & | & | \\ 2c_1 & c_2 & c_3 & c_4 \\ | & | & | & | \end{bmatrix}$

② halve row 3

We can build a 4×4 matrix $M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

s.t. $M_2 \cdot B = \begin{bmatrix} - r_1 - \\ - r_2 - \\ - 0.5 r_3 - \\ - r_4 - \end{bmatrix}$

③ add row 3 to row 1

We can build a 4×4 matrix $M_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

s.t. $M_3 B = \begin{bmatrix} - r_1 + r_3 - \\ - r_2 - \\ - r_3 - \\ - r_4 - \end{bmatrix}$

④ interchange column 1 & 4

We can build a 4×4 matrix $M_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

s.t. $B M_4 = \begin{bmatrix} | & | & | & | \\ c_4 & c_2 & c_3 & c_1 \\ | & | & | & | \end{bmatrix}$

⑤ substract row 2 from each of the other rows

We can build a 4×4 matrix $M_5 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

s.t. $M_5 B = \begin{bmatrix} - r_1 - r_2 - \\ - 0 - \\ - r_3 - r_2 - \\ - r_4 - r_2 - \end{bmatrix}$

to be contined on next page

2. Textbook exercise 1.1 conti.

⑥ replace column 4 by column 3

We can build a 4x4 matrix $M_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

s.t. $BM_6 = \begin{bmatrix} | & | & | & | \\ c_1 & c_2 & c_3 & c_3 \\ | & | & | & | \end{bmatrix}$

⑦ delete column 1

We can build a 4x3 matrix $M_7 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

s.t. $BM_7 = \begin{bmatrix} | & | & | \\ c_2 & c_3 & c_4 \\ | & | & | \end{bmatrix}$

We can write these 7 operations on B as a product of eight matrices :

$$M_5M_3M_2B\cdot M_1M_4M_6M_7 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

#

(b) $ABC = M_5M_3M_2\cdot B\cdot M_1M_4M_6\cdot M_7$

$\Rightarrow A = M_5M_3M_2 = \begin{bmatrix} 1 & -1 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0.5 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$ $C = M_1M_4M_6M_7 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow ABC = \begin{bmatrix} 1 & -1 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0.5 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ #

2. Textbook exercise 1.3

Ans:

$\because R$ is a nonsingular $m\times m$ square matrix $\therefore \det(R) \neq 0$

By Theorem 1.3 in textbook,

we know that R has an inverse R^{-1}, $\operatorname{rank}(R) = m$, $\operatorname{range}(R) = \mathbb{C}^m$

So the m columns of R for the basis for the whole space $\mathbb{C}^m$

And we can uniquely expess any vector as a linear combination of them

In particular, for the canonical unit vector with 1 in the jth entry and zeros elsewhere, written as e_j

we can express e_j as $\sum_{i=1}^{m} z_{ij} r_j$ ← From (1.8) in the text book (Z be the matrix with entries z_{ij} & r_j is the jth column of R)

$= RZ_j$ (Z_j is the jth column of Z)

So

$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = RZ_1 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1m} \\ 0 & r_{22} & & & \\ 0 & & r_{33} & & \vdots \\ 0 & \cdots & & \ddots & \vdots \\ \vdots & & & & \\ 0 & & \cdots & & r_{mm} \end{bmatrix} \begin{bmatrix} z_{11} \\ z_{21} \\ \vdots \\ \vdots \\ z_{m1} \end{bmatrix}$

$\Rightarrow r_{mm} \times z_{m1} = 0 \Rightarrow z_{m1} = 0$

$r_{m-1\,m} \times z_{m1} + r_{m-1\,m-1} z_{m-1} = 0 \Rightarrow z_{m-1} = 0$

If we keep calculating,

we will find that $z_{21} = z_{31} = \cdots z_{m1} = 0$

Only $z_{11} \neq 0$

contine at next page

2. Textbook exercise 1.3 conti

By the same method, to calculate each z_j from the equation $e_j = R z_j$

We will get $z_j = \begin{bmatrix} z_{1j} \\ z_{2j} \\ \vdots \\ z_{jj} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ for $\begin{cases} z_{ij} \neq 0 & \text{if } i \leq j \\ z_{ij} = 0 & \text{if } i > j \end{cases}$

$$Z = \begin{bmatrix} | & | & | & & | \\ z_1 & z_2 & z_3 & \cdots & z_m \\ | & | & | & & | \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1m} \\ 0 & z_{22} & \ddots & z_{2m} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & z_{nn} \end{bmatrix}$$ is a upper-triangular matrix

Besides, because $e_j = R z_j$, we know that $\begin{bmatrix} e_1 & e_2 & \cdots & e_m \\ | & | & & | \end{bmatrix} = R \begin{bmatrix} z_1 & z_2 & \cdots & z_m \\ | & | & & | \end{bmatrix} = RZ$

$= I$

So, Z is the inverse matrix of $R \Rightarrow Z = R^{-1}$ and is also a upper-triangular matrix

Q.E.D. #

2. Textbook exercise 2.2

(a) $\left\| \sum_{i=1}^{2} x_i \right\|^2 = \|x_1 + x_2\|^2 = \|x_1\|^2 + 2\|x_1 \cdot x_2\| + \|x_2\|^2$

Because $x_1 \perp x_2 \Rightarrow \|x_1 \cdot x_2\| = 0$ $\quad = \|x_1\|^2 + \|x_2\|^2 = \sum_{i=1}^{2} \|x_i\|^2$

Q.E.D. #

(b) When $n=1$ $\quad \|x_1\|^2 = \|x_1\|^2$

When $n=2$ $\quad \|x_1 + x_2\|^2 = \|x_1 + x_2\|^2$

So, we suppose for $n=k$, $\left\| \sum_{i=1}^{k} x_i \right\|^2 = \sum_{i=1}^{k} \|x_i\|^2$

for $n = k+1$:

$$\left\| \sum_{i=1}^{k+1} x_i \right\|^2 = \left\| \sum_{i=1}^{k} x_i + x_{k+1} \right\|^2 = \left\| \sum_{i=1}^{k} x_i \right\|^2 + 2\left\| \sum_{i=1}^{k} x_i \cdot x_{k+1} \right\| + \|x_{k+1}\|^2$$

$= \left\| \sum_{i=1}^{k} x_i \right\|^2 + \|x_{k+1}\|^2$ ($\because$ $x_1 . x_2 .. x_{k+1}$ are all from a set of n orthogonal vectors $\{x_i\}$)

$= \sum_{i=1}^{k} \|x_i\|^2 + \|x_{k+1}\|^2$ ($\because$ The supposition above)

$= \sum_{i=1}^{k+1} \|x_i\|^2$

By induction, we prove that the Pythagorean theorem is true #

3)

3. $u, v \in \mathbb{R}^n$, $A = uv^* \in \mathbb{R}^{n\times n}$

$$u = \begin{bmatrix} u(1) \\ u(2) \\ \vdots \\ u(n) \end{bmatrix} \quad v = \begin{bmatrix} v(1) \\ v(2) \\ \vdots \\ v(n) \end{bmatrix} \qquad A = uv^* = \begin{bmatrix} u(1)v(1) & u(1)v(2) & \cdots & u(1)v(n) \\ u(2)v(1) & u(2)v(2) & & u(2)v(n) \\ \vdots & \vdots & & \vdots \\ u(n)v(1) & u(n)v(2) & \cdots & u(n)v(n) \end{bmatrix}$$

It's easy to find $\frac{a_1}{v(1)} = \frac{a_2}{v(2)} \cdots = \frac{a_n}{v(n)}$ (a_i is the ith column of A)

So, all columns of A are a vector u multiplied by a parameter, eg: $v(1) \cdot v(2)$, etc.

$\Rightarrow$ Range$(A) = [u]$ $\Rightarrow$ $\dim(\text{Range}(A)) = 1$ #

Range (A^*) = Row Space (A) $\Rightarrow$ $\dim(\text{Range}(A^*)) = \dim(\text{Row Space}(A))$

By Theorem : column rank = row rank = rank

$\Rightarrow$ $\dim(\text{Row space}(A)) = \dim(\text{Range}(A)) = 1$ #

Let x be a vector $\in \mathbb{R}^n = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(n) \end{bmatrix}$

$$Ax = \vec{0} = uv^* x = \begin{bmatrix} u(1)v(1)x(1) + u(1)v(2)x(2) + \cdots + u(1)v(n)x(n) \\ u(2)v(1)x(1) + u(2)v(2)x(2) + \cdots + u(2)v(n)x(n) \\ \vdots \\ u(n)v(1)x(1) + u(n)v(2)x(2) + \cdots + u(n)v(n)x(n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\because$ $\dim(\text{Column Space}(A)) + \dim(\text{Null Space}(A)) = \dim(\mathbb{R}^n) = n$

$\Rightarrow$ $\dim(\text{Null Space}(A)) = n - 1$ #

4. (a) $A \in \mathbb{R}^{20\times 10}$ & has linearly independent columns.

Range (A) = 10 linearly independent columns

$\Rightarrow$ $\dim(\text{Range}(A)) = 10$ #

$\dim(\text{Range}(A^*)) = \dim(\text{Row space}(A)) = \dim(\text{Range}(A)) = 10$ #

$\text{Null}(A) = \dim(\mathbb{R}^n) - \dim(\text{Range}(A)) = 10 - 10 = 0$ #

$\text{Null}(A^*) = \dim(\mathbb{R}^m) - \dim(\text{Range}(A^*)) = 20 - 10 = 10$ #

(b) Let $\{c_1, \cdots c_r\}$ be a basis of Range(A)

then let $\{x_1, \cdots, x_r\}$ be the vectors s.t. $Ax_i = c_i$ for $i = 1, 2, \cdots, r$

Let $\{n_1, \cdots, n_p\}$ be a basis of Null(A)

If $\sum_{i=1}^{r} a_i x_i + \sum_{j=1}^{p} b_j n_j = 0$ $\Rightarrow$ $A(\sum_{i=1}^{r} a_i x_i + \sum_{j=1}^{p} b_j n_j) = 0$ $\Rightarrow \sum_{i=1}^{r} a_i c_i \Rightarrow a_i = 0$ for $i = 1, 2, \cdots, r$

($a_i, b_j \in \mathbb{R}^1$) ($\because c_1 \cdot c_2 \cdots c_r$ are linearly independent)

$\Rightarrow A \sum_{i=1}^{r} a_i x_i + A \sum_{j=1}^{p} b_j n_j = 0$

$\Rightarrow A \sum_{i=1}^{r} a_i x_i = 0$ ($\because \sum_{j=1}^{p} b_j n_j \in \text{Null}(A)$)

Next page continues

4. (b) Conti.

$\because a_1 = a_2 = \cdots = a_r = 0$

$\Rightarrow \sum_{i=1} a_i x_i + \sum_{j=1} b_j n_j = 0 \Rightarrow \sum_{j=1} b_j n_j = 0 \Rightarrow b_1 = b_2 = b_3 = \cdots = b_p = 0$ ($\because n_1, n_2, \cdots n_p$ are linearly independent)

$\because$ Only when $a_1 = a_2 = \cdots = a_r = 0$ & $b_1 = b_2 = \cdots = b_p = 0$

s.t. $\sum_{i=1}^{r} a_i x_i + \sum_{j=1}^{p} b_j n_j = 0$

$\Rightarrow \{x_1, x_2, \cdots, x_r\}$ & $\{n_1, n_2, \cdots, n_p\}$ are linearly independent

Finally, for any arbitrary vector $v \in \mathbb{R}^n$

$$Av = \sum_{i=1}^{r} \gamma_i c_i \text{ for some } \gamma_i \in \mathbb{R}^1 \text{ for } i = 1, 2, \cdots, r$$

$$\Rightarrow Av = A \sum_{i=1}^{r} \gamma_i x_i \Rightarrow A\left(v - \sum_{i=1}^{r} \gamma_i x_i\right) = 0 \Rightarrow v - \sum_{i=1}^{r} \gamma_i x_i = \sum_{j=1}^{p} \beta_j n_j \text{ for some } \beta_j$$

$$\Rightarrow v = \sum_{i=1}^{r} \gamma_i x_i + \sum_{j=1}^{p} \beta_j n_j$$

$\because v$ is an arbitrary vector $\in \mathbb{R}^n$ & $\{x_1, x_2, \cdots, x_r\}$ & $\{n_1, n_2, \cdots, n_p\}$ are linearly independent

we know that $\{x_1, x_2, \cdots, x_r, n_1, n_2, \cdots, n_p\}$ span $\mathbb{R}^n$

$$\Rightarrow r + p = \dim(\mathbb{R}^n) = n = \dim(\text{Range}(A)) + \dim(\text{Null}(A))$$

#

(c) $m = n \Rightarrow A \cdot B \cdot C \in \mathbb{R}^{n \times n}$ & $B \cdot C$ are invertible

$\text{Null}(BAC) : \{x \in \mathbb{R}^n : BACx = 0\}$

$$BACx = 0 \Rightarrow B^{-1}BACC^{-1}x = B^{-1} 0 C^{-1} \Rightarrow Ax = 0$$

$\Rightarrow$ So we know that $\text{Null}(BAC) = \text{Null}(A)$

$\because \text{rank}(A) + \dim(\text{Null}(A)) = n = \dim(\text{Null}(BAC)) + \text{rank}(BAC)$

& $\text{Null}(A) = \text{Null}(BAC)$

$$\Rightarrow \text{rank}(A) = \text{rank}(BAC)$$

#

5.

Ans: $x \in V, y \in V \to \xi(x,y) \in \mathbb{C}$

For $V = \mathbb{C}^2$:

$$\xi(x,y) = x \cdot y = y^* x = \sum_{i=1}^{2} \bar{y}_i x_i \ , \quad x \in \mathbb{C}^2, y \in \mathbb{C}^2$$

(1) For $a \in \mathbb{C}$

$$\xi(ax, y) = y^*(ax) = \sum_{i=1}^{2} \bar{y}_i a x_i = a \sum_{i=1}^{2} \bar{y}_i x_i = a\xi(x,y) \quad -① $$

For $a, b \in \mathbb{C}$, $x_a, x_b \in \mathbb{C}^2$

$$\xi(ax_a + bx_b, y) = y^*(ax_a + bx_b) = \sum_{i=1}^{2} \bar{y}_i (ax_{ai} + bx_{bi})$$
$$= a\sum_{i=1}^{2} \bar{y}_i x_{ai} + b\sum_{i=1}^{2} \bar{y}_i x_{bi}$$
$$= a\xi(x_a, y) + b\xi(x_b, y) \quad -②$$

By ① & ②, we know that inner product is linear in its first argument x #

(2) $$\overline{\xi(x,y)} = \overline{\sum_{i=1}^{2} \bar{y}_i x_i} = \overline{(\bar{y}_1 x_1)} + \overline{(\bar{y}_2 x_2)} = \bar{x}_1 y_1 + \bar{x}_2 y_2 = \sum_{i=1}^{2} x_i^* y_i = \xi(y,x)$$ #

$\Rightarrow$ It's Hermitian

(3) $$\xi(x,x) = \sum_{i=1}^{2} \bar{x}_i x_i = \bar{x}_1 x_1 + \bar{x}_2 x_2 = \|x_1\|^2 + \|x_2\|^2 > 0, \forall x \neq 0$$

$\Rightarrow$ It's posieive definite #

6.

Ans: Let $g (\neq 0) \in \mathbb{R}^n$ & $\|g\|_2 \neq 1 \Rightarrow \sqrt{g(1)^2 + g(2)^2 + \cdots g(n)^2} \neq 1$

$$\Rightarrow g^* g = \|g\|_2^2 \neq 1$$

(a) If we consider for any vector $x \neq 0$ s.t. $(I - gg^*)x = 0$

$\Rightarrow (I - gg^*)$ must be zero $\Rightarrow I - gg^* = 0 \Rightarrow I = gg^*$

$$g = \begin{bmatrix} g(1) \\ g(2) \\ \vdots \\ g(n) \end{bmatrix} \Rightarrow I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & 1 & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & 0 & 1 \end{bmatrix} = gg^* = \begin{bmatrix} g(1)^2 & g(1)g(2) & \cdots & g(1)g(n) \\ g(2)g(1) & g(2)^2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ g(n)g(1) & g(n)g(2) & \cdots & g(n)^2 \end{bmatrix}$$

$$\Rightarrow g(i)g(j) = \begin{cases} 1 & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases}$$ $\Rightarrow$ This is impossible to find g satisfying this condition

$\Rightarrow (I - gg^*)x = 0$ only when $x = 0 \Rightarrow Null(I - gg^*) = 0$

$\Rightarrow dim(Null(I - gg^*)) = 0$

$$rank(I - gg^*) = n - 0 = n$$ #

⑥

6.(b)

Ans

$$I - qq^* = \begin{bmatrix} | & | & & | \\ e_1 - q(1)q & e_2 - q(2)q & \cdots & e_n - q(n)q \\ | & | & & | \end{bmatrix}$$

We can express the columns of $I - qq^*$ as $e_j - q(j)q$; $j = 1, 2, \cdots, n$

$$(e_j - q(j)q) \cdot q = (e_j - q(j)q)^* q = q(j) - q(j)\|q\|_2^2$$
$$= (1 - \|q\|_2^2)\, q(j)$$
$$\because \|q\| \neq 1 \;\&\; q(\neq 0)$$
$$\therefore (1 - \|q\|_2^2)\, q(j) \neq 0$$

So q isn't orthogonal to the columns of $I - qq^*$

$\Rightarrow$ $I - qq^*$ and q are linearly dependent #

(c) $x \in \mathbb{R}^n$

$$(e_j - q(j)q) \cdot (q^* x q) = (e_j - q(j)q)^* (q^* x q)$$
$$= q^* x \, (e_j - q(j)q)^* q \quad (\because q^* x \in \mathbb{R}^1)$$
$$= q^* x \, (q(j) - q(j)\|q\|_2^2)$$
$$= q^* x \, q(j) \, (1 - \|q\|_2^2)$$
$$\neq 0 \quad (\because \|q\|_2^2 \neq 1)$$

$\Rightarrow$ $(q^* x) q$ isn't orthogonal to the columns of $I - qq^*$

#