PAPER

Compactness of Finite Union of Regular Patterns and Regular Patterns without Adjacent Variables

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SUMMARY A regular pattern is a string consisting of constant symbols and distinct variable symbols. The language $L(p)$ of a regular pattern p is the set of all constant strings obtained by replacing all variable symbols in the regular pattern p with constant strings. $\mathcal{RP}^k$ denotes the class of all sets consisting at most k ($k \geq 2$) regular patterns. For sets of regular patterns P and Q which are in the class $\mathcal{RP}^k$, we write $P \sqsubseteq Q$ if for any regular pattern $p \in P$ there exits a regular pattern $q \in Q$ that is a generalization of p. In 1998 Sato et al.[1] showed that the finite set $S_2(P)$ of symbol strings is a characteristic set of $L(P) = \bigcup_{p \in P} L(p)$, where $S_2(P)$ is obtained from $P \in \mathcal{RP}^k$ by substituting variables with symbol strings of at most length 2. Sato et al.[1] also showed that $\mathcal{RP}^k$ has compactness with respect to containment, if the number of constant symbols is greater than or equal to $2k - 1$. In this paper, we check the results of Sato et al.[1] and correct the error of the proof of their theorem. Further, we consider the set $\mathcal{RP}_{NAV}$ of all non-adjacent regular patterns, which are regular patterns without adjacent variables, and show that the set $S_2(P)$ obtained from a set P in the class $\mathcal{RP}_{NAV}^k$ of at most k ($k \geq 1$) non-adjacent regular patterns is a characteristic set of $L(P)$. Further we show that $\mathcal{RP}_{NAV}^k$ has compactness with respect to containment if the number of constant symbols is greater than or equal to $k+2$. Thus we show that we can design an efficient learning algorithm of a finite union of pattern languages of non-adjacent regular patterns with the number of constant symbols which is smaller than the case of regular patterns.

key words: *Regular Pattern Language, Compactness*

1. Introduction

A pattern is a string consisting of constant symbols and variable symbols. For example, we consider constant symbols a, b, c and variable symbols x, y, then $axbxcy$ is a pattern. $\mathcal{P}$ denotes the set of all patterns. For a pattern $p \in \mathcal{P}$, *the pattern language generated by p*, denoted by $L(p)$, or simply called a pattern language, is the set of all strings obtained by replacing all variable symbols with constant symbol strings, where the same variable symbol is replaced by the same constant string. For example the pattern language $L(axbxcy)$ generated by the above pattern $axbxcy$ denotes $\{aubucw \mid$ u and w are constant strings that are not $\varepsilon\}$. A pattern where each variable symbol appears at most once is called a *regular pattern*. For example, a pattern $axbxcy$ is not a regular pattern, but a pattern $axbzcy$ with variable symbols x, y, z is a regular pattern. $\mathcal{RP}$ denotes the set of

all regular patterns. If a pattern $p \in \mathcal{P}$ is obtained from a pattern $q \in \mathcal{P}$ by replacing variable symbols in q with patterns, we say that q is a *generalization* of p and denote this by $p \preceq q$. For example, a pattern $q = axz$ is a generalization of a pattern $p = axbxcy$, because p is obtained from q by replacing the variable z in q with a pattern $bxcy$. So we write $p \preceq q$. For patterns $p, q \in \mathcal{P}$, it is obvious that $p \preceq q$ implies $L(p) \subseteq L(q)$. But, the converse, that is, the statement that $L(p) \subseteq L(q)$ implies $p \preceq q$ does not always hold. With respect to this statement, Mukouchi[2] showed that if the number of constant symbols is greater than or equal to 3, for any regular pattern $p, q \in \mathcal{RP}$, $L(p) \subseteq L(q)$ implies $p \preceq q$.

We denote by $\mathcal{RP}^+$ the class of all non-empty finite sets of regular patterns and by $\mathcal{RP}^k$ the class of at most k ($k \geq 2$) regular patterns. For a set of regular patterns $P \in \mathcal{RP}^k$ we define $L(P) = \bigcup_{p \in P} L(p)$ and consider the class $\mathcal{RPL}^k$ of regular pattern languages of $\mathcal{RP}^k$, where $\mathcal{RPL}^k = \{L(P) \mid P \in \mathcal{RP}^k\}$. Let $P, Q \in \mathcal{RP}^k$ and $Q = \{q_1, \ldots, q_k\}$. We denote by $P \sqsubseteq Q$ that for any regular pattern $p \in P$ there exists a regular pattern q_i such that $p \preceq q_i$ holds. From definition, it is obvious that $P \sqsubseteq Q$ implies $L(P) \subseteq L(Q)$. Then Sato et al.[1] shows that if $k \geq 3$ and the number of constant symbols is $2k - 1$ then the finite set $S_2(P)$ of constant symbols obtained from $P \in \mathcal{RP}^k$ by substituting variable symbols with constant strings of at most 2 length is a characteristic set of $L(P)$, that is, for any regular pattern language $L' \in \mathcal{RPL}^k$, $S_2(P) \subseteq L'$ implies $L(P) \subseteq L'$. Thus they shows that the following three statements: (i) $S_2(P) \subseteq L(Q)$, (ii) $P \sqsubseteq Q$ and (iii) $L(P) \subseteq L(Q)$ are equivalent. But the Lemma14 [1], which is used in this results, contains an error. In this paper we correct this lemma and give a correct proof showing the equivalence of the three statements shown in [1]. Sato et al.[1] shows that $\mathcal{RP}^k$ has compactness with respect to containment if the number of constant symbols is greater than or equal to $2k - 1$. On the contrary to this result, we show that the set $S_2(P)$ obtained from a set P in the class $\mathcal{RP}_{NAV}^k$ of at most k ($k \geq 1$) regular patterns having non-adjacent variables is a characteristic set of $L(P)$. Further, we show that if the number of constant symbols is greater than or equal to $k + 2$ then $\mathcal{RP}_{NAV}^k$ has compactness with respect to containment. In Table 1 we summarize the all results in this paper.

The results of this paper suggest efficient learning algorithms for the sets of regular patterns representing finite

Table 1 The conditions of the number of constant symbols with respect to the compactness of inclusion

k	2	≥ 3
$\mathcal{RP}^k$	≥ 4	$\geq 2k-1$
$\mathcal{RP}_{NAV}^k$		$\geq k+2$

unions of languages and the sets of regular patterns having non-adjacent variables.

This paper is organized as follows. In Sect.2 as preparations, we give definitions of pattern languages, regular pattern languages and compactness, and then introduce the results of Sato et al.[1]. In Sect.3, we show that $S_2(P)$ is a characteristic set of $L(P)$ in $\mathcal{RPL}^k$ and $\mathcal{RP}^k$ has compactness with respect to containment. In Sect.4, we propose regular patterns having non-adjacent variables, show that $S_2(P)$ obtained from a set P in $\mathcal{RP}_{NAV}^k$ is a characteristic set of $L(P)$, and $\mathcal{RP}_{NAV}^k$ has compactness with respect to containment.

2. Preliminaries

Let Σ be a non-empty finite set of constant symbols. Let X be an infinite set of variable symbols such that $\Sigma \cap X = \emptyset$ holds. Then, a *string* on $\Sigma \cup X$ is a sequence of symbols in $\Sigma \cup X$. Particularly, the string having no symbol is called the *empty string* and is denoted by ε. We denote by $(\Sigma \cup X)^*$ the set of all strings on $\Sigma \cup X$ and by $(\Sigma \cup X)^+$ the set of all strings on $\Sigma \cup X$ except ε, i.e., $(\Sigma \cup X)^+ = (\Sigma \cup X)^* \setminus \{\varepsilon\}$. A *pattern* on $\Sigma \cup X$ is a string in $(\Sigma \cup X)^*$. Note that the empty string ε is a pattern on $\Sigma \cup X$. A pattern p is said to be *regular* if each variable symbol appears at most once in p. The length of p, denote by $|p|$, is the number of symbols in p. Note that $|\varepsilon| = 0$ holds. The set of all patterns and regular patterns are denoted by $\mathcal{P}$ and $\mathcal{RP}$, respectively. For a set S, we denote by $\sharp S$ the number of elements in S. Let p, q be strings. If p and q are equal as strings, we denote it by $p = q$. We denote by $p \cdot q$ the string obtained from p and q by concatenating q after p. Note that for a string p and the empty string ε, $p \cdot \varepsilon = \varepsilon \cdot p = p$. A substitution θ is a mapping from $(\Sigma \cup X)^*$ to $(\Sigma \cup X)^*$ such that (1) θ is a homomorphism with respect to string concatenation, i.e., $\theta(p \cdot q) = \theta(p) \cdot \theta(q)$ holds for patterns p and q, (2) $\theta(\varepsilon) = \varepsilon$ holds, (3) for each constant symbol $a \in \Sigma$, $\theta(a) = a$ holds, and (4) for each variable symbol $x \in X$, $|\theta(x)| \geq 1$ holds. Let $x_1, \ldots, x_n$ are variable symbols and $p_1, \ldots, p_n$ non-empty patterns. The notation $\{x_1 := p_1, \ldots, x_n := p_n\}$ denotes a substitution that replaces each variable symbol x_i with a non-empty pattern p_i for $i \in \{1, \ldots, n\}$. For a pattern p and a substitution $\theta = \{x_1 := p_1, \ldots, x_n := p_n\}$, we denote by $p\theta$ a new pattern obtained from p by replacing variable symbols $x_1, \ldots, x_n$ in p with patterns $p_1, \ldots, p_n$ according to θ, respectively. For a pattern p and q, the pattern q is a *generalization* of p, or p is an *instance* of q, denoted by $p \preceq q$, if there exists a substitution θ such that $p = q\theta$ holds. If $p \preceq q$ and $p \succeq q$ hold, we denote it by $p \equiv q$. The notation $p \equiv q$ means that p and q are equal as strings except for variable symbols. For a pattern p, the *pattern language*

of p, denoted by $L(p)$, is the set $\{w \in \Sigma^* \mid w \preceq p\}$. For patterns p and q, it is clear that $L(p) = L(q)$ if $p \equiv q$, and $L(p) \subseteq L(q)$ if $p \preceq q$. Note that $L(\varepsilon) = \{\varepsilon\}$. In particular, if p is a regular pattern, we say that $L(p)$ is a *regular pattern language*. The set of all pattern languages and regular patterns languages are denoted by $\mathcal{PL}$ and $\mathcal{RPL}$, respectively.

Lemma 1 (Mukouchi[2]): Let p and q be regular patterns. Then $p \preceq q$ if and only if $L(p) \subseteq L(q)$.

Next, we consider unions of pattern languages. The class of all non-empty finite subsets of $\mathcal{P}$ is denoted by $\mathcal{P}^+$, i.e., $\mathcal{P}^+ = \{P \subseteq \mathcal{P} \mid 0 < \sharp P < \infty\}$. For a positive integer k ($k > 0$), the class of non-empty sets consisting of at most k patterns, i.e., $\mathcal{P}^k = \{P \subseteq \mathcal{P} \mid 0 < \sharp P \leq k\}$. We denote by $\mathcal{PL}^k$ the class of unions of at most k pattern languages, i.e., $\mathcal{PL}^k = \{L(P) \mid P \in \mathcal{P}^k\}$, where $L(P) = \bigcup_{p \in P} L(p)$. In a similar way, we also define $\mathcal{RP}^+$, $\mathcal{RP}^k$ and $\mathcal{RPL}^k$. For P, Q in $\mathcal{P}^+$, the notation $P \sqsubseteq Q$ means that for any $p \in P$ there is a pattern $q \in Q$ such that $p \preceq q$ holds. It is clear that $P \sqsubseteq Q$ implies $L(P) \subseteq L(Q)$. However, the converse is not valid in general.

Definition 1: Let C be a class of languages, L a language in C and S a non-empty finite subset of L. We say that S is a *characteristic* set of L within C if for any $L' \in C$, $S \subseteq L'$ implies $L \subseteq L'$.

Let n be a positive integer and p a regular pattern. We denote by $S_n(p)$ the set of all strings in Σ^* obtained by replacing all variable symbols in p with strings in Σ^+ of length at most n. Moreover, for a positive integer n and a set $P \in \mathcal{RP}^+$, let $S_n(P) = \bigcup_{p \in P} S_n(p)$. It is clear that $S_n(P) \subseteq S_{n+1}(P) \subseteq L(P)$ for any positive integer n.

Theorem 1 (Sato et al.[1]): Let k be a positive integer and $P \in \mathcal{RP}^k$. Then, there exists a positive integer n such that $S_n(P)$ is a characteristic set of $L(P)$ within $\mathcal{RPL}^k$.

Sato et al.[1] showed that 2 is sufficient for the number n in the theorem above, under the assumption that the number of constants is not less than $2k - 1$. Hence, in this paper, we consider a characteristic set $S_2(P)$ of $L(P)$ within $\mathcal{RPL}^k$.

Theorem 2 (Sato et al.[1]): Let p, q, p_1, p_2, q_1, q_2, q_3 be regular patterns and x a variable symbol with $p = p_1 x p_2$ and $q = q_1 q_2 q_3$. Then $p \preceq q$ if the following three conditions are holds:

(i) $p_1 \preceq q_1 q_2$, (ii) $p_2 \preceq q_2 q_3$,
(iii) q_2 contains at least one variable symbol.

Lemma 2 (Sato et al.[1]): Suppose $\sharp \Sigma \geq 3$. Let p, p_1, p_2, q be regular patterns and x a variable symbol with $p = p_1 x p_2$. Let a, b and c be mutually distinct constant symbols. If $p_1 a p_2 \preceq q$, $p_1 b p_2 \preceq q$ and $p_1 c p_2 \preceq q$, then $p \preceq q$ holds.

Lemma 3 (Sato et al.[1]): Suppose $\sharp \Sigma \geq 3$. Let p_1, p_2, q_1, q_2 be regular patterns and x a variable symbol. Let a, b be constant symbols with $a \neq b$ and w a string in Σ^*. Let $p = p_1 A w x w B p_2$ and $q = q_1 A w B q_2$ be regular patterns that

satisfy the following three conditions:

(i) $p_1 \preceq q_1$,
(ii) $p_2 \preceq q_2$,
(iii) $(A, B) \in \{(a, b), (b, a)\}$.

If $p\{x := a\} \preceq q$ and $p\{x := b\} \preceq q$, then we have $p \not\preceq q$.

From Lemma 2, the following lemma holds.

Theorem 3 (Sato et al.[1]): Let $\sharp\Sigma \geq 2k + 1$, $P \in \mathcal{RP}^+$ and $Q \in \mathcal{RP}^k$. Then, the following (i), (ii) and (iii) are equivalent:

(i) $S_1(P) \subseteq L(Q)$, (ii) $P \sqsubseteq Q$, (iii) $L(P) \subseteq L(Q)$.

Example 1 in [1] is given as a counter-example of Theorem 3.

From Theorem 3, we have the following corollary.

Corollary 1 (Sato et al.[1]): Let $\sharp\Sigma \geq 3$ and p, q regular patterns. Then, the following (i), (ii) and (iii) are equivalent:

(i) $S_1(p) \subseteq L(q)$, (ii) $p \preceq q$, (iii) $L(p) \subseteq L(q)$.

3. Compactness for Sets of Regular Patterns

In this section, we define the compactness of sets of regular patterns, formally. Then, if $\sharp\Sigma \geq 2k - 1$ holds, we show that $\mathcal{RP}^k$ has compactness with respect to the containment.

Definition 2: Let C be a subset of $\mathcal{RP}^+$. For any regular pattern $p \in \mathcal{RP}$ and any set $Q \in C$, the set C said to have *compactness with respect to containment* if there exists a regular pattern $q \in Q$ such that $L(p) \subseteq L(q)$ holds if $L(p) \subseteq L(Q)$ holds.

Lemma 4 (Sato et al.[1]): Let Σ be an alphabet with $\sharp\Sigma \geq 3$ and p, q regular patterns on Σ. Let D be the set of either (i) or (ii) of regular patterns on Σ below: Assume that $a \neq b$ and that a variable symbol y does not appear in p.

(i) $\{ay, by\}$ (ii) $\{ya, yb\}$.

Then, if $p\{x := r\} \preceq q$ for all $r \in D$, then $p\{x := xy\} \preceq q$.

Proof. It is obvious if no variable symbol appears in p. Therefore, let $p = p_1 x p_2$, where p_1, p_2 are regular patterns and x is a variable symbol. We assume that $p\{x := xy\} \not\preceq q$ in order to derive the contradictions.
(i) Case of $D = \{ay, by\}$ $(a \neq b)$:
Since $p\{x := xy\} \not\preceq q$, $p_1 a y p_2 \preceq q$ and $p_1 b y p_2 \preceq q$, there exist regular patterns q_1, q_2 on Σ such that $q = q_1 a y_1 w b y_2 q_2$ or $q = q_1 b y_1 w a y_2 q_2$ for some variable symbols y_1, y_2 $(y_1 \neq y_2)$ and a constant string w $(|w| \geq 0)$ from Theorem 2. When $q = q_1 a y_1 w b y_2 q_2$ holds, the following four conditions (1), (2), (1'), (2') holds:

(1) $p_1 \preceq q_1$
(1') $p_2 \preceq w b y_2 q_2$ or $p_2 \preceq y' w b y_2 q_2$ $(y' \in X)$
(2) $p_1 \preceq q_1 a y_1 w$

(2') $p_2 \preceq q_2$ or $p_2 \preceq y'' q_2$ $(y'' \in X)$

From the above condition (2), there exist regular patterns p'_1, p''_1 such that $p_1 = p'_1 p''_1$, $p'_1 \preceq q_1 a$ and $p''_1 \preceq y_1 w$ hold. Therefore, since $p = p_1 x p_2 = p'_1 p''_1 x p_2$, if $p_2 \preceq w b y_2 q_2$ holds, $p \preceq q_1 a p''_1 x w b y_2 q_2 \equiv q\{y_1 := p''_1 x\}$ holds. Otherwise $p_2 \preceq y' w b y_2 q_2$, $p \preceq q_1 a p''_1 x y' w b y_2 q_2 = q\{y_1 := p''_1 x y'\}$ holds. Hence, $p \preceq q$ holds. This contradicts the assumption.
(ii) Case of $D = \{ya, yb\}$ $(a \neq b)$: By reversing the strings of p and q, we can prove that $p\{x := xy\} \preceq q$ holds, in a similar way as (i). $\square$

Proposition 1: Let w be a string of constant symbols in Σ and a, b constant symbols in Σ. If

$$wa = bw \tag{1}$$

holds, then $a = b$ holds.

Proof. Trivial. $\square$

Proposition 2: Let w be a string of constant symbols in Σ and a, b, c, d constant symbols in Σ. If

$$wda = bcw \tag{2}$$

holds, then $(b, c) \in \{(a, d), (d, a)\}$ holds.

Proof. We will prove this proposition by induction on $|w|$. For $i = 1, \ldots, |w|$, we refer to the i-th symbol of w as $w[i-1]$.

- $|w| = 0$: Directly from Eq. (2), $(b, c) = (d, a)$ holds.
- $|w| = 1$: From $w[0]da = bcw[0]$, we have $w[0] = b$, $d = c$, and $a = w[0]$. Thus, $(b, c) = (a, d)$ holds.
- $|w| = 2$: From $w[0]w[1]da = bcw[0]w[1]$, we have $w[0] = b$, $w[1] = c$, $d = w[0]$, $a = w[1]$. Thus, $(b, c) = (d, a)$ holds.
- $|w| = 3$: From $w[0]w[1]w[2]da = bcw[0]w[1]w[2]$, we have $w[0] = b$, $w[1] = c$, $w[2] = w[0]$, $d = w[1]$, $a = w[2]$. Thus, $(b, c) = (a, d)$ holds.
- $|w| \geq 4$: We assume that for any string u with $0 \leq |u| < n$, if $uda = bcu$ holds, $(b, c) \in \{(a, d), (d, a)\}$ holds. Since the string w has a prefix bc and a suffix da, there exists a string u with $|u| = |w| - 4 < |w|$ such that $w = bcuda$ holds. Since $wda = bcw$, we have $bcudada = bcbcuda$, and then $uda = bcu$. Thus, from the assumption, we get $(b, c) \in \{(a, d), (d, a)\}$.

From the above, we conclude that if $wda = bcw$ holds, then $(b, c) \in \{(a, d), (d, a)\}$ holds. $\square$

The conclusion from Proposition 2 shows that $(a, d) \in \{(b, c), (c, b)\}$. Therefore, if the equation $daw = wbd$ holds, we arrive at the same conclusion.

Proposition 3: Let w, w' be strings of constant symbols in Σ and a, b, c, d constant symbols in Σ. If

$$wdaw' = w'bcw \tag{3}$$

holds, then $(b, c) \in \{(a, d), (d, a)\}$ holds.

Proof. We will prove this proposition by an induction on $|w| + |w'|$. Without loss of generality, we assume that $|w| \geq |w'|$ because, if $|w| > |w'|$, we arrive at the same conclusion that $(a, d) \in \{(b, c), (c, b)\}$ holds.

- $|w| \geq 0$ and $|w'| = 0$: Eq. (3) reduces to $wda = bcw$. By Proposition 2, $(b, c) \in \{(a, d), (d, a)\}$ holds.

We assume that for constant strings u and u' with $|u| + |u'| < |w| + |w'|$, if $udau' = u'bcu$ holds, then $(b, c) \in \{(a, d), (d, a)\}$ holdss. We divide the relations between $|w|$ and $|w'|$ into the following four cases:

- $0 < |w'| \leq |w| \leq |w'| + 1$: When either $|w| = |w'|$ or $|w| = |w'| + 1$, Eq. (3) is illustrated in Figs. 1 and 2, respectively. If $|w| = |w'|$, $(b, c) = (d, a)$ holds. If $|w| = |w'| + 1$, $a = c$ and $w = w'b = dw'$ hold. From Proposition 1, we deduce that $b = d$. Therefore, $(b, c) \in \{(a, d), (d, a)\}$ holds.
- $|w'| + 2 \leq |w| \leq 2|w'| - 1$: In Eq. 3, since $|wdaw'| = |w'bcw| = |w| + |w'| + 2$, a suffix of w overlaps with a prefix of w, as illustrated in Fig. 3. That is, there exists a constant string u of length $2|w| - (|w| + |w'| + 2) = |w| - |w'| - 2$ such that u is both a prefix and a suffix of w. Since uda has a length of $|w| - |w'|$, it is also a prefix of w. Similarly, bcu is a suffix of w. Because $|w| - (|uda| + |bcu|) = 2|w| - |w'| \geq 1$, there exist a constant string u' of length $2|w'| - |w|$ such that $w = udavbcu$ holds. Since w' is a suffix of w and $|u'bcu| = (2|w'| - |w|) + 2 + (|w| - |w'| - 2) = |w'|$, we have $w' = u'bcu$. Similarly, $w' = udau'$. Thus, we derive the equation $u'bcu = udau'$. Since $|u| = |w| - |w'| - 2 \leq |w| - 3 < |w|$ and $|u'| = 2|w'| - |w| < |w|$, i.e., $|u| + |u'| < |w| + |w'|$, the induction hypothesis on $|u| + |u'|$ implies that $(b, c) \in \{(a, d), (d, a)\}$ holds.
- $2|w'| \leq |w| \leq 2|w'| + 3$: When $|w| = 2|w'|$, it is straightforward to observe that $w = w'w'$. Therefore, $w'da = bcw'$ holds, as illustrated in Fig. 4. From Proposition 2, $(b, c) \in \{(a, d), (d, a)\}$ holds. When $|w| = 2|w'| + i$ ($i = 1, 2, 3$), Eq. (3) is depicted in Figs. 5, 6, and 7, respectively. When $|w| = 2|w'| + 2$, it is clear that $(b, c) = (d, a)$. When $|w| = 2|w'| + 1$ and $|w| = 2|w'| + 3$, Proposition 1 implies that $(b, c) = (a, d)$ holds.
- $2|w'| + 4 \leq |w|$: Since the strings $w'bc$ and adw' are a prefix and a suffix of w, respectively, and $|w'bc| + |adw'| = 2|w'| + 4$, there exists a string u with $|u| \geq 0$ such that $w = w'bcudaw'$ holds. From Eq. (3), $w'bcudaw'daw' = w'bcw'bcudaw'$, i.e., $udaw' = w'bcu$ holds, as illustrated in Fig. 8. Let $u' = w'$. Since $|u| + |u'| = |w| - (2|w'| + 4) + |w'| < |w| + |w'|$, the induction hypothesis on $|u| + |u'|$ implies that $(b, c) \in \{(a, d), (d, a)\}$ holds.

From the above, we conclude that if $wdaw' = w'bcw$, then $(b, c) \in \{(a, d), (d, a)\}$ holds. $\square$

Lemma 5: Let Σ be an alphabet with $\sharp\Sigma \geq 3$ and p, q regular patterns on $\Sigma \cup X$. Let D be the following set of
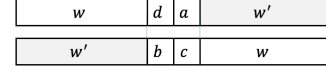
Fig. 1 Case $|w| = |w'|$ in Proposition 3
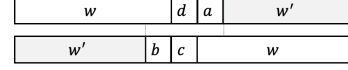
Fig. 2 Case $|w| = |w'| + 1$ in Proposition 3
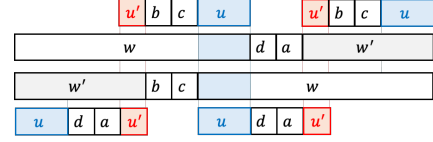
Fig. 3 Case $|w'| + 2 \leq |w| \leq 2|w'| - 1$ in Proposition 3
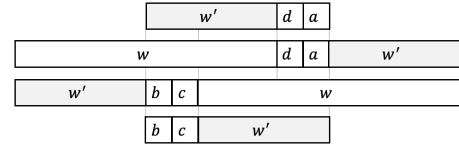
Fig. 4 Case $|w| = 2|w'|$ in Proposition 3
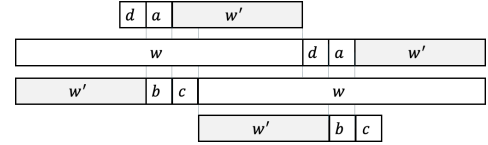
Fig. 5 Case $|w| = 2|w'| + 1$ in Proposition 3

Fig. 6 Case $|w| = 2|w'| + 2$ in Proposition 3
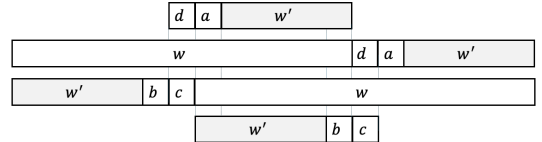
Fig. 7 Case $|w| = 2|w'| + 3$ in Proposition 3

regular patterns on $\Sigma \cup X$. Then, if $p\{x := r\} \preceq q$ for all $r \in D$, then $p\{x := xy\} \preceq q$:

$$D = \{ya, bc, dy\} \ (b \notin \{a, d\} \text{ and } c \notin \{a, d\}).$$

Proof. If no variable symbol appears in p, the statement holds trivially. Thus, for a variable symbol $x \in X$, let $p = p_1 x p_2$, where each p_i ($i = 1, 2$) is either an empty symbol or a regular pattern on $\Sigma \cup X$. We assume that $p\{x := xy\} \npreceq q$ in order to derive a contradiction.

Since $p\{x := r\} \preceq q$ for all $r \in D$, there are three
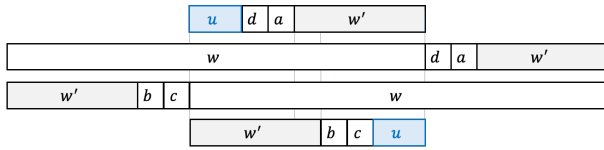
Fig. 8 Case $2|w'| + 4 \le |w|$ in Proposition 3

strings of length 2 in q corresponding to ya, bc, dy. Note that the three strings may appear with partial overlaps. The symbols in D correspond to either a variable or a constant symbol in q. Let y_1, y_2, y_3 be variable symbols appearing in q. The strings ya and dy must correspond to the strings y_1a and dy_2 in q, respectively. There are three possible strings in q that correspond to bc in $p\{x := bc\}$, as follows:

$$\text{(a)} \quad bc, \quad \text{(b)} \quad y_3c, \quad \text{(c)} \quad by_3.$$

Suppose that there exists (b) y_3c in q that corresponds to bc in $p\{x := bc\}$, i.e., there exist q_1 and q_2, each of which is an empty string or a regular pattern on $\Sigma \cup X$, such that:

(1) $p_1bcp_2 \preceq q_1y_3cq_2$,
(2) either $p_1 \preceq q_1$ or $p_1 \preceq q_1y_3'$ for some variable symbol $y_3' \in X$, and
(3) $p_2 \preceq q_2$.

In this case, it is straightforward to see that $p\{x := yc\} = p_1ycp_2 \preceq q_1y_3cq_2$ also holds. Thus, both $p\{x := ya\} \preceq q$ and $p\{x := yc\} \preceq q$ hold. Since $c \ne a$, by (ii) in Lemma 4, $p\{x := xy\} \preceq q$ also holds. This contradicts the assumption. Similarly, the case (c) leads to a contradiction by (i) in Lemma 4. Therefore, in the following, we consider only case (a).

Since $p\{x := xy\} \not\preceq q$ and the condition $b \notin \{a, d\}$ and $c \notin \{a, d\}$ hold, the regular pattern q can be expressed in one of the following forms: Let y_1, y_2 be distinct variable symbols in X and q_1, q_2, w, w' be either an empty string or a regular pattern on $\Sigma \cup X$.

(a1) $q = q_1AwBw'Cq_2$, where $\{A, B, C\} = \{y_1a, bc, dy_2\}$,
(a2) $q = q_1AwBq_2$, where $\{A, B\} = \{dy_1a, bc\}$,
(a3) $q = q_1AwBq_2$, where $\{A, B\} = \{y_1ay_2, bc\}$ $(a = d)$.

First, we consider case (a1).

Claim 1. $B \notin \{y_1a, dy_2\}$.

Proof of Claim 1. Suppose that $(A, B, C) = (dy_2, y_1a, bc)$. For some $y_1', y_2' \in X$, the following conditions must be satisfied:

(1) $p_1 \preceq q_1$,
(1') $p_2 \preceq wy_1aw'bcq_2$ or $p_2 \preceq y_2'wy_1aw'bcq_2$,
(2) $p_1 \preceq q_1dy_2w$ or $p_1 \preceq q_1dy_2wy_1'$,
(2') $p_2 \preceq w'bcq_2$,
(3) $p_1 \preceq q_1dy_2wy_1aw'$,
(3') $p_2 \preceq q_2$.

When $p_2 \preceq wy_1aw'bcq_2$ in (1') holds, let $q_1' = q_1dy_2$, $q_2' = wy_1aw'$, $q_3' = bcq_2$. Since $p_1 \preceq q_1dy_2wy_1aw'$ holds from (3), both $p_1 \preceq q_1'q_2'$ and $p_2 \preceq q_2'q_3'$ hold, and q_2' contains a

variable symbol. When $p_2 \preceq y_2'wy_1aw'bcq_2$ in (1') holds, let $q_1' = q_1d$, $q_2' = y_2wy_1aw'$, $q_3' = bcq_2$. Since $p_1 \preceq q_1dy_2wy_1aw'$ holds from (3), both $p_1 \preceq q_1'q_2'$ and $p_2 \preceq q_2'q_3'$ hold, and q_2' contains a variable symbol. In both cases, by Theorem 2, $p \preceq q$ holds. This contradicts the assumption that $p\{x := xy\} \not\preceq q$.

Similarly, we can show that any case where $(A, B, C) = (y_1a, dy_2, bc)$, (bc, y_1a, dy_2), or (bc, dy_2, y_1a) also contradicts the assumption. Therefore, we have $B \notin \{y_1a, dy_2\}$. (*End of Proof of Claim*)

Claim 2. $(A, B, C) = (y_1a, bc, dy_2)$.

Proof of Claim 2. From *Claim* 1, we have $B = bc$. Suppose that $(A, B, C) = (dy_2, bc, y_1a)$, i.e., $q = q_1dy_2wbcw'y_1aq_2$ holds. Then, the following conditions must be satisfied: for $y_1', y_2' \in X$:

(1) $p_1 \preceq q_1$,
(1') $p_2 \preceq wbcw'y_1aq_2$ or $p_2 \preceq y_2'wbcw'y_1aq_2$,
(2) $p_1 \preceq q_1dy_2w$,
(2') $p_2 \preceq w'y_1aq_2$,
(3) $p_1 \preceq q_1dy_2wbcw'$ or $p_1 \preceq q_1dy_2wbcw'y_1'$,
(3') $p_2 \preceq q_2$.

From $p_1 \preceq q_1dy_2w$ in (2), p_1 is expressed as $p_1'p_1''$ for some p_1' and p_1'', where $p_1' \preceq q_1d$ and $p_1'' \preceq y_2w$. When $p_2 \preceq wbcw'y_1aq_2$ in (1'), we have $p = p_1xp_2 = p_1'p_1''xp_2 \preceq q_1dp_1''xwbcw'y_1aq_2 = q\{y_2 := p_1''x\}$. Thus, $p\{x := xy\} \preceq q\{y_2 := p_1''xy\}$ holds. This contradicts the assumption that $p\{x := xy\} \not\preceq q$. When $p_2 \preceq y_2'wbcw'y_1aq_2$ in (1'), we similarly have $p = p_1xp_2 = p_1'p_1''xp_2 \preceq q_1dp_1''xy_2'wbcw'y_1aq_2 = q\{y_2 := p_1''xy_2'\}$. Thus, $p\{x := xy\} \preceq q\{y_2 := p_1''xyy_2'\}$ holds. This also contradicts the assumption. Therefore, we conclude that $(A, B, C) = (y_1a, bc, dy_2)$. (*End of Proof of Claim*)

From *Claim* 2, The regular pattern q is expressed as $q_1y_1awbcw'dy_2q_2$, where $b \notin \{a, d\}$ and $c \notin \{a, d\}$. If $p\{x := xy\} \not\preceq q$ holds, the following conditions must be satisfied: For $y_1', y_2' \in X$,

(1) $p_1 \preceq q_1$ or $p_1 \preceq q_1y_1'$,
(1') $p_2 \preceq wbcw'dy_2q_2$,
(2) $p_1 \preceq q_1y_1aw$,
(2') $p_2 \preceq w'dy_2q_2$,
(3) $p_1 \preceq q_1y_1awbcw'$,
(3') $p_2 \preceq q_2$ or $p_2 \preceq y_2'q_2$.

Claim 3. w and w' contain no variable symbols.

Proof of Claim 3. Let $q_1' = q_1y_1a$, $q_2' = wbcw'$, and $q_3' = dy_2q_2$. From (1') and (3), $p_1 \preceq q_1'q_2'$ and $p_2 \preceq q_2'q_3'$. If q_2' contains a variable symbol, then by Theorem 2, $p \preceq q$ holds. This contradicts the assumption. Therefore, w and w' contain no variable symbols. (*End of Proof of Claim*)

From *Claim* 3, w and w' are strings consisting of symbols in Σ. From (1') and (2'), $wbcw'd$ and $w'd$ are prefixes of p_2, and from (2) and (3), $awbcw'$ and aw are suffixes of p_1. From these facts:

- $|w| = |w'|$: $b = d$ and $a = c$ hold.
- $|w| = |w'| + 1$: $a = b$ holds.
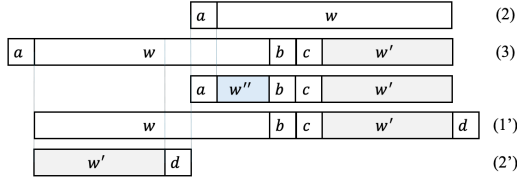
Fig. 9 Case (a1) in Lemma 5: Relation of strings w, w', and w''

- $|w| = |w'| + 2$ Since $awbcw'$ and aw are suffixes of p_1, and $|w| \geq 2$, a is a suffix of w. From (1') and (2'), we have $w = w'da$. Furthermore, since $awbcw'$ and aw are suffixes of p_1, it follows that $w = bcw'$. Thus, $w'da = bcw'$ holds. From Proposition 2, $(b, c) \in \{(a, d), (d, a)\}$ holds. Therefore, these cases contradict the conditions $b \notin \{a, d\}$ and $c \notin \{a, d\}$.

- $|w| \geq |w'| + 3$: From (2) and (3), there exists a string w'' of length $|w| - |w'| - 2$ such that $w = w''bcw'$ holds. Moreover, from (2) and (3), since $|aw| < |wbcw'|$ and $aw = aw''bcw'$, it follows that aw'' is a suffix of w. On the other hand, from (1') and (2'), $w'd$ is a prefix of w. Since $|w'd| + |aw''| = |w'| + |w''| + 2 = |w|$, it follows that $w = w'daw''$ (Fig. 9). Therefore, $w'daw'' = w''bcw'$ holds. From Proposition 3, $(b, c) \in \{(a, d), (d, a)\}$ holds. This contradicts the conditions $b \notin \{a, d\}$ and $c \notin \{a, d\}$.

From the above, we conclude that all cases of (a1) contradict the assertion that $p\{x := xy\} \npreceq q$ and the conditions $b \notin \{a, d\}$ and $c \notin \{a, d\}$.

Second, for the case (a2), we suppose that $(A, B) = (dy_1a, bc)$, i.e., $q = q_1 dy_1 awbcq_2$ holds. Then, the following conditions must be satisfied for $y_1' \in X$:

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq awbcq_2$ or
 $p_2 \preceq y_1'awbcq_2$,
(2) $p_1 \preceq q_1 d$ or (2') $p_2 \preceq wbcq_2$,
 $p_1 \preceq q_1 dy_1'$,
(3) $p_1 \preceq q_1 dy_1 aw$, (3') $p_2 \preceq q_2$.

From $p_1 \preceq q_1 dy_1 aw$ in (3), p_1 can be expressed as $p_1' p_1''$ for some p_1' and p_1'', where $p_1' \preceq q_1 d$ and $p_1'' \preceq y_1 aw$. When $p_2 \preceq awbcq_2$ in (1'), we have

$$p = p_1 x p_2 = p_1' p_1'' x p_2 \preceq q_1 d p_1'' x awbcq_2 = q\{y_1 := p_1''x\}.$$

Thus, $p\{x := xy\} \preceq q\{y_1 := p_1''xy\}$ holds. This contradicts the assumption. When $p_2 \preceq y_1'awbcq_2$ in (1'), we similarly have

$$p = p_1 x p_2 = p_1' p_1'' x p_2 \preceq q_1 d p_1'' x y_1' wbcq_2 = q\{y_1 := p_1''xy_1'\}.$$

Thus, $p\{x := xy\} \preceq q\{y_1 := p_1''xyy_1'\}$ holds. This contradicts the assumption that $p\{x := xy\} \npreceq q$. Similarly, we can show that the case $(A, B) = (bc, dy_1a)$ also contradicts the assumption.

Finally, we will prove that for the case (a3), $p\{x := xy\} \preceq q$ holds. Suppose that $(A, B) = (y_1 ay_2, bc)$, i.e.,

$q = q_1 y_1 a y_2 wbcq_2$ holds. Then, the following conditions must be satisfied for $y_1' \in X$:

(1) $p_1 \preceq q_1$ or (1') $p_2 \preceq y_2 wbcq_2$,
 $p_1 \preceq q_1 y_1'$,
(2) $p_1 \preceq q_1 dy_1$, (2') $p_2 \preceq wbcq_2$ or
 $p_2 \preceq y_2' wbcq_2$,
(3) $p_1 \preceq q_1 y_1 a y_2 w$, (3') $p_2 \preceq q_2$.

Let $q_1' = q_1 y_1 a$, $q_2' = y_2 w$, $q_3' = bcq_2$. From (3) and (1'), we have $p_1 \preceq q_1' q_2'$ and $p_2 \preceq q_2' q_3'$, respectively. Since q_2' contains a variable symbol, Theorem 2 implies that $p \preceq q$ holds. This contradicts the assumption. Similarly, we can show that the case $(A, B) = (bc, y_1 a y_2)$ also contradicts the assumption.

From the above, we conclude that if $p\{x := r\} \preceq q$ for all $r = \{ya, bc, dy\}$ ($b \notin \{a, d\}$ and $c \notin \{a, d\}$), then $p\{x := xy\} \preceq q$ holds. □

Lemma 6: Let Σ be an alphabet with $\sharp \Sigma \geq 3$ and let p, q be regular patterns on $\Sigma \cup X$. Let D be one of the following sets of regular patterns on $\Sigma \cup X$:

(i) $D = \{ya, bc, dy\}$ ($b \neq d$ and $c \notin \{a, d\}$),
(ii) $D = \{ya, bc, dy\}$ ($b \notin \{a, d\}$ and $c \neq a$).

Then, if $p\{x := r\} \preceq q$ for all $r \in D$, it follows that $p\{x := xy\} \preceq q$.

In (i), we note that if $b = d$, then, because $p\{x := dy\} \preceq q$, $p\{x := bc\} \preceq q$ is always satisfied. In this sense, D essentially consists of only two elements. To avoid this, we assume $b \neq d$. In (ii), for the same reason, we assume $c \neq a$.

Proof. It is obvious if no variable symbol appears in p. Therefore, let $p = p_1 x p_2$, where p_i (for $i = 1, 2$) is either an empty string or a regular pattern, and x is a variable symbol. We assume that $p\{x := xy\} \npreceq q$ in order to derive a contradiction. In the case of (ii), by reversing the strings p and q, we can prove that the assumption $p\{x := xy\} \preceq q$ leads to a contradiction, as in the case of (i). Therefore, in the following, we consider only the case of (i): $D = \{ya, bc, dy\}$ ($b \neq d$ and $c \notin \{a, d\}$).

The proof of this lemma is almost the same as the proof of Lemma 5. Since $p\{x := r\} \preceq q$ for all $r \in D$, there are three strings of length 2 corresponding to ya, bc, dy in q The symbols appearing in D correspond to either a variable or a constant symbol in q. Let y_1 and y_2 be variable symbols appearing in q. The strings ya and dy must correspond to the strings $y_1 a$ and dy_2 in q, respectively. For the same reasons stated at the beginning of Lemma 5, the string bc corresponds to the string bc in q as well. Let A, B, C be regular patterns on $\Sigma \cup X$, where $\{A, B, C\} = \{y_1 a, ac, dy_3\}$. Since $p\{x := xy\} \npreceq q$, q can be expressed in one of the following four forms:

Let y_1, y_2 be distinct variable symbols in X, and q_1, q_2, w, w' either an empty string or a regular pattern on $\Sigma \cup X$. From

Fig. 10 Substitutions for p and each correspondence to q.

the condition $b \neq d$, if $b = a$, it follows that $a \neq d$.

(i1) $q = q_1 A w B w' C q_2$, where $\{A, B, C\} = \{y_1 a, bc, dy_2\}$.
(i2) $q = q_1 A w B q_2$, where $\{A, B\} = \{y_1 ac, dy_2\}$ ($b = a$).
(i3) $q = q_1 A w B q_2$, where $\{A, B\} = \{y_1 a y_2, bc\}$ ($a = d$).
(i4) $q = q_1 A q_2$, where $\{A\} = \{dy_1 ac\}$ ($b = a$).

In these cases, just as in cases (a2) and (a3) in Lemma 5, cases (i3) and (i4) also lead to contradictions based on Theorem 2. In cases (i1) and (i2), similar to Lemma 5, it is shown that $q = q_1 y_1 awacw' dy_2 q_2$ and $q = q_1 y_1 acwdy_2 q_2$, respectively, where w and w' contain no variable symbols.

Firstly, we will consider the case (i1) when $b = a$. Since $p\{x := r\} \preceq q$ for all $r \in \{ya, ac, dy\}$ and $p\{x := xy\} \npreceq q$, the following conditions must be satisfied:

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq wacw' dy_2 q_2$,
(2) $p_1 \preceq q_1 y_1 aw$, (2') $p_2 \preceq w' dy_2 q_2$,
(3) $p_1 \preceq q_1 y_1 awacw'$, (3') $p_2 \preceq q_2$.

From (1') and (2'), $wacw' d$ and $w' d$ are prefixes of p_2, and from (2) and (3), $awacw'$ and aw are suffixes of p_1. From these facts:

- $|w| = |w'|$: $c = a$ holds.
- $|w| = |w'| + 1$: $w = w' d = cw'$ holds. Thus, from Proposition 1, $c = d$ holds.
- $|w| = |w'| + 2$: $w = w' da = acw'$ holds. From Proposition 2, $c \in \{a, d\}$ holds.
- $|w| \geq |w'| + 3$: From (2) and (3), there exists a string w'' of length $|w| - |w'| - 2$ such that $w = w'' acw'$ holds. Moreover, from (2) and (3), since $|aw| < |wacw'|$ and $aw = aw'' acw'$, it follows that aw'' is a suffix of w. On the other hand, from (1') and (2'), $w' d$ is a prefix of w. Since $|w' d| + |aw''| = |w'| + |w''| + 2 = |w|$, we have $w = w' daw''$. Therefore, $w' daw'' = w'' acw'$ holds (Fig. 11). From Proposition 3, we have $c \in \{a, d\}$.
- $|w'| = |w| + 1$: From (1') and (2'), $c = d$ holds.
- $|w'| = |w| + 2$: From (1') and (2'), d is a prefix of w'. Thus, from (2) and (3), $w' = wac = daw$ holds. From Proposition 2, $c \in \{a, d\}$ holds.
- $|w'| \geq |w| + 3$: From (1') and (2'), there exists a string w'' of length $|w| - |w'| - 2$ such that $w' = wacw''$ holds. Moreover, from (1') and (2'), since $|w' d| < |wacw'|$ and $w' d = wacw'' d$, $w' d$ is a prefix of w'. On the other hand, from (1') and (2'), $aw' w$ is a suffix of w'.

$$|w| \geq |w'|$$

$$|w| < |w'|$$

Fig. 11 Case (i1) in Lemma 6: Relation of strings w, w', and w''

Since $|w'' d| + |aw| = |w'| + |w| + 2 = |w'|$, we have $w' = w'' daw$. Therefore, $w'' daw = wacw''$ holds. From Proposition 3, we have $c \in \{a, d\}$.

All the cases contradict the condition $c \notin \{a, d\}$. Therefore, if $b \neq d$ and $c \notin \{a, d\}$ are satisfied, the case (i1) is impossible.

Second, we will consider the case (i2). For this $q = q_1 y_1 acwdy_2 q_2$, the following conditions must be satisfied:

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq cwdy_3 q_2$,
(2) $p_1 \preceq q_1 y_1$, (2') $p_2 \preceq wdy_3 q_2$,
(3) $p_1 \preceq q_1 y_1 acwdy_3$, (3') $p_2 \preceq q_2$.

- If $|w| = 0$, from (1') and (2'), the prefix of p_2 is cd and d. Thus, we have $c = d$.
- If $|w| = 1$, from (1') and (2'), the prefix of p_2 is cwd and wd. Thus, we have $w = c = d$.
- If $|w| \geq 2$, then from (1') and (2'), cwd and wd are prefixes of p_2. Thus, we have $cw = wd$. From Proposition 2, $c = d$ holds.

All of these cases do not meet $b \neq d$ and $c \notin \{a, d\}$. Therefore, if $b \neq d$ and $c \notin \{a, d\}$ are satisfied, the case (i2) is also impossible.

From the above, we conclude that if $p\{x := r\} \preceq q$ for all $r = \{ya, bc, dy\}$ ($b \neq d$ and $c \notin \{a, d\}$), then $p\{x := xy\} \preceq q$ holds. □

When the conditions of both Lemmas 5 and 6 are not satisfied, counterexamples can be constructed as follows:

Proposition 4: Let Σ be an alphabet with $\sharp\Sigma \geq 3$. For a variable symbol y, let $D = \{ya, bc, dy\}$ ($b = a$ and $c = d$). There exist regular patterns p and q on $\Sigma \cup X$ such that $p\{x := r\} \preceq q$ for any $r \in D$, but $p\{x := xy\} \not\preceq q$.

Proof. We provide an example to demonstrate this proposition. Let a, b, c, d, e be constant symbols in Σ, and let x, y, y_1, y_2 be variable symbols in X. Define the regular patterns p and q as follows:

$$p = eabcbcadabcbcadaxbcadadabcbcadade,$$
$$q = y_1abcbcadabcbcadady_2 \quad (b = a \text{ and } c = d).$$

Obviously $p\{x := xy\} \not\preceq q$ holds. For these p and q, the condition for Proposition 4 holds as follows (see also Fig. 10):

$p \ \{x := ya\}$
$= (eabcbcadabcbcaday)abcadadabcbcadade$
$= q\{y_1 := eabcbcadabcbcaday, \ y_2 := e\}$
$\preceq q,$

$p \ \{x := bc\}$
$= (eabcbcad)abcbcadabcbcadad(abcbcadade)$
$= q\{y_1 := eabcbcad, \ y_2 := abcbcadade\}$
$\preceq q,$

$p \ \{x := dy\}$
$= eabcbcadabcbcadad(ybcadadabcbcadade)$
$= q\{y_1 := e, \ y_2 := ybcadadabcbcadade\}$
$\preceq q.$

$\square$

In Lemma 14 (ii) of [1], they stated that, when $\sharp\Sigma \geq 3$, for regular patterns p, q, if $p\{x := r\} \preceq q$ for any $r \in D$, then $p\{x := xy\} \preceq q$ holds, where $D = \{a_1b_1, a_2b_2, a_3b_3\}$ ($a_i \neq a_j$ and $b_i \neq b_j$ for each i, j ($i \neq j, 1 \leq i, j \leq 3$)). Unfortunately, there exist the following counterexamples of Lemma 14 (ii) of [1].

Example 1: Assume that $a_1 = b_2$ and $a_3 = b_1$ hold.

(1) Let $p = ca_1x'a_3c$ and $q = xa_1a_3y$. It is clear that $\{x := xy\} \not\preceq q$ holds. However, we can see that $p\{x' := a_1b_1\} \preceq q$, $p\{x' := a_2b_2\} \preceq q$ and $p\{x' := a_3b_3\} \preceq q$ hold, since $p\{x' := a_1b_1\} = ca_1a_1b_1a_3c = q\{x := ca_1, y := a_3c\}$, $p\{x' := a_2b_2\} = ca_1a_2b_2a_3c = q\{x := ca_1a_2, y := c\}$ and $p\{x' := a_3b_3\} = ca_1a_3b_3a_3c = q\{x := c, y := b_3a_3c\}$ hold.

(2) Let $p = cb_2a_1b_1b_2x'a_1b_1b_2a_3c$ and $q = xb_2a_1b_1b_2a_3y$. It is clear that $p\{x := xy\} \not\preceq q$ holds. However, we have $p\{x' := a_1b_1\} \preceq q$, $p\{x' := a_2b_2\} \preceq q$, and $p\{x' := a_3b_3\} \preceq q$, since $p\{x' := a_1b_1\} = cb_2a_1b_1b_2a_1b_1a_1b_1b_2a_3c = q\{x := cb_2a_1b_1, y := b_2a_3c\}$, $p\{x' := a_2b_2\} = cb_2a_1b_1b_2a_2b_2a_1b_1b_2a_3c = q\{x := cb_2a_1b_1b_2a_2, y := c\}$, and $p\{x' := a_3b_3\} = cb_2a_1b_1b_2a_3b_3a_1b_1b_2a_3c = q\{x := c, y := $

$b_3a_1b_1b_2a_3c\}$ hold.

Lemma 7: Let $D = \{a_1b_1, a_2b_2, a_3b_3\}$, where $a_i \neq a_j$ and $b_i \neq b_j$ for each i, j ($i \neq j, 1 \leq i, j \leq 3$), $b_1 \neq a_2$ and $b_2 \neq a_3$. Let p, q be regular patterns such that a variable symbol y does not appear in p. Then, if $p\{x := r\} \preceq q$ for all $r \in D$, then $p\{x := xy\} \preceq q$.

Proof. It is obvious if the variable symbol x does not appear in p. Therefore, let $p = p_1xp_2$, where p_1, p_2 are regular patterns. We assume that $p\{x := xy\} \not\preceq q$ in order to derive the contradictions. Since $p\{x := r\} \preceq q$ holds for any $r \in D$, the regular pattern q contains a_1b_1, a_2b_2 and a_3b_3. We remark that a_i and b_j may be the same for i, j ($1 \leq i, j \leq 3$). Since $p\{x := r\} \preceq q$ for all $r \in D$ holds, there exist the following 10 cases (i)–(xv) for three regular patterns on Σ contained in q that correspond to three constant strings in D: Here, y_1, y_2, y_3 are variable symbols.

(i)	a_1b_1, a_2b_2, a_3b_3	(vi)	a_1b_1, y_1b_2, y_2b_3
(ii)	a_1b_1, a_2b_2, a_3y_1	(vii)	y_1b_1, y_2b_2, y_3b_3
(iii)	a_1b_1, a_2b_2, y_1b_3	(viii)	y_1b_1, y_2b_2, a_3y_3
(iv)	a_1b_1, y_1b_2, a_3y_2	(ix)	y_1b_1, a_2y_2, a_3y_3
(v)	a_1b_1, a_2y_1, a_3y_2	(x)	a_1y_1, a_2y_2, a_3y_3

For the case (iv), we can prove that $p\{x := xy\} \preceq q$ holds from Lemma **??**. For the cases (v)–(x), we can prove that $p\{x := xy\} \preceq q$ holds from Lemma 4. Hence, for the cases (i)–(iii), we will prove that $p\{x := xy\} \preceq q$ holds.

(I) Case of (i), that are the cases that q contains a_1b_1, a_2b_2 and a_3b_3:
We consider the following four cases (I-1)-(I-4) of q for some regular patterns q_1, q_2 and some constant strings w, w' ($|w| \geq 0$ and $|w'| \geq 0$):

(I-1) $q = q_1a_1b_1wa_2b_2w'a_3b_3q_2$,
(I-2) $q = q_1a_1b_1a_3b_3q_2$ ($b_1 = a_2$ and $a_3 = b_2$),
(I-3) $q = q_1a_1b_1b_2wa_3b_3q_2$ ($b_1 = a_2$),
(I-4) $q = q_1a_1b_1wa_2b_2b_3q_2$ ($b_2 = a_3$).

(I-1) Case of $q = q_1a_1b_1wa_2b_2w'a_3b_3q_2$: Assume that the following six conditions (1),(2),(3),(1'),(2'),(3') are hold.

(1) $p_1 \preceq q_1$	(1') $p_2 \preceq wa_2b_2w'a_3b_3q_2$
(2) $p_1 \preceq q_1a_1b_1w$	(2') $p_2 \preceq w'a_3b_3q_2$
(3) $p_1 \preceq q_1a_1b_1wa_2b_2w'$	(3') $p_2 \preceq q_2$

If $|w| = |w'|$ holds, $a_1b_1wa_2b_2w'$ and a_1b_1w are the suffix of p_1 from the above conditions (2) and (3). Then, $a_1b_1w = a_2b_2w'$. Hence, $a_1b_1 = a_2b_2$. This contracts the assumption of $a_1 \neq a_2$ and $b_1 \neq b_2$.

If $|w| + 1 = |w'|$ holds, $wa_2b_2w'a_3b_3$ and $w'a_3b_3$ are the prefix of p_2. If there exists a constant symbol w_1 such that $w'a_3b_3 = ww_1a_3b_3$, then b_2 and a_3 are the same symbol from $wa_2b_2 = ww_1a_3$. From the above conditions (2) and (3), $a_1b_1wa_2b_2w'$ and a_1b_1w are the suffix of p_1. Then, there exists a constant symbol w_2 such that $w' = w_2w$, then b_2 and a_1 are the same symbol from $b_2w_2w = a_1b_1w$. Hence, from $b_2 = a_3$, a_3 and a_1 are same symbol. This contradicts the assumption of $a_3 \neq a_1$.

If $|w| + 1 < |w'|$, from the above (2) and (3), $a_1b_1wa_2b_2w'$ and a_1b_1w are the suffix of p_1. If there exists a constant string w_1 ($|w_1| \geq 2$) such that $w' = w_1w$, then a_1b_1 is the suffix of w_1. From the above conditions (1′) and (2′), $wa_2b_2w'a_3b_3$ and $w'a_3b_3$ are the prefix of p_2. If there exist constant strings w_1 and w_2 such that $w' = w_1w = ww_2$ holds, then a_2b_2 and a_3b_3 are the suffix of w_1 from $|w_1| = |w_2|$ and $|ww_2a_3b_3| = |wa_2b_2w_1|$. Hence, $a_1b_1 = a_3b_3$. This contradicts the assumption of $a_1 \neq a_3$ and $b_1 \neq b_3$.

If $|w| > |w'|$, we can prove the contradiction in a similar way as $|w| \leq |w'|$.

(I-2) Case of $q = q_1a_1b_1a_3b_3q_2$ ($b_1 = a_2$ and $a_3 = b_2$): Assume that the following six conditions (1),(2),(3),(1′),(2′),(3′) are hold.

(1) $p_1 \preceq q_1$	(1′) $p_2 \preceq a_3b_3q_2$
(2) $p_1 \preceq q_1a_1$	(2′) $p_2 \preceq b_3q_2$
(3) $p_1 \preceq q_1a_1b_1$	(3′) $p_2 \preceq q_2$

From the above conditions (2) and (3), since a_1b_1 and a_1 are the suffix of p_1, $b_1 = a_1$ holds. From the assumption of $b_1 = a_2$, $a_1 = a_2$. This contradicts the assumption of $a_1 \neq a_2$.

(I-3) Case of $q = q_1a_1b_1b_2wa_3b_3q_2$ ($b_1 = a_2$): Assume that the following six conditions (1),(2),(3),(1′),(2′),(3′) are hold.

(1) $p_1 \preceq q_1$	(1′) $p_2 \preceq b_2wa_3b_3q_2$
(2) $p_1 \preceq q_1a_1$	(2′) $p_2 \preceq wa_3b_3q_2$
(3) $p_1 \preceq q_1a_1b_1b_2w$	(3′) $p_2 \preceq q_2$

If $|w| = 0$, i.e., w is the empty string, then a_1 and $a_1b_1b_2$ are the suffix of p_1 from the above conditions (2) and (3) and $b_2a_3b_3$ and a_3b_3 are the prefix of p_2 from the above conditions (1′) and (2′). Since $b_2 = a_1$ and $b_2a_3 = a_3b_3$, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.

If $|w| \geq 1$, a_1 and $a_1b_1b_2w$ are the suffix of p_1 from the above conditions (2) and (3). Hence, the last symbol of w is a_1. Moreover, $b_2wa_3b_3$ and wa_3b_3 are the prefix of p_2 from the above conditions (1′) and (2′). Hence, the last symbol of w is a_3. Therefore, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.

(I-4) Case of $q = q_1a_1b_1wa_2b_2b_3q_2$ ($b_2 = a_3$): Assume that the following six conditons (1),(2),(3),(1′),(2′),(3′) are hold.

(1) $p_1 \preceq q_1$	(1′) $p_2 \preceq wa_2b_2b_3q_2$
(2) $p_1 \preceq q_1a_1b_1w$	(2′) $p_2 \preceq b_3q_2$
(3) $p_1 \preceq q_1a_1b_1wa_2$	(3′) $p_2 \preceq q_2$

If $|w| = 0$, i.e., w is the empty string, then a_1b_1 and $a_1b_1a_2$ are the suffix of p_1 from the above conditions (2) and (3) and $a_2b_2b_3$ and b_3 are the prefix of p_2 from the above conditions (1′) and (2′). Since $b_1 = a_2$ and $a_2 = b_3$, then $b_1 = b_3$ holds. This contradicts the assumption of $b_1 \neq b_3$.

If $|w| \geq 1$, since a_1b_1w and $a_1b_1wa_2$ are the suffix of

p_1 from the above conditions (2) and (3), the first symbol of w is b_1. Moreover, since $wa_2b_2b_3$ and b_3 are the prefix of p_2 from the above conditions (1′) and (2′), the first symbol of w is b_3. Therefore, $b_1 = b_3$ holds. This contradicts the assumption of $b_1 \neq b_3$.

(II) Case of (ii) that q contains a_1b_1, a_2b_2 and a_3y: Let A, B, C be distinct regular patterns in $\{a_1b_1, a_2b_2, a_3y\}$ such that $q = q_1AwBw'Cq_2$. Assume that the following six conditions (1),(2),(3),(1′),(2′),(3′) are hold.

(1) $p_1 \preceq q_1$	(1′) $p_2 \preceq wBw'Cq_2$
(2) $p_1 \preceq q_1Aw$	(2′) $p_2 \preceq w'Cq_2$
(3) $p_1 \preceq q_1AwBw'$	(3′) $p_2 \preceq q_2$

If $|w| = |w'|$, then Aw and $AwBw'$ are the suffix of p_1 from the above conditions (2) and (3). Hence, $Aw = Bw'$ holds. This contradicts the assumption of $A \neq B$.

If $|w| \neq |w'|$, then we consider the two cases $A = a_3y$ and $B = a_3y$: In the case of $A = a_3y$, without losing generality, we assume that $B = a_1b_1$ and $C = a_2b_2$. Then, there exist regular patterns p_1', p_1'' such that $p_1 = p_1'p_1''$, $p_1' \preceq q_1a_3$ and $p_1'' \preceq yw$ from the above condition (2). Moreover, from the above condition (1′), $p = p_1xp_2 = p_1'p_1''xp_2 \preceq q_1a_3p_1''xwa_1b_1w'a_2b_2q_2 = q_1a_3ywa_1b_1w'a_2b_2q_2\{y := p_1''x\} = q\{y := p_1''x\}$ holds. Hence, $p \preceq q$ holds. This contracts the assumption. In the case of $B = a_3y$, without losing generality, we assume that $A = a_1b_1$ and $C = a_2b_2$. Let $q_1' = q_1a_1b_1$, $q_2' = wa_3yw'$, and $q_3' = a_2b_2q_2$ such that q_2' contains at most one variable symbol. Then, the above conditions (3) and (1′) are represented by $p_1 \preceq q_1'q_2'$ and $p_2 \preceq q_2'q_3'$, respectively. From Theorem 2, $p \preceq q$ holds. This contradicts the assumption.

Next, in the case of $C = a_3y$, we consider the following five cases (II-1)–(II-5):

(II-1) $q = q_1a_1b_1wa_2b_2w'a_3yq_2$,
(II-2) $q = q_1a_1b_1b_2yq_2$ ($a_2 = b_1$ and $a_3 = b_2$),
(II-3) $q = q_1a_1b_1b_2wa_3yq_2$ ($b_1 = a_2$),
(II-4) $q = q_1a_3ywa_1b_1b_2q_2$ ($b_1 = a_2$),
(II-5) $q = q_1a_1b_1ywa_2b_2q_2$ ($b_1 = a_3$).

(II-1) Case of $q = q_1a_1b_1wa_2b_2w'a_3yq_2$: Assume that the following six conditions (1),(2),(3),(1′),(2′),(3′) are hold.

(1) $p_1 \preceq q_1$	(1′) $p_2 \preceq wa_2b_2w'a_3yq_2$
(2) $p_1 \preceq q_1a_1b_1w$	(2′) $p_2 \preceq w'a_3yq_2$
(3) $p_1 \preceq q_1a_1b_1wa_2b_2w'$	(3′) $p_2 \preceq q_2$

If $|w| + 1 = |w'|$, then $a_1b_1wa_2b_2w'$ and a_1b_1w are the suffix of p_1 from the above conditions (2) and (3). Since there exists a constant symbol w_1 such that $w' = w_1w$ and $b_2w_1w = a_1b_1w$ hold, then $b_2 = a_1$. Moreover, $wa_2b_2w'a_3$ and $w'a_3$ are the prefix of p_2 from the above conditions (1′) and (2′). Since there exists a constant symbol w_2 such that $w' = ww_2$ and $wa_2b_2 = ww_2a_3$ hold, then $b_2 = a_3$. Thus, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.

If $|w| + 1 < |w'|$, then $a_1b_1wa_2b_2w'$ and a_1b_1w are the suffix of p_1 from the above conditions (2) and (3). Hence, a_1b_1 is the suffix of $w_,$. Moreover, $wa_2b_2w'a_3$ and $w'a_3$

are the prefix of p_2 from the above conditions (1') and (2'). Hence, there exist constant symbols w_1 and w_2 such that $w' = w_1 w$, $w' = w w_2$ and $|a_2 b_2 w_1| = |w_2 a_3| + 1$ hold. Thus, since the second-to-last symbol of w_1 is a_3, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.

If $|w| = |w'| + 1$, then $w a_2 b_2 w' a_3$ and $w' a_3$ are the prefix of p_2 from the above conditions (1') and (2'). Since there exists a constant symbol w_1 such that $w = w' w_1$ and $w' w_1 = w' a_3$ hold, then $w_1 = a_3$ holds. Moreover, since $a_1 b_1 w a_2 b_2 w'$ and $a_1 b_1 w$ are the suffix of p_1 from the above conditions (2) and (3), there exists a constant symbol w_2 such that $w = w_2 w'$ and $|w_1 a_2 b_2 w'| = |a_1 b_1 w_2 w'|$ hold. Hence, $w_1 = a_1$ holds. Thus, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.

If $|w| > |w'| + 1$, since $w a_2 b_2 w' a_3$ and $w' a_3$ are the prefix of p_2 from the above conditions (1') and (2'), there exists a constant string w_1 such that $w = w' w_1$ and the first symbol of w_1 is a_3. Moreover, since there exists a constant string w_2 such that $w = w_2 w'$ and $|w_1 a_2 b_2| = |a_1 b_1 w_2|$ hold, $a_1 b_1$ is the prefix of w_1. Thus, $a_3 = a_1$ holds. This contradicts the assumption of $a_1 \neq a_3$.

(II-2) Case of $q = q_1 a_1 b_1 b_2 y q_2$ ($a_2 = b_1$ and $a_3 = b_2$): Assume that the following six conditions (1),(2),(3),(1'),(2'),(3') are hold.

(1) $p_1 \preceq q_1$	(1') $p_2 \preceq b_2 y q_2$
(2) $p_1 \preceq q_1 a_1$	(2') $p_2 \preceq y q_2$
(3) $p_1 \preceq q_1 a_1 b_1$	(3') $p_2 \preceq q_2$

From the above conditions (2) and (3), $a_1 b_1$ and a_1 are the suffix of p_1. Hence, $b_1 = a_1$ holds. Thus, from the assumption of $b_1 = a_2$, $a_1 = a_2$ holds. This contradicts the assumption of $a_1 \neq a_2$.

(II-3) Case of $q = q_1 a_1 b_1 b_2 w a_3 y q_2$ ($b_1 = a_2$): Assume that the following six conditions (1),(2),(3),(1'),(2'),(3') are hold.

(1) $p_1 \preceq q_1$	(1') $p_2 \preceq b_2 w a_3 y q_2$
(2) $p_1 \preceq q_1 a_1$	(2') $p_2 \preceq w a_3 y q_2$
(3) $p_1 \preceq q_1 a_1 b_1 b_2 w$	(3') $p_2 \preceq q_2$

If $|w| = 0$, i.e., w is the empty string, then a_1 and $a_1 b_1 b_2$ are the suffix of p_1 from the above conditions (2) and (3). Hence, $a_1 = b_2$ holds. Moreover, since $b_2 a_3$ and a_3 is the prefix of p_2, $b_2 = a_3$ holds. Thus, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.

If $|w| \geq 1$, since a_1 and $a_1 b_1 b_2 w$ are the suffix of p_1 from the above conditions (2) and (3), the last symbol of w is a_1. Moreover, since $b_2 w a_3$ and $w a_3$ are the prefix of p_2 from the above conditions (1') and (2'), the last symbol of w is a_3. Thus, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.

(II-4) Case of $q = q_1 a_3 y w a_1 b_1 b_2 q_2$ ($b_1 = a_2$): Assume that the following six conditions (1),(2),(3),(1'),(2'),(3') are hold.

(1) $p_1 \preceq q_1$	(1') $p_2 \preceq w a_1 b_1 b_2 q_2$
(2) $p_1 \preceq q_1 a_3 y w$	(2') $p_2 \preceq b_2 q_2$
(3) $p_1 \preceq q_1 a_3 y w a_1$	(3') $p_2 \preceq q_2$

From the above condition (3), there exist regular patterns p_1' と p_1'' such that $p_1 = p_1' p_1''$, $p_1' \preceq q_1 a_3$ and $p_1'' \preceq y w a_1$ hold. Hence, since $p = p_1 x p_2 = p_1' p_1'' x p_2 \preceq q_1 a_3 p_1'' x w a_1 b_1 b_2 q_2 = q_1 a_3 y x w a_1 b_1 b_2 q_2 \{y := p_1'' x\} = q\{y := p_1'' x\}$, then $p \preceq q$ holds. Thus, this contradicts the assumption.

(II-5) Case of $q = q_1 a_1 b_1 y w a_2 b_2 q_2$ ($b_1 = a_3$): Assume that the following six conditions (1),(2),(3),(1'),(2'),(3') are hold.

(1) $p_1 \preceq q_1$	(1') $p_2 \preceq y w a_2 b_2 q_2$
(2) $p_1 \preceq q_1 a_1$	(2') $p_2 \preceq w a_2 b_2 q_2$
(3) $p_1 \preceq q_1 a_1 b_1 y w$	(3') $p_2 \preceq q_2$

There exist regular patterns q_1', q_2', q_3' such that $q_1' = q_1 a_1 b_1$, $q_2' = y w$, $q_3' = a_2 b_2 q_2$, from the above condition (3) $p_1 \preceq q_1' q_2'$ and from the above condition (1') $p_2 \preceq q_2' q_3'$ hold. Moreover, since q_2' contains the variable symbol y, $p \preceq q$ holds from Theorem 2. This contradicts the assumption.

(III) Case of (iii) that q contains $a_1 b_1$, $a_2 b_2$ and $y b_3$: Let A, B, C be distinct regular patterns in $\{a_1 b_1, a_2 b_2, y b_3\}$ such that $q = q_1 A w B w' C q_2$. Assume that the following six conditions (1),(2),(3),(1'),(2'),(3') are hold.

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(1) $p_1 \preceq q_1$	(1') $p_2 \preceq w B w' C q_2$
(2) $p_1 \preceq q_1 A w$	(2') $p_2 \preceq w' C q_2$
(3) $p_1 \preceq q_1 A w B w'$	(3') $p_2 \preceq q_2$

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□

Lemma 8: Let k be an integer with $k \geq 1$. Let Σ be an alphabet with $\sharp\Sigma = k + 2$. Let $p \in \mathcal{RP}$ in which a variable symbol x appears, and let $Q \in \mathcal{RP}^k$. If for any string $w \in \Sigma^*$ with $|w| = 2$, there exists a regular pattern $q_w \in Q$ such that $p\{x := w\} \preceq q_w$ holds, then there exists a regular pattern $q \in Q$ such that $p\{x := xy\} \preceq q$ holds, where y is a variable symbol that does not appear in q.

Proof. Without loss of generality, we suppose that $\sharp Q = k$ holds. Otherwise, for some regular pattern q already in Q, we can add a new regular pattern q' equivalent to q, i.e., $q' \equiv q$, to Q repeatedly until $\sharp Q = k$ is satisfied. For any $q \in Q$, we define the sets $A(q), B(q) \subseteq \Sigma$ as follows:

$$A(q) = \{a \in \Sigma \mid p\{x := ay\} \preceq q, \ y \in X\},$$
$$B(q) = \{b \in \Sigma \mid p\{x := yb\} \preceq q, \ y \in X\}.$$

If there exists $q \in Q$ such that $|A(q)| \geq 2$ or $|B(q)| \geq 2$, from Lemma 4, $p\{x := xy\} \preceq q$ holds. Below, we suppose that $|A(q)| \leq 1$ and $|B(q)| \leq 1$. Let $\varnothing$ be a constant symbol that is not a member in Σ. We define the functions $\sigma_A : Q \to \Sigma \cup \{\varnothing\}$ and $\sigma_B : Q \to \Sigma \cup \{\varnothing\}$ as follows:
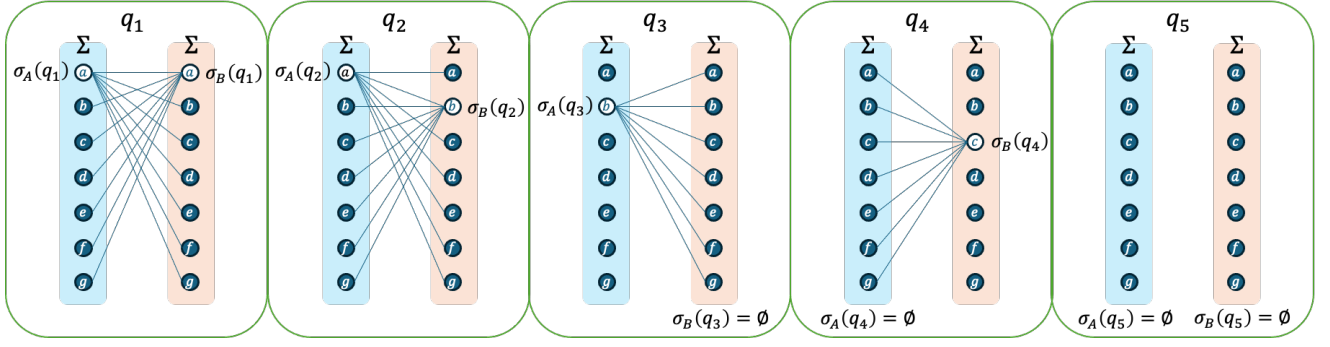
Fig. 12 Let $\Sigma = \{a, b, c, d, e, f, g\}$, $Q = \{q_1, q_2, q_3, q_4, q_5\}$. We set $A(q_1) = \{a\}$ and $B(q_1) = \{a\}$, and then $\sigma_A(q_1) = a$ and $\sigma_B(q_1) = a$, and so on. For each regular pattern q_i ($i = 1, \dots, 5$), we represent a string $w \in \Sigma \cdot \Sigma$ satisfying that $p\{x := w\} \preceq q_i$ by the line between the left (first) and right (second) symbols of w. For example, the leftmost figure shows that $p\{x := ay\} \preceq q_1$ and $p\{x := ya\} \preceq q_1$ for a variable symbol y. We note that these figures may contain more lines than those depicted. From these figures, we get $\ell_A = 1$, $\ell_B = 0$, and $Q^{(\varnothing,\varnothing)} = \{q_5\}$, $Q^{(\varnothing,\cdot)} = \{q_4\}$, $Q^{(\cdot,\varnothing)} = \{q_3\}$, $Q^{(\cdot,\cdot)} = \{q_1, q_2\}$.
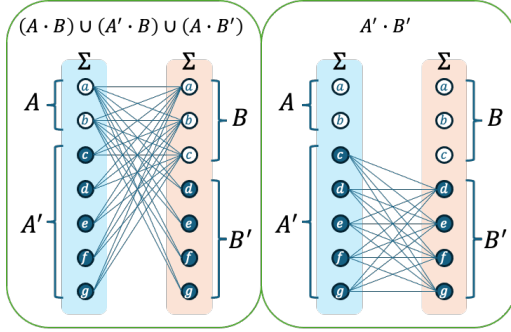
Fig. 13 In the left figure, we aggregate all of the lines appearing in Fig. 12. For all $w = a'b' \in A' \cdot B'$, there must be a regular pattern q_i ($1 \le i \le 5$) that satisfies that $p\{x := w\} \preceq q_i$.

$$\sigma_A(q) = \begin{cases} a & \text{if } A(q) = \{a\}, \\ \varnothing & \text{if } A(q) = \emptyset. \end{cases}$$

$$\sigma_B(q) = \begin{cases} b & \text{if } B(q) = \{b\}, \\ \varnothing & \text{if } B(q) = \emptyset. \end{cases}$$

The inverse functions of σ_A and σ_B are denoted by σ_A^{-1} and σ_B^{-1}, respectively. That is, for $a, b \in \Sigma \cup \{\varnothing\}$, let $\sigma_A^{-1}(a) = \{q \in Q \mid \sigma_A(q) = a\}$ and $\sigma_B^{-1}(b) = \{q \in Q \mid \sigma_B(q) = b\}$. We give an example in Fig. 12.

A and B denotes the following subsets of Σ:

$$A = \bigcup_{q \in Q \setminus \sigma_A^{-1}(\varnothing)} A(q), \quad B = \bigcup_{q \in Q \setminus \sigma_B^{-1}(\varnothing)} B(q).$$

Then, let $A' = \Sigma \setminus A$ and $B' = \Sigma \setminus B$. For any $a, b \in \Sigma$, we use the following notations:

$$\ell_A = \sum_{a \in A} (\sharp\sigma_A^{-1}(a) - 1), \quad \ell_B = \sum_{b \in B} (\sharp\sigma_B^{-1}(b) - 1).$$

These ℓ_A and ℓ_B represent the numbers of excess duplicate symbols in A and B. We easily see the following claim:

Claim 1.

(i) $\sharp A + \sharp A' = \sharp B + \sharp B' = k + 2$,
(ii) $\sharp A + \ell_A + \sharp\sigma_A^{-1}(\varnothing) = \sharp B + \ell_B + \sharp\sigma_B^{-1}(\varnothing) = k$.

Since $\sharp\Sigma = k + 2$ and $\sharp Q = k$, $\sharp A' \ge 2$ and $\sharp B' \ge 2$ hold. We partition Q into the following subsets:

$$Q^{(\varnothing,\varnothing)} = \sigma_A^{-1}(\varnothing) \cap \sigma_B^{-1}(\varnothing),$$
$$Q^{(\varnothing,\cdot)} = \sigma_A^{-1}(\varnothing) \cap (Q \setminus \sigma_B^{-1}(\varnothing)),$$
$$Q^{(\cdot,\varnothing)} = (Q \setminus \sigma_A^{-1}(\varnothing)) \cap \sigma_B^{-1}(\varnothing),$$
$$Q^{(\cdot,\cdot)} = (Q \setminus \sigma_A^{-1}(\varnothing)) \cap (Q \setminus \sigma_B^{-1}(\varnothing)).$$

From the condition of this lemma, for any string $w \in \Sigma^*$ with $|w| = 2$, there exists a regular pattern $q_w \in Q$ such that $p\{x := w\} \preceq q_w$ holds. In particular, for $w = a'b' \in A' \cdot B'$, we must have $q_w \in Q$ that satisfies that $p\{x := w\} \preceq q_w$ (Fig. 13). It is easy to see that if $w \in (A \cdot B) \cup (A' \cdot B) \cup (A \cdot B')$, there exists a regular pattern $q_w \in Q^{(\varnothing,\cdot)} \cup Q^{(\cdot,\varnothing)} \cup Q^{(\cdot,\cdot)}$ such that $p\{x := w\} \preceq q_w$ holds. The following two claims are proven from Lemmas 4 and 7:

Claim 2. If there exist $q \in Q^{(\varnothing,\varnothing)}$ and distinct 5 strings $w_i \in A' \cdot B'$ ($1 \le i \le 5$) such that $p\{x := w_i\} \preceq q$ holds ($1 \le i \le 5$), then $p\{x := xy\} \preceq q$ holds.

Claim 3. If there exist $q \in Q^{(\varnothing,\cdot)} \cup Q^{(\cdot,\varnothing)}$ and distinct 3 strings $w_i \in A' \cdot B'$ ($1 \le i \le 3$) such that $p\{x := w_i\} \preceq q$ holds ($1 \le i \le 3$), then $p\{x := xy\} \preceq q$ holds.

If there exist a regular pattern $q \in Q^{(\varnothing,\varnothing)} \cup Q^{(\varnothing,\cdot)} \cup Q^{(\cdot,\varnothing)}$ and enough strings $w \in A' \cdot B'$ such that either of the conditions of *Claims* 2 and 3 is satisfied, this lemma holds. Then, we assume that it is not the case.

Assumption 1. There is no regular pattern $q \in Q^{(\varnothing,\varnothing)}$ and 5 strings $w \in A' \cdot B'$ such that the condition of *Claim* 2 is satisfied and there is no regular pattern $q \in Q^{(\varnothing,\cdot)} \cup Q^{(\cdot,\varnothing)}$ and 3 strings $w \in A' \cdot B'$ such that the condition of *Claim* 3 is satisfied.

Let $\mathcal{L}_1 = \sharp\{w \in A' \cdot B' \mid \exists q \in Q^{(\varnothing,\varnothing)} \cup Q^{(\varnothing,\cdot)} \cup$

$Q^{(\cdot,\varnothing)}$ s.t. $p\{x := w\} \preceq q\}$. Under *Assumption* 1, each $q \in Q^{(\varnothing,\varnothing)}$ has at most 4 strings $w \in A' \cdot B'$ such that the condition of *Claim* 2 is satisfied, and each $q \in Q^{(\varnothing,\cdot)} \cup Q^{(\cdot,\varnothing)}$ has at most 2 strings $w \in A' \cdot B'$ such that the condition of *Claim* 3 is satisfied. Then, by *Claim* 1,

$$\begin{aligned}
\mathcal{L}_1 &\leq 4 \sharp Q^{(\varnothing,\varnothing)} + 2 \sharp Q^{(\varnothing,\cdot)} + 2 \sharp Q^{(\cdot,\varnothing)} \\
&= 2(\sharp Q^{(\varnothing,\varnothing)} + \sharp Q^{(\varnothing,\cdot)}) + 2(\sharp Q^{(\varnothing,\varnothing)} + \sharp Q^{(\cdot,\varnothing)}) \\
&= 2 \sharp \sigma_A^{-1}(\varnothing) + 2 \sharp \sigma_B^{-1}(\varnothing) \\
&= 2(k - \sharp A - \ell_A) + 2(k - \sharp B - \ell_B) \\
&= 2(\sharp A' - \ell_A - 2) + 2(\sharp B' - \ell_B - 2) \\
&= 2(\sharp A' + \sharp B') - 2(\ell_A + \ell_B) - 8.
\end{aligned}$$

Next, we partition $Q^{(\cdot,\cdot)}$ into the following two subsets:

$$\begin{aligned}
Q_1^{(\cdot,\cdot)} &= \{q \in Q^{(\cdot,\cdot)} \mid \sigma_A(q) \in B \text{ or} \sigma_B(q) \in A\}, \\
Q_2^{(\cdot,\cdot)} &= \{q \in Q^{(\cdot,\cdot)} \mid \sigma_A(q) \in B' \text{ and } \sigma_B(q) \in A'\}.
\end{aligned}$$

We show the next two claims on $Q_1^{(\cdot,\cdot)}$ and $Q_2^{(\cdot,\cdot)}$:

Claim 4. If there exist $q \in Q_1^{(\cdot,\cdot)}$ and a string $a'b' \in A' \cdot B'$ such that $p\{x := a'b'\} \preceq q$ holds, then $p\{x := xy\} \preceq q$ holds.

Proof of Claim 4. Suppose that both $\sigma_A(q) \in B$ and $\sigma_B(q) \in A$ hold. Then, since $a' \notin \{\sigma_A(q), \sigma_B(q)\} \subseteq A \cap B$ and $b' \notin \{\sigma_A(q), \sigma_B(q)\} \subseteq A \cap B$, from Lemma 5, $p\{x := xy\} \preceq q$ holds. Suppose that $\sigma_A(q) \in B$ and $\sigma_B(q) \in A'$. If $a' = \sigma_B(q)$, since $a' \in B$, $a' \neq b'$ holds. Since $\sigma_A(q) \in B$, $b' \neq \sigma_A(q)$ holds. That is, $a' = \sigma_B(q)$, $a' \neq \sigma_A(q)$, and $b' \notin \{\sigma_A(q), \sigma_B(q)\}$ hold. Therefore, from Lemma 6, $p\{x := xy\} \preceq q$ holds. If $a' \neq \sigma_B(q)$, since $b' \neq \sigma_A(q)$, from Lemma 5, $p\{x := xy\} \preceq q$ holds. Similarly, the case that $\sigma_A(q) \in B'$ and $\sigma_B(q) \in A$ is proven. (*End of Proof of Claim*)

Claim 5. If there exist $q \in Q_2^{(\cdot,\cdot)}$ and a string $a'b' \in A' \cdot B'$ such that $(a' \neq \sigma_B(q)$ or $b' \neq \sigma_A(q))$ and $p\{x := a'b'\} \preceq q$ hold, then $p\{x := xy\} \preceq q$ holds.

Proof of Claim 5. When $a' = b'$, since $\sigma_A(q) \neq \sigma_B(q)$, from Lemma 5, this claim holds. Similarly, when $a' \neq b'$, from Lemma 5 or Lemma 6, this holds. (*End of Proof of Claim*)

If there exist a regular pattern $q \in Q_2^{(\cdot,\cdot)}$ and a string $w \in A' \cdot B'$ such that the condition of *Claim* 5 is satisfied, this lemma holds. Then, we also assume that it is not the case.

Assumption 2. There is no $q \in Q_2^{(\cdot,\cdot)}$ and a string $a'b' \in A' \cdot B'$ such that the condition of *Claim* 5 is satisfied.

Let $\mathcal{L}_2 = \sharp\{a'b' \in A' \cdot B' \mid \exists q \in Q_2^{(\cdot,\cdot)} \text{ s.t. } p\{x := a'b'\} \preceq q\}$. For any $a'b' \in A' \cdot B'$ and $q \in Q_2^{(\cdot,\cdot)}$, if $a' = \sigma_B(q)$ and $b' = \sigma_A(q)$ hold (it is the condition of Proposition **??**), by considering the duplicate numbers ℓ_A and ℓ_B, we have the following inequality:

$$\mathcal{L}_2 \leq \min\{\sharp A' + \ell_B, \sharp B' + \ell_A\}.$$

We show the last claim:

Claim 6. $\sharp A' \times \sharp B' - \mathcal{L}_1 - \mathcal{L}_2 \geq 2$.

Proof of Claim 6. First we prove the inequality when $\sharp A \leq k - 1$ and $\sharp B \leq k - 1$, i.e., $\sharp A' \geq 3$ and $\sharp B' \geq 3$ hold. Since $\mathcal{L}_2 \leq \frac{1}{2}(\sharp A' + \sharp B' + \ell_A + \ell_B)$,

$$\begin{aligned}
&\sharp A' \times \sharp B' - \mathcal{L}_1 - \mathcal{L}_2 \\
&\geq \sharp A' \times \sharp B' - (2(\sharp A' + \sharp B') - 2(\ell_A + \ell_B) - 8) \\
&\quad - \frac{1}{2}(\sharp A' + \sharp B' + \ell_A + \ell_B) \\
&= \sharp A' \times \sharp B' - \frac{5}{2}(\sharp A' + \sharp B') + \frac{3}{2}(\ell_A + \ell_B) + 8 \\
&= (\sharp A' - \frac{5}{2})(\sharp B' - \frac{5}{2}) + \frac{3}{2}(\ell_A + \ell_B) + \frac{7}{4} \geq 2.
\end{aligned}$$

When $\sharp A = k$ and $\sharp B \leq k$, i.e., $\sharp A' = 2$ and $\sharp B' \geq 2$ hold, since $\ell_A = 0$, $\mathcal{L}_1 \leq 2\sharp B' - 2\ell_B - 4$ holds. Moreover, $\mathcal{L}_2 \leq \min\{\sharp B', \ell_B + 2\}$ holds. From *Claim* 1, $\ell_B + 2 = k - \sharp \sigma_B^{-1}(\varnothing) - \sharp B = \sharp B' - \sharp \sigma_B^{-1}(\varnothing)$ holds. Therefore, $\mathcal{L}_2 \leq \ell_B + 2$ holds. Thus,

$$\begin{aligned}
&\sharp A' \times \sharp B' - \mathcal{L}_1 - \mathcal{L}_2 \\
&\geq 2\sharp B' - (2\sharp B' - 2\ell_B - 4) - (\ell_B + 2) \\
&= \ell_B + 2 \geq 2.
\end{aligned}$$

Similarly, the case when $\sharp A \leq k$ and $\sharp B = k$ is proven. (*End of Proof of Claim*)

Under *Assumptions* 1 and 2, from *Claim* 6, there exist at least two $w \in A' \cdot B'$ and a regular pattern $q \in Q_1^{(\cdot,\cdot)}$ such that the condition of *Claim* 4 is satisfied. Therefore, for such a regular pattern q, $p\{x := xy\} \preceq q$ holds. □

Lemma 9 (Sato et al.[1]): Let Σ be a finite alphabet with $\sharp \Sigma \geq 3$ and p, q regular patterns. If there exists a constant symbol $a \in \Sigma$ such that $p\{x := a\} \preceq q$ and $p\{x := xy\} \preceq q$, then $p \preceq q$ holds, where y is a variable symbol that does not appear in q.

From the Lemma 8 and Lemma 9, we have the following theorem.

Theorem 4: Let $k \geq 3$, $\sharp \Sigma \geq 2k - 1$, $P \in \mathcal{RP}^+$ and $Q \in \mathcal{RP}^k$. Then, the following (i),(ii) and (iii) are equivalent:

(i) $S_2(P) \subseteq L(Q)$, (ii) $P \sqsubseteq Q$, (iii) $L(P) \subseteq L(Q)$.

Proof. it is clear that (ii) implies (iii) and (iii) implies (i). From Theorem3, if $\sharp \Sigma \geq 2k + 1$, then (i) implies (ii). Let $\sharp Q = k$, $p \in P$, $\sharp \Sigma = 2k - 1$ or $2k$. Then, we show that (i) implies (ii). It suffices to show that $S_2(p) \subseteq L(Q)$ implies $P \sqsubseteq Q$ for any regular pattern $p \in \mathcal{RP}$. The proof is done by mathematical induction on n, where n is the number of variable symbols appears in p.

In case $n = 0$, $S_2(p) = \{p\}$. By (i), we have $\{p\} = L(Q)$. Thus, $p \preceq q$ for some $q \in Q$.

For $n \geq 0$, we assume that it is valid for any regular pattern p with n variable symbols. Let p be a regular pattern such that $n + 1$ variable symbols appear in p and $S_2(p) \subseteq L(Q)$.

We assume that $p \not\sqsubseteq Q$, that is, $p \not\preceq q_i$ for any $i \in \{1, \ldots, k\}$. Let $Q = \{q_1, \ldots, q_k\}$ and p_1, p_2 regular patterns, x a variable symbol with $p = p_1 x p_2$. For $a, b \in \Sigma$, let $p_a = p\{x := a\}$ and $p_{ab} = p\{x := ab\}$. Both p_a and p_{ab} have n variable symbols, respectively. Thus, $S_2(p_a) \subseteq L(Q)$ and $S_2(p_{ab}) \subseteq L(P)$ hold. By the induction hypothesis, there exist $i, i' \in \{1, \ldots, k\}$ such that $p_a \preceq q_i$ and $p_{ab} \preceq q_{i'}$. Let $D_i = \{a \in \Sigma \mid p\{x := a\} \preceq q_i\}$ $(i = 1, \ldots, k)$. We assume that $\sharp D_i \geq 3$ for some $i \in \{1, \ldots, k\}$. By Lemma **??**, we have $p \preceq q_i$. This contradicts the assumption. Thus, we have $\sharp D_i \leq 2$ for any $i \in \{1, \ldots, k\}$. If $\sharp \Sigma = 2k - 1$, then $\sharp D_i = 2$ or $\sharp D_i = 1$ for any $i \in \{1, \ldots, k\}$. Moreover, If $\sharp \Sigma = 2k$, then $\sharp D_i = 2$ for any $i \in \{1, \ldots, k\}$. Since $k \geq 3$, $2k + 1 \geq k + 2$ holds. By Lemma 8, there exists $i \in \{1, \ldots, k\}$ such that $p\{x := xy\} \preceq q_i$. Therefore, by Lemma 9, we have $p \preceq q_i$. This contradicts the assumption. Thus, (i) implies (ii). □

From Theorem 4, the following corollary holds.

Corollary 2: Let $k \geq 3$, $\sharp \Sigma \geq 2k - 1$ and $P \in \mathcal{RP}^+$. Then, $S_2(P)$ is a characteristic set for $L(P)$ within $\mathcal{RPL}^k$.

Lemma 10 (Sato et al.[1]): Let $k \geq 3$ and $\sharp \Sigma \leq 2k - 2$. Then, $\mathcal{RP}^k$ does not have compactness with respect to containment.

Proof. Let $\Sigma = \{a_1, \ldots, a_{k-1}, b_1, \ldots, b_{k-1}\}$ and p, q_i regular patterns, $w_i \in \Sigma^*$ $(i = 1, \ldots, k - 1)$ defined in a similar way to Example **??**. Let $q_k = x_1 a_1 w_1 xy w_1 b_1 x_2$. Since $p\{x := a_i\} = x_1 a_1 w_1 a_i w_1 b_1 x_2 \preceq q_i$ and $p\{x := b_i\} = x_1 a_1 w_1 b_i w_1 b_1 x_2 \preceq q_i$ for any $i \in \{1, \ldots, k - 1\}$, we have $S_1(p) \subseteq \bigcup_{i=1}^{k-1} L(q_i)$. For any $w \in \{s \in \Sigma^+ \mid |s| \geq 2\}$, $p\{x := w\} = x_1 a_1 w_1 w w_1 b_1 x_2 \preceq q_k$. Thus, we have $L(p) \subseteq L(Q)$. By Theorem 1, since $p \not\preceq q_i$, $L(p) \not\subseteq L(q_i)$ for any $i \in \{1, \ldots, k\}$. Therefore, $\mathcal{RP}^k$ does not have compactness with respect to containment. □

From Theorem 4 and Lemma 10, we have the following thorem.

Theorem 5: Let $k \geq 3$ and $\sharp \Sigma \geq 2k - 1$. Then, $\mathcal{RP}^k$ has compactness with respect to containment.

In case $k = 2$, we have the following theorem.

Theorem 6: Let $\sharp \Sigma \geq 4$, $P \in \mathcal{RP}^+$ and $Q \in \mathcal{RP}^2$. The following (i), (ii) and (iii) are equivalent:

(i) $S_2(P) \subseteq L(Q)$, (ii) $P \sqsubseteq Q$, (iii) $L(P) \subseteq L(Q)$.

Proof. It is clear that (ii) implies (iii), and (iii) implies (i). Thus, we show that (i) implies (ii). It suffices to show that $S_2(p) \subseteq L(Q)$ implies $P \sqsubseteq Q$ for any regular pattern $p \in \mathcal{RP}$. Let $Q = \{q_1, q_2\}$. The proof is done by mathematical induction on n, where n is the number of variable symbols appearing in p. In case $n = 0$, $p \in \Sigma^+$. Since $S_2(p) = \{p\} \subseteq L(Q)$, we have $p \preceq q$ for some $q \in Q$. For $n \geq 0$, we assume that it is valid for any regular pattern p with n variable symbols. Let p be a regular pattern such that $n + 1$ variable symbols appear in p, ans $S_2(p) \subseteq L(Q)$. We assume that $p \not\preceq q_i$ $(i = 1, 2)$. Let p_1, p_2 be regular patterns

and x a variable symbol with $p = p_1 x p_2$. For $a, b \in \Sigma$, let $p_a = p\{x := a\}$ and $p_{ab} = p\{x := ab\}$. Note that p_a and p_{ab} have n variable symbols. Thus, by the assumption, $S_2(p_a) \subseteq L(Q)$ and $S_2(p_{ab}) \subseteq L(Q)$ implies $p_a \preceq q_i$ and $p_{ab} \preceq q_{i'}$ for some $i, i' \in \{1, 2\}$. Let $D_i = \{a \in \Sigma \mid p\{x := a\} \preceq q_i\}$ $(i = 1, 2)$. By Lemma **??**, if $\sharp D_i \geq 3$ for some $i \in \{1, 2\}$, then $p \preceq q_i$. This contradicts that $p \not\preceq q_i$ $(i = 1, 2)$. Thus, we have $\sharp D_i \leq 2$ for any $i \in \{1, 2\}$. Since $\sharp \Sigma \geq 4$, We consider that $\sharp D_1 = 2$ and $\sharp D_2 = 2$. From Lemma 8, $p\{x := xy\} \preceq q_i$ for some $i \in \{1, 2\}$. From Lemma 9, we have $p \preceq q_i$ for some $i \in \{1, 2\}$. This contradicts that $p \not\preceq q_i$ $(i = 1, 2)$. Therefore, (i) implies (ii). □

The next example is a counter-example of Theorem 6.

Example 2: Let $\Sigma = \{a, b, c\}$, p, q_1, q_2 regular patterns and x, x', x'' variable symbols such that $p = x' a x b x''$, $q_1 = x' a b x''$ and $q_2 = x' c x''$. Let $w \in \Sigma^+$. If w contains c, then $p\{x := w\} \preceq q_2$. On the other hand, if w does not contain c, then $p\{x := w\} \preceq q_1$. Thus, $L(p) \subseteq L(q_1) \cup L(q_2)$. However, $p \not\preceq q_1$ and $p \not\preceq q_2$.

From Theorem 6, we have that following two corollaries.

Corollary 3: Let $\sharp \Sigma \geq 4$ and $P \in \mathcal{RP}^+$. Then, $S_2(P)$ is a characteristic set for $L(P)$ within $\mathcal{RPL}^2$.

Corollary 4: Let $\sharp \Sigma \geq 4$. Then, $\mathcal{RP}^2$ has compactness with respect to containment.

4. Regular Pattern without Adjacent Variable Symbols

A regular pattern p is said to be a *non-adjacent variable regular pattern* (*NAV* regular pattern) if p does not contain consecutive variable symbols. For example, the regular pattern $p = axybc$ is not a *NAV* regular pattern because xy is appeared in p. Let $\mathcal{RP}_{NAV}$ be the set of all *NAV* regular patterns. Let $\mathcal{RP}_{NAV}^+$ be the set of all finite subsets S of $\mathcal{RP}_{NAV}$ such that S is not the empty set, i.e., $\mathcal{RP}_{NAV}^+ = \{S \subseteq \mathcal{RP}_{NAV} \mid \sharp S \leq 1\}$, and $\mathcal{RP}_{NAV}^k$ the set of all subsets P of $\mathcal{RP}_{NAV}^+$ such that P consists of at most k $(k \geq 1)$ *NAV* regular patterns, i.e., $\mathcal{RP}_{NAV}^k = \{P \in \mathcal{RP}_{NAV}^+ \mid \sharp P \leq k\}$. We can define the compactness with respect to containment for $\mathcal{RP}_{NAV}^k$ in a similar way as Def.2. For any *NAV* regular pattern $p \in \mathcal{RP}_{NAV}$ and any set $Q \in \mathcal{RP}_{NAV}^k$ with k $(k \geq 1)$, the set $\mathcal{RP}_{NAV}^k$ said to have *compactness with respect to containment* if there exists a *NAV* regular pattern $q \in Q$ such that $L(p) \subseteq L(q)$ holds if $L(p) \subseteq L(Q)$ holds. Then, we have the following Theorem 7.

Theorem 7: For an integer k $(k \geq 2)$, let $\sharp \Sigma \geq k + 2$, $P \in \mathcal{RP}_{NAV}^+$, $Q \in \mathcal{RP}_{NAV}^k$. Then, the following (i), (ii) and (iii) are equivalent:

(i) $S_2(P) \subseteq L(Q)$, (ii) $P \sqsubseteq Q$, (iii) $L(P) \subseteq L(Q)$.

Proof. From the definitions of $\mathcal{RP}_{NAV}^+$ and $\mathcal{RP}_{NAV}^k$, it is clear that (ii) implies (iii) and (iii) implies (i). Hence, we will show

$p = x'cadadaadacbadadaadaxadadaadacbadadaadabx''$,
$q_1 = x'cadadaadacbadadaadacx''$,
$q_2 = x'badadaadacx''$,
$q_3 = x'aadadx''$.

Fig. 14 *NAV* regular patterns p, q_1, q_2, and q_3

Table 2 The conditions on the number $\sharp\Sigma$ of constant symbols in Σ required for compactness with respect to containment.

Class	$k = 2$	$k \geq 3$
$\mathcal{RP}^k$	$\sharp\Sigma \geq 4$	$\sharp\Sigma \geq 2k - 1$
$\mathcal{RP}^k_{NAV}$	$\sharp\Sigma \geq k + 2$	

that (i) implies (ii) by mathematical induction on the number n of variable symbols that appear in a *NAV* regular pattern $p \in P$ as follows: If $n = 0$, then we have $S_2(\{p\}) = \{p\}$. Hence, $p \in L(Q)$. Therefore, there exists $q \in Q$ such that $p \preceq q$ holds.

If $n \geq 0$, we assume that the proposition holds for any regular *NAV* regular pattern containing $n \geq 0$ variable symbols. Let p be a *NAV* regular pattern containing $n + 1$ variable symbols such that $S_2(\{p\}) \subseteq L(Q)$ holds and p contains a variable symbol x. There exist two *NAV* regular patterns p_1, p_2 such that $p = p_1 x p_2$ holds. By the induction hypothesis, for any constant string $w \in \Sigma^*$ with $|w| = 2$, $\{p\{x := w\}\} \preceq Q$ holds because $p\{x := w\}$ contains n variable symbols. Hence, there exists a *NAV* regular pattern $q_w \in Q$ such that $p\{x := w\} \preceq q_w$ holds. From Lemma 8, there exists a regular pattern $q \in Q$ such that $p\{x := xy\} \preceq q$ holds, where y is a variable symbol that does not appear in q. This contradicts the condition $Q \in \mathcal{RP}^k_{NAV}$. Thus, we have that (i) implies (ii). □

Corollary 5: Let $k \geq 2$, $\sharp\Sigma \geq k + 2$ and $P \in \mathcal{RP}^+_{NAV}$. Then, $S_2(P)$ is a characteristic set of $\mathcal{RPL}^k_{NAV}$.

Lemma 11: Let $k \geq 2$ and $\sharp\Sigma \leq k + 1$. Then, $\mathcal{RP}^k_{NAV}$ does not have compactness with respect to containment.

Proof. Let Σ be the set of $k + 1$ constant symbols $a_1, \ldots, a_{k+1}$, i.e., $\Sigma = \{a_1, \ldots, a_{k+1}\}$. We assume that for $i = 1, 2, \ldots, k$, $p\{x := a_i y\} \preceq q_i$ and $p\{x := y a_{i+1}\} \preceq q_i$ ($i = 1, 2, \ldots, k$) hold. If $p\{x := a_{k+1} a_1\} \preceq q_1$ holds, $S_2(p) \backslash S_1(p) \subseteq \bigcup_{i=1}^{k} L(q_i)$ holds. This show that $L(p) \subseteq L(Q)$ holds. However, for $i = 1, 2, \ldots, k$, since $p \not\preceq q_i$ holds, we have that $L(p) \not\subseteq L(q_i)$ holds. Hence, $\mathcal{RP}^k_{NAV}$ does not have compactness with respect to containment. □

Next, we give an example for Lemma 11 in Example 3.

Example 3: Let Σ be the set of four constant symbols a, b, c, d, i.e., $\Sigma = \{a, b, c, d\}$ and x, x', x'' three distinct variable symbols. Let p, q_1, q_2, q_3 be the *NAV* regular patterns given in Fig. 14. Then, we have $L(p) \subseteq L(q_1) \cup L(q_2) \cup L(q_3)$. This show that for $P = \{p\}$, $Q = \{q_1, q_2, q_3\}$, (iii) of Theorem 7 holds. However, since $p \not\preceq q_1$, $p \not\preceq q_2$ and $p \not\preceq q_3$ hold, we have $P \not\sqsubseteq Q$, that is, (ii) of Theorem 7 does not hold.

From Theorem 7 and Lemma 11, we have the following theorem.

Theorem 8: Let $k \geq 2$ and $\sharp\Sigma \geq k + 2$. Then, the set $\mathcal{RPL}^k_{NAV}$ has compactness with respect to containment.

5. Conclusion

In this paper, for an integer k ($k \geq 2$), we have shown the conditions on the number of constant symbols in Σ, summarized in Table 2, required for the classes $\mathcal{RP}^k$ of all the set of k regular pattern languages and $\mathcal{RP}^k_{NAV}$ of all the set of k *NAV* regular patterns to have compactness with respect to containment.

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