PAPER

Compactness in Finite Unions of Regular Patterns and Regular Patterns without Adjacent Variables

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SUMMARY

A *regular pattern* is a string consisting of constant and distinct variable symbols. The language $L(p)$ of a pattern p is defined as the set of all constant strings obtained by replacing each variable with a constant string. Let $\mathcal{RP}^k$ denote the class of all sets containing at most k $(k \geq 2)$ regular patterns. Sato et al. (Proc. ALT'98, 1998) showed that the finite set $S_2(P)$, obtained from $P \in \mathcal{RP}^k$ by replacing variables with constant strings of length at most two, serves as a characteristic set for the language $L(P) = \bigcup_{p \in P} L(p)$. They also claimed that $\mathcal{RP}^k$ is compact with respect to language containment when the number of constant symbols is at least $2k-1$. In this paper, we revisit their results and identify an error in the original proof of their theorem. We then present a new and correct proof by introducing additional conditions that guarantee the validity of their claim. Furthermore, we study the subclass $\mathcal{RP}^k_{\mathrm{NAV}}$, consisting of at most k $(k \geq 1)$ *non-adjacent regular patterns*, in which no two variable symbols occur consecutively. For any $P \in \mathcal{RP}^k_{\mathrm{NAV}}$, we prove that the set $S_2(P)$ serves as a characteristic set of $L(P)$ and that $\mathcal{RP}^k_{\mathrm{NAV}}$ is compact with respect to language containment if the number of constant symbols is at least $k+2$. These results demonstrate that finite unions of non-adjacent regular pattern languages can be learned efficiently under weaker constraints on constant symbols than those required in the general case. Our analysis thus refines and extends the compactness properties of regular pattern languages originally discussed by Sato et al., providing a corrected theoretical foundation for subsequent studies on the learnability of pattern languages.

key words: *Regular Pattern Language, Compactness with respect to Language Containment, Non-adjacent Regular Patterns Language*

1. Introduction

A pattern is a string consisting of constant symbols and variable symbols [1], [2]. For example, we consider constant symbols a, b, c and variable symbols x, y, then $axbxcy$ is a pattern. $\mathcal{P}$ denotes the set of all patterns. For a pattern $p \in \mathcal{P}$, *the pattern language generated by* p, denoted by $L(p)$, or simply called a pattern language, is the set of all strings obtained by replacing all variable symbols with constant symbol strings, where the same variable symbol is replaced by the same constant string. For example, the pattern language $L(axbxcy)$, generated by the above pattern $axbxcy$, denotes $\{aubucw \mid u \text{ and } w \text{ are constant strings that are not } \varepsilon\}$. A pattern where each variable symbol appears at most once is called a *regular pattern*. For example, $axbxcy$ is not a regular pattern, but $axbzcy$ with variable symbols x, y, z is a regular pattern. $\mathcal{RP}$ denotes the set of all regular patterns. If a pattern $p \in \mathcal{P}$ is obtained from a pattern $q \in \mathcal{P}$ by replacing variable symbols in q with patterns, we say that q is a *generalization* of p and denote this by $p \preceq q$. For example, a pattern $q = axz$ is a generalization of a pattern $p = axbxcy$, because p is obtained from q by replacing the variable z in q with a pattern $bxcy$. So we write $p \preceq q$. For patterns $p, q \in \mathcal{P}$, it is obvious that $p \preceq q$ implies $L(p) \subseteq L(q)$. But, the converse, that is, the statement that $L(p) \subseteq L(q)$ implies $p \preceq q$ does not always hold. Mukouchi [3] demonstrated that if the number of constant symbols is at least three, for any regular pattern $p, q \in \mathcal{RP}$, $L(p) \subseteq L(q)$ implies $p \preceq q$.

We denote by $\mathcal{RP}^+$ the class of all non-empty finite sets of regular patterns and by $\mathcal{RP}^k$ the class of at most k $(k \geq 2)$ regular patterns. For a set of regular patterns $P \in \mathcal{RP}^k$ we define $L(P) = \bigcup_{p \in P} L(p)$ and consider the class $\mathcal{RPL}^k$ of regular pattern languages of $\mathcal{RP}^k$, where $\mathcal{RPL}^k = \{L(P) \mid P \in \mathcal{RP}^k\}$. Let $P, Q \in \mathcal{RP}^k$ and $Q = \{q_1, \ldots, q_k\}$. We denote by $P \sqsubseteq Q$ that for any regular pattern $p \in P$ there exists a regular pattern q_i such that $p \preceq q_i$ holds. From definition, it is obvious that $P \sqsubseteq Q$ implies $L(P) \subseteq L(Q)$. Then, Sato et al. [4] shows that if $k \geq 3$ and the number of constant symbols is at least $2k-1$ then the finite set $S_2(P)$ of constant symbols obtained from $P \in \mathcal{RP}^k$ by substituting variable symbols with constant strings of at most two length is a characteristic set of $L(P)$, that is, for any regular pattern language $L' \in \mathcal{RPL}^k$, $S_2(P) \subseteq L'$ implies $L(P) \subseteq L'$. Thus they shows that the following three statements: (i) $S_2(P) \subseteq L(Q)$, (ii) $P \sqsubseteq Q$ and (iii) $L(P) \subseteq L(Q)$ are equivalent. Nevertheless, Lemma 14 presented in [4], upon which these results rely, is found to contain an error. In this paper, we revisit their results and correct an error in the proof of their theorem by introducing additional conditions. Specifically, we show that any generalization of the strings in $S_2(P)$ would violate the condition $p\{x := r\} \preceq q$ for all $r \in S_2(P)$ where p is a regular pattern in P and q is a regular pattern.

Sato et al. [4] shows that $\mathcal{RP}^k$ has compactness with respect to language containment if the number of constant symbols is greater than or equal to $2k-1$. On the contrary to this result, we show that the set $S_2(P)$ obtained from a set P in the class $\mathcal{RP}^k_{NAV}$ of at most k $(k \geq 1)$ regular patterns having non-adjacent variables is a characteristic set of $L(P)$. Further, we show that if the number of constant

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Table 1 The conditions of the number of constant symbols with respect to the compactness of inclusion

k	2	$\geqq 3$
$\mathcal{RP}^k$	$\geqq 4$	$\geqq 2k-1$
$\mathcal{RP}^k_{NAV}$	$\geqq k+2$	

symbols is greater than or equal to $k+2$ then $\mathcal{RP}^k_{NAV}$ has compactness with respect to language containment. In Table 1, we summarize the all results in this paper.

Mukouchi [5] examined the decision problem of determining whether a containment relation exists between the languages generated by two given patterns. The inductive inference of formal languages—specifically, pattern languages [2] and unions of pattern languages [6], [7] from positive data has been extensively investigated. Arimura et al. [8] introduced a formal framework for the efficient generalization of unions of pattern languages, presenting a polynomial-time algorithm to identify the minimal set of patterns whose union encompasses a given set of positive examples. In a subsequent study, Arimura et al. [9] proposed the concept of strong compactness of language containment for unions of regular pattern languages. Day et al. [10] established that pattern languages are, in general, not closed under standard language operations such as union, intersection, and complement. Matsumoto et al. [11] developed an efficient query learning algorithm for regular pattern languages that requires only a single positive example and a linear number of membership queries. More recently, Takeda et al. [12] proposed a query learning algorithm that utilizes a deep learning model trained on a set of strings as an oracle, enabling the learned features to be visualized as regular patterns. Subsequent research extended the study of regular patterns to Elementary Formal Systems (EFS) [13], thereby broadening the theoretical foundation of pattern languages. This extension inspired further work on tree patterns [14], [15] for generating tree languages, as well as on the development of Formal Graph Systems [16]. These advancements have facilitated the formalization and efficient learning of increasingly complex structured data beyond strings, fostering applications in domains such as grammatical inference and graph-based learning.

This paper is organized as follows. In Section 2, we formally define pattern languages and regular pattern languages, and subsequently present results concerning characteristic sets composed of symbols associated with regular pattern languages. In Sect.3, we provide characteristic sets consisting of strings of length two for $\mathcal{RPL}^k$. In Sect.4, we demonstrate that $\mathcal{RP}^k$ exhibit compactness with respect to language containment. In Sect.5, we propose regular patterns with non-adjacent variables, show that $S_2(P)$ derived from a set P in $\mathcal{RP}^k_{NAV}$ constitutes a characteristic set of $L(P)$, and establish that also $\mathcal{RP}^k_{NAV}$ exhibits compactness with respect to language containment.

2. Preliminaries

2.1 Basic Definitions and Notations

Let Σ be a non-empty finite set of constant symbols. Let X be an infinite set of variable symbols such that $\Sigma \cap X = \emptyset$ holds. Then, a *string* over $\Sigma \cup X$ is a sequence of symbols in $\Sigma \cup X$. Particularly, the string having no symbol is called the *empty string* and is denoted by ε. We denote by $(\Sigma \cup X)^*$ the set of all strings over $\Sigma \cup X$ and by $(\Sigma \cup X)^+$ the set of all strings over $\Sigma \cup X$ except ε, i.e., $(\Sigma \cup X)^+ = (\Sigma \cup X)^* \setminus \{\varepsilon\}$.

A *pattern* over $\Sigma \cup X$ is a string in $(\Sigma \cup X)^*$. Note that the empty string ε is a pattern over $\Sigma \cup X$. A pattern p is said to be *regular* if each variable symbol appears at most once in p. The length of p, denote by $|p|$, is the number of symbols in p. Note that $|\varepsilon| = 0$ holds. The sets of all patterns and regular patterns over $\Sigma \cup X$ are denoted by $\mathcal{P}_{\Sigma\cup X}$ and $\mathcal{RP}_{\Sigma\cup X}$, respectively. When Σ and X are clear from the context, we omit them in the notation and simply write $\mathcal{P}$ and $\mathcal{RP}$, respectively. For a set S, we denote by $\sharp S$ the number of elements in S. Let p, q be strings. If p and q are equal as strings, we denote it by $p = q$. We denote by $p \cdot q$ the string obtained from p and q by concatenating q after p. Note that for a string p and the empty string ε, $p \cdot \varepsilon = \varepsilon \cdot p = p$.

A *substitution* θ is a mapping from $(\Sigma \cup X)^*$ to $(\Sigma \cup X)^*$ such that (1) θ is a homomorphism with respect to string concatenation, i.e., $\theta(p \cdot q) = \theta(p) \cdot \theta(q)$ holds for patterns p and q, (2) $\theta(\varepsilon) = \varepsilon$ holds, (3) for each constant symbol $a \in \Sigma$, $\theta(a) = a$ holds, and (4) for each variable symbol $x \in X$, $|\theta(x)| \geq 1$ holds. Let $x_1, \ldots, x_n$ are variable symbols and $p_1, \ldots, p_n$ non-empty patterns. The notation $\{x_1 := p_1, \ldots, x_n := p_n\}$ denotes a substitution that replaces each variable symbol x_i with a non-empty pattern p_i for each $i \in \{1, \ldots, n\}$. For a pattern p and a substitution $\theta = \{x_1 := p_1, \ldots, x_n := p_n\}$, we denote by $p\theta$ a new pattern obtained from p by replacing variable symbols $x_1, \ldots, x_n$ in p with patterns $p_1, \ldots, p_n$ according to θ, respectively.

For a pattern p and q, the pattern q is a *generalization* of p, or p is an *instance* of q, denoted by $p \preceq q$, if there exists a substitution θ such that $p = q\theta$ holds. If $p \preceq q$ and $p \succeq q$ hold, we denote it by $p \equiv q$. The notation $p \equiv q$ means that p and q are equal as strings except for variable symbols. For a pattern p, the *pattern language* of p, denoted by $L(p)$, is the set $\{w \in \Sigma^* \mid w \preceq p\}$. For patterns p and q, it is clear that $L(p) = L(q)$ if $p \equiv q$, and $L(p) \subseteq L(q)$ if $p \preceq q$. Note that $L(\varepsilon) = \{\varepsilon\}$. In particular, if p is a regular pattern, we say that $L(p)$ is a *regular pattern language*. The sets of all pattern languages and regular patterns languages are denoted by $\mathcal{PL}$ and $\mathcal{RPL}$, respectively.

Lemma 1 (Mukouchi(Theorem 6.1, [3])): Suppose $\sharp\Sigma \geq 3$. Let p and q be regular patterns. Then $p \preceq q$ if and only if $L(p) \subseteq L(q)$.

Next, we consider unions of pattern languages. The class of all non-empty finite subsets of $\mathcal{P}$ is denoted by $\mathcal{P}^+$, i.e., $\mathcal{P}^+ = \{P \subseteq \mathcal{P} \mid 0 < \sharp P < \infty\}$. For a positive integer k (i.e., $k > 0$), we denote that the class of non-empty sets consisting of at most k patterns, i.e., $\mathcal{P}^k = \{P \subseteq \mathcal{P} \mid 0 < \sharp P \leq k\}$. For a set P of patterns, the pattern language of P, denoted by $L(P)$, is the set $\bigcup_{p \in P} L(p)$. We denote by $\mathcal{PL}^k$ the class of unions of at most k pattern languages, i.e., $\mathcal{PL}^k = \{L(P) \mid P \in \mathcal{P}^k\}$. In a similar way, we also define

$\mathcal{RP}^+$, $\mathcal{RP}^k$ and $\mathcal{RPL}^k$. For P, Q in $\mathcal{P}^+$, the notation $P \sqsubseteq Q$ means that for any $p \in P$ there is a pattern $q \in Q$ such that $p \preceq q$ holds. It is clear that $P \sqsubseteq Q$ implies $L(P) \subseteq L(Q)$. However, the converse is not valid in general.

2.2 Characteristic Sets Consisting of Symbols

Definition 1: Let C be a class of languages, L a language in C and S a non-empty finite subset of L. We say that S is a *characteristic set* of L within C if for any $L' \in C$, $S \subseteq L'$ implies $L \subseteq L'$.

Let n be a positive integer and p a regular pattern. We denote by $S_n(p)$ the set of all strings in Σ^* obtained by replacing all variable symbols in p with strings in Σ^+ of length at most n. Moreover, for a positive integer n and a set $P \in \mathcal{RP}^+$, let $S_n(P) = \bigcup_{p \in P} S_n(p)$. It is clear that $S_n(P) \subseteq S_{n+1}(P) \subseteq L(P)$ for any positive integer n.

Theorem 1 (Sato et al.(Theorem 8, [4])): Let k be a positive integer and $P \in \mathcal{RP}^k$. Then, there exists a positive integer n such that $S_n(P)$ is a characteristic set of $L(P)$ within $\mathcal{RPL}^k$.

Theorem 2 (Sato et al.(Lemma 9, [4])): Let p, q, p_1, p_2, q_1, q_2, q_3 be regular patterns in $\mathcal{RP}$ and x a variable symbol such that $p = p_1 x p_2$ and $q = q_1 q_2 q_3$ hold. Then $p \preceq q$ if the following three conditions (i), (ii) and (iii) are holds:

(i) $p_1 \preceq q_1 q_2$, (ii) $p_2 \preceq q_2 q_3$,
(iii) q_2 contains at least one variable symbol.

Lemma 2 (Sato et al.(Lemma 10, [4])): Suppose $\sharp\Sigma \geq 3$. Let p, q be regular patterns in $\mathcal{RP}_{\Sigma \cup X}$ and x a variable symbol. Let a, b and c be mutually distinct constant symbols in Σ. If $p\{x := a\} \preceq q$, $p\{x := b\} \preceq q$, and $p\{x := c\} \preceq q$ hold, then $p \preceq q$.

From Lemma 2, the following theorem holds.

Theorem 3 (Sato et al.(Theorem 11, [4])): Let k be a positive integer. Suppose $\sharp\Sigma \geq 2k+1$. For $P \in \mathcal{RP}^+$ and $Q \in \mathcal{RP}^k$, the following (i), (ii) and (iii) are equivalent:

(i) $S_1(P) \subseteq L(Q)$, (ii) $P \sqsubseteq Q$, (iii) $L(P) \subseteq L(Q)$.

From Theorem 3, we have the following corollary.

Corollary 1 (Sato et al.(Corollary 12, [4])): Suppose $\sharp\Sigma \geq 3$. For two regular patterns p, $q \in \mathcal{RP}_{\Sigma \cup X}$, the following (i), (ii) and (iii) are equivalent:

(i) $S_1(p) \subseteq L(q)$, (ii) $p \preceq q$, (iii) $L(p) \subseteq L(q)$.

The following lemma demonstrates that Theorem 3 does not hold in general when $\sharp\Sigma \leq 2k$. That is, the following lemma specifies the minimal cardinality of Σ required for Theorem 3 to hold.

Lemma 3 (Sato et al.(Lemma 13, [4])): Suppose $\sharp\Sigma \geq 3$. Let p_1, p_2, q_1, q_2 be regular patterns in $\mathcal{RP}_{\Sigma \cup X}$ and x a variable symbol. Let a, b be constant symbols in Σ with $a \neq b$ and w a string in Σ^*. Let $p = p_1 A w x w B p_2$ and $q = q_1 A w B q_2$ be regular patterns in $\mathcal{RP}_{\Sigma \cup X}$ satisfying the following three conditions:

(i) $p_1 A w \preceq q_1$,
(ii) $w B p_2 \preceq q_2$,
(iii) $(A, B) \in \{(a, b), (b, a)\}$.

Then, we have that $p\{x := a\} \preceq q$ and $p\{x := b\} \preceq q$ hold but $p \npreceq q$.

The following example 1 in [4] illustrates the failure of Theorem 3 under the condition of $\sharp\Sigma \leq 2k$, in accordance with Lemma 3.

Example 1: Let k be a positive integer and $\Sigma = \{a_1, \ldots, a_k, b_1, \ldots, b_k\}$. Let $w_1, \ldots, w_k$ be regular patterns in $\mathcal{RP}_{\Sigma \cup X}$ such that $w_k = \varepsilon$ and for $i = 1, 2, \ldots, k-1$, $w_i = w_{i+1} b_{i+1} a_{i+1} w_{i+1}$ hold. Let p, $q_1, \ldots, q_k$ be regular patterns in $\mathcal{RP}_{\Sigma \cup X}$ such that $p = x_1 a_1 w_1 x w_1 b_1 x_2$ and for $i = 1, 2, \ldots, k$, $q_i = x_1 a_i w_i b_i x_2$ hold. Let Q be a set $\{q_1, \ldots, q_k\}$ in $\mathcal{RP}^k$. For $i = 1$, we have $p\{x := a_1\} = (x_1 a_1 w_1) a_1 (w_1 b_1 x_2) = q_1\{x_1 := x_1 a_1 w_1\} \preceq q_1$. For $i \geq 2$, from the definition of w_i, we easily see that $w_1 = (w_i b_i) w^{(i)} = w'^{(i)} (a_i w_i)$ for some strings $w^{(i)}$ and $w'^{(i)}$. Then, for each $i \geq 2$,

$$\begin{aligned}
p\{x := a_i\} &= (x_1 a_1 w_1) a_i (w_1 b_1 x_2) \\
&= (x_1 a_1 w_1) a_i (w_i b_i w^{(i)}) b_1 x_2 \\
&= (x_1 a_1 w_1)(a_i w_i b_i)(w^{(i)} b_1 x_2) \\
&= q_i\{x_1 := x_1 a_1 w_1, x_2 := w^{(i)} b_1 x_2\} \\
&\preceq q_i, \\
p\{x := b_i\} &= (x_1 a_1 w_1) b_i (w_1 b_1 x_2) \\
&= x_1 a_1 (w'^{(i)} a_i w_i) b_i (w_1 b_1 x_2) \\
&= (x_1 a_1 w'^{(i)}) a_i w_i b_i (w_1 b_1 x_2) \\
&= q_i\{x_1 := x_1 a_1 w'^{(i)}, x_2 := w_1 b_1 x_2\} \\
&\preceq q_i.
\end{aligned}$$

Hence, $S_1(p) \subseteq L(Q)$ holds. However, from $p \npreceq q_i$, $L(p) \nsubseteq L(q_i)$ holds for each $i = 1, 2, \ldots, k$.

2.3 Basic word equations

Proposition 1: Let w be a string in Σ^* and a, b constant symbols in Σ. If

$$wa = bw \tag{1}$$

holds, then $a = b$ holds.

Proof. Since it is trivial, we omit the proof. □

Proposition 2: Let w be a string in Σ^* and a, b, c, d constant symbols in Σ. If

$$wda = bcw \tag{2}$$

holds, then $(b, c) \in \{(a, d), (d, a)\}$ holds.

Proof. We will prove this proposition by induction on the length of w (i.e., $|w|$).

- $|w| = 0, 1, 2,$ or 3: it is straightforward to observe that $(b, c) \in \{(a, d), (d, a)\}$ holds.
- $|w| \geq 4$: We assume that for any string u with $0 \leq |u| < n$, if $uda = bcu$ holds, $(b, c) \in \{(a, d), (d, a)\}$ holds. Since the string w has a prefix bc and a suffix da, there exists a string u with $|u| = |w| - 4 < |w|$ such that $w = bcuda$ holds. Since $wda = bcw$, we have $bcudada = bcbcuda$, and then $uda = bcu$. Thus, from the assumption, we get $(b, c) \in \{(a, d), (d, a)\}$.

From the above, we conclude that if $wda = bcw$ holds, then $(b, c) \in \{(a, d), (d, a)\}$ holds. □

The conclusion from Proposition 2 shows that $(a, d) \in \{(b, c), (c, b)\}$. Therefore, if the equation $daw = wbc$ holds, we arrive at the same conclusion.

Proposition 3: Let w, w' be strings of constant symbols in Σ and a, b, c, d constant symbols in Σ. If

$$wdaw' = w'bcw \tag{3}$$

holds, then $(b, c) \in \{(a, d), (d, a)\}$ holds.

Proof. We will prove this proposition by an induction on $|w| + |w'|$. Without loss of generality, we assume that $|w| \geq |w'|$ holds, since the case $|w| < |w'|$ similarly leads to the same conclusion that $(a, d) \in \{(b, c), (c, b)\}$ holds.

- $|w| \geq 0$ and $|w'| = 0$: Eq. (3) reduces to $wda = bcw$. By Proposition 2, $(b, c) \in \{(a, d), (d, a)\}$ holds.

We assume that for constant strings u and u' with $|u| + |u'| < |w| + |w'|$, if $udau' = u'bcu$ holds, then $(b, c) \in \{(a, d), (d, a)\}$ holds. We divide the relations between $|w|$ and $|w'|$ into the following four cases:

- $0 < |w'| \leq |w| \leq |w'| + 1$: When either $|w| = |w'|$ or $|w| = |w'| + 1$, Eq. (3) is illustrated in Figs. 1 and 2, respectively. If $|w| = |w'|$, $(b, c) = (d, a)$ holds. If $|w| = |w'| + 1$, $d = c$ and $w = w'b = aw'$ hold. From Proposition 1, we deduce that $b = a$. Therefore, $(b, c) \in \{(a, d), (d, a)\}$ holds.
- $|w'| + 2 \leq |w| \leq 2|w'| - 1$: In Eq. 3, since $|wdaw'| = |w'bcw| = |w| + |w'| + 2$, a suffix of w overlaps with a prefix of w, as illustrated in Fig. 3. That is, there exists a constant string u of length $2|w| - (|w| + |w'| + 2) = |w| - |w'| - 2$ such that u is both a prefix and a suffix of w. Since uda has a length of $|w| - |w'|$, it is also a prefix of w. Similarly, bcu is a suffix of w. Because $|w| - (|uda| + |bcu|) = 2|w| - |w'| \geq 1$, there exists a constant string u' of length $2|w'| - |w|$ such that $w = udau'bcu$ holds. Since w' is a suffix of w and $|u'bcu| = (2|w'| - |w|) + 2 + (|w| - |w'| - 2) = |w'|$, we have $w' = u'bcu$. Similarly, $w' = udau'$. Thus, we derive the equation $u'bcu = udau'$. Since $|u| = |w| - |w'| - 2 \leq |w| - 3 < |w|$ and $|u'| = 2|w'| - |w| < |w'|$, i.e., $|u| + |u'| < |w| + |w'|$, the induction hypothesis on $|u| + |u'|$ implies that $(b, c) \in \{(a, d), (d, a)\}$ holds.

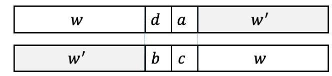

Fig. 1 Case $|w| = |w'|$ in Proposition 3

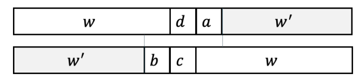

Fig. 2 Case $|w| = |w'| + 1$ in Proposition 3

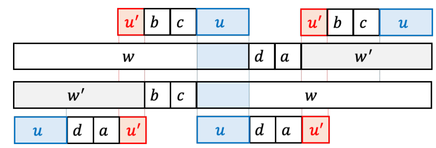

Fig. 3 Case $|w'| + 2 \leq |w| \leq 2|w'| - 1$ in Proposition 3

- $2|w'| \leq |w| \leq 2|w'| + 3$: When $|w| = 2|w'|$, it is straightforward to observe that $w = w'w'$. Therefore, $w'da = bcw'$ holds, as illustrated in Fig. 4. From Proposition 2, $(b, c) \in \{(a, d), (d, a)\}$ holds. When $|w| = 2|w'| + i$ $(i = 1, 2, 3)$, Eq. (3) is depicted in Figs. 5, 6, and 7, respectively. When $|w| = 2|w'| + 2$, it is clear that $(b, c) = (d, a)$. When $|w| = 2|w'| + 1$ and $|w| = 2|w'| + 3$, Proposition 1 implies that $(b, c) = (a, d)$ holds.
- $2|w'| + 4 \leq |w|$: Since the strings $w'bc$ and adw' are a prefix and a suffix of w, respectively, and $|w'bc| + |adw'| = 2|w'| + 4$, there exists a string u with $|u| \geq 0$ such that $w = w'bcudaw'$ holds. From Eq. (3), $w'bcudaw'daw' = w'bcw'bcudaw'$, i.e., $udaw' = w'bcu$ holds, as illustrated in Fig. 8. Let $u' = w'$. Since $|u| + |u'| = |w| - (2|w'| + 4) + |w'| < |w| + |w'|$, the induction hypothesis on $|u| + |u'|$ implies that $(b, c) \in \{(a, d), (d, a)\}$ holds.

From the above, we conclude that if $wdaw' = w'bcw$, then $(b, c) \in \{(a, d), (d, a)\}$ holds. □

3. Characteristic Sets Regular Patterns with Maximum Length Two

Let $D \subsetneq \mathcal{RP}_{\Sigma \cup X}$ with $\sharp D = 2$ or 3, and let p, q be regular patterns in $\mathcal{RP}_{\Sigma \cup X}$. In the following subsections (Subsecs. 3.1–3.4), we provide the conditions on D under which the implication holds: if $p\{x := r\} \preceq q$ for all $r \in D$, then $p\{x := xy\} \preceq q$. It is obvious if the variable symbol x does not appear in p. Therefore, in the following lemmas and propositions, let $p = p_1 x p_2$, where $p_i \in \mathcal{RP}$ $(i = 1, 2)$ and x is a variable symbol.

In Lemma 14 (ii) of [4], they stated that, when $\sharp\Sigma \geq 3$, for regular patterns p, q, if $p\{x := r\} \preceq q$ for any $r \in D$, then $p\{x := xy\} \preceq q$ holds, where $D = \{a_1b_1, a_2b_2, a_3b_3\}$ $(a_i \neq a_j$ and $b_i \neq b_j$ for each i, j $(i \neq j, 1 \leq i, j \leq 3))$. Unfortunately, there exist the following counterexamples of Lemma 14 (ii) of [4].

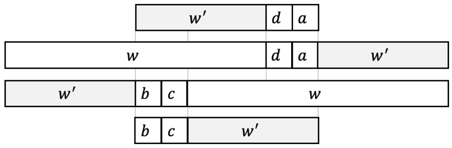

Fig. 4 Case $|w| = 2|w'|$ in Proposition 3

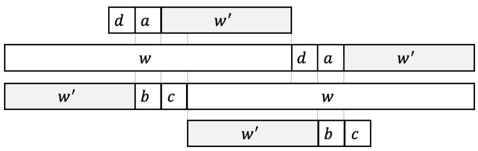

Fig. 5 Case $|w| = 2|w'| + 1$ in Proposition 3

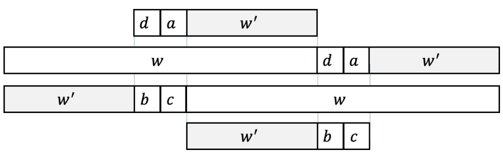

Fig. 6 Case $|w| = 2|w'| + 2$ in Proposition 3

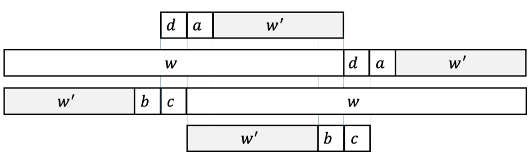

Fig. 7 Case $|w| = 2|w'| + 3$ in Proposition 3

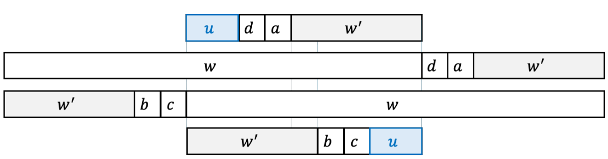

Fig. 8 Case $2|w'| + 4 \leq |w|$ in Proposition 3

Example 2: Assume that $a_1 = b_2$ and $a_3 = b_1$ hold. In the following two examples, we have that $p\{x := r\} \preceq q$ holds for $r \in D$.

(1) Let $p = ca_1xa_3c$ and $q = ya_1a_3z$ where c is a symbol in Σ. It is clear that $p\{x := xy\} \npreceq q$ holds. However, we can see that $p\{x := a_1b_1\} \preceq q$, $p\{x := a_2b_2\} \preceq q$ and $p\{x := a_3b_3\} \preceq q$ hold, since $p\{x := a_1b_1\} = ca_1a_1b_1a_3c = q\{y := ca_1, z := a_3c\}$, $p\{x := a_2b_2\} = ca_1a_2b_2a_3c = q\{y := ca_1a_2, z := c\}$ and $p\{x := a_3b_3\} = ca_1a_3b_3a_3c = q\{y := c, z := b_3a_3c\}$ hold.

(2) Let $p = cb_2a_1b_1b_2xa_1b_1b_2a_3c$ and $q = yb_2a_1b_1b_2a_3z$ where c is a symbol in Σ. It is clear that $p\{x := xy\} \npreceq q$ holds. However, we have $p\{x := a_1b_1\} \preceq q$, $p\{x := a_2b_2\} \preceq q$, and $p\{x := a_3b_3\} \preceq q$, since $p\{x := a_1b_1\} = cb_2a_1b_1b_2a_1b_1a_1b_1b_2a_3c = q\{y := cb_2a_1b_1, z := b_2a_3c\}$, $p\{x := a_2b_2\} = cb_2a_1b_1b_2a_2b_2a_1b_1b_2a_3c = q\{y := cb_2a_1b_1b_2a_2, z := c\}$, and $p\{x := a_3b_3\} = cb_2a_1b_1b_2a_3b_3a_1b_1b_2a_3c = q\{y := c, z := b_3a_1b_1b_2a_3c\}$ hold.

We consider the correspondence from $r \in D$ to some string in q when $p\{x := r\} \preceq q$ holds. The symbols in D correspond to either a variable or a constant symbol in q. If D has a constant string ab of length 2 for $a, b \in \Sigma$, there are three possible strings in q that correspond to ab in $p\{x := ab\}$ as follows: For $y_1 \in X$,

$$\text{(a)} \quad ab, \quad \text{(b)} \quad ay_1, \quad \text{(c)} \quad y_1b.$$

If there exists ay_1 in q that corresponds to ab, i.e., there exist q_1 and $q_2 \in \mathcal{RP}$ such that

(1) $p_1abp_2 \preceq q_1ay_1q_2$,
(2) $p_1 \preceq q_1$, and
(3) either $p_2 \preceq q_2$ or $p_2 \preceq y_1'q_2$ for $y_1' \in X$.

Let $D' = (D \setminus \{ab\}) \cup \{ay\}$. It is straightforward to see that $p\{x := ay\} = p_1ayp_2 \preceq q_1ay_1q_2$ holds. Thus, $p\{x := r\} \preceq q$ for all $r \in D'$ holds. Let $D'' = (D \setminus \{ab\}) \cup \{yb\}$. By a similar discussion, if there exists y_1b in q that corresponds to ab, $p\{x := r\} \preceq q$ for all $r \in D''$ holds. Therefore, in this paper, we make the following definition on D:

Definition 2: Let $p, q \in \mathcal{RP}$ with $p \npreceq q$. Let $D \subsetneq \mathcal{RP}$ such that for all $r \in D$, $|r| = 2$ and $p\{x := r\} \preceq q$ hold. Then, if for any $ab \in D$ $(a, b \in \Sigma)$, $ay, yb \notin D$, $p\{x := ay\} \npreceq q$ and $p\{x := yb\} \npreceq q$ hold for any $y \in X$ that does not appear in q, the regular pattern q is said to *minimally support D for p*, in the sense that any generalization of the strings in D would invalidate the conditions $p\{x := r\} \preceq q$ for all $r \in D$.

In Subsecs. 3.2–3.4, we consider a subset $D \subsetneq \mathcal{RP}_{\Sigma\cup X}$ such that a regular pattern $q \in \mathcal{RP}_{\Sigma\cup X}$ minimally support D for a regular pattern p.

3.1 $D = \{ay, by\}$ and $D = \{ya, yb\}$

Lemma 4: Let Σ be an alphabet with $\sharp\Sigma \geq 3$. Let p, q be regular patterns in $\mathcal{RP}_{\Sigma\cup X}$. Let D be the following set of regular patterns in $\mathcal{RP}_{\Sigma\cup X}$, where y is a variable symbol that does not appear in p and q:

(i) $D = \{ay, by\}$ $(a \neq b)$,
(ii) $D = \{ya, yb\}$ $(a \neq b)$.

Then, if $p\{x := r\} \preceq q$ holds for all $r \in D$, then $p\{x := xy\} \preceq q$ holds.

Proof. Case (ii) follows from case (i) by symmetry, upon reversing the strings p and q. Therefore, in the following, we consider only the case of (i): $D = \{ay, by\}$ $(a \neq b)$. We may assume $p \npreceq q$, since the case $p \preceq q$ is trivial. Since $p \npreceq q$, but $p_1ayp_2 \preceq q$ and $p_1byp_2 \preceq q$ hold, it follows from Theorem 2 that there exist regular patterns q_1, q_2 over $\Sigma \cup X$ such that $q = q_1ay_1wby_2q_2$ or $q = q_1by_1way_2q_2$ for some variable symbols y_1, y_2 with $y_1 \neq y_2$, and a constant string $w \in \Sigma^*$ with $|w| \geq 0$.

When $q = q_1ay_1wby_2q_2$, the following four conditions

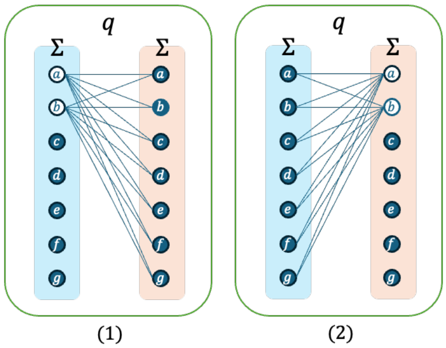

Fig. 9 Let $\Sigma = \{a, b, c, d, e, f, g\}$ and $p, q \in \mathcal{RP}_{\Sigma\cup X}$. We assume that the symbols in Σ are mutually distinct. These figures (1) and (2) express two cases $D = \{ay, by\}$ and $D = \{ya, yb\}$, respectively. In these cases, if $p\{x := r\} \preceq q$ for all $r \in D$, then $p\{x := xy\} \preceq q$ holds.

hold: For $y'_1, y'_2 \in X$,

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq wby_2q_2$ or $p_2 \preceq y'_1wby_2q_2$,
(2) $p_1 \preceq q_1ay_1w$, (2') $p_2 \preceq q_2$ or $p_2 \preceq y'_2q_2$.

From (2), there exist regular patterns p'_1, p''_1 such that $p_1 = p'_1p''_1$, $p'_1 \preceq q_1a$ and $p''_1 \preceq y_1w$ hold. Therefore, since $p = p_1xp_2 = p'_1p''_1xp_2$, if $p_2 \preceq wby_2q_2$ of (1') holds, $p \preceq q_1ap''_1xwby_2q_2 \equiv q\{y_1 := p''_1x\}$ holds. If $p_2 \preceq y'_1wby_2q_2$ of (1') holds, $p \preceq q_1ap''_1xy'_1wby_2q_2 = q\{y_1 := p''_1xy'_1\}$ holds. Thus, $p\{x := xy\} \preceq q\{y_1 := p''_1xyy'_1\}$ holds. Hence, $p\{x := xy\} \preceq q$ holds. Therefore, we conclude that if $p\{x := r\} \preceq q$ for all $r \in \{ay, by\}$ with $a \neq b$, then $p\{x := xy\} \preceq q$ holds. □

Let p, q be regular patterns in $\mathcal{RP}_{\Sigma\cup X}$. In this paper, the statement like Lemma 4 is illustrated by a bipartite graph (Σ, Σ, E) where $E = \{(a, b) \in \Sigma \times \Sigma \mid p\{x := ab\} \preceq q\}$. For example, the conditions (i) and (ii) in Lemma 4 are illustrated in (1) and (2) in Fig. 9, respectively.

3.2 $D = \{ya, bc, dy\}$

In this subsection, for a subset $D = \{ya, bc, dy\} \subsetneq \mathcal{RP}_{\Sigma\cup X}$, we consider a regular pattern q which minimally supports D for a regular pattern p under some conditions with the symbols a, b, c and d in D. Obviously, we remark that $a \neq c$ and $b \neq d$ hold, since q minimally supports D for p.

Lemma 5: Let Σ be an alphabet with $\sharp\Sigma \geq 3$ and p, q regular patterns in $\mathcal{RP}_{\Sigma\cup X}$. Let $D = \{ya, bc, dy\} \subsetneq \mathcal{RP}_{\Sigma\cup X}$ where $b \notin \{a, d\}, c \notin \{a, d\}$ and y is a variable symbol in X that does not appear in either p or q. Then, if q minimally supports D for p, then $p\{x := xy\} \preceq q$ holds.

Proof. We may assume $p \not\preceq q$, since the case $p \preceq q$ is trivial. We assume that $p\{x := xy\} \not\preceq q$ in order to derive a contradiction with the conditions that the symbols a, b, c and d satisfy in D. Since q minimally supported D for p, i.e., $p\{x := r\} \preceq q$ for all $r \in D$, the regular pattern q can be expressed in one of the following forms: Let y_1, y_2 be distinct variable symbols in X and q_1, q_2, w, w' be either the empty string or a regular pattern over $\Sigma \cup X$.

(5-1) $q = q_1AwBw'Cq_2$,
where $\{A, B, C\} = \{y_1a, bc, dy_2\}$,
(5-2) $q = q_1AwBq_2$,
where $\{A, B\} = \{dy_1a, bc\}$,
(5-3) $q = q_1AwBq_2$,
where $\{A, B\} = \{y_1ay_2, bc\}$ $(a = d)$.

We note the following observations regarding the behavior of substrings in the sequence q: In (5-1), each string in D occurs independently within q. In (5-2), the substring dya appears in q as a result of either variable sharing or adjacency between ya and dy in D. In (5-3), when $a = d$, the substring y_1ay_2, formed by a one-character overlap between ya and dy in D, is observed within q.

(5-1) Case of $q = q_1AwBw'Cq_2$, where $\{A, B, C\} = \{y_1a, bc, dy_2\}$: At first, we prove the following three claims:

Claim 1. $B \notin \{y_1a, dy_2\}$.

Proof of Claim 1. Suppose that $(A, B, C) = (dy_2, y_1a, bc)$. The following conditions must be satisfied: For $y'_1, y'_2 \in X$,

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq wy_1aw'bcq_2$ or $p_2 \preceq y'_2wy_1aw'bcq_2$,
(2) $p_1 \preceq q_1dy_2w$ or $p_1 \preceq q_1dy_2wy'_1$, (2') $p_2 \preceq w'bcq_2$,
(3) $p_1 \preceq q_1dy_2wy_1aw'$, (3') $p_2 \preceq q_2$.

When $p_2 \preceq wy_1aw'bcq_2$ in (1') holds, let $q'_1 = q_1dy_2$, $q'_2 = wy_1aw'$, $q'_3 = bcq_2$. Since $p_1 \preceq q_1dy_2wy_1aw'$ holds from (3) and y_2, y'_2 in X, both $p_1 \preceq q'_1q'_2$ and $p_2 \preceq q'_2q'_3$ hold, and q'_2 contains a variable symbol. When $p_2 \preceq y'_2wy_1aw'bcq_2$ in (1') holds, let $q'_1 = q_1d$, $q'_2 = y_2wy_1aw'$, $q'_3 = bcq_2$. Since $p_1 \preceq q_1dy_2wy_1aw'$ holds from (3), both $p_1 \preceq q'_1q'_2$ and $p_2 \preceq q'_2q'_3$ hold, and q'_2 contains a variable symbol. In both cases, by Theorem 2, $p \preceq q$ holds. This contradicts the assumption that p and q satisfy $p \not\preceq q$.

Similarly, we can show that any case where $(A, B, C) = (y_1a, dy_2, bc)$, (bc, y_1a, dy_2), or (bc, dy_2, y_1a) also contradicts the assumption. Therefore, we have $B \notin \{y_1a, dy_2\}$. (*End of Proof of Claim* 1)

Claim 2. $(A, B, C) = (dy_2, bc, y_1a)$.

Proof of Claim 2. From *Claim* 1, we have $B = bc$. Suppose that $(A, B, C) = (dy_2, bc, y_1a)$, i.e., $q = q_1dy_2wbcw'y_1aq_2$ holds. Then, the following conditions must be satisfied: For $y'_1, y'_2 \in X$,

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq wbcw'y_1aq_2$ or $p_2 \preceq y'_2wbcw'y_1aq_2$,
(2) $p_1 \preceq q_1dy_2w$, (2') $p_2 \preceq w'y_1aq_2$,
(3) $p_1 \preceq q_1dy_2wbcw'$ or $p_1 \preceq q_1dy_2wbcw'y'_1$, (3') $p_2 \preceq q_2$.

From $p_1 \preceq q_1dy_2w$ in (2), p_1 is expressed as $p'_1p''_1$

for some p'_1 and p''_1, where $p'_1 \preceq q_1 d$ and $p''_1 \preceq y_2 w$. When $p_2 \preceq wbcw'y_1aq_2$ in (1'), we have $p = p_1xp_2 = p'_1p''_1xp_2 \preceq q_1dp''_1xwbcw'y_1aq_2 = q\{y_2 := p''_1x\}$. Thus, $p\{x := xy\} \preceq q\{y_2 := p''_1xy\}$ holds. This contradicts the assumption that p and q satisfy $p \npreceq q$. When $p_2 \preceq y'_2wbcw'y_1aq_2$ in (1'), we similarly have $p = p_1xp_2 = p'_1p''_1xp_2 \preceq q_1dp''_1xy'_2wbcw'y_1aq_2 = q\{y_2 := p''_1xy'_2\}$. Thus, $p\{x := xy\} \preceq q\{y_2 := p''_1xyy'_2\}$ holds. This also contradicts the assumption that p and q satisfy $p \npreceq q$. Therefore, we conclude that $(A, B, C) = (y_1a, bc, dy_2)$. (*End of Proof of Claim* 2)

From *Claim* 2, The regular pattern q is expressed as $q_1y_1awbcw'dy_2q_2$, where $b \notin \{a, d\}$ and $c \notin \{a, d\}$. If $p\{x := xy\} \npreceq q$ holds, the following conditions must be satisfied: For $y'_1, y'_2 \in X$,

(1) $p_1 \preceq q_1$ or $p_1 \preceq q_1y'_1$, (1') $p_2 \preceq wbcw'dy_2q_2$,
(2) $p_1 \preceq q_1y_1aw$, (2') $p_2 \preceq w'dy_2q_2$,
(3) $p_1 \preceq q_1y_1awbcw'$, (3') $p_2 \preceq q_2$ or $p_2 \preceq y'_2q_2$.

Claim 3. w and w' contain no variable symbols.

Proof of Claim 3. Let $q'_1 = q_1y_1a$, $q'_2 = wbcw'$, and $q'_3 = dy_2q_2$. From (1') and (3), $p_1 \preceq q'_1q'_2$ and $p_2 \preceq q'_2q'_3$. If q'_2 contains a variable symbol, then by Theorem 2, $p \preceq q$ holds. This contradicts the assumption that p and q satisfy $p \npreceq q$. Therefore, w and w' contain no variable symbols. (*End of Proof of Claim* 3)

From *Claim* 3, w and w' are strings consisting of symbols in Σ. From (1') and (2'), both $wbcw'd$ and $w'd$ are prefixes of p_2, and from (2) and (3), both $awbcw'$ and aw are suffixes of p_1. It implies a contradiction in the following inductive way:

- $|w| = |w'|$: Directly, $b = d$ and $a = c$ hold.
- $|w| = |w'| + 1$: Also, $a = b$ holds.
- $|w| = |w'| + 2$: Since both $awbcw'$ and aw are suffixes of p_1, and $|w| \geq 2$, a is a suffix of w. From (1') and (2'), we have $w = w'da$. Furthermore, since both $awbcw'$ and aw are suffixes of p_1, it follows that $w = bcw'$. Thus, $w'da = bcw'$ holds. From Proposition 2, $(b, c) \in \{(a, d), (d, a)\}$ holds. Therefore, these cases contradict the conditions $b \notin \{a, d\}$ and $c \notin \{a, d\}$.
- $|w| \geq |w'| + 3$: From (2) and (3), there exists a string w'' of length $|w| - |w'| - 2$ such that $w = w''bcw'$ holds. Moreover, from (2) and (3), since $|aw| < |wbcw'|$ and $aw = aw''bcw'$, it follows that aw'' is a suffix of w. On the other hand, from (1') and (2'), $w'd$ is a prefix of w. Since $|w'd| + |aw''| = |w'| + |w''| + 2 = |w|$, it follows that $w = w'daw''$ (Fig. 10). Therefore, $w'daw'' = w''bcw'$ holds. From Proposition 3, $(b, c) \in \{(a, d), (d, a)\}$ holds. This contradicts the conditions $b \notin \{a, d\}$ and $c \notin \{a, d\}$.
- $|w| < |w'|$: The proof can be established in a manner analogous to the case where $|w| > |w'|$.

From the above, we conclude that all cases of (5-1) contradict the assertion that $p \npreceq q$ and the conditions $b \notin \{a, d\}$ and $c \notin \{a, d\}$.

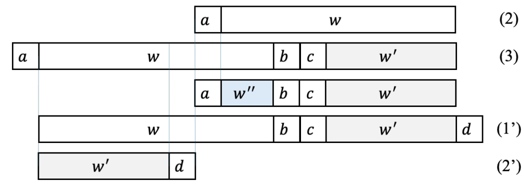

Fig. 10 Case (5-1) in Lemma 5: Relation of strings w, w', and w''

(5-2) Case of $q = q_1AwBq_2$, where $\{A, B\} = \{dy_1a, bc\}$: We suppose that $(A, B) = (dy_1a, bc)$, i.e., $q = q_1dy_1awbcq_2$ holds. Then, the following conditions must be satisfied for $y'_1 \in X$:

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq awbcq_2$ or $p_2 \preceq y'_1awbcq_2$,
(2) $p_1 \preceq q_1d$ or $p_1 \preceq q_1dy'_1$ (2') $p_2 \preceq wbcq_2$,
(3) $p_1 \preceq q_1dy_1aw$, (3') $p_2 \preceq q_2$.

From $p_1 \preceq q_1dy_1aw$ in (3), p_1 can be expressed as $p'_1p''_1$ for some p'_1 and p''_1, where $p'_1 \preceq q_1d$ and $p''_1 \preceq y_1aw$. When $p_2 \preceq awbcq_2$ in (1'), we have

$$p = p'_1p''_1xp_2 \preceq q_1dp''_1xawbcq_2 = q\{y_1 := p''_1x\}.$$

Thus, $p\{x := xy\} \preceq q\{y_1 := p''_1xy\}$ holds. This contradicts the assumption that p and q satisfy $p \npreceq q$. When $p_2 \preceq y'_1awbcq_2$ in (1'), we similarly have

$$p = p'_1p''_1xp_2 \preceq q_1dp''_1xy'_1awbcq_2 = q\{y_1 := p''_1xy'_1\}.$$

This contradicts the assumption that p and q satisfy $p \npreceq q$. Similarly, we can show that the case $(A, B) = (bc, dy_1a)$ also contradicts the assumption.

(5-3) Case of $q = q_1AwBq_2$, where $\{A, B\} = \{y_1ay_2, bc\}$ ($a = d$): Suppose that $(A, B) = (y_1ay_2, bc)$, i.e., $q = q_1y_1ay_2wbcq_2$ holds. Then, the following conditions must be satisfied: For $y'_1, y'_2 \in X$,

(1) $p_1 \preceq q_1$ or $p_1 \preceq q_1y'_1$ (1') $p_2 \preceq y_2wbcq_2$,
(2) $p_1 \preceq q_1y_1ay_2$, (2') $p_2 \preceq wbcq_2$ or $p_2 \preceq y'_2wbcq_2$,
(3) $p_1 \preceq q_1y_1ay_2w$, (3') $p_2 \preceq q_2$.

Let $q'_1 = q_1y_1a$, $q'_2 = y_2w$, $q'_3 = bcq_2$. From (3) and (1'), we have $p_1 \preceq q'_1q'_2$ and $p_2 \preceq q'_2q'_3$, respectively. Since q'_2 contains a variable symbol, Theorem 2 implies that $p \preceq q$ holds. This contradicts the assumption that p and q satisfy $p \npreceq q$. Similarly, we can show that the case $(A, B) = (bc, y_1ay_2)$ also contradicts the assumption.

From the above, we conclude that if $p\{x := r\} \preceq q$ for all $r \in \{ya, bc, dy\}$ ($b \notin \{a, d\}$ and $c \notin \{a, d\}$), then $p\{x := xy\} \preceq q$ holds. □

Note that Lemma 5 is valid under the condition that $\sharp\Sigma \geq 2$. The condition in Lemma 5 is illustrated in four cases (3)–(6)

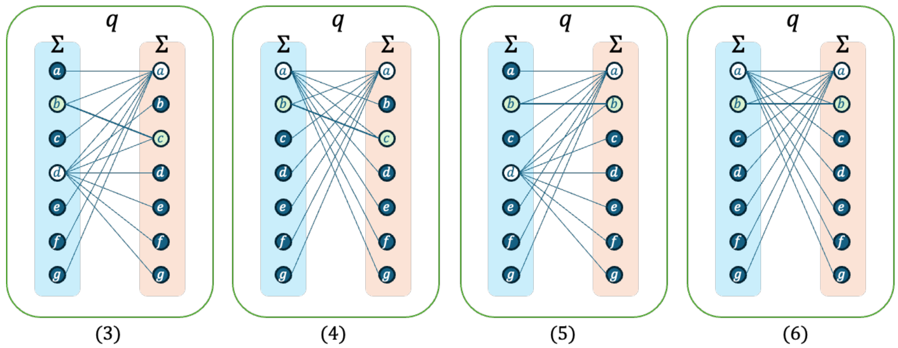

Fig. 11 Let $\Sigma = \{a, b, c, d, e, f, g\}$ and $p, q \in \mathcal{RP}_{\Sigma\cup X}$. We assume that the symbols in Σ are mutually distinct. The figure (3) expresses case $D = \{ya, bc, dy\}$ in Lemma 5. The figures (4), (5), and (6) express three cases $D = \{ya, bc, ay\}$, $D = \{ya, bb, dy\}$, and $D = \{ya, bb, ay\}$, respectively. In these cases, if $p\{x := r\} \preceq q$ for all $r \in D$, then $p\{x := xy\} \preceq q$ holds.

in Fig. 11.

Lemma 6: Let Σ be an alphabet with $\sharp\Sigma \geq 3$, p, q regular patterns in $\mathcal{RP}_{\Sigma\cup X}$ and $D = \{ya, bc, dy\} \subsetneq \mathcal{RP}_{\Sigma\cup X}$ where $b = a$, $b \neq d, c \notin \{a, d\}$, and y is a variable symbol in X that does not appear in p and q. Then, if q minimally supports D for p, then $p\{x := xy\} \preceq q$ holds.

Proof. We may assume $p \npreceq q$, since the case $p \preceq q$ is trivial. We assume that $p\{x := xy\} \npreceq q$ in order to derive a contradiction. The proof is almost the same as the proof of Lemma 5. Since q minimally supports D for p, i.e., $p\{x := r\} \preceq q$ for all $r \in D$, there are three strings of length 2 corresponding to ya, bc, dy in q. The symbols appearing in D correspond to either a variable or a constant symbol in q. Let y_1 and y_2 be variable symbols appearing in q. The strings ya and dy must correspond to the strings y_1a and dy_2 in q, respectively. For the same reasons stated at the beginning of Lemma 5, the string bc corresponds to the string bc in q as well. Let A, B, C be regular patterns over $\Sigma \cup X$, where $\{A, B, C\} = \{y_1a, ac, dy_3\}$. Since $p\{x := xy\} \npreceq q$, q can be expressed in one of the following four forms: Let y_1, y_2 be distinct variable symbols in X, and q_1, q_2, w, w' either the empty string or a regular pattern over $\Sigma \cup X$. From the conditions $b = a$ and $b \neq d$, it follows that $a \neq d$.

(6-1) $q = q_1AwBw'Cq_2$, where $\{A, B, C\} = \{y_1a, ac, dy_2\}$,
(6-2) $q = q_1AwBq_2$, where $\{A, B\} = \{y_1ac, dy_2\}$,
(6-3) $q = q_1Aq_2$, where $A = dy_1ac$.

In cases (6-1) and (6-2), similar to Lemma 5, it is shown that $q = q_1y_1awacw'dy_2q_2$ and $q = q_1y_1acwdy_2q_2$, respectively, where w and w' contain no variable symbols.

(6-1) Case of $q = q_1AwBw'Cq_2$, where $(A, B, C) = (y_1a, ac, dy_2)$: The following conditions must be satisfied:

(1) $p_1 \preceq q_1$,	(1') $p_2 \preceq wacw'dy_2q_2$,
(2) $p_1 \preceq q_1y_1aw$,	(2') $p_2 \preceq w'dy_2q_2$,
(3) $p_1 \preceq q_1y_1awacw'$,	(3') $p_2 \preceq q_2$.

From (1') and (2'), both $wacw'd$ and $w'd$ are prefixes of p_2, and from (2) and (3), both $awacw'$ and aw are suffixes of p_1. It implies a contradiction in the following inductive way:

- $|w| = |w'|$: $c = a$ holds.
- $|w| = |w'| + 1$: $w = w'd = cw'$ holds. Thus, from Proposition 1, $c = d$ holds.
- $|w| = |w'| + 2$: $w = w'da = acw'$ holds. From Proposition 2, $c \in \{a, d\}$ holds.
- $|w| \geq |w'| + 3$: From (2) and (3), there exists a string w'' of length $|w| - |w'| - 2$ such that $w = w''acw'$ holds. Moreover, from (2) and (3), since $|aw| < |wacw'|$ and $aw = aw''acw'$, it follows that aw'' is a suffix of w. On the other hand, from (1') and (2'), $w'd$ is a prefix of w. Since $|w'd| + |aw''| = |w'| + |w''| + 2 = |w|$, we have $w = w'daw''$. Therefore, $w'daw'' = w''acw'$ holds (Fig. 12). From Proposition 3, we have $c \in \{a, d\}$.
- $|w'| = |w| + 1$: From (1') and (2'), $c = d$ holds.
- $|w'| = |w| + 2$: From (1') and (2'), d is a prefix of w'. Thus, from (2) and (3), $w' = wac = daw$ holds. From Proposition 2, $c \in \{a, d\}$ holds.
- $|w'| \geq |w| + 3$: From (1') and (2'), there exists a string w'' of length $|w| - |w'| - 2$ such that $w' = wacw''$ holds. Moreover, from (1') and (2'), since $|w'd| < |wacw'|$ and $w'd = wacw''d$, $w''d$ is a prefix of w'. On the other hand, from (2) and (3), aw' is a suffix of w'. Since $|w''d| + |aw| = |w''| + |w| + 2 = |w'|$, we have $w' = w''daw$. Therefore, $w''daw = wacw''$ holds. From Proposition 3, we have $c \in \{a, d\}$.

All the cases contradict the condition $c \notin \{a, d\}$. Therefore, if $b = a$, $b \neq d$, and $c \notin \{a, d\}$ are satisfied, case (6-1) is impossible.

(6-2) Case of $q = q_1AwBq_2$, where $(A, B) = (y_1ac, dy_2)$: For $q = q_1y_1acwdy_2q_2$, the following conditions must be satisfied:

(1) $p_1 \preceq q_1$,	(1') $p_2 \preceq cwdy_3q_2$,
(2) $p_1 \preceq q_1y_1$,	(2') $p_2 \preceq wdy_3q_2$,
(3) $p_1 \preceq q_1y_1acwd$,	(3') $p_2 \preceq q_2$.

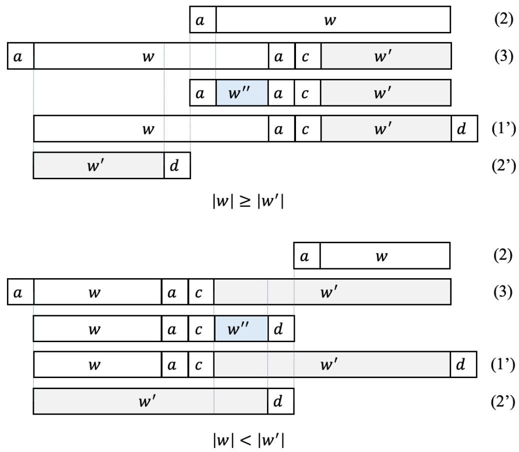

Fig. 12 Case (6-1) in Lemma 6: Relation of strings w, w', and w''

This leads to a contradiction, as demonstrated by the following inductive argument:

- If $|w| = 0$, from (1') and (2'), both cd and d are prefixes of p_2. Thus, we have $c = d$.
- If $|w| = 1$, from (1') and (2'), both cwd and wd are prefixes of p_2. Thus, we have $w = c = d$.
- If $|w| \geq 2$, then from (1') and (2'), both cwd and wd are prefixes of p_2. Thus, we have $cw = wd$. From Proposition 2, $c = d$ holds.

All of these cases do not meet $b = a$, $b \neq d$, and $c \notin \{a, d\}$. Therefore, if $b = a$, $b \neq d$, and $c \notin \{a, d\}$ are satisfied, case (6-2) is also impossible.

(6-3) Case of $q = q_1 A q_2$, where $A = dy_1ac$: For $q = q_1 dy_1 ac q_2$, the following conditions must be satisfied for $y_1', y_1'' \in X$:

(1) $p_1 \preceq q_1 d$ or $p_1 \preceq q_1 dy_1'$	(1') $p_2 \preceq acq_2$,
(2) $p_1 \preceq q_1 dy_1$,	(2') $p_2 \preceq q_2$,
(3) $p_1 \preceq q_1$,	(3') $p_2 \preceq acq_2$ or $p_2 \preceq y_1'' acq_2$.

For $p_1 \preceq q_1 d$ in (1) and $p_2 \preceq acq_2$ in (3'), $p = p_1 x p_2 \preceq q_1 dxacq_2 \preceq q\{y_1 := x\}$ holds. From this, we have $p \preceq q$. This contradicts the assumption that p and q satisfy $p \not\preceq q$. Similarly, we can show that the other cases of (1) and (3') also contradict the assumption.

From the above, we conclude that if $p\{x := r\} \preceq q$ for all $r \in \{ya, bc, dy\}$ ($b = a$, $b \neq d$, and $c \notin \{a, d\}$), then $p\{x := xy\} \preceq q$ holds. □

Lemma 7: Let Σ be an alphabet with $\sharp\Sigma \geq 3$, p, q be regular patterns in $\mathcal{RP}_{\Sigma \cup X}$ and $D = \{ya, bc, dy\} \subsetneq \mathcal{RP}_{\Sigma \cup X}$ where $b \notin \{a, d\}$, $c \neq a, c = d$, and y is a variable symbol in X that does not appear in p and q. Then, if q minimally supports D for p, then $p\{x := xy\} \preceq q$ holds.

Proof. The proof follows by reversing p and q and subsequently applying Lemma 6. □

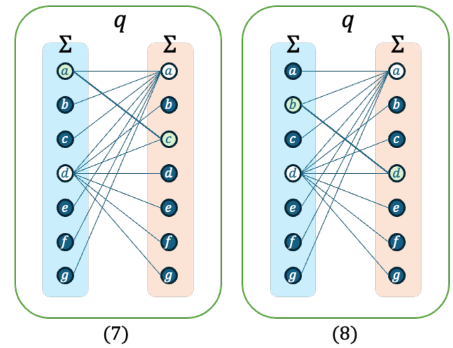

Fig. 13 Let $\Sigma = \{a, b, c, d, e, f, g\}$ and $p, q \in \mathcal{RP}_{\Sigma \cup X}$. We assume that the symbols in Σ are mutually distinct. The figures (7) and (8) express two cases $D = \{ya, ac, dy\}$ and $D = \{ya, bd, dy\}$ in Lemmas 6 and 7, respectively. In these cases, if $p\{x := r\} \preceq q$ for all $r \in D$, then $p\{x := xy\} \preceq q$ holds.

The conditions in Lemmas 6 and 7 are illustrated in (7) and (8) in Fig. 13, respectively.

When the conditions of Lemmas 5, 6, and 7 are not satisfied, counterexamples can be constructed as follows:

Proposition 4: Let Σ be an alphabet. For a variable symbol y, let $D = \{ya, bc, dy\}$ ($b = a$ and $c = d$). There exist regular patterns p and q in $\mathcal{RP}_{\Sigma \cup X}$ such that q minimally supports D for p (i.e., $p \not\preceq q$, $p\{x := r\} \preceq q$ for any $r \in D$, $p\{x := by\} \not\preceq q$ and $p\{x := yc\} \not\preceq q$), but $p\{x := xy\} \not\preceq q$.

Proof. We provide an example to demonstrate this proposition. Let a, b, c, d, e be constant symbols in Σ, and let x, y, y_1, y_2 be variable symbols in X. Define the regular patterns p and q as follows:

$$p = eabcbcadabcbcadaxbcadadabcbcadade,$$
$$q = y_1 abcbcadabcbcadady_2 \quad (b = a \text{ and } c = d).$$

Obviously $p\{x := xy\} \not\preceq q$ holds. For these p and q, the condition for Proposition 4 holds as follows (see also Fig. 14):

$$\begin{aligned}
& p\ \{x := ya\} \\
&= (eabcbcadabcbcaday)abcadadabcbcadade \\
&= q\{y_1 := eabcbcadabcbcaday,\ y_2 := e\} \\
&\preceq q, \\
& p\ \{x := bc\} \\
&= (eabcbcad)abcbcadabcbcadad(abcbcadade) \\
&= q\{y_1 := eabcbcad,\ y_2 := abcbcadade\} \\
&\preceq q, \\
& p\ \{x := dy\} \\
&= eabcbcadabcbcadad(ybcadadabcbcadade) \\
&= q\{y_1 := e,\ y_2 := ybcadadabcbcadade\} \\
&\preceq q.
\end{aligned}$$

□

3.3 $D = \{a_1b_1, a_2b_2, a_3y\}$ and $D = \{a_1b_1, a_2b_2, yb_3\}$

Fig. 14 Substitutions for p and each correspondence to q.

In this subsection, for $D = \{a_1b_1, a_2b_2, a_3y\} \subsetneq \mathcal{RP}_{\Sigma\cup X}$ or $D = \{a_1b_1, a_2b_2, yb_3\} \subsetneq \mathcal{RP}_{\Sigma\cup X}$, we consider a regular pattern q in $\mathcal{RP}_{\Sigma\cup X}$ such that q minimally supports D for a regular pattern p. Obviously, we remark that $D = \{a_1b_1, a_2b_2, a_3y\}$ and $D = \{a_1b_1, a_2b_2, yb_3\}$ satisfy that $a_i \neq a_3$ and $b_i \neq b_3$ for $i = 1, 2$, respectively.

Lemma 8: Let Σ be an alphabet with $\sharp\Sigma \geq 3$, p, q regular patterns in $\mathcal{RP}_{\Sigma\cup X}$ and $D = \{a_1b_1, a_2b_2, a_3y\} \subsetneq \mathcal{RP}_{\Sigma\cup X}$ where $a_i \neq a_j$ for $i, j (1 \leq i, j \leq 3, i \neq j)$,$b_1 \neq b_2$ and y is a variable symbol in X that does not appear in p and q. Then, if q minimally supports D for p, then $p\{x := xy\} \preceq q$ holds.

Proof. We may assume $p \npreceq q$, since the case $p \preceq q$ is trivial. We assume that $p\{x := xy\} \npreceq q$ holds. Since q minimally supports D for p, i.e., $p\{x := r\} \preceq q$ for all $r \in D$, from the same argument as in the proof of Lemma 6, it is sufficient to consider the following five cases (8-1)–(8-5) of q: For $y_1 \in X$,

(8-1) $q = q_1a_1b_1wa_2b_2w'a_3y_1q_2$,
(8-2) $q = q_1a_1b_1b_2y_1q_2$ ($a_2 = b_1$ and $a_3 = b_2$),
(8-3) $q = q_1a_1b_1b_2wa_3y_1q_2$ ($b_1 = a_2$),
(8-4) $q = q_1a_3y_1wa_1b_1b_2q_2$ ($b_1 = a_2$),
(8-5) $q = q_1a_1b_1y_1wa_2b_2q_2$ ($b_1 = a_3$),

where no variable symbol appears in both w and w'.

(8-1) Case of $q = q_1a_1b_1wa_2b_2w'a_3y_1q_2$: The following conditions must be satisfied: For $y'_1 \in X$,

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq wa_2b_2w'a_3y_1q_2$,
(2) $p_1 \preceq q_1a_1b_1w$, (2') $p_2 \preceq w'a_3y_1q_2$,
(3) $p_1 \preceq q_1a_1b_1wa_2b_2w'$, (3') $p_2 \preceq q_2$ or $p_2 \preceq y'_1q_2$.

This leads to a contradiction, as demonstrated by the following inductive argument:

- $|w| = |w'|$: From (1') and (2'), we have $a_2 = a_3$. This contradicts that $a_2 \neq a_3$ holds.
- $|w|+1 = |w'|$: From (2) and (3), both $a_1b_1wa_2b_2w'$ and a_1b_1w are suffixes of p_1. Since there exists a constant symbol w_1 such that $w' = w_1w$ and $b_2w_1w = a_1b_1w$ hold, then $b_2 = a_1$. Moreover, both $wa_2b_2w'a_3$ and $w'a_3$ are prefixes of p_2 from (1') and (2'). Since there exists a constant symbol w_2 such that $w' = ww_2$ and $wa_2b_2 = ww_2a_3$ hold, then $b_2 = a_3$. Thus, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.
- $|w|+1 < |w'|$: From (2) and (3), both $a_1b_1wa_2b_2w'$ and a_1b_1w are suffixes of p_1. Hence, a_1b_1w is the suffix of w'. Moreover, both $wa_2b_2w'a_3$ and $w'a_3$ are prefixes of p_2 from (1') and (2'). Hence, there exist constant strings w_1 and w_2 such that $w' = w_1w$, $w' = ww_2$ and $|a_2b_2w_1| = |w_2a_3| + 1$ hold. Thus, since the second-to-last symbol of w_1 is a_3, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.
- $|w| = |w'| + 1$: From (1') and (2'), both $wa_2b_2w'a_3$ and $w'a_3$ are prefixes of p_2. Since there exists a constant symbol w_1 such that $w = w'w_1$ and $w'w_1 = w'a_3$ hold, then $w_1 = a_3$ holds. Moreover, since both $a_1b_1wa_2b_2w'$ and a_1b_1w are suffixes of p_1 from (2) and (3), there exists a constant symbol w_2 such that $w = w_2w'$ and $w_1a_2b_2w' = a_1b_1w_2w'$ hold. Hence, $w_1 = a_1$ holds. Thus, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.
- $|w| > |w'| + 1$: Since both $wa_2b_2w'a_3$ and $w'a_3$ are prefixes of p_2 from (1') and (2'), there exists a constant string w_1 such that $w = w'w_1$ and the first symbol of w_1 is a_3. Moreover, since there exists a constant string w_2 such that $w = w_2w'$ and $w_1a_2b_2 = a_1b_1w_2$ hold, a_1b_1 is a prefix of w_1. Thus, $a_3 = a_1$ holds. This contradicts the assumption of $a_1 \neq a_3$.

(8-2) Case of $q = q_1a_1b_1b_2y_1q_2$ ($a_2 = b_1$ and $a_3 = b_2$): The following conditions must be satisfied: For $y'_1 \in X$,

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq b_2y_1q_2$,
(2) $p_1 \preceq q_1a_1$, (2') $p_2 \preceq y_1q_2$,
(3) $p_1 \preceq q_1a_1b_1$, (3') $p_2 \preceq q_2$ or $p_2 \preceq y'_1q_2$.

From (2) and (3), both a_1b_1 and a_1 are suffixes of p_1. Hence, $b_1 = a_1$ holds. Thus, from the assumption of $b_1 = a_2$, $a_1 = a_2$ holds. This contradicts the assumption of $a_1 \neq a_2$.

(8-3) Case of $q = q_1a_1b_1b_2wa_3y_1q_2$ ($b_1 = a_2$): The following conditions must be satisfied: For $y'_1 \in X$,

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq b_2wa_3y_1q_2$,
(2) $p_1 \preceq q_1a_1$, (2') $p_2 \preceq wa_3y_1q_2$,
(3) $p_1 \preceq q_1a_1b_1b_2w$, (3') $p_2 \preceq q_2$ or $p_2 \preceq y'_1q_2$.

This leads to a contradiction, as demonstrated by the following inductive argument:

- $|w| = 0$: From (2) and (3), both a_1 and $a_1b_1b_2$ are suffixes of p_1. Hence, $a_1 = b_2$ holds. Moreover, since both b_2a_3 and a_3 is prefixes of p_2, $b_2 = a_3$ holds. Thus, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.

- $|w| \geq 1$: Since both a_1 and $a_1b_1b_2w$ are suffixes of p_1 from (2) and (3), the last symbol of w is a_1. Moreover, since both b_2wa_3 and wa_3 are prefixes of p_2 from (1') and (2'), the last symbol of w is a_3. Thus, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.

(8-4) Case of $q = q_1a_3y_1wa_1b_1b_2q_2$ ($b_1 = a_2$): The following conditions must be satisfied: For $y_1' \in X$,

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq wa_1b_1b_2q_2$ or $p_2 \preceq y_1'wa_1b_1b_2q_2$,
(2) $p_1 \preceq q_1a_3y_1w$, (2') $p_2 \preceq b_2q_2$,
(3) $p_1 \preceq q_1a_3y_1wa_1$, (3') $p_2 \preceq q_2$.

From (3), there exist regular patterns p_1' and p_1'' such that $p_1 = p_1'p_1''$, $p_1' \preceq q_1a_3$, and $p_1'' \preceq y_1wa_1$ hold. Hence, if $p_2 \preceq wa_1b_1b_2q_2$ of (1') holds, since $p = p_1xp_2 = p_1'p_1''xp_2 \preceq q_1a_3p_1''xwa_1b_1b_2q_2 = q\{y_1 := p_1''x\}$, then $p \preceq q$ holds. Thus, this contradicts the assumption that p and q satisfy $p \npreceq q$. Similarly, $p_2 \preceq y_1'wa_1b_1b_2q_2$ of (1') leads to a contradiction.

(8-5) Case of $q = q_1a_1b_1y_1wa_2b_2q_2$ ($b_1 = a_3$): The following conditions must be satisfied: For $y_1' \in X$,

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq y_1wa_2b_2q_2$,
(2) $p_1 \preceq q_1a_1$, (2') $p_2 \preceq wa_2b_2q_2$ or $p_2 \preceq y_1'wa_2b_2q_2$,
(3) $p_1 \preceq q_1a_1b_1y_1w$, (3') $p_2 \preceq q_2$.

Let $q_1' = q_1a_1b_1$, $q_2' = y_1w$, $q_3' = a_2b_2q_2$. From (3), $p_1 \preceq q_1'q_2'$ holds, and from (1'), $p_2 \preceq q_2'q_3'$ holds. Since q_2' contains a variable symbol y_1, $p \preceq q$ holds from Theorem 2. This contradicts the assumption that p and q satisfy $p \npreceq q$. □

Lemma 9: Let Σ be an alphabet $\sharp\Sigma \geq 3$, p, q regular patterns in $\mathcal{RP}_{\Sigma\cup X}$ and $D = \{a_1b_1, a_2b_2, yb_3\} \subsetneq \mathcal{RP}_{\Sigma\cup X}$ where $a_1 \neq a_2$, $b_i \neq b_j$ for $1 \leq i, j \leq 3$ with $i \neq j$ and y is a variable symbol in X that does not appear in p and q. Then, if q minimally supports D for p, then $p\{x := xy\} \preceq q$ holds.

Proof. The proof follows by reversing p and q and subsequently applying Lemma 8. □

3.4 $D = \{a_1b_1, a_2b_2, a_3b_3\}$

In this subsection, for $D = \{a_1b_1, a_2b_2, a_3b_3\} \subsetneq \mathcal{RP}_{\Sigma\cup X}$, we consider a regular pattern q in $\mathcal{RP}_{\Sigma\cup X}$ such that q minimally supports D for a regular pattern p in $\mathcal{RP}_{\Sigma\cup X}$ under some conditions with the symbols a_i, $b_i \in \Sigma$ for i ($1 \leq i \leq k$).

The conditions in Lemmas 8, 9, and 10 are illustrated in the cases (9), (10), and (11) in Fig. 15.

Lemma 10: Let Σ be an alphabet with $\sharp\Sigma \geq 3$, p, q regular pattern in $\mathcal{RP}_{\Sigma\cup X}$ and $D = \{a_1b_1, a_2b_2, a_3b_3\} \subsetneq \mathcal{RP}_{\Sigma\cup X}$ where $a_i \neq a_j$ and $b_i \neq b_j$ with $i \neq j$ ($1 \leq i, j \leq 3$). If q minimally supports D for a regular pattern p, then $p\{x := xy\} \preceq q$.

Proof. We may assume $p \npreceq q$, since the case $p \preceq q$ is trivial. We assume that $p\{x := xy\} \npreceq q$ holds. Since q minimally supports D for p, it is sufficient to consider the following four cases (10-1)-(10-4) of q for some regular patterns q_1, q_2 and some constant strings w, w' ($|w| \geq 0$ and $|w'| \geq 0$):

(10-1) $q = q_1a_1b_1wa_2b_2w'a_3b_3q_2$,
(10-2) $q = q_1a_1b_1a_3b_3q_2$ ($b_1 = a_2$ and $a_3 = b_2$),
(10-3) $q = q_1a_1b_1b_2wa_3b_3q_2$ ($b_1 = a_2$),
(10-4) $q = q_1a_1b_1wa_2b_2b_3q_2$ ($b_2 = a_3$).

(10-1) Case of $q = q_1a_1b_1wa_2b_2w'a_3b_3q_2$: The following conditions must be satisfied:

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq wa_2b_2w'a_3b_3q_2$,
(2) $p_1 \preceq q_1a_1b_1w$, (2') $p_2 \preceq w'a_3b_3q_2$,
(3) $p_1 \preceq q_1a_1b_1wa_2b_2w'$, (3') $p_2 \preceq q_2$.

This leads to a contradiction, as demonstrated by the following inductive argument:

- $|w| = |w'|$: From (2) and (3), both $a_1b_1wa_2b_2w'$ and a_1b_1w are suffixes of p_1. Then, $a_1b_1w = a_2b_2w'$. Hence, $a_1b_1 = a_2b_2$. This contracts the assumption of $a_1 \neq a_2$ and $b_1 \neq b_2$.
- $|w|+1 = |w'|$: The two strings $wa_2b_2w'a_3b_3$ and $w'a_3b_3$ are prefixes of p_2. If there exists a constant symbol w_1 such that $w'a_3b_3 = ww_1a_3b_3$, then b_2 and a_3 are the same symbol from $wa_2b_2 = ww_1a_3$. From (2) and (3), both $a_1b_1wa_2b_2w'$ and a_1b_1w are suffixes of p_1. Then, there exists a constant symbol w_2 such that $w' = w_2w$, then b_2 and a_1 are the same symbol from $b_2w_2w = a_1b_1w$. Hence, from $b_2 = a_3$, a_3 and a_1 are same symbol. This contradicts the assumption of $a_3 \neq a_1$.
- $|w| + 1 < |w'|$: From (2) and (3), both $a_1b_1wa_2b_2w'$ and a_1b_1w are suffixes of p_1. If there exists a constant string w_1 ($|w_1| \geq 2$) such that $w' = w_1w$, then a_1b_1 is a suffix of w_1. From conditions (1') and (2'), both $wa_2b_2w'a_3b_3$ and $w'a_3b_3$ are prefixes of p_2. If there exist constant strings w_1 and w_2 such that $w' = w_1w = ww_2$ holds, then a_3b_3 is a suffix of w_1 from $|w_1| = |w_2|$ and $ww_2a_3b_3 = wa_2b_2w_1$. Hence, $a_1b_1 = a_3b_3$. This contradicts the assumption of $a_1 \neq a_3$ and $b_1 \neq b_3$.
- $|w| > |w'|$: We can prove the contradiction in a similar way as $|w| \leq |w'|$.

(10-2) Case of $q = q_1a_1b_1a_3b_3q_2$ ($b_1 = a_2$ and $a_3 = b_2$): The following conditions must be satisfied:

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq a_3b_3q_2$,
(2) $p_1 \preceq q_1a_1$, (2') $p_2 \preceq b_3q_2$,
(3) $p_1 \preceq q_1a_1b_1$, (3') $p_2 \preceq q_2$.

From (2) and (3), since both a_1b_1 and a_1 are suffixes of p_1, $b_1 = a_1$ holds. From the assumption of $b_1 = a_2$, the

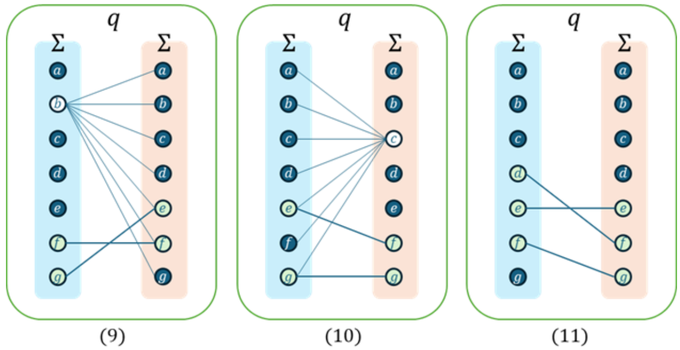

Fig. 15 Let $\Sigma = \{a, b, c, d, e, f, g\}$ and $p, q \in \mathcal{RP}$. We assume that the symbols in Σ are mutually distinct. The figures (9), (10) and (11) express cases of Ds in Lemmas 8, 9, and 10, respectively. In these cases, if q minimally supports D for p, then $p\{x := xy\} \preceq q$ holds.

equation $a_1 = a_2$ holds. This contradicts the assumption of $a_1 \neq a_2$.

(10-3) Case of $q = q_1a_1b_1b_2wa_3b_3q_2$ ($b_1 = a_2$): The following conditions must be satisfied:

(1) $p_1 \preceq q_1$,	(1') $p_2 \preceq b_2wa_3b_3q_2$,
(2) $p_1 \preceq q_1a_1$,	(2') $p_2 \preceq wa_3b_3q_2$,
(3) $p_1 \preceq q_1a_1b_1b_2w$,	(3') $p_2 \preceq q_2$.

This leads to a contradiction, as demonstrated by the following inductive argument:

- $|w| = 0$: From (2) and (3), both a_1 and $a_1b_1b_2$ are suffixes of p_1. Moreover, from (1') and (2'), both $b_2a_3b_3$ and a_3b_3 are prefixes of p_2. Since $b_2 = a_1$ and $b_2 = a_3$, we have $a_1 = a_3$. This contradicts the assumption of $a_1 \neq a_3$.
- $|w| \geq 1$: From (2) and (3), both a_1 and $a_1b_1b_2w$ are suffixes of p_1. Hence, the last symbol of w is a_1. Moreover, both $b_2wa_3b_3$ and wa_3b_3 are prefixes of p_2 from (1') and (2'). Hence, the last symbol of w is a_3. Therefore, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.

(10-4) Case of $q = q_1a_1b_1wa_2b_2b_3q_2$ ($b_2 = a_3$): The following conditions must be satisfied:

(1) $p_1 \preceq q_1$,	(1') $p_2 \preceq wa_2b_2b_3q_2$,
(2) $p_1 \preceq q_1a_1b_1w$,	(2') $p_2 \preceq b_3q_2$,
(3) $p_1 \preceq q_1a_1b_1wa_2$,	(3') $p_2 \preceq q_2$.

This leads to a contradiction, as demonstrated by the following inductive argument:

- $|w| = 0$: From (2) and (3), both a_1b_1 and $a_1b_1a_2$ are suffixes of p_1. And from (1') and (2'), both $a_2b_2b_3$ and b_3 are prefixes of p_2. Since $b_1 = a_2$ and $a_2 = b_3$, then $b_1 = b_3$ holds. This contradicts the assumption of $b_1 \neq b_3$.
- $|w| \geq 1$: Since both a_1b_1w and $a_1b_1wa_2$ are suffixes of p_1 from (2) and (3), the first symbol of w is b_1. Moreover, since both $wa_2b_2b_3$ and b_3 are prefixes of p_2 from (1') and (2'), the first symbol of w is b_3. Therefore, $b_1 = b_3$ holds. This contradicts the assumption of $b_1 \neq b_3$.

□

4. Compactness for Sets of Regular Patterns

In this section, we define the compactness of sets of regular patterns, formally. Then, if $\sharp\Sigma \geq 2k - 1$ holds, we show that $\mathcal{RP}^k$ has compactness with respect to language containment.

Definition 3: Let C be a subset of $\mathcal{RP}^+$. For any regular pattern $p \in \mathcal{RP}$ and any set $Q \in C$, the set C is said to have *compactness with respect to language containment* if there exists a regular pattern $q \in Q$ such that $L(p) \subseteq L(q)$ holds if $L(p) \subseteq L(Q)$ holds.

As a preliminary step toward the main theorem that $\mathcal{RP}^k$ has compactness with respect to language containment, we establish some Lemmas, Theorem and Corollaries.

Lemma 11: Let k be an integer with $k \geq 1$. Let Σ be an alphabet with $\sharp\Sigma = k + 2$. Let $p \in \mathcal{RP}$ in which a variable symbol x appears, and let $Q \in \mathcal{RP}^k$. If for any string $w \in \Sigma^*$ with $|w| = 2$, there exists a regular pattern $q_w \in Q$ such that $p\{x := w\} \preceq q_w$ holds, then there exists a regular pattern $q \in Q$ such that $p\{x := xy\} \preceq q$ holds, where y is a variable symbol that does not appear in q.

Proof. Without loss of generality, we suppose that $\sharp Q = k$ holds. Otherwise, for some regular pattern q already in Q, we can add a new regular pattern q' equivalent to q, i.e., $q' \equiv q$, to Q repeatedly until $\sharp Q = k$ is satisfied. For any $q \in Q$, we define the sets $A(q), B(q) \subseteq \Sigma$ as follows:

$$A(q) = \{a \in \Sigma \mid p\{x := ay\} \preceq q,\ y \in X\},$$
$$B(q) = \{b \in \Sigma \mid p\{x := yb\} \preceq q,\ y \in X\}.$$

If there exists $q \in Q$ such that $\sharp A(q) \geq 2$ or $\sharp B(q) \geq 2$, from Lemma 4, $p\{x := xy\} \preceq q$ holds. Below, we suppose that $\sharp A(q) \leq 1$ and $\sharp B(q) \leq 1$. Let $\perp$ be a constant symbol that is not a member in Σ. We define the functions $\sigma_A : Q \to \Sigma \cup \{\perp\}$ and $\sigma_B : Q \to \Sigma \cup \{\perp\}$ as follows:

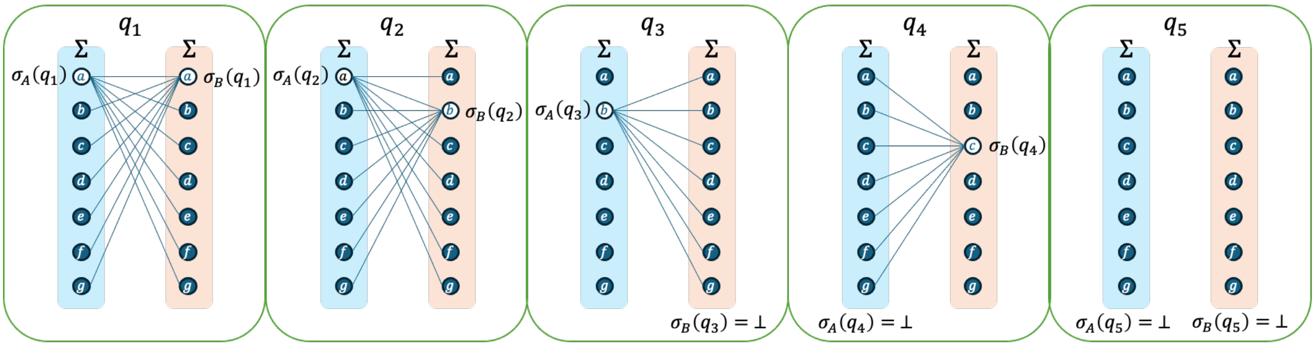

Fig. 16 Let $\Sigma = \{a, b, c, d, e, f, g\}$ and $Q = \{q_1, q_2, q_3, q_4, q_5\}$. We set $A(q_1) = \{a\}$ and $B(q_1) = \{a\}$, and then $\sigma_A(q_1) = a$ and $\sigma_B(q_1) = a$, and so on. For each regular pattern q_i $(i = 1, \ldots, 5)$, we represent a string $w \in \Sigma \cdot \Sigma$ satisfying that $p\{x := w\} \preceq q_i$ by the edge between the left (first) and right (second) symbols of w. For example, the leftmost figure shows that $p\{x := ay\} \preceq q_1$ and $p\{x := ya\} \preceq q_1$ for a variable symbol y. We note that these figures may contain more edges than those illustrated. From these figures, we get $\ell_A = 1, \ell_B = 0$, and $Q^{(\perp,\perp)} = \{q_5\}, Q^{(\perp,\cdot)} = \{q_4\}, Q^{(\cdot,\perp)} = \{q_3\}, Q^{(\cdot,\cdot)} = \{q_1, q_2\}$.

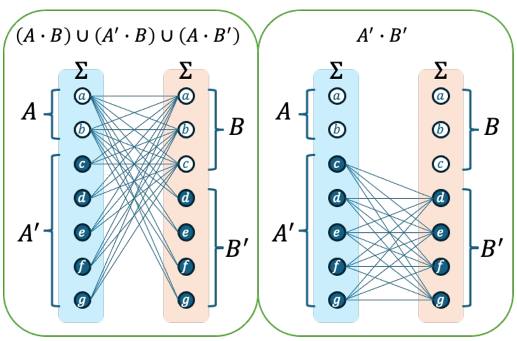

Fig. 17 In the left figure, we aggregate all edges appearing in Fig. 16. For all $w = a'b' \in A' \cdot B'$, there must be a regular pattern q_i $(1 \leq i \leq 5)$ satisfying $p\{x := w\} \preceq q_i$.

$$\sigma_A(q) = \begin{cases} a & \text{if } A(q) = \{a\}, \\ \perp & \text{if } A(q) = \emptyset. \end{cases}$$

$$\sigma_B(q) = \begin{cases} b & \text{if } B(q) = \{b\}, \\ \perp & \text{if } B(q) = \emptyset. \end{cases}$$

The inverse functions of σ_A and σ_B are denoted by σ_A^{-1} and σ_B^{-1}, respectively. That is, for $a, b \in \Sigma \cup \{\perp\}$, let $\sigma_A^{-1}(a) = \{q \in Q \mid \sigma_A(q) = a\}$ and $\sigma_B^{-1}(b) = \{q \in Q \mid \sigma_B(q) = b\}$. We give an example in Fig. 16.

A and B denotes the following subsets of Σ:

$$A = \bigcup_{q \in Q \setminus \sigma_A^{-1}(\perp)} A(q), \quad B = \bigcup_{q \in Q \setminus \sigma_B^{-1}(\perp)} B(q).$$

Then, let $A' = \Sigma \setminus A$ and $B' = \Sigma \setminus B$. For any $a, b \in \Sigma$, we use the following notations:

$$\ell_A = \sum_{a \in A} (\sharp\sigma_A^{-1}(a) - 1), \quad \ell_B = \sum_{b \in B} (\sharp\sigma_B^{-1}(b) - 1).$$

These ℓ_A and ℓ_B represent the numbers of excess duplicate symbols in A and B. We easily see the following claim:

Claim 1.

(i) $\sharp A + \sharp A' = \sharp B + \sharp B' = k + 2$,
(ii) $\sharp A + \ell_A + \sharp\sigma_A^{-1}(\perp) = \sharp B + \ell_B + \sharp\sigma_B^{-1}(\perp) = k$.

Since $\sharp\Sigma = k + 2$ and $\sharp Q = k$, $\sharp A' \geq 2$ and $\sharp B' \geq 2$ hold. We partition Q into the following subsets:

$$Q^{(\perp,\perp)} = \sigma_A^{-1}(\perp) \cap \sigma_B^{-1}(\perp),$$
$$Q^{(\perp,\cdot)} = \sigma_A^{-1}(\perp) \cap (Q \setminus \sigma_B^{-1}(\perp)),$$
$$Q^{(\cdot,\perp)} = (Q \setminus \sigma_A^{-1}(\perp)) \cap \sigma_B^{-1}(\perp),$$
$$Q^{(\cdot,\cdot)} = (Q \setminus \sigma_A^{-1}(\perp)) \cap (Q \setminus \sigma_B^{-1}(\perp)).$$

From the condition of this lemma, for any string $w \in \Sigma^*$ with $|w| = 2$, there exists a regular pattern $q_w \in Q$ such that $p\{x := w\} \preceq q_w$ holds. In particular, for $w = a'b' \in A' \cdot B'$, we must have $q_w \in Q$ satisfying $p\{x := w\} \preceq q_w$ (Fig. 17). It is easy to see that if $w \in (A \cdot B) \cup (A' \cdot B) \cup (A \cdot B')$, there exists a regular pattern $q_w \in Q^{(\perp,\cdot)} \cup Q^{(\cdot,\perp)} \cup Q^{(\cdot,\cdot)}$ such that $p\{x := w\} \preceq q_w$ holds. We have the following two claims:

Claim 2. If there exist $q \in Q^{(\perp,\perp)}$ and distinct 5 strings $w_i \in A' \cdot B'$ $(1 \leq i \leq 5)$ such that $p\{x := w_i\} \preceq q$ holds $(1 \leq i \leq 5)$, then $p\{x := xy\} \preceq q$ holds.

Proof of Claim 2. Let $W = \{a_1b_1, \ldots, a_5b_5\} \subseteq A' \cdot B'$. For any i $(1 \leq i \leq 5)$, if $\sharp(W \cap \{a_ic \mid c \in \Sigma\}) \geq 3$ or $\sharp(W \cap \{cb_i \mid c \in \Sigma\}) \geq 3$ holds, then $a_i \in A(q)$ or $b_i \in B(q)$ holds. From the definitions of A' and B', it can see that, for any i $(1 \leq i \leq 5)$, $\sharp(W \cap \{a_ic \mid c \in \Sigma\}) \leq 2$ and $\sharp(W \cap \{cb_i \mid c \in \Sigma\}) \leq 2$. Then, it can be proven that there are 3 strings $a_{i_1}b_{i_1}, a_{i_2}b_{i_2}, a_{i_3}b_{i_3} \in W$ such that $a_{i_j} \neq a_{i_{j'}}$ and $b_{i_j} \neq b_{i_{j'}}$ for any $i_j, i_{j'}$ $(i_j \neq i_{j'}, 1 \leq j, j' \leq 3)$. Therefore, from Lemma 10, this claim holds. (*End of Proof of Claim* 2)

Claim 3. If there exist $q \in Q^{(\perp,\cdot)} \cup Q^{(\cdot,\perp)}$ and distinct 3 strings $w_i \in A' \cdot B'$ $(1 \leq i \leq 3)$ such that $p\{x := w_i\} \preceq q$ holds $(1 \leq i \leq 3)$, then $p\{x := xy\} \preceq q$ holds.

Proof of Claim 3. Let $W = \{a_1b_1, a_2b_2, a_3b_3\} \subseteq A' \cdot B'$.

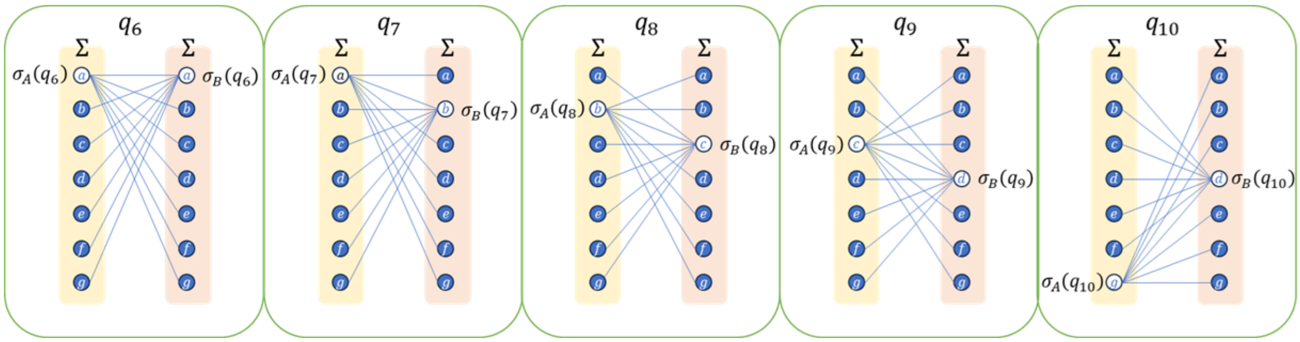

Fig. 18 Let $\Sigma = \{a, b, c, d, e, f, g\}$ and $Q = \{q_6, q_7, q_8, q_9, q_{10}\}$. From these figures, we get $\ell_A = 1$, $\ell_B = 1$, $Q^{(\perp,\perp)} = Q^{(\perp,\cdot)} = Q^{(\cdot,\perp)} = \emptyset$, and $Q^{(\cdot,\cdot)} = Q$.

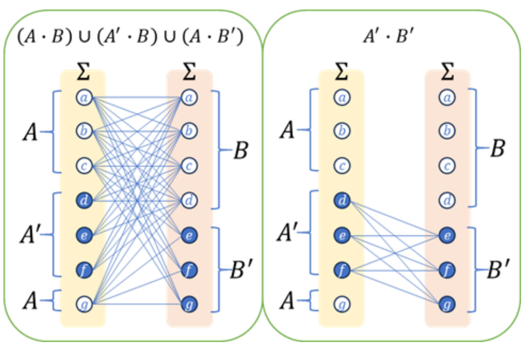

Fig. 19 In the left figure, we aggregate all edges appearing in Fig. 18. From Fig. 18 and this right figure, we get $Q_1^{(\cdot,\cdot)} = \{q_6, q_7, q_8, q_9\}$ and $Q_2^{(\cdot,\cdot)} = \{q_{10}\}$. From Proposition 4, even if the string $dg \in A' \cdot B'$ satisfies $p\{x := dg\} \preceq q_{10}$, it does not imply that $p\{x := xy\} \preceq q_{10}$.

Because, for any i ($1 \le i \le 3$), $\sharp(W \cap \{a_i c \mid c \in \Sigma\}) \le 2$ and $\sharp(W \cap \{cb_i \mid c \in \Sigma\}) \le 2$, it can be proven that there are 2 strings $a_{i_1}b_{i_1}, a_{i_2}b_{i_2} \in W$ such that $a_{i_1} \ne a_{i_2}$ and $b_{i_1} \ne b_{i_2}$. Therefore, from Lemmas 8 and 9, this claim holds. (*End of Proof of Claim* 3)

If there exist a regular pattern $q \in Q^{(\perp,\perp)} \cup Q^{(\perp,\cdot)} \cup Q^{(\cdot,\perp)}$ and enough strings $w \in A' \cdot B'$ such that either of the conditions of *Claims* 2 and 3 is satisfied, this lemma holds. Then, we assume that it is not the case.

Assumption 1. There is no regular pattern $q \in Q^{(\perp,\perp)}$ and 5 strings $w \in A' \cdot B'$ such that the condition of *Claim* 2 is satisfied and there is no regular pattern $q \in Q^{(\perp,\cdot)} \cup Q^{(\cdot,\perp)}$ and 3 strings $w \in A' \cdot B'$ such that the condition of *Claim* 3 is satisfied.

Let $\mathcal{L}_1 = \sharp\{w \in A' \cdot B' \mid \exists q \in Q^{(\perp,\perp)} \cup Q^{(\perp,\cdot)} \cup Q^{(\cdot,\perp)}$ s.t. $p\{x := w\} \preceq q\}$. Under *Assumption* 1, each $q \in Q^{(\perp,\perp)}$ has at most 4 strings $w \in A' \cdot B'$ such that the condition of *Claim* 2 is satisfied, and each $q \in Q^{(\perp,\cdot)} \cup Q^{(\cdot,\perp)}$ has at most 2 strings $w \in A' \cdot B'$ such that the condition of *Claim* 3 is satisfied. Then, by *Claim* 1,

$$\begin{aligned}
\mathcal{L}_1 &\le 4\sharp Q^{(\perp,\perp)} + 2\sharp Q^{(\perp,\cdot)} + 2\sharp Q^{(\cdot,\perp)} \\
&= 2(\sharp Q^{(\perp,\perp)} + \sharp Q^{(\perp,\cdot)}) + 2(\sharp Q^{(\perp,\perp)} + \sharp Q^{(\cdot,\perp)}) \\
&= 2\sharp\sigma_A^{-1}(\perp) + 2\sharp\sigma_B^{-1}(\perp) \\
&= 2(k - \sharp A - \ell_A) + 2(k - \sharp B - \ell_B) \\
&= 2(\sharp A' - \ell_A - 2) + 2(\sharp B' - \ell_B - 2) \\
&= 2(\sharp A' + \sharp B') - 2(\ell_A + \ell_B) - 8.
\end{aligned}$$

Next, we partition $Q^{(\cdot,\cdot)}$ into the following two subsets:

$$Q_1^{(\cdot,\cdot)} = \{q \in Q^{(\cdot,\cdot)} \mid \sigma_A(q) \in B \text{ or } \sigma_B(q) \in A\},$$
$$Q_2^{(\cdot,\cdot)} = \{q \in Q^{(\cdot,\cdot)} \mid \sigma_A(q) \in B' \text{ and } \sigma_B(q) \in A'\}.$$

We show the following two claims concerning $Q_1^{(\cdot,\cdot)}$ and $Q_2^{(\cdot,\cdot)}$:

Claim 4. If there exist $q \in Q_1^{(\cdot,\cdot)}$ and a string $a'b' \in A' \cdot B'$ such that $p\{x := a'b'\} \preceq q$ holds, then $p\{x := xy\} \preceq q$ holds.

Proof of Claim 4. Suppose that both $\sigma_A(q) \in B$ and $\sigma_B(q) \in A$ hold. Then, since $a' \notin \{\sigma_A(q), \sigma_B(q)\} \subseteq A \cap B$ and $b' \notin \{\sigma_A(q), \sigma_B(q)\} \subseteq A \cap B$, from Lemma 5, $p\{x := xy\} \preceq q$ holds. Suppose that $\sigma_A(q) \in B$ and $\sigma_B(q) \in A'$. If $a' = \sigma_B(q)$, since $a' \in B$, $a' \ne b'$ holds. Since $\sigma_A(q) \in B$, $b' \ne \sigma_A(q)$ holds. That is, $a' = \sigma_B(q)$, $a' \ne \sigma_A(q)$, and $b' \notin \{\sigma_A(q), \sigma_B(q)\}$ hold. Therefore, from Lemmas 6 and 7, $p\{x := xy\} \preceq q$ holds. If $a' \ne \sigma_B(q)$, since $b' \ne \sigma_A(q)$, from Lemma 5, $p\{x := xy\} \preceq q$ holds. Similarly, the case that $\sigma_A(q) \in B'$ and $\sigma_B(q) \in A$ is proven. (*End of Proof of Claim* 4)

Claim 5. If there exist $q \in Q_2^{(\cdot,\cdot)}$ and a string $a'b' \in A' \cdot B'$ such that ($a' \ne \sigma_B(q)$ or $b' \ne \sigma_A(q)$) and $p\{x := a'b'\} \preceq q$ hold, then $p\{x := xy\} \preceq q$ holds.

Proof of Claim 5. When $a' = b'$, since $\sigma_A(q) \ne \sigma_B(q)$, from Lemma 5, this claim holds. Similarly, when $a' \ne b'$, from Lemmas 5, 6, and 7, this holds. (*End of Proof of Claim* 5)

We give an example in Fig. 18 and Fig. 19.

If there exist a regular pattern $q \in Q_2^{(\cdot,\cdot)}$ and a string $w \in A' \cdot B'$ such that the condition of *Claim* 5 is satisfied, this lemma holds. Then, we also assume that it is not the case.

Assumption 2. There is no $q \in Q_2^{(\cdot,\cdot)}$ and a string $a'b' \in A' \cdot B'$ such that the condition of *Claim* 5 is satisfied.

Let $\mathcal{L}_2 = \sharp\{a'b' \in A' \cdot B' \mid \exists q \in Q_2^{(\cdot,\cdot)}$ s.t. $p\{x := a'b'\} \preceq q\}$. For any $a'b' \in A' \cdot B'$ and $q \in Q_2^{(\cdot,\cdot)}$, if $a' = \sigma_B(q)$ and

$b' = \sigma_A(q)$ hold (it is the condition of Proposition 4), by considering the duplicate numbers ℓ_A and ℓ_B, we have the following inequality:

$$\mathcal{L}_2 \leq \min\{\sharp A' + \ell_B, \sharp B' + \ell_A\}.$$

We give an example of the above inequality in Fig. 20 and Fig. 21.

We show the last claim:

Claim 6. $\sharp A' \times \sharp B' - \mathcal{L}_1 - \mathcal{L}_2 \geq 2$.

Proof of Claim 6. First we prove the inequality when $\sharp A \leq k-1$ and $\sharp B \leq k-1$, i.e., $\sharp A' \geq 3$ and $\sharp B' \geq 3$ hold. Since $\mathcal{L}_2 \leq \frac{1}{2}(\sharp A' + \sharp B' + \ell_A + \ell_B)$,

$$\begin{aligned}
&\sharp A' \times \sharp B' - \mathcal{L}_1 - \mathcal{L}_2 \\
\geq &\sharp A' \times \sharp B' - (2(\sharp A' + \sharp B') - 2(\ell_A + \ell_B) - 8) \\
&- \frac{1}{2}(\sharp A' + \sharp B' + \ell_A + \ell_B) \\
= &\sharp A' \times \sharp B' - \frac{5}{2}(\sharp A' + \sharp B') + \frac{3}{2}(\ell_A + \ell_B) + 8 \\
= &(\sharp A' - \frac{5}{2})(\sharp B' - \frac{5}{2}) + \frac{3}{2}(\ell_A + \ell_B) + \frac{7}{4} \geq 2.
\end{aligned}$$

When $\sharp A = k$ and $\sharp B \leq k$, i.e., $\sharp A' = 2$ and $\sharp B' \geq 2$ hold, since $\ell_A = 0$, $\mathcal{L}_1 \leq 2\sharp B' - 2\ell_B - 4$ holds. Moreover, $\mathcal{L}_2 \leq \min\{\sharp B', \ell_B + 2\}$ holds. From *Claim* 1, $\ell_B + 2 = k - \sharp\sigma_B^{-1}(\perp) - \sharp B = \sharp B' - \sharp\sigma_B^{-1}(\perp)$ holds. Therefore, $\mathcal{L}_2 \leq \ell_B + 2$ holds. Thus,

$$\begin{aligned}
&\sharp A' \times \sharp B' - \mathcal{L}_1 - \mathcal{L}_2 \\
\geq &2\sharp B' - (2\sharp B' - 2\ell_B - 4) - (\ell_B + 2) \\
= &\ell_B + 2 \geq 2.
\end{aligned}$$

Similarly, the case when $\sharp A \leq k$ and $\sharp B = k$ is proven. (*End of Proof of Claim* 6)

Under *Assumptions* 1 and 2, from *Claim* 6, there exist at least two $w \in A' \cdot B'$ and a regular pattern $q \in Q_1^{(\cdot,\cdot)}$ such that the condition of *Claim* 4 is satisfied. Therefore, for such a regular pattern q, $p\{x := xy\} \preceq q$ holds. □

Lemma 12 (Sato et al.[4]): Let Σ be an alphabet with $\sharp\Sigma \geq 3$ and p, q regular patterns. If there exists a constant symbol $a \in \Sigma$ such that $p\{x := a\} \preceq q$ and $p\{x := xy\} \preceq q$, then $p \preceq q$ holds, where y is a variable symbol that does not appear in q.

From Lemma 11 and Lemma 12, we have the following theorem.

Theorem 4: Let $k \geq 3$, $\sharp\Sigma \geq 2k-1$, $P \in \mathcal{RP}^+$ and $Q \in \mathcal{RP}^k$. Then, the following (i),(ii) and (iii) are equivalent:

$$\text{(i) } S_2(P) \subseteq L(Q), \text{ (ii) } P \sqsubseteq Q, \text{ (iii) } L(P) \subseteq L(Q).$$

Proof. It is clear that (ii) implies (iii) and (iii) implies (i). From Theorem 3, if $\sharp\Sigma \geq 2k+1$, then (i) implies (ii). Let $\sharp Q = k$, $p \in P$, $\sharp\Sigma = 2k-1$ or $2k$. Then, we show that (i) implies (ii). It suffices to show that $S_2(p) \subseteq L(Q)$ implies $\{p\} \sqsubseteq Q$ for any regular pattern $p \in P$. The proof is done by mathematical induction on n, where n is the number of variable symbols appears in p.

In case $n = 0$, $S_2(p) = \{p\}$ holds. By (i), we have $\{p\} \subseteq L(Q)$. Thus, $p \preceq q$ for some $q \in Q$.

For $n \geq 0$, we assume that it is valid for any regular pattern p with n variable symbols. Let p be a regular pattern such that $n+1$ variable symbols appear in p and $S_2(p) \subseteq L(Q)$. Let $Q = \{q_1, \ldots, q_k\}$. We assume that $p \not\sqsubseteq Q$, that is, $p \npreceq q_i$ for any $i \in \{1, \ldots, k\}$. Let p_1, p_2 be regular patterns and x a variable symbol with $p = p_1 x p_2$. For $a, b \in \Sigma$, let $p_a = p\{x := a\}$ and $p_{ab} = p\{x := ab\}$. Both p_a and p_{ab} have n variable symbols, respectively. Thus, $S_2(p_a) \subseteq L(Q)$ and $S_2(p_{ab}) \subseteq L(P)$ hold. By the induction hypothesis, there exist $i, i' \in \{1, \ldots, k\}$ such that $p_a \preceq q_i$ and $p_{ab} \preceq q_{i'}$. Let $D_i = \{a \in \Sigma \mid p\{x := a\} \preceq q_i\}$ $(i = 1, \ldots, k)$. We assume that $\sharp D_i \geq 3$ for some $i \in \{1, \ldots, k\}$. By Lemma 2, we have $p \preceq q_i$. This contradicts the assumption. Thus, we have $\sharp D_i \leq 2$ for any $i \in \{1, \ldots, k\}$. If $\sharp\Sigma = 2k-1$, then $\sharp D_i = 2$ or $\sharp D_i = 1$ for any $i \in \{1, \ldots, k\}$. Moreover, If $\sharp\Sigma = 2k$, then $\sharp D_i = 2$ for any $i \in \{1, \ldots, k\}$. Since $k \geq 3$, $2k-1 \geq k+2$ holds. By Lemma 11, there exists $i \in \{1, \ldots, k\}$ such that $p\{x := xy\} \preceq q_i$. Therefore, by Lemma 12, we have $p \preceq q_i$. This contradicts the assumption. Thus, (i) implies (ii). □

From Theorem 4, the following Corollary holds.

Corollary 2: Let $k \geq 3$, $\sharp\Sigma \geq 2k-1$ and $P \in \mathcal{RP}^+$. Then, $S_2(P)$ is a characteristic set for $L(P)$ within $\mathcal{RPL}^k$.

From Theorem 4, we have the following Theorem.

Theorem 5: Let $k \geq 3$ and $\sharp\Sigma \geq 2k-1$. Then, $\mathcal{RP}^k$ has compactness with respect to language containment.

The following lemma demonstrates that Theorem 5 fails to hold under the condition $\sharp\Sigma \leq 2k-2$, thereby establishing the minimum cardinality of Σ required for the validity of Theorem 5.

Lemma 13 (Sato et al.[4]): Let $k \geq 3$ and $\sharp\Sigma \leq 2k-2$. Then, $\mathcal{RP}^k$ does not have compactness with respect to language containment.

Proof. Let $\Sigma = \{a_1, \ldots, a_{k-1}, b_1, \ldots, b_{k-1}\}$, p, q_i regular patterns, $w_i = w_{i+1} b_{i+1} a_{i+1} w_{i+1} \in \Sigma^*$ $(i = 1, \ldots, k-2)$ and $w_{k-1} = \varepsilon$ defined in a similar way to Example 1. Let $q_k = x_1 a_1 w_1 x y w_1 b_1 x_2$. Since $p\{x := a_i\} = x_1 a_1 w_1 a_i w_1 b_1 x_2 \preceq q_i$ and $p\{x := b_i\} = x_1 a_1 w_1 b_i w_1 b_1 x_2 \preceq q_i$ for any $i \in \{1, \ldots, k-1\}$, we have $S_1(p) \subseteq \bigcup_{i=1}^{k-1} L(q_i)$. For any $w \in \{s \in \Sigma^+ \mid |s| \geq 2\}$, $p\{x := w\} = x_1 a_1 w_1 w w_1 b_1 x_2 \preceq q_k$. Thus, we have $L(p) \subseteq L(Q)$. By Theorem 1, since $p \npreceq q_i$, $L(p) \nsubseteq L(q_i)$ for any $i \in \{1, \ldots, k\}$. Therefore, $\mathcal{RP}^k$ does not have compactness with respect to language containment. □

In case $k = 2$, we have the following theorem.

Theorem 6: Let $\sharp\Sigma \geq 4$, $P \in \mathcal{RP}^+$ and $Q \in \mathcal{RP}^2$. The following (i), (ii) and (iii) are equivalent:

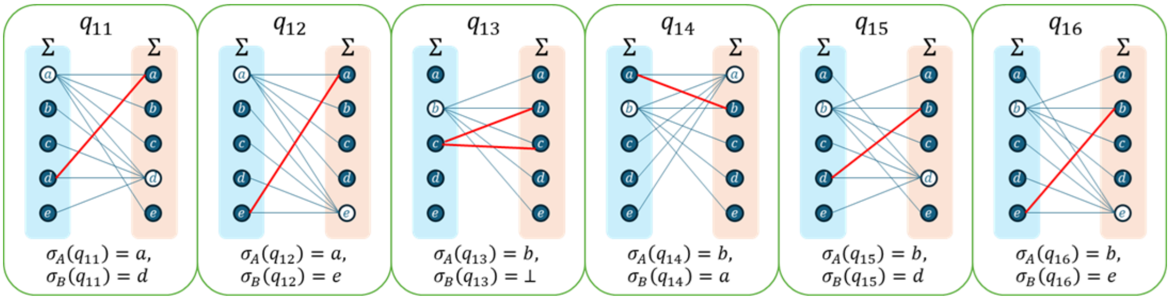

Fig. 20 Let $\Sigma = \{a, b, c, d, e\}$ and $Q = \{q_{11}, q_{12}, q_{13}, q_{14}, q_{15}, q_{16}\}$. From these figures, we get $\ell_A = 4$, $\ell_B = 1$, $Q^{(\perp,\perp)} = Q^{(\perp,\cdot)} = \emptyset$, $Q^{(\cdot,\perp)} = \{q_{13}\}$, and $Q^{(\cdot,\cdot)} = \{q_{11}, q_{12}, q_{14}, q_{15}, q_{16}\}$. From Proposition 4, we note again that for example, even if $p\{x := db\} \preceq q_{15}$ holds, it does not imply that $p\{x := xy\} \preceq q_{15}$.

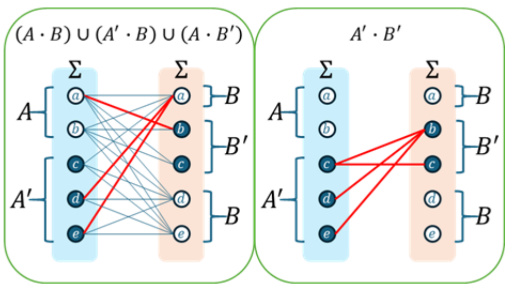

Fig. 21 In the left and right figures, we aggregate all edges corresponding to $(A \cdot B) \cup (A' \cdot B) \cup (A \cdot B')$ and $A' \cdot B'$ in Fig. 20, respectively. From these figures, we get $Q_1^{(\cdot,\cdot)} = \{q_{11}, q_{12}, q_{14}\}$ and $Q_2^{(\cdot,\cdot)} = \{q_{15}, q_{16}\}$. Then, $\mathcal{L}_1 = 2$ and $\mathcal{L}_2 = 2 \le \min\{\sharp A' + \ell_B, \sharp B' + \ell_A\} = 4$ holds.

$$\text{(i) } S_2(P) \subseteq L(Q), \quad \text{(ii) } P \sqsubseteq Q, \quad \text{(iii) } L(P) \subseteq L(Q).$$

Proof. It is clear that (ii) implies (iii), and (iii) implies (i). Thus, we show that (i) implies (ii). It suffices to show that $S_2(p) \subseteq L(Q)$ implies $\{p\} \sqsubseteq Q$ for any regular pattern $p \in P$. Let $Q = \{q_1, q_2\}$. The proof is done by mathematical induction on n, where n is the number of variable symbols appearing in p. In case $n = 0$, $p \in \Sigma^+$. Since $S_2(p) = \{p\} \subseteq L(Q)$, we have $p \preceq q$ for some $q \in Q$. For $n \ge 0$, we assume that it is valid for any regular pattern p with n variable symbols. Let p be a regular pattern such that $n + 1$ variable symbols appear in p, and $S_2(p) \subseteq L(Q)$ holds. We assume that $p \not\preceq q_i$ $(i = 1, 2)$. Let p_1, p_2 be regular patterns and x a variable symbol with $p = p_1 x p_2$. For $a, b \in \Sigma$, let $p_a = p\{x := a\}$ and $p_{ab} = p\{x := ab\}$. Note that p_a and p_{ab} have n variable symbols. Thus, by the assumption, $S_2(p_a) \subseteq L(Q)$ and $S_2(p_{ab}) \subseteq L(Q)$ imply $p_a \preceq q_i$ and $p_{ab} \preceq q_{i'}$ for some $i, i' \in \{1, 2\}$. Let $D_i = \{a \in \Sigma \mid p\{x := a\} \preceq q_i\}$ $(i = 1, 2)$. By Lemma 2, if $\sharp D_i \ge 3$ for some $i \in \{1, 2\}$, then $p \preceq q_i$. This contradicts that $p \not\preceq q_i$ $(i = 1, 2)$. Thus, we have $\sharp D_i \le 2$ for any $i \in \{1, 2\}$. Since $\sharp\Sigma \ge 4$, we consider that $\sharp D_1 = 2$ and $\sharp D_2 = 2$. From Lemma 11, $p\{x := xy\} \preceq q_i$ for some $i \in \{1, 2\}$. From Lemma 12, we have $p \preceq q_i$ for some $i \in \{1, 2\}$. This contradicts that $p \not\preceq q_i$ $(i = 1, 2)$. Hence, (i) implies (ii). □

The following example provides a set of regular patterns $P \in \mathcal{RP}^+$ and a set of regular patterns $Q \in \mathcal{RP}^2$ demonstrating that, when $\sharp\Sigma = 3$, the three conditions (i), (ii), and (iii) stated in Theorem 6 are not equivalent.

Example 3: Let $\Sigma = \{a, b, c\}$, p, q_1, q_2 regular patterns and x, x', x'' variable symbols such that $p = x'axbx''$, $q_1 = x'abx''$ and $q_2 = x'cx''$. Let $w \in \Sigma^+$. If w contains c, then $p\{x := w\} \preceq q_2$. On the other hand, if w does not contain c, then $p\{x := w\} \preceq q_1$. Thus, $L(p) \subseteq L(q_1) \cup L(q_2)$. However, $p \not\preceq q_1$ and $p \not\preceq q_2$.

From Theorem 6, the following two corollaries holds.

Corollary 3: Let $\sharp\Sigma \ge 4$ and $P \in \mathcal{RP}^+$. Then, $S_2(P)$ is a characteristic set for $L(P)$ within $\mathcal{RPL}^2$.

Corollary 4: Let $\sharp\Sigma \ge 4$. Then, $\mathcal{RP}^2$ has compactness with respect to language containment.

5. Regular Pattern without Adjacent Variable Symbols

A regular pattern p is said to be a *non-adjacent variable regular pattern* (*NAV* regular pattern) if p does not contain consecutive variable symbols. For example, the regular pattern $p = axybc$ is not an *NAV* regular pattern because xy is appeared in p. Let $\mathcal{RP}_{NAV}$ be the set of all *NAV* regular patterns. Let $\mathcal{RP}^+_{NAV}$ be the set of all finite subsets S of $\mathcal{RP}_{NAV}$ such that S is not the empty set, i.e., $\mathcal{RP}^+_{NAV} = \{S \subseteq \mathcal{RP}_{NAV} \mid \sharp S \ge 1\}$, and $\mathcal{RP}^k_{NAV}$ the set of all subsets P of $\mathcal{RP}^+_{NAV}$ such that P consists of at most k $(k \ge 1)$ *NAV* regular patterns, i.e., $\mathcal{RP}^k_{NAV} = \{P \in \mathcal{RP}^+_{NAV} \mid \sharp P \le k\}$. We define the compactness with respect to language containment for $\mathcal{RP}^k_{NAV}$ in a similar way as $\mathcal{RP}^k$.For any *NAV* regular pattern $p \in \mathcal{RP}_{NAV}$ and any set $Q \in \mathcal{RP}^k_{NAV}$ with k $(k \ge 1)$, the set $\mathcal{RP}^k_{NAV}$ is said to have *compactness with respect to language containment* if there exists an *NAV* regular pattern $q \in Q$ such that $L(p) \subseteq L(q)$ holds if $L(p) \subseteq L(Q)$ holds. Then, the following Theorem 7 holds.

Theorem 7: For an integer k $(k \ge 2)$, let $\sharp\Sigma \ge k + 2$, $P \in \mathcal{RP}^+_{NAV}$, $Q \in \mathcal{RP}^k_{NAV}$. Then, the following (i), (ii) and (iii) are equivalent:

$$\text{(i) } S_2(P) \subseteq L(Q), \text{(ii) } P \sqsubseteq Q, \text{(iii) } L(P) \subseteq L(Q).$$

$p = x'cadadaadacbadadaadaxadadaadacbadadaadabx''$,
$q_1 = x'cadadaadacbadadaadacx''$,
$q_2 = x'badadaadacx''$,
$q_3 = x'aadadx''$.

Fig. 22 *NAV* regular patterns p, q_1, q_2, and q_3

Proof. From the definitions of $\mathcal{RP}^+_{NAV}$ and $\mathcal{RP}^k_{NAV}$, it is clear that (ii) implies (iii) and (iii) implies (i). Hence, we will show that (i) implies (ii) by mathematical induction on the number n of variable symbols that appear in an *NAV* regular pattern $p \in P$ as follows: If $n = 0$, then we have $S_2(\{p\}) = \{p\}$. Hence, $p \in L(Q)$. Therefore, there exists $q \in Q$ such that $p \preceq q$ holds.

If $n \geq 0$, we assume that the proposition holds for any regular *NAV* regular pattern containing $n \geq 0$ variable symbols. Let p be an *NAV* regular pattern containing $n+1$ variable symbols such that $S_2(\{p\}) \subseteq L(Q)$ holds and p contains a variable symbol x. There exist two *NAV* regular patterns p_1, p_2 such that $p = p_1 x p_2$ holds. By the induction hypothesis, for any constant string $w \in \Sigma^*$ with $|w| = 2$, $\{p\{x := w\}\} \subseteq Q$ holds because $p\{x := w\}$ contains n variable symbols. Hence, there exists an *NAV* regular pattern $q_w \in Q$ such that $p\{x := w\} \preceq q_w$ holds. From Lemma 11, there exists a regular pattern $q \in Q$ such that $p\{x := xy\} \preceq q$ holds, where y is a variable symbol that does not appear in q. This contradicts the condition $Q \in \mathcal{RP}^k_{NAV}$. Thus, we have that (i) implies (ii). □

Corollary 5: Let $k \geq 2$, $\sharp\Sigma \geq k+2$ and $P \in \mathcal{RP}^+_{NAV}$. Then, $S_2(P)$ is a characteristic set of $\mathcal{RPL}^k_{NAV}$.

Lemma 14: Let $k \geq 2$ and $\sharp\Sigma \leq k+1$. Then, $\mathcal{RP}^k_{NAV}$ does not have compactness with respect to language containment.

Proof. Let $\Sigma = \{a_1, \ldots, a_{k+1}\}$. We assume that for $i = 1, 2, \ldots, k$, $p\{x := a_i y\} \preceq q_i$ and $p\{x := y a_{i+1}\} \preceq q_i$ holds. If $p\{x := a_{k+1} a_1\} \preceq q_1$ holds, $S_2(p) \backslash S_1(p) \subseteq \bigcup_{i=1}^k L(q_i)$ holds. This show that $L(p) \subseteq L(Q)$ holds. However, for $i = 1, 2, \ldots, k$, since $p \npreceq q_i$ holds, we have that $L(p) \nsubseteq L(q_i)$ holds. Hence, $\mathcal{RP}^k_{NAV}$ does not have compactness with respect to language containment. □

Next, in Example 4, we give an example for Lemma 14.

Example 4: Let Σ be the set of four constant symbols a, b, c, d, i.e., $\Sigma = \{a, b, c, d\}$ and x, x', x'' three distinct variable symbols. Let p, q_1, q_2, q_3 be the *NAV* regular patterns given in Fig. 22. Then, we have $L(p) \subseteq L(q_1) \cup L(q_2) \cup L(q_3)$. This show that for $P = \{p\}$, $Q = \{q_1, q_2, q_3\}$, (iii) of Theorem 7 holds. However, since $p \npreceq q_1$, $p \npreceq q_2$ and $p \npreceq q_3$ hold, we have $P \nsubseteq Q$, that is, (ii) of Theorem 7 does not hold.

From Theorem 7 and Lemma 14, we have the following theorem.

Theorem 8: Let $k \geq 2$ and $\sharp\Sigma \geq k+2$. Then, the set $\mathcal{RPL}^k_{NAV}$ has compactness with respect to language containment.

Table 2 The conditions on the number $\sharp\Sigma$ of constant symbols in Σ required for compactness with respect to language containment.

Class	$k = 2$	$k \geq 3$
$\mathcal{RP}^k$	$\sharp\Sigma \geq 4$	$\sharp\Sigma \geq 2k-1$
$\mathcal{RP}^k_{NAV}$	$\sharp\Sigma \geq k+2$	

6. Conclusion

In this paper, for an integer k ($k \geq 2$), we have shown the conditions on the number of constant symbols in Σ, summarized in Table 2, required for the classes $\mathcal{RP}^k$ of all the set of k regular pattern languages and $\mathcal{RP}^k_{NAV}$ of all the set of k non-adjacent variable regular patterns in $\mathcal{RP}_{NAV}$ to have compactness with respect to language containment. This result leads to design an efficient learning algorithm for finite unions of languages of non-adjacent variable regular patterns in $\mathcal{RP}_{NAV}$, based on the learning algorithm for $\mathcal{RP}^k$ proposed by Arimura et al. [8].

Extending the notion of strong compactness, as introduced by Arimura et al. [9], to finite unions of regular pattern languages with non-adjacent variables remains as a topic for future research. Furthermore, based on the characteristic set for $\mathcal{RP}^k_{NAV}$, we plan to propose a polynomial-time inductive inference algorithm that identifies finite unions of regular pattern languages with non-adjacent variables in the limit from positive examples. Ishinada et al. [17] investigated a query learning model that employs high-precision Graph Convolution Networks (GCNs) as oracles for tree patterns. Applying the findings of the present study to tree pattern languages, with the aim of enabling the extension of their work to finite unions of tree pattern languages, remains an important direction for future research.

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