PAPER

# Compactness of Finite Union of Regular Patterns and Regular Patterns without Adjacent Variables

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**SUMMARY** A regular pattern is a string consisting of constant symbols and distinct variable symbols. The language $L(p)$ of a regular pattern $p$ is the set of all constant strings obtained by replacing all variable symbols in the regular pattern $p$ with constant strings. $\mathcal{RP}^k$ denotes the class of all sets consisting at most $k$ ($k \geq 2$) regular patterns. Sato et al. (Proc. ALT'98, 1998) showed that the finite set $S_2(P)$ of symbol strings is a characteristic set of $L(P) = \bigcup_{p \in P} L(p)$, where $S_2(P)$ is obtained from $P \in \mathcal{RP}^k$ by substituting variables with symbol strings of at most length 2. They also showed that $\mathcal{RP}^k$ has compactness with respect to containment, if the number of constant symbols is greater than or equal to $2k-1$. In this paper, we check the their results and correct the error of the proof of their theorem. Further, we consider the set $\mathcal{RP}_{NAV}$ of all non-adjacent regular patterns, which are regular patterns without adjacent variables, and show that the set $S_2(P)$ obtained from a set $P$ in the class $\mathcal{RP}^k_{NAV}$ of at most $k$ ($k \geq 1$) non-adjacent regular patterns is a characteristic set of $L(P)$. Further we show that $\mathcal{RP}^k_{NAV}$ has compactness with respect to containment if the number of constant symbols is greater than or equal to $k+2$. Thus we show that we can design an efficient learning algorithm of a finite union of pattern languages of non-adjacent regular patterns with the number of constant symbols which is smaller than the case of regular patterns.
***key words:*** *Regular Pattern Language, Compactness w.r.t. Containment, Non-adjacent Regular Patterns Language*

## 1. Introduction

A pattern is a string consisting of constant symbols and variable symbols [1], [2]. For example, we consider constant symbols $a, b, c$ and variable symbols $x, y$, then $axbxcy$ is a pattern. $\mathcal{P}$ denotes the set of all patterns. For a pattern $p \in \mathcal{P}$, *the pattern language generated by* $p$, denoted by $L(p)$, or simply called a pattern language, is the set of all strings obtained by replacing all variable symbols with constant symbol strings, where the same variable symbol is replaced by the same constant string. For example the pattern language $L(axbxcy)$ generated by the above pattern $axbxcy$ denotes $\{aubucw \mid$ u and w are constant strings that are not $\varepsilon\}$. A pattern where each variable symbol appears at most once is called a *regular pattern*. For example, a pattern $axbxcy$ is not a regular pattern, but a pattern $axbzcy$ with variable symbols $x, y, z$ is a regular pattern. $\mathcal{RP}$ denotes the set of all regular patterns.

If a pattern $p \in \mathcal{P}$ is obtained from a pattern $q \in \mathcal{P}$ by replacing variable symbols in $q$ with patterns, we say that $q$ is a *generalization* of $p$ and denote this by $p \preceq q$. For example, a pattern $q = axz$ is a generalization of a pattern $p = axbxcy$, because $p$ is obtained from $q$ by replacing the variable $z$ in $q$ with a pattern $bxcy$. So we write $p \preceq q$. For patterns $p, q \in \mathcal{P}$, it is obvious that $p \preceq q$ implies $L(p) \subseteq L(q)$. But, the converse, that is, the statement that $L(p) \subseteq L(q)$ implies $p \preceq q$ does not always hold. With respect to this statement, Mukouchi [3] showed that if the number of constant symbols is greater than or equal to 3, for any regular pattern $p, q \in \mathcal{RP}$, $L(p) \subseteq L(q)$ implies $p \preceq q$.

We denote by $\mathcal{RP}^+$ the class of all non-empty finite sets of regular patterns and by $\mathcal{RP}^k$ the class of at most $k$ ($k \geq 2$) regular patterns. For a set of regular patterns $P \in \mathcal{RP}^k$ we define $L(P) = \bigcup_{p \in P} L(p)$ and consider the class $\mathcal{RPL}^k$ of regular pattern languages of $\mathcal{RP}^k$, where $\mathcal{RPL}^k = \{L(P) \mid P \in \mathcal{RP}^k\}$. Let $P, Q \in \mathcal{RP}^k$ and $Q = \{q_1, \ldots, q_k\}$. We denote by $P \sqsubseteq Q$ that for any regular pattern $p \in P$ there exists a regular pattern $q_i$ such that $p \preceq q_i$ holds. From definition, it is obvious that $P \sqsubseteq Q$ implies $L(P) \subseteq L(Q)$. Then Sato et al. [4] shows that if $k \geq 3$ and the number of constant symbols is $2k-1$ then the finite set $S_2(P)$ of constant symbols obtained from $P \in \mathcal{RP}^k$ by substituting variable symbols with constant strings of at most 2 length is a characteristic set of $L(P)$, that is, for any regular pattern language $L' \in \mathcal{RPL}^k$, $S_2(P) \subseteq L'$ implies $L(P) \subseteq L'$. Thus they shows that the following three statements: (i) $S_2(P) \subseteq L(Q)$, (ii) $P \sqsubseteq Q$ and (iii) $L(P) \subseteq L(Q)$ are equivalent. But the Lemma 14 in [4], which is used in this results, contains an error. In this paper we correct this lemma and give a correct proof showing the equivalence of the three statements shown in [4]. Sato et al. [4] shows that $\mathcal{RP}^k$ has compactness with respect to containment if the number of constant symbols is greater than or equal to $2k-1$. On the contrary to this result, we show that the set $S_2(P)$ obtained from a set $P$ in the class $\mathcal{RP}^k_{NAV}$ of at most $k$ ($k \geq 1$) regular patterns having non-adjacent variables is a characteristic set of $L(P)$. Further, we show that if the number of constant symbols is greater than or equal to $k+2$ then $\mathcal{RP}^k_{NAV}$ has compactness with respect to containment. In Table 1 we summarize the all results in this paper.

**Table 1** The conditions of the number of constant symbols with respect to the compactness of inclusion

| $k$ | 2 | $\geqq 3$ |
|---|---|---|
| $\mathcal{RP}^k$ | $\geqq 4$ | $\geqq 2k-1$ |
| $\mathcal{RP}^k_{NAV}$ | $\geqq k+2$ | |

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Mukouchi [5] examined the decision problem of determining whether a containment relation exists between the languages generated by two given patterns. The inductive inference of formal languages—specifically, pattern languages [2] and unions of pattern languages [6], [7] from positive data has been extensively investigated. Arimura et al. [8] introduced a formal framework for the efficient generalization of unions of pattern languages, presenting a polynomial-time algorithm to identify the minimal set of patterns whose union encompasses a given set of positive examples. In a subsequent study, Arimura et al. [9] proposed the concept of strong compactness of containment for unions of regular pattern languages. Day et al. [10] established that pattern languages are, in general, not closed under standard language operations such as union, intersection, and complement. Matsumoto et al. [11] developed an efficient query learning algorithm for regular pattern languages that requires only a single positive example and a linear number of membership queries. More recently, Takeda et al. [12] proposed a query learning algorithm that utilizes a deep learning model trained on a set of strings as an oracle, enabling the learned features to be visualized as regular patterns. Subsequent research extended the study of regular patterns to Elementary Formal Systems (EFS) [13], thereby broadening the theoretical foundation of pattern languages. This extension inspired further work on tree patterns [14], [15] for generating tree languages, as well as on the development of Formal Graph Systems [16]. These advancements have facilitated the formalization and efficient learning of increasingly complex structured data beyond strings, fostering applications in domains such as grammatical inference and graph-based learning.

This paper is organized as follows. In Sect.2 as preparations, we give definitions of pattern languages, regular pattern languages and compactness, and then introduce the results of Sato et al.[4]. In Sect.3, we show that $S_2(P)$ is a characteristic set of $L(P)$ in $\mathcal{RPL}^k$ and $\mathcal{RP}^k$ has compactness with respect to containment. In Sect.4, we propose regular patterns having non-adjacent variables, show that $S_2(P)$ obtained from a set $P$ in $\mathcal{RP}^k_{NAV}$ is a characteristic set of $L(P)$, and $\mathcal{RP}^k_{NAV}$ has compactness with respect to containment.

## 2. Preliminaries

### 2.1 Basic definitions and notations

Let $\Sigma$ be a non-empty finite set of constant symbols. Let $X$ be an infinite set of variable symbols such that $\Sigma \cap X = \emptyset$ holds. Then, a *string* over $\Sigma \cup X$ is a sequence of symbols in $\Sigma \cup X$. Particularly, the string having no symbol is called the *empty string* and is denoted by $\varepsilon$. We denote by $(\Sigma \cup X)^*$ the set of all strings over $\Sigma \cup X$ and by $(\Sigma \cup X)^+$ the set of all strings over $\Sigma \cup X$ except $\varepsilon$, i.e., $(\Sigma \cup X)^+ = (\Sigma \cup X)^* \setminus \{\varepsilon\}$.

A *pattern* over $\Sigma \cup X$ is a string in $(\Sigma \cup X)^*$. Note that the empty string $\varepsilon$ is a pattern over $\Sigma \cup X$. A pattern $p$ is said to be *regular* if each variable symbol appears at most once in $p$. The length of $p$, denote by $|p|$, is the number of symbols in $p$. Note that $|\varepsilon| = 0$ holds. The set of all patterns and regular patterns over $\Sigma \cup X$ are denoted by $\mathcal{P}_{\Sigma \cup X}$ and $\mathcal{RP}_{\Sigma \cup X}$, respectively. When $\Sigma$ and $X$ are clear from the context, we omit them in the notation and simply write $\mathcal{P}$ and $\mathcal{RP}$, respectively. For a set $S$, we denote by $\sharp S$ the number of elements in $S$. Let $p, q$ be strings. If $p$ and $q$ are equal as strings, we denote it by $p = q$. We denote by $p \cdot q$ the string obtained from $p$ and $q$ by concatenating $q$ after $p$. Note that for a string $p$ and the empty string $\varepsilon$, $p \cdot \varepsilon = \varepsilon \cdot p = p$.

A substitution $\theta$ is a mapping from $(\Sigma \cup X)^*$ to $(\Sigma \cup X)^*$ such that (1) $\theta$ is a homomorphism with respect to string concatenation, i.e., $\theta(p \cdot q) = \theta(p) \cdot \theta(q)$ holds for patterns $p$ and $q$, (2) $\theta(\varepsilon) = \varepsilon$ holds, (3) for each constant symbol $a \in \Sigma$, $\theta(a) = a$ holds, and (4) for each variable symbol $x \in X$, $|\theta(x)| \geq 1$ holds. Let $x_1, \ldots, x_n$ are variable symbols and $p_1, \ldots, p_n$ non-empty patterns. The notation $\{x_1 := p_1, \ldots, x_n := p_n\}$ denotes a substitution that replaces each variable symbol $x_i$ with a non-empty pattern $p_i$ for each $i \in \{1, \ldots, n\}$. For a pattern $p$ and a substitution $\theta = \{x_1 := p_1, \ldots, x_n := p_n\}$, we denote by $p\theta$ a new pattern obtained from $p$ by replacing variable symbols $x_1, \ldots, x_n$ in $p$ with patterns $p_1, \ldots, p_n$ according to $\theta$, respectively.

For a pattern $p$ and $q$, the pattern $q$ is a *generalization* of $p$, or $p$ is an *instance* of $q$, denoted by $p \preceq q$, if there exists a substitution $\theta$ such that $p = q\theta$ holds. If $p \preceq q$ and $p \succeq q$ hold, we denote it by $p \equiv q$. The notation $p \equiv q$ means that $p$ and $q$ are equal as strings except for variable symbols. For a pattern $p$, the *pattern language* of $p$, denoted by $L(p)$, is the set $\{w \in \Sigma^* \mid w \preceq p\}$. For patterns $p$ and $q$, it is clear that $L(p) = L(q)$ if $p \equiv q$, and $L(p) \subseteq L(q)$ if $p \preceq q$. Note that $L(\varepsilon) = \{\varepsilon\}$. In particular, if $p$ is a regular pattern, we say that $L(p)$ is a *regular pattern language*. The set of all pattern languages and regular patterns languages are denoted by $\mathcal{PL}$ and $\mathcal{RPL}$, respectively.

**Lemma 1** (Mukouchi(Theorem 6.1, [3])): Suppose $\sharp\Sigma \geq 3$. Let $p$ and $q$ be regular patterns. Then $p \preceq q$ if and only if $L(p) \subseteq L(q)$.

Next, we consider unions of pattern languages. The class of all non-empty finite subsets of $\mathcal{P}$ is denoted by $\mathcal{P}^+$, i.e., $\mathcal{P}^+ = \{P \subseteq \mathcal{P} \mid 0 < \sharp P < \infty\}$. For a positive integer $k$ i.e., $k > 0$, the class of non-empty sets consisting of at most $k$ patterns, i.e., $\mathcal{P}^k = \{P \subseteq \mathcal{P} \mid 0 < \sharp P \leq k\}$. For a set $P$ of patterns, the pattern language of $P$, denoted by $L(P)$, is the set $\bigcup_{p \in P} L(p)$. We denote by $\mathcal{PL}^k$ the class of unions of at most $k$ pattern languages, i.e., $\mathcal{PL}^k = \{L(P) \mid P \in \mathcal{P}^k\}$. In a similar way, we also define $\mathcal{RP}^+$, $\mathcal{RP}^k$ and $\mathcal{RPL}^k$. For $P$, $Q$ in $\mathcal{P}^+$, the notation $P \sqsubseteq Q$ means that for any $p \in P$ there is a pattern $q \in Q$ such that $p \preceq q$ holds. It is clear that $P \sqsubseteq Q$ implies $L(P) \subseteq L(Q)$. However, the converse is not valid in general.

### 2.2 Characteristic sets

**Definition 1:** Let $C$ be a class of languages, $L$ a language in $C$ and $S$ a non-empty finite subset of $L$. We say that $S$ is a *characteristic* set of $L$ within $C$ if for any $L' \in C$, $S \subseteq L'$ implies $L \subseteq L'$.

Let $n$ be a positive integer and $p$ a regular pattern. We denote by $S_n(p)$ the set of all strings in $\Sigma^*$ obtained by replacing all variable symbols in $p$ with strings in $\Sigma^+$ of length at most $n$. Moreover, for a positive integer $n$ and a set $P \in \mathcal{RP}^+$, let $S_n(P) = \bigcup_{p \in P} S_n(p)$. It is clear that $S_n(P) \subseteq S_{n+1}(P) \subseteq L(P)$ for any positive integer $n$.

**Theorem 1** (Sato et al.(Theorem 8, [4])): Let $k$ be a positive integer and $P \in \mathcal{RP}^k$. Then, there exists a positive integer $n$ such that $S_n(P)$ is a characteristic set of $L(P)$ within $\mathcal{RPL}^k$.

**Theorem 2** (Sato et al.(Lemma 9, [4])): Let $p$, $q$, $p_1$, $p_2$, $q_1, q_2, q_3$ be regular patterns in $\mathcal{RP}$ and $x$ a variable symbol such that $p = p_1 x p_2$ and $q = q_1 q_2 q_3$ hold. Then $p \preceq q$ if the following three conditions (i), (ii) and (iii) are holds:

(i) $p_1 \preceq q_1 q_2$, (ii) $p_2 \preceq q_2 q_3$,
(iii) $q_2$ contains at least one variable symbol.

**Lemma 2** (Sato et al.(Lemma 10, [4])): Suppose $\sharp\Sigma \geq 3$. Let $p_1$, $p_2$, $q$ be regular patterns in $\mathcal{RP}$ and $x$ a variable symbol. Let $a$, $b$ and $c$ be mutually distinct constant symbols in $\Sigma$. If $p_1 a p_2 \preceq q$, $p_1 b p_2 \preceq q$ and $p_1 c p_2 \preceq q$, then $p_1 x p_2 \preceq q$ holds.

**Lemma 3** (Sato et al.(Lemma 13, [4])): Suppose $\sharp\Sigma \geq 3$. Let $p_1, p_2, q_1, q_2$ be regular patterns in $\mathcal{RP}$ and $x$ a variable symbol. Let $a, b$ be constant symbols in $\Sigma$ with $a \neq b$ and $w$ a string in $\Sigma^*$. Let $p = p_1 AwxwB p_2$ and $q = q_1 AwB q_2$ be regular patterns that satisfy the following three conditions:

(i) $p_1 Aw \preceq q_1$,
(ii) $wB p_2 \preceq q_2$,
(iii) $(A, B) \in \{(a, b), (b, a)\}$.

Then, we have that $p\{x := a\} \preceq q$ and $p\{x := b\} \preceq q$ hold but $p \not\preceq q$.

From Lemma 2, the following theorem holds.

**Theorem 3** (Sato et al.(Theorem 10, [4])): Let $k$ be a positive integer. Suppose $\sharp\Sigma \geq 2k + 1$. For $P \in \mathcal{RP}^+$ and $Q \in \mathcal{RP}^k$, the following (i), (ii) and (iii) are equivalent:

(i) $S_1(P) \subseteq L(Q)$, (ii) $P \sqsubseteq Q$, (iii) $L(P) \subseteq L(Q)$.

The following Example 1 in [4] shows that Theorem 3 does not hold if $\sharp\Sigma \leq 2k$ holds.

**Example 1:** Let $k$ be a positive integer and $\Sigma = \{a_1, \ldots, a_k, b_1, \ldots, b_k\}$. Let $w_1, \ldots, w_k$ be regular patterns in $\mathcal{RP}$ such that $w_k = \varepsilon$ and for $i = 1, 2, \ldots, k-1$, $w_i = w_{i+1} b_{i+1} a_{i+1} w_{i+1}$ hold. Let $p$, $q_1, \ldots, q_k$ be regular patterns in $\mathcal{RP}$ such that $p = x_1 a_1 w_1 x w_1 b_1 x_2$ and for $i = 1, 2, \ldots, k$, $q_i = x_1 a_i w_i b_i x_2$ hold. Let $Q$ be a set $\{q_1, \ldots, q_k\}$ in $\mathcal{RP}^k$. For $i = 1$, we have $p\{x := a_1\} = (x_1 a_1 w_1) a_1 (w_1 b_1 x_2) = q_1\{x_1 := x_1 a_1 w_1\} \preceq q_1$. For $i \geq 2$, from the definition of $w_i$, we easily see that $w_1 = (w_i b_i) w^{(i)} = w'(i)(a_i w_i)$ for some strings $w^{(i)}$ and $w'^{(i)}$. Then, for each $i \geq 2$,

$$\begin{aligned}
p\{x := a_i\} &= (x_1 a_1 w_1) a_i (w_1 b_1 x_2) \\
&= (x_1 a_1 w_1) a_i (w_i b_i w^{(i)}) b_1 x_2 \\
&= (x_1 a_1 w_1)(a_i w_i b_i)(w^{(i)} b_1 x_2) \\
&= q_i\{x_1 := x_1 a_1 w_1, x_2 := w^{(i)} b_1 x_2\} \\
&\preceq q_i, \\
p\{x := b_i\} &= (x_1 a_1 w_1) b_i (w_1 b_1 x_2) \\
&= x_1 a_1 (w'^{(i)} a_i w_i) b_i (w_1 b_1 x_2) \\
&= (x_1 a_1 w'^{(i)}) a_i w_i b_i (w_1 b_1 x_2) \\
&= q_i\{x_1 := x_1 a_1 w'^{(i)}, x_2 := w_1 b_1 x_2\} \\
&\preceq q_i.
\end{aligned}$$

Hence, $S_1(p) \subseteq L(Q)$ holds. However, from $p \not\preceq q_i$, $L(p) \not\subseteq L(q_i)$ holds for each $i = 1, 2, \ldots, k$.

From Theorem 3, we have the following corollary.

**Corollary 1** (Sato et al.(Corollary 12, [4])): Suppose $\sharp\Sigma \geq 3$. For two regular patterns $p$ and $q$, the following (i), (ii) and (iii) are equivalent:

(i) $S_1(p) \subseteq L(q)$, (ii) $p \preceq q$, (iii) $L(p) \subseteq L(q)$.

### 2.3 Basic word equations

**Proposition 1:** Let $w$ be a string in $\Sigma^*$ and $a, b$ constant symbols in $\Sigma$. If

$$wa = bw \tag{1}$$

holds, then $a = b$ holds.

**Proof.** Since it is trivial, we omit the proof. □

**Proposition 2:** Let $w$ be a string in $\Sigma^*$ and $a, b, c, d$ constant symbols in $\Sigma$. If

$$wda = bcw \tag{2}$$

holds, then $(b, c) \in \{(a, d), (d, a)\}$ holds.

**Proof.** We will prove this proposition by induction on the length of $w$, i.e., $|w|$.

- $|w| = 0, 1, 2,$ or $3$: it is straightforward to observe that $(b, c) \in \{(a, d), (d, a)\}$ holds.
- $|w| \geq 4$: We assume that for any string $u$ with $0 \leq |u| < n$, if $uda = bcu$ holds, $(b, c) \in \{(a, d), (d, a)\}$ holds. Since the string $w$ has a prefix $bc$ and a suffix $da$, there exists a string $u$ with $|u| = |w| - 4 < |w|$ such that $w = bcuda$ holds. Since $wda = bcw$, we have $bcudada = bcbcuda$, and then $uda = bcu$. Thus, from the assumption, we get $(b, c) \in \{(a, d), (d, a)\}$.

From the above, we conclude that if $wda = bcw$ holds, then $(b, c) \in \{(a, d), (d, a)\}$ holds. □

The conclusion from Proposition 2 shows that $(a, d) \in \{(b, c), (c, b)\}$. Therefore, if the equation $daw = wbc$ holds, we arrive at the same conclusion.

**Proposition 3:** Let $w, w'$ be strings of constant symbols in $\Sigma$ and $a, b, c, d$ constant symbols in $\Sigma$. If

$$wdaw' = w'bcw \qquad (3)$$

holds, then $(b, c) \in \{(a, d), (d, a)\}$ holds.

**Proof.** We will prove this proposition by an induction on $|w| + |w'|$. Without loss of generality, we assume that $|w| \geq |w'|$ because, if $|w| < |w'|$, we arrive at the same conclusion that $(a, d) \in \{(b, c), (c, b)\}$ holds.

- $|w| \geq 0$ and $|w'| = 0$: Eq. (3) reduces to $wda = bcw$. By Proposition 2, $(b, c) \in \{(a, d), (d, a)\}$ holds.

We assume that for constant strings $u$ and $u'$ with $|u| + |u'| < |w| + |w'|$, if $udau' = u'bcu$ holds, then $(b, c) \in \{(a, d), (d, a)\}$ holds. We divide the relations between $|w|$ and $|w'|$ into the following four cases:

- $0 < |w'| \leq |w| \leq |w'| + 1$: When either $|w| = |w'|$ or $|w| = |w'| + 1$, Eq. (3) is illustrated in Figs. 1 and 2, respectively. If $|w| = |w'|$, $(b, c) = (d, a)$ holds. If $|w| = |w'| + 1$, $d = c$ and $w = w'b = aw'$ hold. From Proposition 1, we deduce that $b = a$. Therefore, $(b, c) \in \{(a, d), (d, a)\}$ holds.
- $|w'| + 2 \leq |w| \leq 2|w'| - 1$: In Eq. 3, since $|wdaw'| = |w'bcw| = |w| + |w'| + 2$, a suffix of $w$ overlaps with a prefix of $w$, as illustrated in Fig. 3. That is, there exists a constant string $u$ of length $2|w| - (|w| + |w'| + 2) = |w| - |w'| - 2$ such that $u$ is both a prefix and a suffix of $w$. Since $uda$ has a length of $|w| - |w'|$, it is also a prefix of $w$. Similarly, $bcu$ is a suffix of $w$. Because $|w| - (|uda| + |bcu|) = 2|w| - |w'| \geq 1$, there exists a constant string $u'$ of length $2|w'| - |w|$ such that $w = udau'bcu$ holds. Since $w'$ is a suffix of $w$ and $|u'bcu| = (2|w'| - |w|) + 2 + (|w| - |w'| - 2) = |w'|$, we have $w' = u'bcu$. Similarly, $w' = udau'$. Thus, we derive the equation $u'bcu = udau'$. Since $|u| = |w| - |w'| - 2 \leq |w| - 3 < |w|$ and $|u'| = 2|w'| - |w| < |w'|$, i.e., $|u| + |u'| < |w| + |w'|$, the induction hypothesis on $|u| + |u'|$ implies that $(b, c) \in \{(a, d), (d, a)\}$ holds.
- $2|w'| \leq |w| \leq 2|w'| + 3$: When $|w| = 2|w'|$, it is straightforward to observe that $w = w'w'$. Therefore, $w'da = bcw'$ holds, as illustrated in Fig. 4. From Proposition 2, $(b, c) \in \{(a, d), (d, a)\}$ holds. When $|w| = 2|w'| + i$ $(i = 1, 2, 3)$, Eq. (3) is depicted in Figs. 5, 6, and 7, respectively. When $|w| = 2|w'| + 2$, it is clear that $(b, c) = (d, a)$. When $|w| = 2|w'| + 1$ and $|w| = 2|w'| + 3$, Proposition 1 implies that $(b, c) = (a, d)$ holds.
- $2|w'| + 4 \leq |w|$: Since the strings $w'bc$ and $adw'$ are a prefix and a suffix of $w$, respectively, and $|w'bc| + |adw'| = 2|w'| + 4$, there exists a string $u$ with $|u| \geq 0$ such that $w = w'bcudaw'$ holds. From Eq. (3), $w'bcudaw'daw' = w'bcw'bcudaw'$, i.e.,

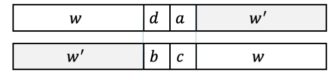



**Fig. 1** Case $|w| = |w'|$ in Proposition 3

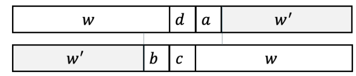



**Fig. 2** Case $|w| = |w'| + 1$ in Proposition 3

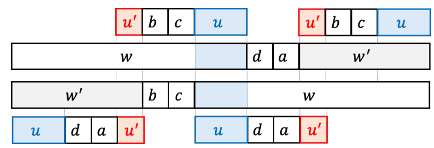



**Fig. 3** Case $|w'| + 2 \leq |w| \leq 2|w'| - 1$ in Proposition 3

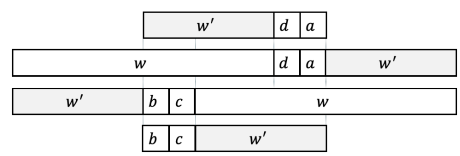



**Fig. 4** Case $|w| = 2|w'|$ in Proposition 3

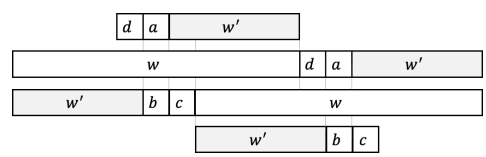



**Fig. 5** Case $|w| = 2|w'| + 1$ in Proposition 3

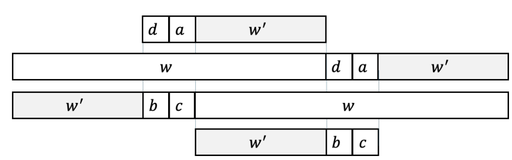



**Fig. 6** Case $|w| = 2|w'| + 2$ in Proposition 3

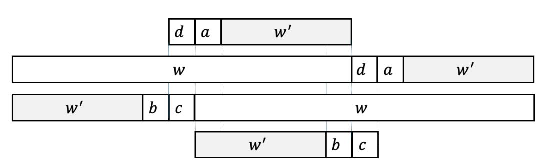



**Fig. 7** Case $|w| = 2|w'| + 3$ in Proposition 3

$udaw' = w'bcu$ holds, as illustrated in Fig. 8. Let $u' = w'$. Since $|u| + |u'| = |w| - (2|w'| + 4) + |w'| < |w| + |w'|$, the induction hypothesis on $|u| + |u'|$ implies that $(b, c) \in \{(a, d), (d, a)\}$ holds.

From the above, we conclude that if $wdaw' = w'bcw$, then $(b, c) \in \{(a, d), (d, a)\}$ holds. □

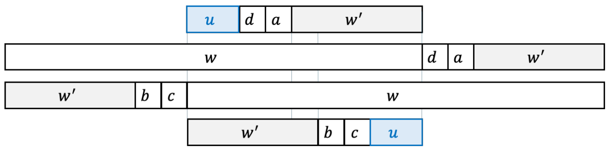



**Fig. 8** Case $2|w'| + 4 \le |w|$ in Proposition 3

## 3. Compactness for Sets of Regular Patterns

### 3.1 Compactness

In this section, we define the compactness of sets of regular patterns, formally. Then, if $\sharp\Sigma \ge 2k-1$ holds, we show that $\mathcal{RP}^k$ has compactness with respect to the containment.

**Definition 2:** Let $C$ be a subset of $\mathcal{RP}^+$. For any regular pattern $p \in \mathcal{RP}$ and any set $Q \in C$, the set $C$ said to have *compactness with respect to containment* if there exists a regular pattern $q \in Q$ such that $L(p) \subseteq L(q)$ holds if $L(p) \subseteq L(Q)$ holds.

Let $D \subset \mathcal{RP}$ with $\sharp D = 2$ or 3, and let $p, q$ be regular patterns in $\mathcal{RP}$. In the following subsections (Subsecs. 3.2–3.5), we provide the conditions on $D$ under which the implication holds: if $p\{x := r\} \preceq q$ for all $r \in D$, then $p\{x := xy\} \preceq q$. It is obvious if the variable symbol $x$ does not appear in $p$. Therefore, in the following lemmas and propositions, let $p = p_1 x p_2$, where $p_i \in \mathcal{RP}$ $(i = 1, 2)$ and $x$ is a variable symbol.

First of all, we consider the correspondence from $r \in D$ to some string in $q$ when $p\{x := r\} \preceq q$ holds. The symbols in $D$ correspond to either a variable or a constant symbol in $q$. If $D$ has a constant string $ab$ of length 2 for $a, b \in \Sigma$, there are three possible strings in $q$ that correspond to $ab$ in $p\{x := ab\}$ as follows: For $y_1 \in X$,

$$\text{(a)} \quad ab, \quad \text{(b)} \quad ay_1, \quad \text{(c)} \quad y_1 b.$$

If there exists (b) $ay_1$ in $q$ that corresponds to $ab$, i.e., there exist $q_1$ and $q_2 \in \mathcal{RP}$ such that

(1) $p_1 ab p_2 \preceq q_1 a y_1 q_2$,
(2) $p_1 \preceq q_1$, and
(3) either $p_2 \preceq q_2$ or $p_2 \preceq y'_1 q_2$ for $y'_1 \in X$.

Let $D' = (D \setminus \{ab\}) \cup \{ay\}$. It is straightforward to see that $p\{x := ay\} = p_1 ay p_2 \preceq q_1 a y_1 q_2$ holds. Thus, $p\{x := r\} \preceq q$ for all $r \in D'$ holds. Let $D'' = (D \setminus \{ab\}) \cup \{yb\}$. By a similar discussion, if there exists (c) $y_1 b$ in $q$ that corresponds to $ab$, $p\{x := r\} \preceq q$ for all $r \in D''$ holds. Therefore, in this paper, we make the following definition on $D$:

**Definition 3:** Let $p, q \in \mathcal{RP}$ with $p \npreceq q$. Let $D \subset \mathcal{RP}$ such that for all $r \in D$, $|r| = 2$ and $p\{x := r\} \preceq q$ hold. Then, if for any $ab \in D$ $(a, b \in \Sigma)$, $p\{x := ay\} \npreceq q$ and $p\{x := yb\} \npreceq q$ hold for any $y \in X$ that does not appear in $q$, the set $D$ is said to be *maximally generalized with respect to* $(p, q)$.

### 3.2 $D = \{ay, by\}$ and $D = \{ya, yb\}$

**Lemma 4** (Sato et al.[4]): Let $\Sigma$ be an alphabet with $\sharp\Sigma \ge 3$. Let $p, q \in \mathcal{RP}_{\Sigma \cup X}$. Let $D$ be the following set of regular patterns over $\Sigma \cup X$, where $y$ is a variable symbol that does not appear in $p$ and $q$:

(i) $D = \{ay, by\}$ $(a \ne b)$,
(ii) $D = \{ya, yb\}$ $(a \ne b)$.

Then, if $p\{x := r\} \preceq q$ holds for all $r \in D$, then $p\{x := xy\} \preceq q$ holds.

**Proof.** We assume that $p\{x := xy\} \npreceq q$ in order to derive a contradiction. In the case of (ii), by reversing the strings $p$ and $q$, we can prove that the assumption $p\{x := xy\} \preceq q$ leads to a contradiction, as in the case of (i). Therefore, in the following, we consider only the case of (i): $D = \{ay, by\}$ $(a \ne b)$.

Since $p\{x := xy\} \npreceq q$, $p_1 ay p_2 \preceq q$ and $p_1 by p_2 \preceq q$, there exist regular patterns $q_1, q_2$ over $\Sigma \cup X$ such that $q = q_1 a y_1 w b y_2 q_2$ or $q = q_1 b y_1 w a y_2 q_2$ for some variable symbols $y_1, y_2$ $(y_1 \ne y_2)$ and a constant string $w$ $(|w| \ge 0)$ from Theorem 2. When $q = q_1 a y_1 w b y_2 q_2$ holds, the following four conditions hold: For $y'_1, y'_2 \in X$,

| | |
|---|---|
| (1) $p_1 \preceq q_1$, | (1') $p_2 \preceq w b y_2 q_2$ or $p_2 \preceq y'_1 w b y_2 q_2$, |
| (2) $p_1 \preceq q_1 a y_1 w$, | (2') $p_2 \preceq q_2$ or $p_2 \preceq y'_2 q_2$. |

From (2), there exist regular patterns $p'_1, p''_1$ such that $p_1 = p'_1 p''_1$, $p'_1 \preceq q_1 a$ and $p''_1 \preceq y_1 w$ hold. Therefore, since $p = p_1 x p_2 = p'_1 p''_1 x p_2$, if $p_2 \preceq w b y_2 q_2$ of (1') holds, $p \preceq q_1 a p''_1 x w b y_2 q_2 \equiv q\{y_1 := p''_1 x\}$ holds. If $p_2 \preceq y'_1 w b y_2 q_2$ of (1') holds, $p \preceq q_1 a p''_1 x y'_1 w b y_2 q_2 = q\{y_1 := p''_1 x y'_1\}$ holds. Thus, $p\{x := xy\} \preceq q\{y_1 := p''_1 x y y'_1\}$ holds. Hence, $p \preceq q$ holds. This contradicts the assumption. Therefore, we conclude that if $p\{x := r\} \preceq q$ for all $r \in \{ay, by\}$ $(a \ne b)$, then $p\{x := xy\} \preceq q$ holds. □

Let $p, q$ be regular patterns in $\mathcal{RP}$. In this paper, the statement like Lemma 4 is illustrated by a bipartite graph $(\Sigma, \Sigma, E)$ where $E = \{(a, b) \in \Sigma \times \Sigma \mid p\{x := ab\} \preceq q\}$. For example, the conditions (i) and (ii) in Lemma 4 are illustrated in (1) and (2) in Fig. 9, respectively.

### 3.3 $D = \{ya, bc, dy\}$

**Lemma 5:** Let $\Sigma$ be an alphabet with $\sharp\Sigma \ge 3$ and $p$, $q$ regular patterns over $\Sigma \cup X$. Let $D$ be the following set of regular patterns over $\Sigma \cup X$ such that $D$ is maximally generalized with respect to $(p, q)$, where $b \ne a$ and $c \ne d$ (that is, $b \notin \{a, d\}$ and $c \notin \{a, d\}$), and where $y$ is a variable symbol in $X$ that does not appear in either $p$ or $q$:

$$D = \{ya, bc, dy\}$$

Then, if $p\{x := r\} \preceq q$ for all $r \in D$, then $p\{x := xy\} \preceq q$.

**Proof.** We assume that $p\{x := xy\} \npreceq q$ in order to derive

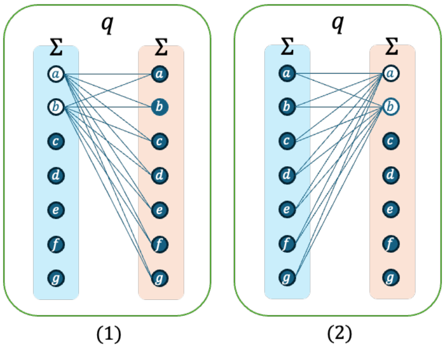



**Fig. 9** Let $\Sigma = \{a, b, c, d, e, f, g\}$ and $p, q \in \mathcal{RP}$. We assume that the symbols in $\Sigma$ are mutually distinct. These figures (1) and (2) express two cases $D = \{ay, by\}$ and $D = \{ya, yb\}$, respectively. In these cases, if $p\{x := r\} \preceq q$ for all $r \in D$, then $p\{x := xy\} \preceq q$ holds.

a contradiction. Since $D$ is maximally generalized over $(p, q)$, the regular pattern $q$ can be expressed in one of the following forms: Let $y_1, y_2$ be distinct variable symbols in $X$ and $q_1, q_2, w, w'$ be either the empty string or a regular pattern over $\Sigma \cup X$.

(5-1) $q = q_1 A w B w' C q_2$, where $\{A, B, C\} = \{y_1 a, bc, dy_2\}$,
(5-2) $q = q_1 A w B q_2$, where $\{A, B\} = \{dy_1 a, bc\}$,
(5-3) $q = q_1 A w B q_2$, where $\{A, B\} = \{y_1 a y_2, bc\}$ $(a = d)$.

(5-1) Case of $q = q_1 A w B w' C q_2$, where $\{A, B, C\} = \{y_1 a, bc, dy_2\}$: At first, we prove the following three claims:

*Claim* 1. $B \notin \{y_1 a, dy_2\}$.

*Proof of Claim* 1. Suppose that $(A, B, C) = (dy_2, y_1 a, bc)$. The following conditions must be satisfied: For $y_1', y_2' \in X$,

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq wy_1 aw'bcq_2$ or $p_2 \preceq y_2' wy_1 aw'bcq_2$,
(2) $p_1 \preceq q_1 dy_2 w$ or $p_1 \preceq q_1 dy_2 wy_1'$, (2') $p_2 \preceq w'bcq_2$,
(3) $p_1 \preceq q_1 dy_2 wy_1 aw'$, (3') $p_2 \preceq q_2$.

When $p_2 \preceq wy_1 aw'bcq_2$ in (1') holds, let $q_1' = q_1 dy_2$, $q_2' = wy_1 aw'$, $q_3' = bcq_2$. Since $p_1 \preceq q_1 dy_2 wy_1 aw'$ holds from (3) and $y_2, y_2'$ in $X$, both $p_1 \preceq q_1' q_2'$ and $p_2 \preceq q_2' q_3'$ hold, and $q_2'$ contains a variable symbol. When $p_2 \preceq y_2' wy_1 aw'bcq_2$ in (1') holds, let $q_1' = q_1 d$, $q_2' = y_2 wy_1 aw'$, $q_3' = bcq_2$. Since $p_1 \preceq q_1 dy_2 wy_1 aw'$ holds from (3), both $p_1 \preceq q_1' q_2'$ and $p_2 \preceq q_2' q_3'$ hold, and $q_2'$ contains a variable symbol. In both cases, by Theorem 2, $p \preceq q$ holds. This contradicts the assumption that $p\{x := xy\} \not\preceq q$.

Similarly, we can show that any case where $(A, B, C) = (y_1 a, dy_2, bc)$, $(bc, y_1 a, dy_2)$, or $(bc, dy_2, y_1 a)$ also contradicts the assumption. Therefore, we have $B \notin \{y_1 a, dy_2\}$. (*End of Proof of Claim* 1)

*Claim* 2. $(A, B, C) = (dy_2, bc, y_1 a)$.

*Proof of Claim* 2. From *Claim* 1, we have $B = bc$. Suppose that $(A, B, C) = (dy_2, bc, y_1 a)$, i.e., $q = q_1 dy_2 wbcw' y_1 aq_2$ holds. Then, the following conditions must be satisfied: For $y_1', y_2' \in X$,

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq wbcw' y_1 aq_2$ or $p_2 \preceq y_2' wbcw' y_1 aq_2$,
(2) $p_1 \preceq q_1 dy_2 w$, (2') $p_2 \preceq w' y_1 aq_2$,
(3) $p_1 \preceq q_1 dy_2 wbcw'$ or $p_1 \preceq q_1 dy_2 wbcw' y_1'$, (3') $p_2 \preceq q_2$.

From $p_1 \preceq q_1 dy_2 w$ in (2), $p_1$ is expressed as $p_1' p_1''$ for some $p_1'$ and $p_1''$, where $p_1' \preceq q_1 d$ and $p_1'' \preceq y_2 w$. When $p_2 \preceq wbcw' y_1 aq_2$ in (1'), we have $p = p_1 x p_2 = p_1' p_1'' x p_2 \preceq q_1 d p_1'' x wbcw' y_1 aq_2 = q\{y_2 := p_1'' x\}$. Thus, $p\{x := xy\} \preceq q\{y_2 := p_1'' xy\}$ holds. This contradicts the assumption that $p\{x := xy\} \not\preceq q$. When $p_2 \preceq y_2' wbcw' y_1 aq_2$ in (1'), we similarly have $p = p_1 x p_2 = p_1' p_1'' x p_2 \preceq q_1 d p_1'' x y_2' wbcw' y_1 aq_2 = q\{y_2 := p_1'' x y_2'\}$. Thus, $p\{x := xy\} \preceq q\{y_2 := p_1'' xyy_2'\}$ holds. This also contradicts the assumption. Therefore, we conclude that $(A, B, C) = (y_1 a, bc, dy_2)$. (*End of Proof of Claim* 2)

From *Claim* 2, The regular pattern $q$ is expressed as $q_1 y_1 awbcw' dy_2 q_2$, where $b \notin \{a, d\}$ and $c \notin \{a, d\}$. If $p\{x := xy\} \not\preceq q$ holds, the following conditions must be satisfied: For $y_1', y_2' \in X$,

(1) $p_1 \preceq q_1$ or $p_1 \preceq q_1 y_1'$, (1') $p_2 \preceq wbcw' dy_2 q_2$,
(2) $p_1 \preceq q_1 y_1 aw$, (2') $p_2 \preceq w' dy_2 q_2$,
(3) $p_1 \preceq q_1 y_1 awbcw'$, (3') $p_2 \preceq q_2$ or $p_2 \preceq y_2' q_2$.

*Claim* 3. $w$ and $w'$ contain no variable symbols.

*Proof of Claim* 3. Let $q_1' = q_1 y_1 a$, $q_2' = wbcw'$, and $q_3' = dy_2 q_2$. From (1') and (3), $p_1 \preceq q_1' q_2'$ and $p_2 \preceq q_2' q_3'$. If $q_2'$ contains a variable symbol, then by Theorem 2, $p \preceq q$ holds. This contradicts the assumption. Therefore, $w$ and $w'$ contain no variable symbols. (*End of Proof of Claim* 3)

From *Claim* 3, $w$ and $w'$ are strings consisting of symbols in $\Sigma$. From (1') and (2'), both $wbcw'd$ and $w'd$ are prefixes of $p_2$, and from (2) and (3), both $awbcw'$ and $aw$ are suffixes of $p_1$. It implies a contradiction in the following inductive way:

- $|w| = |w'|$: Directly, $b = d$ and $a = c$ hold.
- $|w| = |w'| + 1$: Also, $a = b$ holds.
- $|w| = |w'| + 2$: Since both $awbcw'$ and $aw$ are suffixes of $p_1$, and $|w| \geq 2$, $a$ is a suffix of $w$. From (1') and (2'), we have $w = w'da$. Furthermore, since both $awbcw'$ and $aw$ are suffixes of $p_1$, it follows that $w = bcw'$. Thus, $w'da = bcw'$ holds. From Proposition 2, $(b, c) \in \{(a, d), (d, a)\}$ holds. Therefore, these cases contradict the conditions $b \notin \{a, d\}$ and $c \notin \{a, d\}$.
- $|w| \geq |w'| + 3$: From (2) and (3), there exists a string $w''$ of length $|w| - |w'| - 2$ such that $w = w''bcw'$ holds. Moreover, from (2) and (3), since $|aw| < |wbcw'|$ and $aw = aw''bcw'$, it follows that $aw''$ is a suffix of $w$.

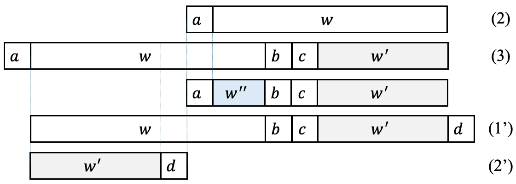



**Fig. 10** Case (5-1) in Lemma 5: Relation of strings $w$, $w'$, and $w''$

On the other hand, from (1’) and (2’), $w'd$ is a prefix of $w$. Since $|w'd| + |aw''| = |w'| + |w''| + 2 = |w|$, it follows that $w = w'daw''$ (Fig. 10). Therefore, $w'daw'' = w''bcw'$ holds. From Proposition 3, $(b, c) \in \{(a, d), (d, a)\}$ holds. This contradicts the conditions $b \notin \{a, d\}$ and $c \notin \{a, d\}$.

From the above, we conclude that all cases of (5-1) contradict the assertion that $p\{x := xy\} \npreceq q$ and the conditions $b \notin \{a, d\}$ and $c \notin \{a, d\}$.

(5-2) Case of $q = q_1 AwBq_2$, where $\{A, B\} = \{dy_1 a, bc\}$: We suppose that $(A, B) = (dy_1 a, bc)$, i.e., $q = q_1 dy_1 awbcq_2$ holds. Then, the following conditions must be satisfied for $y_1' \in X$:

| | |
|---|---|
| (1) $p_1 \preceq q_1$, | (1’) $p_2 \preceq awbcq_2$ or $p_2 \preceq y_1' awbcq_2$, |
| (2) $p_1 \preceq q_1 d$ or $p_1 \preceq q_1 dy_1'$ | (2’) $p_2 \preceq wbcq_2$, |
| (3) $p_1 \preceq q_1 dy_1 aw$, | (3’) $p_2 \preceq q_2$. |

From $p_1 \preceq q_1 dy_1 aw$ in (3), $p_1$ can be expressed as $p_1' p_1''$ for some $p_1'$ and $p_1''$, where $p_1' \preceq q_1 d$ and $p_1'' \preceq y_1 aw$. When $p_2 \preceq awbcq_2$ in (1’), we have

$$p = p_1' p_1'' x p_2 \preceq q_1 d p_1'' x awbcq_2 = q\{y_1 := p_1'' x\}.$$

Thus, $p\{x := xy\} \preceq q\{y_1 := p_1'' xy\}$ holds. This contradicts the assumption. When $p_2 \preceq y_1' awbcq_2$ in (1’), we similarly have

$$p = p_1' p_1'' x p_2 \preceq q_1 d p_1'' x y_1' awbcq_2 = q\{y_1 := p_1'' x y_1'\}.$$

Thus, $p\{x := xy\} \preceq q\{y_1 := p_1'' xyy_1'\}$ holds. This contradicts the assumption that $p\{x := xy\} \npreceq q$. Similarly, we can show that the case $(A, B) = (bc, dy_1 a)$ also contradicts the assumption.

(5-3) Case of $q = q_1 AwBq_2$, where $\{A, B\} = \{y_1 ay_2, bc\}$ ($a = d$): Suppose that $(A, B) = (y_1 ay_2, bc)$, i.e., $q = q_1 y_1 ay_2 wbcq_2$ holds. Then, the following conditions must be satisfied: For $y_1', y_2' \in X$,

| | |
|---|---|
| (1) $p_1 \preceq q_1$ or $p_1 \preceq q_1 y_1'$ | (1’) $p_2 \preceq y_2 wbcq_2$, |
| (2) $p_1 \preceq q_1 y_1 ay_2$, | (2’) $p_2 \preceq wbcq_2$ or $p_2 \preceq y_2' wbcq_2$, |
| (3) $p_1 \preceq q_1 y_1 ay_2 w$, | (3’) $p_2 \preceq q_2$. |

Let $q_1' = q_1 y_1 a$, $q_2' = y_2 w$, $q_3' = bcq_2$. From (3) and (1’), we have $p_1 \preceq q_1' q_2'$ and $p_2 \preceq q_2' q_3'$, respectively. Since $q_2'$ contains a variable symbol, Theorem 2 implies that $p \preceq q$ holds. This contradicts the assumption. Similarly, we can show that the case $(A, B) = (bc, y_1 ay_2)$ also contradicts the assumption.

From the above, we conclude that if $p\{x := r\} \preceq q$ for all $r \in \{ya, bc, dy\}$ ($b \notin \{a, d\}$ and $c \notin \{a, d\}$), then $p\{x := xy\} \preceq q$ holds. □

The condition in Lemma 5 is illustrated in four cases (3)–(6) in Fig. 11.

**Lemma 6:** Let $\Sigma$ be an alphabet with $\sharp\Sigma \geq 3$ and let $p$, $q$ be regular patterns over $\Sigma \cup X$. Let $D$ be one of the following sets of regular patterns over $\Sigma \cup X$ such that and $D$ is maximally generalized with respect to $(p, q)$, where $y$ is a variable symbol in $X$ that does not appear in $p$ and $q$.

$$D = \{ya, bc, dy\} \ (b = a,\ b \neq d,\ \text{and } c \notin \{a, d\}),$$

Then, if $p\{x := r\} \preceq q$ for all $r \in D$, then $p\{x := xy\} \preceq q$.

**Proof.** We assume that $p\{x := xy\} \npreceq q$ in order to derive a contradiction. The proof is almost the same as the proof of Lemma 5. Since $p\{x := r\} \preceq q$ for all $r \in D$ and $D$ is maximally generalized with respect to $(p, q)$, there are three strings of length 2 corresponding to $ya, bc, dy$ in $q$. The symbols appearing in $D$ correspond to either a variable or a constant symbol in $q$. Let $y_1$ and $y_2$ be variable symbols appearing in $q$. The strings $ya$ and $dy$ must correspond to the strings $y_1 a$ and $dy_2$ in $q$, respectively. For the same reasons stated at the beginning of Lemma 5, the string $bc$ corresponds to the string $bc$ in $q$ as well. Let $A, B, C$ be regular patterns over $\Sigma \cup X$, where $\{A, B, C\} = \{y_1 a, ac, dy_3\}$. Since $p\{x := xy\} \npreceq q$, $q$ can be expressed in one of the following four forms: Let $y_1, y_2$ be distinct variable symbols in $X$, and $q_1, q_2, w, w'$ either the empty string or a regular pattern over $\Sigma \cup X$. From the conditions $b = a$ and $b \neq d$, it follows that $a \neq d$.

(6-1) $q = q_1 AwBw'Cq_2$, where $\{A, B, C\} = \{y_1 a, ac, dy_2\}$,
(6-2) $q = q_1 AwBq_2$, where $\{A, B\} = \{y_1 ac, dy_2\}$,
(6-3) $q = q_1 Aq_2$, where $A = dy_1 ac$.

In cases (6-1) and (6-2), similar to Lemma 5, it is shown that $q = q_1 y_1 awacw' dy_2 q_2$ and $q = q_1 y_1 acwdy_2 q_2$, respectively, where $w$ and $w'$ contain no variable symbols.

(6-1) Case of $q = q_1 AwBw'Cq_2$, where $(A, B, C) = (y_1 a, ac, dy_2)$: The following conditions must be satisfied:

| | |
|---|---|
| (1) $p_1 \preceq q_1$, | (1’) $p_2 \preceq wacw' dy_2 q_2$, |
| (2) $p_1 \preceq q_1 y_1 aw$, | (2’) $p_2 \preceq w' dy_2 q_2$, |
| (3) $p_1 \preceq q_1 y_1 awacw'$, | (3’) $p_2 \preceq q_2$. |

From (1’) and (2’), both $wacw'd$ and $w'd$ are prefixes of $p_2$, and from (2) and (3), both $awacw'$ and $aw$ are suffixes of $p_1$. It implies a contradiction in the following inductive way:

- $|w| = |w'|$: $c = a$ holds.

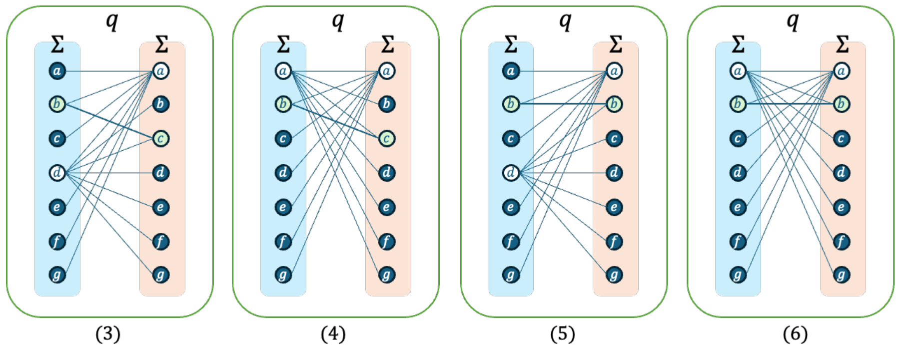



**Fig. 11** Let $\Sigma = \{a, b, c, d, e, f, g\}$ and $p, q \in \mathcal{RP}$. We assume that the symbols in $\Sigma$ are mutually distinct. The figure (3) expresses case $D = \{ya, bc, dy\}$ in Lemma 5. The figures (4), (5), and (6) express three cases $D = \{ya, bc, ay\}$, $D = \{ya, bb, dy\}$, and $D = \{ya, bb, ay\}$, respectively. In these cases, if $p\{x := r\} \preceq q$ for all $r \in D$ and $D$ is maximally generalized with respect to $(p, q)$, then $p\{x := xy\} \preceq q$ holds.

- $|w| = |w'| + 1$: $w = w'd = cw'$ holds. Thus, from Proposition 1, $c = d$ holds.
- $|w| = |w'| + 2$: $w = w'da = acw'$ holds. From Proposition 2, $c \in \{a, d\}$ holds.
- $|w| \geq |w'| + 3$: From (2) and (3), there exists a string $w''$ of length $|w| - |w'| - 2$ such that $w = w''acw'$ holds. Moreover, from (2) and (3), since $|aw| < |wacw'|$ and $aw = aw''acw'$, it follows that $aw''$ is a suffix of $w$. On the other hand, from (1') and (2'), $w'd$ is a prefix of $w$. Since $|w'd| + |aw''| = |w'| + |w''| + 2 = |w|$, we have $w = w'daw''$. Therefore, $w'daw'' = w''acw'$ holds (Fig. 12). From Proposition 3, we have $c \in \{a, d\}$.
- $|w'| = |w| + 1$: From (1') and (2'), $c = d$ holds.
- $|w'| = |w| + 2$: From (1') and (2'), $d$ is a prefix of $w'$. Thus, from (2) and (3), $w' = wac = daw$ holds. From Proposition 2, $c \in \{a, d\}$ holds.
- $|w'| \geq |w| + 3$: From (1') and (2'), there exists a string $w''$ of length $|w| - |w'| - 2$ such that $w' = wacw''$ holds. Moreover, from (1') and (2'), since $|w'd| < |wacw'|$ and $w'd = wacw''d$, $w''d$ is a prefix of $w'$. On the other hand, from (2) and (3), $aw'$ is a suffix of $w'$. Since $|w''d| + |aw| = |w''| + |w| + 2 = |w'|$, we have $w' = w''daw$. Therefore, $w''daw = wacw''$ holds. From Proposition 3, we have $c \in \{a, d\}$.

All the cases contradict the condition $c \notin \{a, d\}$. Therefore, if $b = a$, $b \neq d$, and $c \notin \{a, d\}$ are satisfied, case (6-1) is impossible.

(6-2) Case of $q = q_1 AwBq_2$, where $(A, B) = (y_1 ac, dy_2)$: For $q = q_1 y_1 acwdy_2 q_2$, the following conditions must be satisfied:

| | |
|---|---|
| (1) $p_1 \preceq q_1$, | (1') $p_2 \preceq cwdy_3 q_2$, |
| (2) $p_1 \preceq q_1 y_1$, | (2') $p_2 \preceq wdy_3 q_2$, |
| (3) $p_1 \preceq q_1 y_1 acwd$, | (3') $p_2 \preceq q_2$. |

- If $|w| = 0$, from (1') and (2'), both $cd$ and $d$ are prefixes of $p_2$. Thus, we have $c = d$.
- If $|w| = 1$, from (1') and (2'), both $cwd$ and $wd$ are prefixes of $p_2$. Thus, we have $w = c = d$.
- If $|w| \geq 2$, then from (1') and (2'), both $cwd$ and $wd$ are prefixes of $p_2$. Thus, we have $cw = wd$. From Proposition 2, $c = d$ holds.

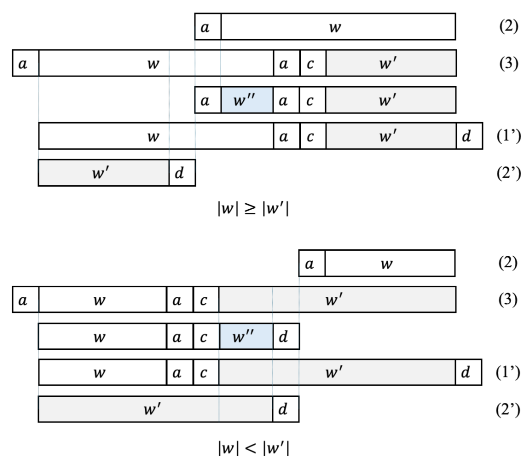


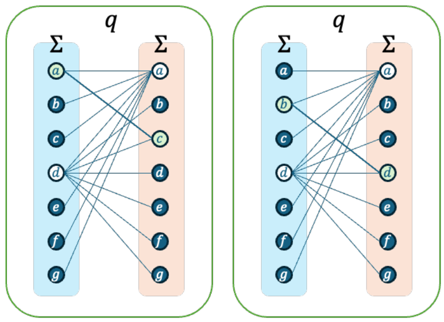

**Fig. 12** Case (6-1) in Lemma 6: Relation of strings $w$, $w'$, and $w''$

All of these cases do not meet $b = a$, $b \neq d$, and $c \notin \{a, d\}$. Therefore, if $b = a$, $b \neq d$, and $c \notin \{a, d\}$ are satisfied, case (6-2) is also impossible.

(6-3) Case of $q = q_1 A q_2$, where $A = dy_1 ac$: For $q = q_1 dy_1 acq_2$, the following conditions must be satisfied for $y_1', y_1'' \in X$:

| | |
|---|---|
| (1) $p_1 \preceq q_1 d$ or $p_1 \preceq q_1 dy_1'$ | (1') $p_2 \preceq acq_2$, |
| (2) $p_1 \preceq q_1 dy_1$, | (2') $p_2 \preceq q_2$, |
| (3) $p_1 \preceq q_1$, | (3') $p_2 \preceq acq_2$ or $p_2 \preceq y_1'' acq_2$. |

For $p_1 \preceq q_1 d$ in (1) and $p_2 \preceq acq_2$ in (3'), $p = p_1 x p_2 \preceq q_1 dxacq_2 \preceq q\{y_1 := x\}$ holds. From this, we have $p\{x := xy\} \preceq q\{y_1 := x\}$. This contradicts the assumption that $p\{x := xy\} \npreceq q$. Similarly, we can show that the other




(7) (8)

**Fig. 13** Let $\Sigma = \{a, b, c, d, e, f, g\}$ and $p, q \in \mathcal{RP}$. We assume that the symbols in $\Sigma$ are mutually distinct. The figures (7) and (8) express two cases $D = \{ya, ac, dy\}$ and $D = \{ya, bd, dy\}$ in Lemmas 6 and 7, respectively. In these cases, if $p\{x := r\} \preceq q$ for all $r \in D$ and $D$ is maximally generalized with respect to $(p, q)$, then $p\{x := xy\} \preceq q$ holds.

cases of (1) and (3') also contradict the assumption.

From the above, we conclude that if $p\{x := r\} \preceq q$ for all $r \in \{ya, bc, dy\}$ ($b = a$, $b \neq d$, and $c \notin \{a, d\}$), then $p\{x := xy\} \preceq q$ holds. □

The conditions in Lemmas 6 and 7 are illustrated in (7) and (8) in Fig. 13, respectively.

**Lemma 7:** Let $\Sigma$ be an alphabet with $\sharp\Sigma \geq 3$ and let $p$, $q$ be regular patterns over $\Sigma \cup X$. Let $D$ be one of the following sets of regular patterns over $\Sigma \cup X$, where $y$ is a variable symbol in $X$ that does not appear in $p$ and $q$.

$$D = \{ya, bc, dy\}\ (b \notin \{a, d\},\ c \neq a,\ \text{and } c = d).$$

Then, if $p\{x := r\} \preceq q$ for all $r \in D$ and $D$ is maximally generalized over $(p, q)$, then $p\{x := xy\} \preceq q$.

**Proof.** The proof follows by reversing $p$ and $q$ and subsequently applying Lemma 6. □

When the conditions of Lemmas 5, 6, and 7 are not satisfied, counterexamples can be constructed as follows:

**Proposition 4:** Let $\Sigma$ be an alphabet with $\sharp\Sigma \geq 3$. For a variable symbol $y$, let $D = \{ya, bc, dy\}$ ($b = a$ and $c = d$). There exist regular patterns $p$ and $q$ over $\Sigma \cup X$ such that $p\{x := r\} \preceq q$ for any $r \in D$, but $p\{x := xy\} \npreceq q$.

**Proof.** We provide an example to demonstrate this proposition. Let $a, b, c, d, e$ be constant symbols in $\Sigma$, and let $x, y, y_1, y_2$ be variable symbols in $X$. Define the regular patterns $p$ and $q$ as follows:

$$p = eabcbcadabcbcadaxbcadadabcbcadade,$$
$$q = y_1abcbcadabcbcadady_2 \qquad (b = a \text{ and } c = d).$$

Obviously $p\{x := xy\} \npreceq q$ holds. For these $p$ and $q$, the condition for Proposition 4 holds as follows (see also Fig. 14):

$$\begin{aligned}
&p\ \{x := ya\}\\
&= (eabcbcadabcbcaday)abcadadabcbcadade\\
&= q\{y_1 := eabcbcadabcbcaday,\ y_2 := e\}\\
&\preceq q,\\
&p\ \{x := bc\}\\
&= (eabcbcad)abcbcadabcbcadad(abcbcadade)\\
&= q\{y_1 := eabcbcad,\ y_2 := abcbcadade\}\\
&\preceq q,\\
&p\ \{x := dy\}\\
&= eabcbcadabcbcadad(ybcadadabcbcadade)\\
&= q\{y_1 := e,\ y_2 := ybcadadabcbcadade\}\\
&\preceq q.
\end{aligned}$$

□

### 3.4 $D = \{a_1b_1, a_2b_2, a_3y\}$ and $D = \{a_1b_1, a_2b_2, yb_3\}$

**Lemma 8:** Let $\Sigma$ be an alphabet with $\sharp\Sigma \geq 3$ and $p$, $q$ regular patterns over $\Sigma \cup X$. Let $D$ be the following set of regular patterns over $\Sigma \cup X$, where $y$ is a variable symbol in $X$ that does not appear in $p$ and $q$:

$$D = \{a_1b_1, a_2b_2, a_3y\},$$
$$\text{where } a_i \neq a_j \text{ and } b_i \neq b_j\ (i \neq j, 1 \leq i, j \leq 3).$$

Then, if $p\{x := r\} \preceq q$ holds for all $r \in D$ and $D$ is maximally generalized with respect to $(p, q)$, then $p\{x := xy\} \preceq q$ holds.

**Proof.** We assume that $p\{x := xy\} \npreceq q$ holds. Since $D$ is maximally generalized with respect to $(p, q)$, from the same argument as in the proof of Lemma 6, it is sufficient to consider the following five cases (8-1)–(8-5) of $q$: For $y_1 \in X$,

(8-1) $q = q_1a_1b_1wa_2b_2w'a_3y_1q_2$,
(8-2) $q = q_1a_1b_1b_2y_1q_2$ ($a_2 = b_1$ and $a_3 = b_2$),
(8-3) $q = q_1a_1b_1b_2wa_3y_1q_2$ ($b_1 = a_2$),
(8-4) $q = q_1a_3y_1wa_1b_1b_2q_2$ ($b_1 = a_2$),
(8-5) $q = q_1a_1b_1y_1wa_2b_2q_2$ ($b_1 = a_3$),

where no variable symbol appears in both $w$ and $w'$.

(8-1) Case of $q = q_1a_1b_1wa_2b_2w'a_3y_1q_2$: The following conditions must be satisfied: For $y_1' \in X$,

| | |
|---|---|
| (1) $p_1 \preceq q_1$, | (1') $p_2 \preceq wa_2b_2w'a_3y_1q_2$, |
| (2) $p_1 \preceq q_1a_1b_1w$, | (2') $p_2 \preceq w'a_3y_1q_2$, |
| (3) $p_1 \preceq q_1a_1b_1wa_2b_2w'$, | (3') $p_2 \preceq q_2$ or $p_2 \preceq y_1'q_2$. |

- $|w|+1 = |w'|$: From (2) and (3), both $a_1b_1wa_2b_2w'$ and $a_1b_1w$ are suffixes of $p_1$. Since there exists a constant symbol $w_1$ such that $w' = w_1w$ and $b_2w_1w = a_1b_1w$ hold, then $b_2 = a_1$. Moreover, both $wa_2b_2w'a_3$ and $w'a_3$ are prefixes of $p_2$ from (1') and (2'). Since there exists a constant symbol $w_2$ such that $w' = ww_2$ and $wa_2b_2 = ww_2a_3$ hold, then $b_2 = a_3$. Thus, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.
- $|w| + 1 < |w'|$: From (2) and (3), both $a_1b_1wa_2b_2w'$ and $a_1b_1w$ are suffixes of $p_1$. Hence, $a_1b_1$ is suffixes of $w_{\prime}$. Moreover, both $wa_2b_2w'a_3$ and $w'a_3$ are prefixes


**Fig. 14** Substitutions for $p$ and each correspondence to $q$.

of $p_2$ from (1') and (2'). Hence, there exist constant symbols $w_1$ and $w_2$ such that $w' = w_1 w$, $w' = w w_2$ and $|a_2 b_2 w_1| = |w_2 a_3| + 1$ hold. Thus, since the second-to-last symbol of $w_1$ is $a_3$, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.

- $|w| = |w'| + 1$: From (1') and (2'), both $wa_2b_2w'a_3$ and $w'a_3$ are prefixes of $p_2$. Since there exists a constant symbol $w_1$ such that $w = w'w_1$ and $w'w_1 = w'a_3$ hold, then $w_1 = a_3$ holds. Moreover, since both $a_1b_1wa_2b_2w'$ and $a_1b_1w$ are suffixes of $p_1$ from (2) and (3), there exists a constant symbol $w_2$ such that $w = w_2w'$ and $|w_1a_2b_2w'| = |a_1b_1w_2w'|$ hold. Hence, $w_1 = a_1$ holds. Thus, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.
- $|w| > |w'| + 1$: Since both $wa_2b_2w'a_3$ and $w'a_3$ are prefixes of $p_2$ from (1') and (2'), there exists a constant string $w_1$ such that $w = w'w_1$ and the first symbol of $w_1$ is $a_3$. Moreover, since there exists a constant string $w_2$ such that $w = w_2w'$ and $|w_1a_2b_2| = |a_1b_1w_2|$ hold, $a_1b_1$ is a prefix of $w_1$. Thus, $a_3 = a_1$ holds. This contradicts the assumption of $a_1 \neq a_3$.

(8-2) Case of $q = q_1a_1b_1b_2y_1q_2$ ($a_2 = b_1$ and $a_3 = b_2$): The following conditions must be satisfied: For $y_1' \in X$,

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq b_2y_1q_2$,
(2) $p_1 \preceq q_1a_1$, (2') $p_2 \preceq y_1q_2$,
(3) $p_1 \preceq q_1a_1b_1$, (3') $p_2 \preceq q_2$ or $p_2 \preceq y_1'q_2$.

From (2) and (3), both $a_1b_1$ and $a_1$ are suffixes of $p_1$. Hence, $b_1 = a_1$ holds. Thus, from the assumption of $b_1 = a_2$, $a_1 = a_2$ holds. This contradicts the assumption of $a_1 \neq a_2$.

(8-3) Case of $q = q_1a_1b_1b_2wa_3y_1q_2$ ($b_1 = a_2$): The following conditions must be satisfied: For $y_1' \in X$,

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq b_2wa_3y_1q_2$,
(2) $p_1 \preceq q_1a_1$, (2') $p_2 \preceq wa_3y_1q_2$,
(3) $p_1 \preceq q_1a_1b_1b_2w$, (3') $p_2 \preceq q_2$ or $p_2 \preceq y_1'q_2$.

- $|w| = 0$: From (2) and (3), both $a_1$ and $a_1b_1b_2$ are suffixes of $p_1$. Hence, $a_1 = b_2$ holds. Moreover, since both $b_2a_3$ and $a_3$ is prefixes of $p_2$, $b_2 = a_3$ holds. Thus, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.
- $|w| \geq 1$: Since both $a_1$ and $a_1b_1b_2w$ are suffixes of $p_1$ from (2) and (3), the last symbol of $w$ is $a_1$. Moreover, since both $b_2wa_3$ and $wa_3$ are prefixes of $p_2$ from (1') and (2'), the last symbol of $w$ is $a_3$. Thus, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.

(8-4) Case of $q = q_1a_3y_1wa_1b_1b_2q_2$ ($b_1 = a_2$): The following conditions must be satisfied: For $y_1' \in X$,

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq wa_1b_1b_2q_2$ or $p_2 \preceq y_1'wa_1b_1b_2q_2$,
(2) $p_1 \preceq q_1a_3y_1w$, (2') $p_2 \preceq b_2q_2$,
(3) $p_1 \preceq q_1a_3y_1wa_1$, (3') $p_2 \preceq q_2$.

From (3), there exist regular patterns $p_1'$ and $p_1''$ such that $p_1 = p_1'p_1''$, $p_1' \preceq q_1a_3$, and $p_1'' \preceq y_1wa_1$ hold. Hence, if $p_2 \preceq wa_1b_1b_2q_2$ of (1') holds, since $p = p_1xp_2 = p_1'p_1''xp_2 \preceq q_1a_3p_1''xwa_1b_1b_2q_2 = q\{y_1 := p_1''x\}$, then $p \preceq q$ holds. Thus, this contradicts the assumption. Similarly, $p_2 \preceq y_1'wa_1b_1b_2q_2$ of (1') leads to a contradiction.

(8-5) Case of $q = q_1a_1b_1y_1wa_2b_2q_2$ ($b_1 = a_3$): The following conditions must be satisfied: For $y_1' \in X$,

(1) $p_1 \preceq q_1$, (1') $p_2 \preceq y_1wa_2b_2q_2$,
(2) $p_1 \preceq q_1a_1$, (2') $p_2 \preceq wa_2b_2q_2$ or $p_2 \preceq y_1'wa_2b_2q_2$,
(3) $p_1 \preceq q_1a_1b_1y_1w$, (3') $p_2 \preceq q_2$.

Let $q_1' = q_1a_1b_1$, $q_2' = y_1w$, $q_3' = a_2b_2q_2$. From (3), $p_1 \preceq q_1'q_2'$ holds, and from (1'), $p_2 \preceq q_2'q_3'$ holds. Since $q_2'$ contains a variable symbol $y_1$, $p \preceq q$ holds from Theorem 2. This contradicts the assumption. □

**Lemma 9:** Let $\Sigma$ be an alphabet with $\sharp\Sigma \geq 3$ and $p$, $q$ regular patterns over $\Sigma \cup X$. Let $D$ be the following set of regular patterns over $\Sigma \cup X$, where $y$ is a variable symbol in $X$ that does not appear in $p$ and $q$:

$$D = \{a_1b_1, a_2b_2, yb_3\},$$
$$\text{where } a_i \neq a_j \text{ and } b_i \neq b_j \ (i \neq j, 1 \leq i, j \leq 3).$$

Then, if $p\{x := r\} \preceq q$ for all $r \in D$ and $D$ is maximally generalized with respect to $(p, q)$, then $p\{x := xy\} \preceq q$.

**Proof.** The proof follows by reversing $p$ and $q$ and subsequently applying Lemma 8. □

### 3.5 $D = \{a_1b_1, a_2b_2, a_3b_3\}$

In Lemma 14 (ii) of [4], they stated that, when $\sharp\Sigma \geq 3$, for regular patterns $p, q$, if $p\{x := r\} \preceq q$ for any $r \in D$, then $p\{x := xy\} \preceq q$ holds, where $D = \{a_1b_1, a_2b_2, a_3b_3\}$ ($a_i \neq a_j$ and $b_i \neq b_j$ for each $i, j$ ($i \neq j, 1 \leq i, j \leq 3$)). Unfortunately, there exist the following counterexamples of Lemma 14 (ii) of [4].

**Example 2:** Assume that $a_1 = b_2$ and $a_3 = b_1$ hold.

(1) Let $p = ca_1x'a_3c$ and $q = xa_1a_3y$. It is clear that $\{x := xy\} \not\preceq q$ holds. However, we can see that $p\{x' := a_1b_1\} \preceq q$, $p\{x' := a_2b_2\} \preceq q$ and $p\{x' := a_3b_3\} \preceq q$ hold, since $p\{x' := a_1b_1\} = ca_1a_1b_1a_3c = q\{x := ca_1, y := a_3c\}$, $p\{x' := a_2b_2\} = ca_1a_2b_2a_3c = q\{x := ca_1a_2, y := c\}$ and $p\{x' := a_3b_3\} = ca_1a_3b_3a_3c = q\{x := c, y := b_3a_3c\}$ hold.
(2) Let $p = cb_2a_1b_1b_2x'a_1b_1b_2a_3c$ and $q = xb_2a_1b_1b_2a_3y$. It is clear that $p\{x := xy\} \not\preceq q$ holds. However, we have $p\{x' := a_1b_1\} \preceq q$, $p\{x' := a_2b_2\} \preceq q$, and $p\{x' := a_3b_3\} \preceq q$, since $p\{x' := a_1b_1\} = cb_2a_1b_1b_2a_1b_1a_1b_1b_2a_3c = q\{x := cb_2a_1b_1, y := b_2a_3c\}$, $p\{x' := a_2b_2\} = cb_2a_1b_1b_2a_2b_2a_1b_1b_2a_3c = q\{x := cb_2a_1b_1b_2a_2, y := c\}$, and $p\{x' := a_3b_3\} = cb_2a_1b_1b_2a_3b_3a_1b_1b_2a_3c = q\{x := c, y := b_3a_1b_1b_2a_3c\}$ hold.

The conditions in Lemmas 8, 9, and 10 are illustrated in the cases (9), (10), and (11) in Fig. 15.

**Lemma 10:** Let $\Sigma$ be an alphabet with $\sharp\Sigma \geq 3$ and $p$, $q$ regular pattern over $\Sigma \cup X$. Let $D$ be the following set of regular patterns over $\Sigma \cup X$, where $y$ is a variable symbol in $X$ that does not appear in $p$ and $q$:

$$D = \{a_1b_1, a_2b_2, a_3b_3\},$$
$$\text{where } a_i \neq a_j \text{ and } b_i \neq b_j \ (i \neq j, 1 \leq i, j \leq 3).$$

Then, if $p\{x := r\} \preceq q$ for all $r \in D$ and $D$ is maximally generalized with respect to $(p, q)$, then $p\{x := xy\} \preceq q$.

**Proof.** We assume that $p\{x := xy\} \not\preceq q$ holds. Since $D$ is maximally generalize with respect to $(p, q)$, it is sufficient to consider the following four cases (10-1)-(10-4) of $q$ for some regular patterns $q_1, q_2$ and some constant strings $w, w'$ ($|w| \geq 0$ and $|w'| \geq 0$):

(10-1) $q = q_1a_1b_1wa_2b_2w'a_3b_3q_2$,
(10-2) $q = q_1a_1b_1a_3b_3q_2$ ($b_1 = a_2$ and $a_3 = b_2$),
(10-3) $q = q_1a_1b_1b_2wa_3b_3q_2$ ($b_1 = a_2$),
(10-4) $q = q_1a_1b_1wa_2b_2b_3q_2$ ($b_2 = a_3$).

(10-1) Case of $q = q_1a_1b_1wa_2b_2w'a_3b_3q_2$: The following conditions must be satisfied:

| | |
|---|---|
| (1) $p_1 \preceq q_1$, | (1') $p_2 \preceq wa_2b_2w'a_3b_3q_2$, |
| (2) $p_1 \preceq q_1a_1b_1w$, | (2') $p_2 \preceq w'a_3b_3q_2$, |
| (3) $p_1 \preceq q_1a_1b_1wa_2b_2w'$, | (3') $p_2 \preceq q_2$. |

- $|w| = |w'|$: From (2) and (3), both $a_1b_1wa_2b_2w'$ and $a_1b_1w$ are suffixes of $p_1$. Then, $a_1b_1w = a_2b_2w'$. Hence, $a_1b_1 = a_2b_2$. This contracts the assumption of $a_1 \neq a_2$ and $b_1 \neq b_2$. $|w| + 1 = |w'|$: The two strings $wa_2b_2w'a_3b_3$ and $w'a_3b_3$ are prefixes of $p_2$. If there exists a constant symbol $w_1$ such that $w'a_3b_3 = ww_1a_3b_3$, then $b_2$ and $a_3$ are the same symbol from $wa_2b_2 = ww_1a_3$. from (2) and (3), both $a_1b_1wa_2b_2w'$ and $a_1b_1w$ are suffixes of $p_1$. Then, there exists a constant symbol $w_2$ such that $w' = w_2w$, then $b_2$ and $a_1$ are the same symbol from $b_2w_2w = a_1b_1w$. Hence, from $b_2 = a_3$, $a_3$ and $a_1$ are same symbol. This contradicts the assumption of $a_3 \neq a_1$.
- $|w|+1 < |w'|$: From (2) and (3), both $a_1b_1wa_2b_2w'$ and $a_1b_1w$ are suffixes of $p_1$. If there exists a constant string $w_1$ ($|w_1| \geq 2$) such that $w' = w_1w$, then $a_1b_1$ is a suffix of $w_1$. From conditions (1′) and (2′), both $wa_2b_2w'a_3b_3$ and $w'a_3b_3$ are prefixes of $p_2$. If there exist constant strings $w_1$ and $w_2$ such that $w' = w_1w = ww_2$ holds, then both $a_2b_2$ and $a_3b_3$ are suffixes of $w_1$ from $|w_1| = |w_2|$ and $|ww_2a_3b_3| = |wa_2b_2w_1|$. Hence, $a_1b_1 = a_3b_3$. This contradicts the assumption of $a_1 \neq a_3$ and $b_1 \neq b_3$.
- $|w| > |w'|$: We can prove the contradiction in a similar way as $|w| \leq |w'|$.

(10-2) Case of $q = q_1a_1b_1a_3b_3q_2$ ($b_1 = a_2$ and $a_3 = b_2$): The following conditions must be satisfied:

| | |
|---|---|
| (1) $p_1 \preceq q_1$, | (1') $p_2 \preceq a_3b_3q_2$, |
| (2) $p_1 \preceq q_1a_1$, | (2') $p_2 \preceq b_3q_2$, |
| (3) $p_1 \preceq q_1a_1b_1$, | (3') $p_2 \preceq q_2$. |

From (2) and (3), since both $a_1b_1$ and $a_1$ are suffixes of $p_1$, $b_1 = a_1$ holds. From the assumption of $b_1 = a_2$, the equation $a_1 = a_2$ holds. This contradicts the assumption of $a_1 \neq a_2$.

(10-3) Case of $q = q_1a_1b_1b_2wa_3b_3q_2$ ($b_1 = a_2$): The following conditions must be satisfied:

| | |
|---|---|
| (1) $p_1 \preceq q_1$, | (1') $p_2 \preceq b_2wa_3b_3q_2$, |
| (2) $p_1 \preceq q_1a_1$, | (2') $p_2 \preceq wa_3b_3q_2$, |
| (3) $p_1 \preceq q_1a_1b_1b_2w$, | (3') $p_2 \preceq q_2$. |

- $|w| = 0$: From (2) and (3), both $a_1$ and $a_1b_1b_2$ are suffixes of $p_1$. Moreover, from (1') and (2'), both $b_2a_3b_3$ and $a_3b_3$ are prefixes of $p_2$. Since $b_2 = a_1$ and $b_2a_3 = a_3b_3$, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.
- $|w| \geq 1$: From (2) and (3), both $a_1$ and $a_1b_1b_2w$ are suffixes of $p_1$. Hence, the last symbol of $w$ is $a_1$. Moreover, both $b_2wa_3b_3$ and $wa_3b_3$ are prefixes of $p_2$ from (1') and (2'). Hence, the last symbol of $w$ is $a_3$. Therefore, $a_1 = a_3$ holds. This contradicts the assumption of $a_1 \neq a_3$.

(10-4) Case of $q = q_1a_1b_1wa_2b_2b_3q_2$ ($b_2 = a_3$): The following conditions must be satisfied:

| | |
|---|---|
| (1) $p_1 \preceq q_1$, | (1') $p_2 \preceq wa_2b_2b_3q_2$, |
| (2) $p_1 \preceq q_1a_1b_1w$, | (2') $p_2 \preceq b_3q_2$, |
| (3) $p_1 \preceq q_1a_1b_1wa_2$, | (3') $p_2 \preceq q_2$. |

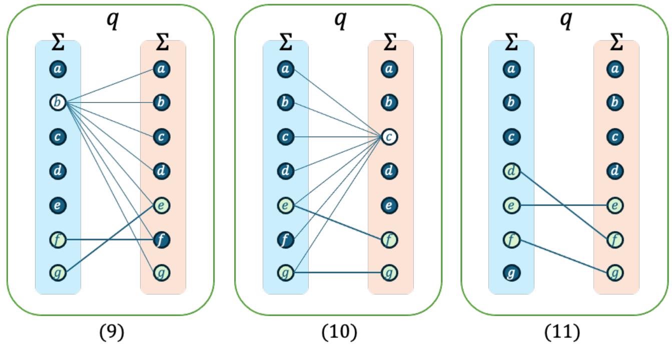



**Fig. 15** Let $\Sigma = \{a, b, c, d, e, f, g\}$ and $p, q \in \mathcal{RP}$. We assume that the symbols in $\Sigma$ are mutually distinct. The figures (9), (10,) and (11) express cases $D = \{a_1b_1, a_2b_2, a_3y\}$, $D = \{a_1b_1, a_2b_2, yb_3\}$, and $D = \{a_1b_1, a_2b_2, a_3b_3\}$ in Lemmas 8, 9, and 10, respectively, where $a_i \neq a_j$ and $b_i \neq b_j$ for each $i, j$ $(i \neq j, 1 \leq i, j \leq 3)$. In these cases, if $p\{x := r\} \preceq q$ for all $r \in D$ and $D$ is maximally generalized with respect to $(p, q)$, then $p\{x := xy\} \preceq q$ holds.

- $|w| = 0$: From (2) and (3), both $a_1b_1$ and $a_1b_1a_2$ are suffixes of $p_1$. And from (1') and (2'), both $a_2b_2b_3$ and $b_3$ are prefixes of $p_2$. Since $b_1 = a_2$ and $a_2 = b_3$, then $b_1 = b_3$ holds. This contradicts the assumption of $b_1 \neq b_3$.
- $|w| \geq 1$: Since both $a_1b_1w$ and $a_1b_1wa_2$ are suffixes of $p_1$ from (2) and (3), the first symbol of $w$ is $b_1$. Moreover, since both $wa_2b_2b_3$ and $b_3$ are prefixes of $p_2$ from (1') and (2'), the first symbol of $w$ is $b_3$. Therefore, $b_1 = b_3$ holds. This contradicts the assumption of $b_1 \neq b_3$.

□

### 3.6 Characteristic sets for finite union of regular patterns

**Lemma 11:** Let $k$ be an integer with $k \geq 1$. Let $\Sigma$ be an alphabet with $\sharp\Sigma = k + 2$. Let $p \in \mathcal{RP}$ in which a variable symbol $x$ appears, and let $Q \in \mathcal{RP}^k$. If for any string $w \in \Sigma^*$ with $|w| = 2$, there exists a regular pattern $q_w \in Q$ such that $p\{x := w\} \preceq q_w$ holds, then there exists a regular pattern $q \in Q$ such that $p\{x := xy\} \preceq q$ holds, where $y$ is a variable symbol that does not appear in $q$.

**Proof.** Without loss of generality, we suppose that $\sharp Q = k$ holds. Otherwise, for some regular pattern $q$ already in $Q$, we can add a new regular pattern $q'$ equivalent to $q$, i.e., $q' \equiv q$, to $Q$ repeatedly until $\sharp Q = k$ is satisfied. For any $q \in Q$, we define the sets $A(q), B(q) \subseteq \Sigma$ as follows:

$$A(q) = \{a \in \Sigma \mid p\{x := ay\} \preceq q,\ y \in X\},$$
$$B(q) = \{b \in \Sigma \mid p\{x := yb\} \preceq q,\ y \in X\}.$$

If there exists $q \in Q$ such that $|A(q)| \geq 2$ or $|B(q)| \geq 2$, from Lemma 4, $p\{x := xy\} \preceq q$ holds. Below, we suppose that $|A(q)| \leq 1$ and $|B(q)| \leq 1$. Let $\perp$ be a constant symbol that is not a member in $\Sigma$. We define the functions $\sigma_A : Q \to \Sigma \cup \{\perp\}$ and $\sigma_B : Q \to \Sigma \cup \{\perp\}$ as follows:

$$\sigma_A(q) = \begin{cases} a & \text{if } A(q) = \{a\}, \\ \perp & \text{if } A(q) = \emptyset. \end{cases}$$

$$\sigma_B(q) = \begin{cases} b & \text{if } B(q) = \{b\}, \\ \perp & \text{if } B(q) = \emptyset. \end{cases}$$

The inverse functions of $\sigma_A$ and $\sigma_B$ are denoted by $\sigma_A^{-1}$ and $\sigma_B^{-1}$, respectively. That is, for $a, b \in \Sigma \cup \{\perp\}$, let $\sigma_A^{-1}(a) = \{q \in Q \mid \sigma_A(q) = a\}$ and $\sigma_B^{-1}(b) = \{q \in Q \mid \sigma_B(q) = b\}$. We give an example in Fig. 16.

$A$ and $B$ denotes the following subsets of $\Sigma$:

$$A = \bigcup_{q \in Q \setminus \sigma_A^{-1}(\perp)} A(q), \quad B = \bigcup_{q \in Q \setminus \sigma_B^{-1}(\perp)} B(q).$$

Then, let $A' = \Sigma \setminus A$ and $B' = \Sigma \setminus B$. For any $a, b \in \Sigma$, we use the following notations:

$$\ell_A = \sum_{a \in A} (\sharp\sigma_A^{-1}(a) - 1), \quad \ell_B = \sum_{b \in B} (\sharp\sigma_B^{-1}(b) - 1).$$

These $\ell_A$ and $\ell_B$ represent the numbers of excess duplicate symbols in $A$ and $B$. We easily see the following claim:

*Claim* 1.

(i) $\sharp A + \sharp A' = \sharp B + \sharp B' = k + 2$,
(ii) $\sharp A + \ell_A + \sharp\sigma_A^{-1}(\perp) = \sharp B + \ell_B + \sharp\sigma_B^{-1}(\perp) = k$.

Since $\sharp\Sigma = k + 2$ and $\sharp Q = k$, $\sharp A' \geq 2$ and $\sharp B' \geq 2$ hold. We partition $Q$ into the following subsets:

$$Q^{(\perp,\perp)} = \sigma_A^{-1}(\perp) \cap \sigma_B^{-1}(\perp),$$
$$Q^{(\perp,\cdot)} = \sigma_A^{-1}(\perp) \cap (Q \setminus \sigma_B^{-1}(\perp)),$$
$$Q^{(\cdot,\perp)} = (Q \setminus \sigma_A^{-1}(\perp)) \cap \sigma_B^{-1}(\perp),$$
$$Q^{(\cdot,\cdot)} = (Q \setminus \sigma_A^{-1}(\perp)) \cap (Q \setminus \sigma_B^{-1}(\perp)).$$

From the condition of this lemma, for any string $w \in \Sigma^*$ with $|w| = 2$, there exists a regular pattern $q_w \in Q$ such that $p\{x := w\} \preceq q_w$ holds. In particular, for $w = a'b' \in A' \cdot B'$, we must have $q_w \in Q$ that satisfies that $p\{x := w\} \preceq q_w$ (Fig. 17). It is easy to see that if $w \in (A \cdot B) \cup (A' \cdot B) \cup (A \cdot B')$, there exists a regular pattern $q_w \in Q^{(\perp,\cdot)} \cup Q^{(\cdot,\perp)} \cup Q^{(\cdot,\cdot)}$

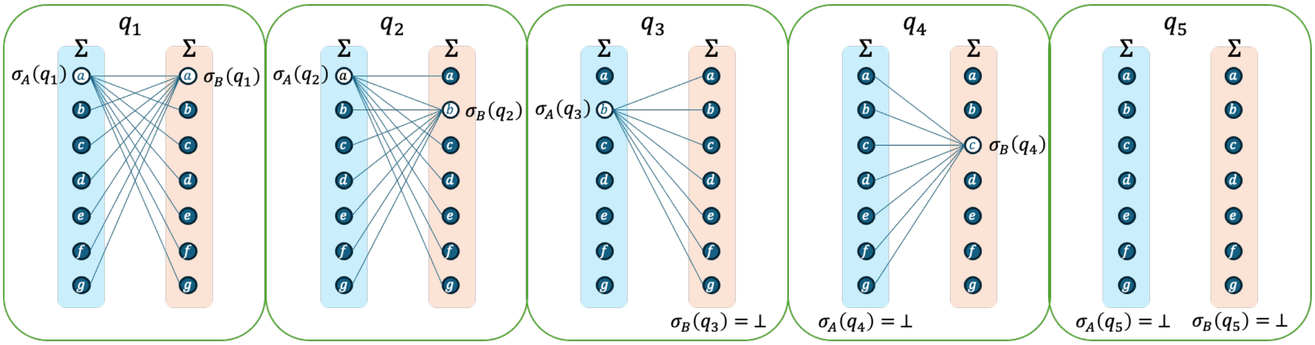



**Fig. 16** Let $\Sigma = \{a, b, c, d, e, f, g\}$, $Q = \{q_1, q_2, q_3, q_4, q_5\}$. We set $A(q_1) = \{a\}$ and $B(q_1) = \{a\}$, and then $\sigma_A(q_1) = a$ and $\sigma_B(q_1) = a$, and so on. For each regular pattern $q_i$ $(i = 1, \ldots, 5)$, we represent a string $w \in \Sigma \cdot \Sigma$ satisfying that $p\{x := w\} \preceq q_i$ by the edge between the left (first) and right (second) symbols of $w$. For example, the leftmost figure shows that $p\{x := ay\} \preceq q_1$ and $p\{x := ya\} \preceq q_1$ for a variable symbol $y$. We note that these figures may contain more edges than those illustrated. From these figures, we get $\ell_A = 1$, $\ell_B = 0$, and $Q^{(\perp,\perp)} = \{q_5\}$, $Q^{(\perp,\cdot)} = \{q_4\}$, $Q^{(\cdot,\perp)} = \{q_3\}$, $Q^{(\cdot,\cdot)} = \{q_1, q_2\}$.

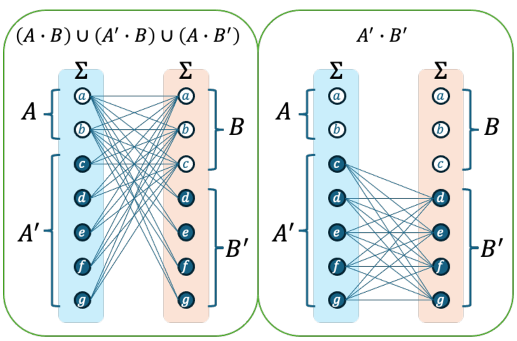



**Fig. 17** In the left figure, we aggregate all of the edges appearing in Fig. 16. For all $w = a'b' \in A' \cdot B'$, there must be a regular pattern $q_i$ $(1 \le i \le 5)$ that satisfies that $p\{x := w\} \preceq q_i$.

such that $p\{x := w\} \preceq q_w$ holds. We have the following two claims:

*Claim* 2. If there exist $q \in Q^{(\perp,\perp)}$ and distinct 5 strings $w_i \in A' \cdot B'$ $(1 \le i \le 5)$ such that $p\{x := w_i\} \preceq q$ holds $(1 \le i \le 5)$, then $p\{x := xy\} \preceq q$ holds.

*Proof of Claim* 2. Let $W = \{a_1b_1, \ldots, a_5b_5\} \subset A' \cdot B'$. Because, for any $i$ $(1 \le i \le 5)$, $|W \cap \{a_ic \mid c \in \Sigma\}| \le 2$ and $|W \cap \{cb_i \mid c \in \Sigma\}| \le 2$, it can be proven that there are 3 strings $a_{i_1}b_{i_1}, a_{i_2}b_{i_2}, a_{i_3}b_{i_3} \in W$ such that $a_{i_j} \ne a_{i_{j'}}$ and $b_{i_j} \ne b_{i_{j'}}$ for any $i_j, i_{j'}$ $(i_j \ne i_{j'}, 1 \le j, j' \le 3)$. Therefore, from Lemma 10, this claim holds. (*End of Proof of Claim* 2)

*Claim* 3. If there exist $q \in Q^{(\perp,\cdot)} \cup Q^{(\cdot,\perp)}$ and distinct 3 strings $w_i \in A' \cdot B'$ $(1 \le i \le 3)$ such that $p\{x := w_i\} \preceq q$ holds $(1 \le i \le 3)$, then $p\{x := xy\} \preceq q$ holds.

*Proof of Claim* 3. Let $W = \{a_1b_1, a_2b_2, a_3b_3\} \subset A' \cdot B'$. Because, for any $i$ $(1 \le i \le 3)$, $|W \cap \{a_ic \mid c \in \Sigma\}| \le 2$ and $|W \cap \{cb_i \mid c \in \Sigma\}| \le 2$, it can be proven that there are 2 strings $a_{i_1}b_{i_1}, a_{i_2}b_{i_2} \in W$ such that $a_{i_1} \ne a_{i_2}$ and $b_{i_1} \ne b_{i_2}$. Therefore, from Lemmas 8 and 9, this claim holds. (*End of Proof of Claim* 3)

If there exist a regular pattern $q \in Q^{(\perp,\perp)} \cup Q^{(\perp,\cdot)} \cup Q^{(\cdot,\perp)}$ and enough strings $w \in A' \cdot B'$ such that either of the conditions of *Claims* 2 and 3 is satisfied, this lemma holds. Then, we assume that it is not the case.

*Assumption* 1. There is no regular pattern $q \in Q^{(\perp,\perp)}$ and 5 strings $w \in A' \cdot B'$ such that the condition of *Claim* 2 is satisfied and there is no regular pattern $q \in Q^{(\perp,\cdot)} \cup Q^{(\cdot,\perp)}$ and 3 strings $w \in A' \cdot B'$ such that the condition of *Claim* 3 is satisfied.

Let $\mathcal{L}_1 = \sharp\{w \in A' \cdot B' \mid \exists q \in Q^{(\perp,\perp)} \cup Q^{(\perp,\cdot)} \cup Q^{(\cdot,\perp)}$ s.t. $p\{x := w\} \preceq q\}$. Under *Assumption* 1, each $q \in Q^{(\perp,\perp)}$ has at most 4 strings $w \in A' \cdot B'$ such that the condition of *Claim* 2 is satisfied, and each $q \in Q^{(\perp,\cdot)} \cup Q^{(\cdot,\perp)}$ has at most 2 strings $w \in A' \cdot B'$ such that the condition of *Claim* 3 is satisfied. Then, by *Claim* 1,

$$\begin{aligned}
\mathcal{L}_1 &\le 4\sharp Q^{(\perp,\perp)} + 2\sharp Q^{(\perp,\cdot)} + 2\sharp Q^{(\cdot,\perp)} \\
&= 2(\sharp Q^{(\perp,\perp)} + \sharp Q^{(\perp,\cdot)}) + 2(\sharp Q^{(\perp,\perp)} + \sharp Q^{(\cdot,\perp)}) \\
&= 2\sharp\sigma_A^{-1}(\perp) + 2\sharp\sigma_B^{-1}(\perp) \\
&= 2(k - \sharp A - \ell_A) + 2(k - \sharp B - \ell_B) \\
&= 2(\sharp A' - \ell_A - 2) + 2(\sharp B' - \ell_B - 2) \\
&= 2(\sharp A' + \sharp B') - 2(\ell_A + \ell_B) - 8.
\end{aligned}$$

Next, we partition $Q^{(\cdot,\cdot)}$ into the following two subsets:

$$Q_1^{(\cdot,\cdot)} = \{q \in Q^{(\cdot,\cdot)} \mid \sigma_A(q) \in B \text{ or } \sigma_B(q) \in A\},$$
$$Q_2^{(\cdot,\cdot)} = \{q \in Q^{(\cdot,\cdot)} \mid \sigma_A(q) \in B' \text{ and } \sigma_B(q) \in A'\}.$$

We show the following two claims concerning $Q_1^{(\cdot,\cdot)}$ and $Q_2^{(\cdot,\cdot)}$:

*Claim* 4. If there exist $q \in Q_1^{(\cdot,\cdot)}$ and a string $a'b' \in A' \cdot B'$ such that $p\{x := a'b'\} \preceq q$ holds, then $p\{x := xy\} \preceq q$ holds.

*Proof of Claim* 4. Suppose that both $\sigma_A(q) \in B$ and $\sigma_B(q) \in A$ hold. Then, since $a' \notin \{\sigma_A(q), \sigma_B(q)\} \subseteq A \cap B$ and $b' \notin \{\sigma_A(q), \sigma_B(q)\} \subseteq A \cap B$, from Lemma 5, $p\{x :=$

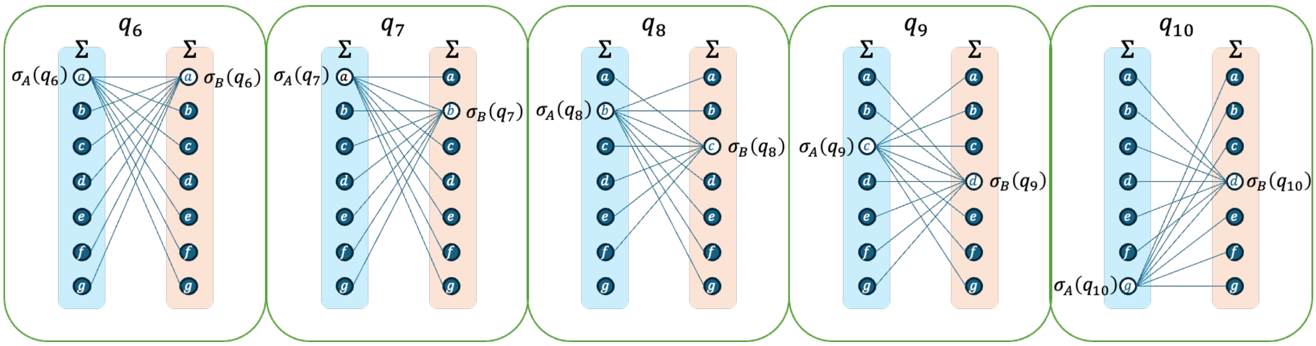



**Fig. 18** Let $\Sigma = \{a, b, c, d, e, f, g\}$, $Q = \{q_6, q_7, q_8, q_9, q_{10}\}$. From these figures, we get $\ell_A = 1$, $\ell_B = 1$, $Q^{(\perp,\perp)} = Q^{(\perp,\cdot)} = Q^{(\cdot,\perp)} = \emptyset$, and $Q^{(\cdot,\cdot)} = Q$.

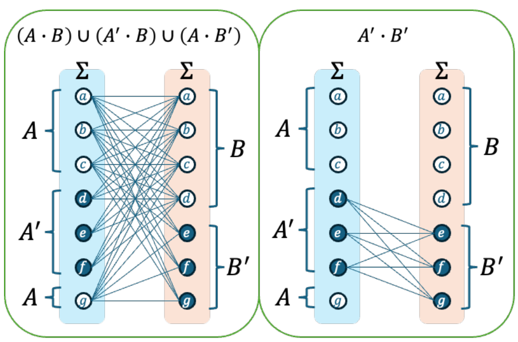



**Fig. 19** In the left figure, we aggregate all of the edges appearing in Fig. 18. From Fig. 18 and this right figure, we get $Q_1^{(\cdot,\cdot)} = \{q_6, q_7, q_8, q_9\}$ and $Q_2^{(\cdot,\cdot)} = \{q_{10}\}$. From Proposition 4, even if the string $dg \in A' \cdot B'$ satisfies $p\{x := gd\} \preceq q_{10}$, it does not imply that $p\{x := xy\} \preceq q_{10}$.

$xy\} \preceq q$ holds. Suppose that $\sigma_A(q) \in B$ and $\sigma_B(q) \in A'$. If $a' = \sigma_B(q)$, since $a' \in B$, $a' \neq b'$ holds. Since $\sigma_A(q) \in B$, $b' \neq \sigma_A(q)$ holds. That is, $a' = \sigma_B(q)$, $a' \neq \sigma_A(q)$, and $b' \notin \{\sigma_A(q), \sigma_B(q)\}$ hold. Therefore, from Lemmas 6 and 7, $p\{x := xy\} \preceq q$ holds. If $a' \neq \sigma_B(q)$, since $b' \neq \sigma_A(q)$, from Lemma 5, $p\{x := xy\} \preceq q$ holds. Similarly, the case that $\sigma_A(q) \in B'$ and $\sigma_B(q) \in A$ is proven. (*End of Proof of Claim* 4)

*Claim* 5. If there exist $q \in Q_2^{(\cdot,\cdot)}$ and a string $a'b' \in A' \cdot B'$ such that ($a' \neq \sigma_B(q)$ or $b' \neq \sigma_A(q)$) and $p\{x := a'b'\} \preceq q$ hold, then $p\{x := xy\} \preceq q$ holds.

*Proof of Claim* 5. When $a' = b'$, since $\sigma_A(q) \neq \sigma_B(q)$, from Lemma 5, this claim holds. Similarly, when $a' \neq b'$, from Lemmas 5, 6, and 7, this holds. (*End of Proof of Claim* 5)

We give an example in Fig. 18 and Fig. 19.

If there exist a regular pattern $q \in Q_2^{(\cdot,\cdot)}$ and a string $w \in A' \cdot B'$ such that the condition of *Claim* 5 is satisfied, this lemma holds. Then, we also assume that it is not the case.

*Assumption* 2. There is no $q \in Q_2^{(\cdot,\cdot)}$ and a string $a'b' \in A' \cdot B'$ such that the condition of *Claim* 5 is satisfied.

Let $\mathcal{L}_2 = \sharp\{a'b' \in A' \cdot B' \mid \exists q \in Q_2^{(\cdot,\cdot)}$ s.t. $p\{x := a'b'\} \preceq q\}$. For any $a'b' \in A' \cdot B'$ and $q \in Q_2^{(\cdot,\cdot)}$, if $a' = \sigma_B(q)$ and $b' = \sigma_A(q)$ hold (it is the condition of Proposition 4), by considering the duplicate numbers $\ell_A$ and $\ell_B$, we have the following inequality:

$$\mathcal{L}_2 \leq \min\{\sharp A' + \ell_B, \sharp B' + \ell_A\}.$$

We show the last claim:

*Claim* 6. $\sharp A' \times \sharp B' - \mathcal{L}_1 - \mathcal{L}_2 \geq 2$.

*Proof of Claim* 6. First we prove the inequality when $\sharp A \leq k-1$ and $\sharp B \leq k-1$, i.e., $\sharp A' \geq 3$ and $\sharp B' \geq 3$ hold. Since $\mathcal{L}_2 \leq \frac{1}{2}(\sharp A' + \sharp B' + \ell_A + \ell_B)$,

$$\begin{aligned}
&\sharp A' \times \sharp B' - \mathcal{L}_1 - \mathcal{L}_2 \\
\geq& \sharp A' \times \sharp B' - (2(\sharp A' + \sharp B') - 2(\ell_A + \ell_B) - 8) \\
&- \frac{1}{2}(\sharp A' + \sharp B' + \ell_A + \ell_B) \\
=& \sharp A' \times \sharp B' - \frac{5}{2}(\sharp A' + \sharp B') + \frac{3}{2}(\ell_A + \ell_B) + 8 \\
=& (\sharp A' - \frac{5}{2})(\sharp B' - \frac{5}{2}) + \frac{3}{2}(\ell_A + \ell_B) + \frac{7}{4} \geq 2.
\end{aligned}$$

When $\sharp A = k$ and $\sharp B \leq k$, i.e., $\sharp A' = 2$ and $\sharp B' \geq 2$ hold, since $\ell_A = 0$, $\mathcal{L}_1 \leq 2\sharp B' - 2\ell_B - 4$ holds. Moreover, $\mathcal{L}_2 \leq \min\{\sharp B', \ell_B + 2\}$ holds. From *Claim* 1, $\ell_B + 2 = k - \sharp\sigma_B^{-1}(\perp) - \sharp B = \sharp B' - \sharp\sigma_B^{-1}(\perp)$ holds. Therefore, $\mathcal{L}_2 \leq \ell_B + 2$ holds. Thus,

$$\begin{aligned}
&\sharp A' \times \sharp B' - \mathcal{L}_1 - \mathcal{L}_2 \\
\geq& 2\sharp B' - (2\sharp B' - 2\ell_B - 4) - (\ell_B + 2) \\
=& \ell_B + 2 \geq 2.
\end{aligned}$$

Similarly, the case when $\sharp A \leq k$ and $\sharp B = k$ is proven. (*End of Proof of Claim* 6)

Under *Assumptions* 1 and 2, from *Claim* 6, there exist at least two $w \in A' \cdot B'$ and a regular pattern $q \in Q_1^{(\cdot,\cdot)}$ such that the condition of *Claim* 4 is satisfied. Therefore, for such a regular pattern $q$, $p\{x := xy\} \preceq q$ holds. □

**Lemma 12** (Sato et al.[4]): Let $\Sigma$ be a finite alphabet with $\sharp\Sigma \geq 3$ and $p, q$ regular patterns. If there exists a constant symbol $a \in \Sigma$ such that $p\{x := a\} \preceq q$ and $p\{x := xy\} \preceq q$, then $p \preceq q$ holds, where $y$ is a variable symbol that does not appear in $q$.

From Lemma 11 and Lemma 12, we have the following theorem.

**Theorem 4:** Let $k \geq 3$, $\sharp\Sigma \geq 2k-1$, $P \in \mathcal{RP}^+$ and $Q \in \mathcal{RP}^k$. Then, the following (i),(ii) and (iii) are equivalent:

$$\text{(i) } S_2(P) \subseteq L(Q), \text{ (ii) } P \sqsubseteq Q, \text{ (iii) } L(P) \subseteq L(Q).$$

**Proof.** It is clear that (ii) implies (iii) and (iii) implies (i). From Theorem 3, if $\sharp\Sigma \geq 2k+1$, then (i) implies (ii). Let $\sharp Q = k$, $p \in P$, $\sharp\Sigma = 2k-1$ or $2k$. Then, we show that (i) implies (ii). It suffices to show that $S_2(p) \subseteq L(Q)$ implies $\{p\} \sqsubseteq Q$ for any regular pattern $p \in P$. The proof is done by mathematical induction on $n$, where $n$ is the number of variable symbols appears in $p$.

In case $n = 0$, $S_2(p) = \{p\}$ holds. By (i), we have $\{p\} \subset L(Q)$. Thus, $p \preceq q$ for some $q \in Q$.

For $n \geq 0$, we assume that it is valid for any regular pattern $p$ with $n$ variable symbols. Let $p$ be a regular pattern such that $n+1$ variable symbols appear in $p$ and $S_2(p) \subseteq L(Q)$. Let $Q = \{q_1, \ldots, q_k\}$. We assume that $p \not\sqsubseteq Q$, that is, $\{p\} \not\preceq q_i$ for any $i \in \{1, \ldots, k\}$. Let $p_1$, $p_2$ be regular patterns, $x$ a variable symbol with $p = p_1 x p_2$. For $a, b \in \Sigma$, let $p_a = p\{x := a\}$ and $p_{ab} = p\{x := ab\}$. Both $p_a$ and $p_{ab}$ have $n$ variable symbols, respectively. Thus, $S_2(p_a) \subseteq L(Q)$ and $S_2(p_{ab}) \subseteq L(P)$ hold. By the induction hypothesis, there exist $i, i' \in \{1, \ldots, k\}$ such that $p_a \preceq q_i$ and $p_{ab} \preceq q_{i'}$. Let $D_i = \{a \in \Sigma \mid p\{x := a\} \preceq q_i\}$ $(i = 1, \ldots, k)$. We assume that $\sharp D_i \geq 3$ for some $i \in \{1, \ldots, k\}$. By Lemma 2, we have $p \preceq q_i$. This contradicts the assumption. Thus, we have $\sharp D_i \leq 2$ for any $i \in \{1, \ldots, k\}$. If $\sharp\Sigma = 2k-1$, then $\sharp D_i = 2$ or $\sharp D_i = 1$ for any $i \in \{1, \ldots, k\}$. Moreover, If $\sharp\Sigma = 2k$, then $\sharp D_i = 2$ for any $i \in \{1, \ldots, k\}$. Since $k \geq 3$, $2k-1 \geq k+2$ holds. By Lemma 11, there exists $i \in \{1, \ldots, k\}$ such that $p\{x := xy\} \preceq q_i$. Therefore, by Lemma 12, we have $p \preceq q_i$. This contradicts the assumption. Thus, (i) implies (ii). □

From Theorem 4, the following Corollary 2 holds.

**Corollary 2:** Let $k \geq 3$, $\sharp\Sigma \geq 2k-1$ and $P \in \mathcal{RP}^+$. Then, $S_2(P)$ is a characteristic set for $L(P)$ within $\mathcal{RPL}^k$.

**Lemma 13** (Sato et al.[4]): Let $k \geq 3$ and $\sharp\Sigma \leq 2k-2$. Then, $\mathcal{RP}^k$ does not have compactness with respect to containment.

**Proof.** Let $\Sigma = \{a_1, \ldots, a_{k-1}, b_1, \ldots, b_{k-1}\}$ and $p, q_i$ regular patterns, $w_i \in \Sigma^*$ $(i = 1, \ldots, k-1)$ defined in a similar way to Example 1. Let $q_k = x_1 a_1 w_1 x y w_1 b_1 x_2$. Since $p\{x := a_i\} = x_1 a_1 w_1 a_i w_1 b_1 x_2 \preceq q_i$ and $p\{x := b_i\} = x_1 a_1 w_1 b_i w_1 b_1 x_2 \preceq q_i$ for any $i \in \{1, \ldots, k-1\}$, we have $S_1(p) \subseteq \bigcup_{i=1}^{k-1} L(q_i)$. For any $w \in \{s \in \Sigma^+ \mid |s| \geq 2\}$, $p\{x := w\} = x_1 a_1 w_1 w w_1 b_1 x_2 \preceq q_k$. Thus, we have $L(p) \subseteq L(Q)$. By Theorem 1, since $p \not\preceq q_i$, $L(p) \not\subseteq L(q_i)$ for any $i \in \{1, \ldots, k\}$. Therefore, $\mathcal{RP}^k$ does not have compactness with respect to containment. □

From Theorem 4 and Lemma 13, we have the following Theorem 5.

**Theorem 5:** Let $k \geq 3$ and $\sharp\Sigma \geq 2k-1$. Then, $\mathcal{RP}^k$ has compactness with respect to containment.

In case $k = 2$, we have the following theorem.

**Theorem 6:** Let $\sharp\Sigma \geq 4$, $P \in \mathcal{RP}^+$ and $Q \in \mathcal{RP}^2$. The following (i), (ii) and (iii) are equivalent:

$$\text{(i) } S_2(P) \subseteq L(Q), \quad \text{(ii) } P \sqsubseteq Q, \quad \text{(iii) } L(P) \subseteq L(Q).$$

**Proof.** It is clear that (ii) implies (iii), and (iii) implies (i). Thus, we show that (i) implies (ii). It suffices to show that $S_2(p) \subseteq L(Q)$ implies $\{p\} \sqsubseteq Q$ for any regular pattern $p \in P$. Let $Q = \{q_1, q_2\}$. The proof is done by mathematical induction on $n$, where $n$ is the number of variable symbols appearing in $p$. In case $n = 0$, $p \in \Sigma^+$. Since $S_2(p) = \{p\} \subseteq L(Q)$, we have $p \preceq q$ for some $q \in Q$. For $n \geq 0$, we assume that it is valid for any regular pattern $p$ with $n$ variable symbols. Let $p$ be a regular pattern such that $n+1$ variable symbols appear in $p$, and $S_2(p) \subseteq L(Q)$ holds. We assume that $p \not\preceq q_i$ $(i = 1, 2)$. Let $p_1$, $p_2$ be regular patterns and $x$ a variable symbol with $p = p_1 x p_2$. For $a, b \in \Sigma$, let $p_a = p\{x := a\}$ and $p_{ab} = p\{x := ab\}$. Note that $p_a$ and $p_{ab}$ have $n$ variable symbols. Thus, by the assumption, $S_2(p_a) \subseteq L(Q)$ and $S_2(p_{ab}) \subseteq L(Q)$ imply $p_a \preceq q_i$ and $p_{ab} \preceq q_{i'}$ for some $i, i' \in \{1, 2\}$. Let $D_i = \{a \in \Sigma \mid p\{x := a\} \preceq q_i\}$ $(i = 1, 2)$. By Lemma 2, if $\sharp D_i \geq 3$ for some $i \in \{1, 2\}$, then $p \preceq q_i$. This contradicts that $p \not\preceq q_i$ $(i = 1, 2)$. Thus, we have $\sharp D_i \leq 2$ for any $i \in \{1, 2\}$. Since $\sharp\Sigma \geq 4$, we consider that $\sharp D_1 = 2$ and $\sharp D_2 = 2$. From Lemma 11, $p\{x := xy\} \preceq q_i$ for some $i \in \{1, 2\}$. From Lemma 12, we have $p \preceq q_i$ for some $i \in \{1, 2\}$. This contradicts that $p \not\preceq q_i$ $(i = 1, 2)$. Hence, (i) implies (ii). □

The following example provides a set of regular patterns $P \in \mathcal{RP}^+$ and a set of regular patterns $Q \in \mathcal{RP}^2$ demonstrating that, when $\sharp\Sigma = 3$, the three conditions (i), (ii), and (iii) stated in Theorem 6 are not equivalent.

**Example 3:** Let $\Sigma = \{a, b, c\}$, $p$, $q_1$, $q_2$ regular patterns and $x, x', x''$ variable symbols such that $p = x' a x b x''$, $q_1 = x' a b x''$ and $q_2 = x' c x''$. Let $w \in \Sigma^+$. If $w$ contains $c$, then $p\{x := w\} \preceq q_2$. On the other hand, if $w$ does not contain $c$, then $p\{x := w\} \preceq q_1$. Thus, $L(p) \subseteq L(q_1) \cup L(q_2)$. However, $p \not\preceq q_1$ and $p \not\preceq q_2$.

From Theorem 6, the following two corollaries holds.

**Corollary 3:** Let $\sharp\Sigma \geq 4$ and $P \in \mathcal{RP}^+$. Then, $S_2(P)$ is a characteristic set for $L(P)$ within $\mathcal{RPL}^2$.

**Corollary 4:** Let $\sharp\Sigma \geq 4$. Then, $\mathcal{RP}^2$ has compactness with respect to containment.

## 4. Regular Pattern without Adjacent Variable Symbols

A regular pattern $p$ is said to be a *non-adjacent variable regular pattern* (*NAV* regular pattern) if $p$ does not contain consecutive variable symbols. For example, the regular pattern $p = axybc$ is not an *NAV* regular pattern because $xy$ is appeared in $p$. Let $\mathcal{RP}_{NAV}$ be the set of all *NAV* regular patterns.

Let $\mathcal{RP}^{+}_{NAV}$ be the set of all finite subsets $S$ of $\mathcal{RP}_{NAV}$ such that $S$ is not the empty set, i.e., $\mathcal{RP}^{+}_{NAV} = \{S \subseteq \mathcal{RP}_{NAV} \mid \sharp S \geq 1\}$, and $\mathcal{RP}^{k}_{NAV}$ the set of all subsets $P$ of $\mathcal{RP}^{+}_{NAV}$ such that $P$ consists of at most $k$ ($k \geq 1$) *NAV* regular patterns, i.e., $\mathcal{RP}^{k}_{NAV} = \{P \in \mathcal{RP}^{+}_{NAV} \mid \sharp P \leq k\}$. We define the compactness with respect to containment for $\mathcal{RP}^{k}_{NAV}$ in a similar way as $\mathcal{RP}^{k}$.For any *NAV* regular pattern $p \in \mathcal{RP}_{NAV}$ and any set $Q \in \mathcal{RP}^{k}_{NAV}$ with $k$ ($k \geq 1$), the set $\mathcal{RP}^{k}_{NAV}$ said to have *compactness with respect to containment* if there exists an *NAV* regular pattern $q \in Q$ such that $L(p) \subseteq L(q)$ holds if $L(p) \subseteq L(Q)$ holds. Then, the following Theorem 7 holds.

**Theorem 7:** For an integer $k$ ($k \geq 2$), let $\sharp\Sigma \geq k+2$, $P \in \mathcal{RP}^{+}_{NAV}$, $Q \in \mathcal{RP}^{k}_{NAV}$. Then, the following (i), (ii) and (iii) are equivalent:

$$\text{(i) } S_2(P) \subseteq L(Q), \text{ (ii) } P \sqsubseteq Q, \text{ (iii) } L(P) \subseteq L(Q).$$

**Proof.** From the definitions of $\mathcal{RP}^{+}_{NAV}$ and $\mathcal{RP}^{k}_{NAV}$, it is clear that (ii) implies (iii) and (iii) implies (i). Hence, we will show that (i) implies (ii) by mathematical induction on the number $n$ of variable symbols that appear in an *NAV* regular pattern $p \in P$ as follows: If $n = 0$, then we have $S_2(\{p\}) = \{p\}$. Hence, $p \in L(Q)$. Therefore, there exists $q \in Q$ such that $p \preceq q$ holds.

If $n \geq 0$, we assume that the proposition holds for any regular *NAV* regular pattern containing $n \geq 0$ variable symbols. Let $p$ be an *NAV* regular pattern containing $n+1$ variable symbols such that $S_2(\{p\}) \subseteq L(Q)$ holds and $p$ contains a variable symbol $x$. There exist two *NAV* regular patterns $p_1, p_2$ such that $p = p_1 x p_2$ holds. By the induction hypothesis, for any constant string $w \in \Sigma^*$ with $|w| = 2$, $\{p\{x := w\}\} \sqsubseteq Q$ holds because $p\{x := w\}$ contains $n$ variable symbols. Hence, there exists an *NAV* regular pattern $q_w \in Q$ such that $p\{x := w\} \preceq q_w$ holds. From Lemma 11, there exists a regular pattern $q \in Q$ such that $p\{x := xy\} \preceq q$ holds, where $y$ is a variable symbol that does not appear in $q$. This contradicts the condition $Q \in \mathcal{RP}^{k}_{NAV}$. Thus, we have that (i) implies (ii). □

**Corollary 5:** Let $k \geq 2$, $\sharp\Sigma \geq k+2$ and $P \in \mathcal{RP}^{+}_{NAV}$. Then, $S_2(P)$ is a characteristic set of $\mathcal{RPL}^{k}_{NAV}$.

**Lemma 14:** Let $k \geq 2$ and $\sharp\Sigma \leq k+1$. Then, $\mathcal{RP}^{k}_{NAV}$ does not have compactness with respect to containment.

**Proof.** Let $\Sigma = \{a_1, \ldots, a_{k+1}\}$. We assume that for $i = 1, 2, \ldots, k$, $p\{x := a_i y\} \preceq q_i$ and $p\{x := y a_{i+1}\} \preceq q_i$ holds. If $p\{x := a_{k+1} a_1\} \preceq q_1$ holds, $S_2(p) \backslash S_1(p) \subseteq \bigcup_{i=1}^{k} L(q_i)$ holds. This show that $L(p) \subseteq L(Q)$ holds. However, for $i = 1, 2, \ldots, k$, since $p \not\preceq q_i$ holds, we have that $L(p) \not\subseteq L(q_i)$ holds. Hence, $\mathcal{RP}^{k}_{NAV}$ does not have compactness with respect to containment. □

Next, in Example 4, we give an example for Lemma 14.

**Example 4:** Let $\Sigma$ be the set of four constant symbols $a$, $b$, $c$, $d$, i.e., $\Sigma = \{a, b, c, d\}$ and $x, x', x''$ three distinct variable symbols. Let $p, q_1, q_2, q_3$ be the *NAV* regular patterns given in Fig. 20. Then, we have $L(p) \subseteq L(q_1) \cup L(q_2) \cup L(q_3)$. This show that for $P = \{p\}$, $Q = \{q_1, q_2, q_3\}$, (iii) of Theorem 7 holds. However, since $p \not\preceq q_1$, $p \not\preceq q_2$ and $p \not\preceq q_3$ hold, we have $P \not\sqsubseteq Q$, that is, (ii) of Theorem 7 does not hold.

$$\begin{aligned}
p &= x' cadadaadacbadadaadaxadadaadacbadadaadabx'', \\
q_1 &= x' cadadaadacbadadaadacx'', \\
q_2 &= x' badadaadacx'', \\
q_3 &= x' aadadx''.
\end{aligned}$$

**Fig. 20** *NAV* regular patterns $p$, $q_1$, $q_2$, and $q_3$

**Table 2** The conditions on the number $\sharp\Sigma$ of constant symbols in $\Sigma$ required for compactness with respect to containment.

| Class | $k = 2$ | $k \geq 3$ |
|---|---|---|
| $\mathcal{RP}^{k}$ | $\sharp\Sigma \geq 4$ | $\sharp\Sigma \geq 2k-1$ |
| $\mathcal{RP}^{k}_{NAV}$ | $\sharp\Sigma \geq k+2$ | |

From Theorem 7 and Lemma 14, we have the following theorem.

**Theorem 8:** Let $k \geq 2$ and $\sharp\Sigma \geq k+2$. Then, the set $\mathcal{RPL}^{k}_{NAV}$ has compactness with respect to containment.

## 5. Conclusion

In this paper, for an integer $k$ ($k \geq 2$), we have shown the conditions on the number of constant symbols in $\Sigma$, summarized in Table 2, required for the classes $\mathcal{RP}^{k}$ of all the set of $k$ regular pattern languages and $\mathcal{RP}^{k}_{NAV}$ of all the set of $k$ non-adjacent variable regular patterns in $\mathcal{RP}_{NAV}$ to have compactness with respect to containment. This result leads to design an efficient learning algorithm for finite unions of languages of non-adjacent variable regular patterns in $\mathcal{RP}_{NAV}$, based on the learning algorithm for $\mathcal{RP}^{k}$ proposed by Arimura et al. [8].

Extending the notion of strong compactness, as introduced by Arimura et al. [9], to finite unions of regular pattern languages with non-adjacent variables remains as a topic for future research. Furthermore, based on the characteristic set for $\mathcal{RP}^{k}_{NAV}$, we plan to propose a polynomial-time inductive inference algorithm that identifies finite unions of regular pattern languages with non-adjacent variables in the limit from positive examples. Ishinada et al. [17] investigated a query learning model that employs high-precision Graph Convolution Networks (GCNs) as oracles for tree patterns. Applying the findings of the present study to tree pattern languages, with the aim of enabling the extension of their work to finite unions of tree pattern languages, remains an important direction for future research.

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