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By

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Abstract

This paper develops a mathematical theory of language identification from a set theoretic viewpoint. We investigate two types of language classes called M-finite thickness and finite elasticity as a hypothesis space of an inductive inference machine. It is known that both of the families of such classes include interesting and important classes and are substantially large.

For a class with M-finite thickness, we first show some equivalences between several key concepts in language identification such as a finite tell-tale and others. We also show that M-finite thickness is preserved under some operations such as intersection, concatenation and so on as well as finite elasticity, except union operation. Then we apply those results to problems of inferability in the criteria of ordinary identification in the limit or inductive refutable identification proposed by Mukouchi and Arikawa as a framework for machine discovery. In particular, we present a characterization theorem and some useful sufficient conditions for inductive refutable inferability from complete data, in case a hypothesis space has M-finite thickness. Furthermore, we discuss inductive inference of length-bounded elementary formal systems as a framework for defining target languages.

1. Introduction

Inductive inference is a process of hypothesizing a general rule from eventually incomplete data. Since Gold [3] has proposed a mathematical model of inductive inference based on the identification in the limit paradigm, there have been introduced various criteria related to the identification in the limit so far. In inductive inference of languages, an inference machine requires data or facts of a target language from time to time and produces hypotheses from time to time. The set of hypotheses the machine may produce is called the hypothesis space.

In the present paper, we assume that the hypothesis space is an indexed family of recursive languages. Angluin [1] gave a theorem characterizing inferability of such classes (families) from positive data. The set theoretic aspect of the inferability shown in [1] is that there exists a finite tell-tale for every language in the class. On the other

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hand, it has been given that if there exists a pair of finite tell-tales for every language in the class, then the class is inferable from positive data (cf. [4], [5] and [11]), and that if a class has M-finite thickness, then the existence of a finite tell-tale for a language in the class is equivalent to that of a pair of finite tell-tales for the language (cf. [10]). It means that the inferability for a class with M-finite thickness depends on only such a set theoretic property of the class.

In the present paper, we will make an approach to inferability in two criteria mentioned below from a set theoretic viewpoint.

We first investigate structural features between a target language and a class of languages as a hypothesis space. We introduce a concept of finite cross property representing some structural relation between a language and a class, and show equivalences between it and several key concepts such as a finite tell-tale and others proposed as necessary or sufficient conditions for inferability in various criteria. Then we consider two types of language classes called M-finite thickness and finite elasticity. Finite elasticity has been introduced by Wright [14] as a sufficient condition for inferability from positive data, and shown in [6] and [14] to be preserved under operations such as union, intersection, concatenation and so on except complement. M-finite thickness due to Sato&Moriyama [10] is a generalization of finite thickness proposed by Angluin [1], but of much weaker condition. We obtain that if a class has M-finite thickness, then equivalences between some concepts mentioned above are valid. Furthermore, we have another result that M-finite thickness is preserved under operations mentioned above as well as finite elasticity, except union operation.

Next, we apply these results to problems of inferability in the criteria of ordinary identification or inductive refutable identification proposed by Mukouchi&Arikawa [9]. Note that every indexed family considered is inferable from complete data (cf. [3]). While, concerning inductive refutable inferability from positive data, the power is very small (cf. [9]). We present a characterization theorem and some useful sufficient conditions for inductive refutable inferability in case a hypothesis space has M-finite thickness.

Finally, we adopt length-bounded elementary formal systems (EFS's, for short) as a framework for defining target languages. Concerning inductive inference from positive data, Shinohara [12] has obtained the result that for any n, the class of languages definable by length-bounded EFS's with at most n axioms is inferable from positive data. It has been shown in [10] that the class has M-finite thickness. In this paper, we obtain a characterization theorem for inductive refutable inferability from complete data. By the result, we derive as a corollary the result obtained by Mukouchi&Arikawa [9] that for any n, the class mentioned above is inductively refutably inferable from complete data.

In §2, we investigate relationships between a language and a class of languages mentioned above. In §3, we discuss inductive refutable inference from complete data as well as inductive inference from positive data. The section 4 is concerned with EFS language identification, provided a hypothesis space is a class of length-bounded EFS's.

2. Formal Language Classes

In study of language identification, there are so many set-theoretic concepts such as a *finite tell-tale* [1], *finite cross property* [11], a *pair of finite tell-tales* [11], [4], a *pair of definite finite tell-tales* [8] and so on. These concepts are defined only for a hypothesis space and a language belonging to the hypothesis space.

In this section, first we define these concepts also for a language not belonging the hypothesis space, and consider what languages do not have, for instance, *finite tell-tale* within the hypothesis space. We give conditions of the non-existence of *finite tell-tale* and so on in terms of the concept of *infinite cross property.*

Next, we investigate properties of two special types of language classes called M-finite thickness [10] and finite elasticity [14].

2.1. Finite Cross Property

Let Σ be a finite alphabet. We denote a language over Σ by $L, L', L_1, L_2, \cdots$, and denote a class of languages over Σ by $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2, \cdots$. Let $N = \{n \mid n \geq 1\}$. For a language L and $n \geq 0$, let $L^{\leq n}$ be the set of strings in L whose lengths are less than or equal to n.

First let us define the following notion which was introduced by Sato&Umayahara [11]:

DEFINITION 2.1. A language L has *infinite cross property within* a class $\mathcal{L}$ if there exists an infinite sequence of languages $(L_n)_{n \in N}$ such that

$$(1)\ S_1 \subsetneq S_2 \subsetneq \cdots, \quad (2)\ \bigcup_{n=1}^{\infty} S_n = L, \quad (3)\ L_n \in \mathcal{L}, \quad n \in N,$$

where

$$S_n = \bigcap_{k=n}^{\infty} (L_k \cap L), \quad n \in N.$$

L has *finite cross property within* $\mathcal{L}$, denoted by $\langle L, \mathcal{L} \rangle$, if L does not have infinite cross property within $\mathcal{L}$.

As easily seen, if $(L_n)_{n \in N}$ is a sequence defined above, then L_n's are mutually different and $L \not\subseteq L_n$ for all $n \in N$, and moreover, L is infinite.

The following is useful to establish the results shown later:

LEMMA 2.2. *A language L has infinite cross property if and only if there exists a sequence of finite sets $(T_n)_{n \in N}$ of strings and a sequence of languages $(L_n)_{n \in N}$, each in $\mathcal{L}$, such that*

$$\textit{(1)}\ T_1 \subsetneq T_2 \subsetneq \cdots, \quad \textit{(2)}\ \bigcup_{n=1}^{\infty} T_n = L, \quad \textit{(3)}\ T_n \subseteq L_n,\ \textit{but}\ T_{n+1} \not\subseteq L_n, \quad n \in N.$$

PROOF. The *if* direction is obvious, so we will prove the *only if* direction.

Assume that L has infinite cross property within $\mathcal{L}$. Then there exists a sequence of languages $(L_n)_{n\in N}$. As mentioned above, L is infinite. Let us define a sequence $(T_t)_{t\in N}$ and a subsequence $(L_{n_t})_{t\in N}$ of $(L_n)_{n\in N}$ recursively as follows: Let $S_n = \bigcap_{k=n}^{\infty}(L_k \cap L)$ for $n \in N$.

For $t = 1$, let

$$\begin{aligned} &T_1 = \Sigma^{\leq l_1}, \quad \text{where} \quad l_1 = \min\{l \in N \mid L^{\leq l} \neq \phi\}, \quad \text{and} \\ &n_1 = \min\{n \in N \mid T_1 \subseteq S_n\}. \end{aligned}$$

For $t \geq 2$, let

$$\begin{aligned} &T_t = \Sigma^{\leq l_t}, \quad \text{where} \quad l_t = \min\{l \in N \mid \Sigma^{\leq l} \not\subseteq L_{n_{t-1}}\}, \quad \text{and} \\ &n_t = \min\{n \in N \mid T_t \subseteq S_n\}. \end{aligned}$$

CLAIM: The sequences $(l_t)_{t\in N}$ and $(n_t)_{t\in N}$ are both defined and strictly increasing.

PROOF OF THE CLAIM. Due to the conditions (1) and (2) in Definition 2.1, for some $n \in N$, $\{w_1\} = T_1 \subseteq S_n$. Thus n_1 is defined. It suffices to show that for any $t \in N$, if l_t and n_t are both defined, then so are both l_{t+1} and n_{t+1}, and moreover $l_t < l_{t+1}$ and $n_t < n_{t+1}$. Assume that l_t and n_t are both defined. As mentioned above, $L \not\subseteq L_{n_t}$ holds. Thus $l_{t+1} = \min\{l \in N \mid L^{\leq l} \not\subseteq L_{n_t}\}$ is bounded. By $T_t \subseteq S_{n_t} \subseteq L_{n_t}$, it follows that $l_t < l_{t+1}$ and $T_t \subsetneq T_{t+1}$. Moreover, $T_{t+1} \not\subseteq L_{n_t}$ implies $T_{t+1} \not\subseteq S_{n_t}$. It implies together with the condition (2) in Definition 2.1 that n_{t+1} is bounded and $n_t < n_{t+1}$.

The above claim yields that the sequences $(T_t)_{t\in N}$ and $(L_{n_t})_{t\in N}$ satisfy the conditions (1)-(3) in our lemma. In fact, clearly the conditions (1) and (2) hold, because $(l_t)_{t\in N}$ is strictly increasing. On the other hand, by the definitions of n_t and l_{t+1}, it follows that $T_t \subseteq S_{n_t} (\subseteq L_{n_t})$ and $T_{t+1} \not\subseteq L_{n_t}$. Thus the condition (3) is also valid. This completes our proof. □

For a language L and a class $\mathcal{L}$, let us define the following subclass:

$$\mathcal{L}_L = \{L' \in \mathcal{L} \mid L' \subseteq L\}.$$

By Definition 2.1, it follows immediately that:

LEMMA 2.3.

(i) *If* $L \not\subseteq L'$, *then* $\langle L, \mathcal{L}_{L'} \rangle$.

(ii) *If* $\langle L, \mathcal{L}_{L'} \rangle$ *and* $L'' \subseteq L'$, *then* $\langle L, \mathcal{L}_{L''} \rangle$.

(iii) *If* L *is finite, then* $\langle L, \mathcal{L} \rangle$.

Kapur [4] has introduced the following notion similar to our *infinite cross property* considered, in order to clarify the notion of finite tell-tale described later.

DEFINITION 2.4. A language L is an *accumulation point* of $\mathcal{L}$ if there exists a sequence of finite sets $(T_n)_{n\in N}$ of strings such that

(1) $T_1 \subseteq T_2 \subseteq \cdots$, (2) $\bigcup_{n=1}^{\infty} T_n = L$,

(3) For every $n \in N$ there exists a language $L_n \in \mathcal{L}$ such that $T_n \subseteq L_n \subsetneq L$.

For a pair of sets $I = (T, F)$ of strings, we denote

$$CON(I, \mathcal{L}) = \{L \in \mathcal{L} \mid T \subseteq L \text{ and } F \subseteq L^C\}$$
$$CON(I) = \{L \subseteq \Sigma^* \mid T \subseteq L \text{ and } F \subseteq L^C\}.$$

For two pairs $I = (T, F)$ and $I' = (T', F')$ of sets of strings, let

$$I \prec I' \iff T \subseteq T',\ F \subseteq F' \text{ and } T \cup F \subsetneq T' \cup F'.$$

DEFINITION 2.5. A language L has *p-infinite cross property within* $\mathcal{L}$ if there exists a sequence of pairs of finite sets $(I_n)_{n\in N}$ and a sequence of languages $(L_n)_{n\in N}$, each in $\mathcal{L}$, such that

(1) $I_1 \prec I_2 \prec \cdots$, (2) $\bigcup_{n=1}^{\infty} T_n = L, \quad \bigcup_{n=1}^{\infty} F_n = L^C$, where $I_n = (T_n, F_n)$

(3) $L_n \in CON(I_n, \mathcal{L})$, but $L_n \notin CON(I_{n+1}, \mathcal{L})$, $\quad n \in N$.

L has *finite cross property within* $\mathcal{L}$ if L does not have p-infinite cross property within $\mathcal{L}$.

2.1.1. A Finite Tell-tale

The following notion has been introduced by Angluin [1] as a necessary condition for inferability from positive data.

DEFINITION 2.6. A set T of strings is *a finite tell-tale* (*ftt*, for short) *of a language L within $\mathcal{L}$* if T is a finite subset of L and there does not exist any language $L' \in \mathcal{L}$ such that $T \subseteq L' \subsetneq L$. L has *ftt within* $\mathcal{L}$ if there is a finite tell-tale of L within $\mathcal{L}$.

In a case of $L \in \mathcal{L}$, it has been shown that the existence of *ftt* of L is equivalent to that (i) L is not an accumulation point of $\mathcal{L}$ (Kapur [4]), and to that (ii) L has finite cross property within $\mathcal{L}_L$ (Sato&Umayahara [11]). These equivalences are also valid for $L \notin \mathcal{L}$.

THEOREM 2.7. *The following four statements are equivalent:*

(i) *L has ftt within $\mathcal{L}$.*
(ii) *L is not an accumulation point of $\mathcal{L}$.*
(iii) *L has finite cross property within $\mathcal{L}_L$, that is $\langle L, \mathcal{L}_L\rangle$.*
(iv) *L has p-finite cross property within $\mathcal{L}_L$.*

PROOF. The proofs of ((i) $\Leftrightarrow$ (ii)) and ((i) $\Leftrightarrow$ (iii)) are found in [4] and [11], respectively. Thus (ii) $\Leftrightarrow$ (iii). It is easy to show the equivalence between (iii) and (iv). □

2.1.2. A Pair of Finite Tell-tales

The present author and Umayahara [11] have introduced the following notion as a sufficient condition for inferability from positive data (by Kapur [4] at the same time, which is called a *test set*).

DEFINITION 2.8. A pair of sets $I = (T, F)$ is *a pair of finite tell-tales* (*pftt*, for short) *of a language L within $\mathcal{L}$* if T and F are finite sets of strings, $L \in CON(I)$ and $L \subseteq L'$ for all $L' \in CON(I, \mathcal{L})$. L has *pftt within $\mathcal{L}$* if there exists *pftt* of L within $\mathcal{L}$.

Note that if $I = (T, F)$ is *pftt* of L within $\mathcal{L}$, then T is *ftt* of L within $\mathcal{L}$. The notion of *pftt* may be characterized by finite cross property as follows:

THEOREM 2.9. *The following two statements are equivalent:*

(i) *L has pftt within $\mathcal{L}$.*
(ii) *There exists a finite set $F \subseteq L^C$ such that $\langle L, \mathcal{L}_{F^C} \rangle$.*

PROOF. ((i) $\Longrightarrow$ (ii)) Assume that L has *pftt* within $\mathcal{L}$, say $I = (T, F)$. For the set F, we show that $\langle L, \mathcal{L}_{F^C} \rangle$.

Assume the converse. By Lemma 2.2, there exist two infinite sequences $(T_n)_{n \in N}$ and $(L_n)_{n \in N}$ satisfying the conditions (1)-(3) in Lemma 2.2, where $L_n \subseteq F^C$ for all $n \in N$. Due to the conditions (1) and (2), for some $n \in N$, $T \subseteq T_n \subseteq L_n$ and $T_{n+1} \not\subseteq L_n$. It implies together with $L_n \subseteq F^C$ that $L_n \in CON(I, \mathcal{L})$. Since I is *pftt* of L, we have $L \subseteq L_n$. This contradicts that $T_{n+1} \not\subseteq L_n$.

((ii) $\Longrightarrow$ (i)) Assume that there exists a finite set $F \subseteq L^C$ such that $\langle L, \mathcal{L}_{F^C} \rangle$. By Lemma 2.3 (ii), $\langle L, \mathcal{L}_L \rangle$ because of $L \subseteq F^C$. Theorem 2.7 implies that L has *ftt* within $\mathcal{L}$, say T.

Assume that L does not have *pftt* within $\mathcal{L}$. Let us define two infinite sequences $(T_n)_{n \in N}$ and $(L_n)_{n \in N}$ recursively as follows:

For $n = 1$, let

$$T_1 = L^{\leq l_1}, \quad \text{where} \quad l_1 = \min\{l \in N \mid T \subseteq L^{\leq l}\}, \quad \text{and}$$
$$L_1 \in \mathcal{F}_1, \quad \text{where} \quad I_1 = (T_1, F) \quad \text{and} \quad \mathcal{F}_1 = \{L' \in CON(I_1, \mathcal{L}) \mid L \not\subseteq L'\}.$$

For $n \geq 2$, let

$$T_n = L^{\leq l_n}, \quad \text{where} \quad l_n = \min\{l \in N \mid L^{\leq l} \not\subseteq L_{n-1}\}, \quad \text{and}$$
$$L_n \in \mathcal{F}_n, \quad \text{where} \quad I_n = (T_n, F) \quad \text{and} \quad \mathcal{F}_n = \{L' \in CON(I_n, \mathcal{L}) \mid L \not\subseteq L'\}.$$

Similar to the proof of Lemma 2.2, we can show that the sequences $(T_n)_{n \in N}$ and $(L_n)_{n \in N}$ defined above satisfy the conditions (1)-(3) in Lemma 2.2. Since $L_n \subseteq F^C$ for all $n \in N$, L does not have finite cross property within $\mathcal{L}_{F^C}$. This contradicts our assumption. □

2.1.3. A Pair of Definite Finite Tell-tales

Mukouchi [8] has introduced the following notion as a necessary condition for finite identification from complete data.

DEFINITION 2.10. A pair of sets $I = (T, F)$ is *a pair of definite finite tell-tales* (*pdftt*, for short) *of L within $\mathcal{L}$* if I is *pftt* of L within $\mathcal{L}$ and

$$CON(I, \mathcal{L}) = \begin{cases} \{L\}, & \text{if } L \in \mathcal{L}, \\ \phi, & \text{o.w.} \end{cases}$$

L has *pdftt within $\mathcal{L}$* if there is *pdftt* of L within $\mathcal{L}$.

The notion of *pdftt* may be characterized by our finite cross property or p-finite cross property as follows:

THEOREM 2.11. *The following three statements are equivalent:*

(i) *L has pdftt within $\mathcal{L}$.*
(ii) *$\langle L^C, \mathcal{L}^C_{L^C}\rangle$ and there exists a finite set $F \subseteq L^C$ such that $\langle L, \mathcal{L}_{F^C}\rangle$, where $\mathcal{L}^C = \{L'^C \mid L' \in \mathcal{L}\}$.*
(iii) *L has p-finite cross property within $\mathcal{L}$.*

PROOF. The proof of (i) $\Leftrightarrow$ (ii) can be done similarly to that of Theorem 2.9. So we will prove the equivalency between (i) and (iii).

((i) $\Longrightarrow$ (iii)) Assume that L has *pdftt* within $\mathcal{L}$, say $I = (T, F)$, and L does not have *p-finite cross property* within $\mathcal{L}$, then there exist infinite sequences $(I_n)_{n\in N}$ and $(L_n)_{n\in N}$ satisfying the conditions (1)-(3) in Definition 2.5. Due to the conditions (1) and (2), for some $n \in N$, $I \prec I_n$. By the condition (3), $L_n \in CON(I_n, \mathcal{L})$, but $L_n \notin CON(I_{n+1}, \mathcal{L})$. These imply that $L_n \in CON(I, \mathcal{L})$ and $L \neq L_n$. This contradicts that I is *pdftt* of L within $\mathcal{L}$.

((iii) $\Longrightarrow$ (i)) Assume that L does not have *pdftt* within $\mathcal{L}$. Let us define two infinite sequences $(I_n)_{n\in N}$ and $(L_n)_{n\in N}$ as follows:

For $n = 1$, let

$$T_1 = L^{\leq l_1},\ F_1 = (L^C)^{\leq l_1},\quad \text{where}\quad l_1 = 1,$$
$$L_1 \in \mathcal{F}_1,\quad \text{where}\quad I_1 = (T_1, F_1)\quad \text{and}\quad \mathcal{F}_1 = \{L' \in CON(I_1, \mathcal{L}) \mid L \neq L'\}.$$

For $n \geq 2$, let

$$T_n = L^{\leq l_n},\ F_n = (L^C)^{\leq l_n},\quad \text{where}\quad l_n = \min\{l \in N \mid L^{\leq l} \not\subseteq L_{n-1} \text{ or } (L^C)^{\leq l} \not\subseteq L^C_{n-1}\},$$
$$L_n \in \mathcal{F}_n,\quad \text{where}\quad I_n = (T_n, F_n)\quad \text{and}\quad \mathcal{F}_n = \{L' \in CON(I_n, \mathcal{L}) \mid L \neq L'\}.$$

Similar to the proof of Lemma 2.2, we can prove that the sequences $(I_n)_{n\in N}$ and $(L_n)_{n\in N}$ defined above satisfy the conditions (1)-(3) in Definition 2.5. □

2.2. A Class with M-finite Thickness

In this subsection, we consider a special type of language classes, called M-finite thickness, introduced by the present author and Moriyama [10]. We show that M-finite thickness has rich properties.

For a set T of strings, L is a *minimal* language of T within $\mathcal{L}$ if $T \subseteq L$ and there does not exist $L' \in \mathcal{L}$ such that $T \subseteq L' \subsetneq L$. If T is *ftt* of L within $\mathcal{L}$, then L is a minimal language of T within $\mathcal{L}$. That is, the notion of *minimal* is dual to that of *ftt*. Let us denote

$$MIN(T, \mathcal{L}) = \{L \in \mathcal{L} \mid L \text{ is a } \textit{minimal} \text{ language of } T \text{ within } \mathcal{L}.\ \}.$$

DEFINITION 2.12. A class $\mathcal{L}$ is of *M-finite thickness* if for any nonempty finite set T of strings, (1) $MIN(T, \mathcal{L})$ is finite and (2) for any $L \in \mathcal{L}$, $T \subseteq L$ implies that there exists a language $L' \in MIN(T, \mathcal{L})$ such that $L' \subseteq L$.

Note that *M-finite thickness* is a generalization of *finite thickness* proposed by Angluin [1], but of much weaker condition. In fact, any classes containing all finite languages have *M-finite thickness.*

The following result is given immediately by the above definition.

LEMMA 2.13. *Let $\mathcal{L}$ be a class of M-finite thickness and L be a language. Then the subclass $\mathcal{L}_L$ has M-finite thickness.*

The following result corresponds to Lemma 2.2 for a class with M-finite thickness.

LEMMA 2.14. *Let $\mathcal{L}$ be a class with M-finite thickness. A language L has infinite cross property within $\mathcal{L}$ if and only if there exists a sequence of finite sets $(T_n)_{n\in N}$ and a sequence of languages $(L_n)_{n\in N}$, each in $\mathcal{L}$, such that*

$$\text{(1) } T_1 \subsetneq T_2 \subsetneq \cdots, \quad \text{(2) } \bigcup_{n=1}^{\infty} T_n = L, \quad \text{(3) } T_n \subseteq L_n, \text{ but } T_{n+1} \not\subseteq L_n, \quad n \in N,$$

$$\text{(4) } L_1 \subsetneq L_2 \subsetneq \cdots, \quad \text{(5) } L_n \in MIN(T_n, \mathcal{L}), \quad n \in N.$$

PROOF. The *if* direction is obvious, so we will prove the *only if* direction.

Assume that L has infinite cross property within $\mathcal{L}$. By Lemma 2.2, there exists a sequence of finite sets $(T_n)_{n\in N}$ of strings and a sequence of languages $(L_n)_{n\in N}$, each in $\mathcal{L}$, satisfying the conditions (1)-(3) in Lemma 2.2. Let us define a subsequence $(T_{n_t})_{t\in N}$ of $(T_n)_{n\in N}$ and a sequence of finite sets $(\mathcal{F}_t)_{t\in N}$ of languages in $\mathcal{L}$ recursively as follows: Without loss of generality, we can assume $T_1 \neq \phi$.

For $t = 1$, let

$$n_1 = 1, \quad \text{and} \quad \mathcal{F}_1 = \{L' \in MIN(T_1, \mathcal{L}) \mid L \not\subseteq L'\}.$$

For $t \geq 2$, let

$$n_t = \min\{n \in N \mid T_n \not\subseteq L' \text{ for every } L' \in \mathcal{F}_{t-1}\}, \quad \text{and}$$
$$\mathcal{F}_t = \{L' \in MIN(T_{n_t}, \mathcal{L}) \mid L \not\subseteq L'\}.$$

CLAIM A: The sequence of integers $(n_t)_{t\in N}$ is defined and strictly increasing, and for every $t \in N$, $\mathcal{F}_t$ is nonempty finite set and $\mathcal{F}_t \cap \mathcal{F}_{t+1} = \phi$.

The proof of the claim A can be done similar to that of Lemma 2.2.

CLAIM B: For every $t \in N$ and every $L' \in \mathcal{F}_{t+1}$, there exists a language $L'' \in \mathcal{F}_t$ such that $L'' \subsetneq L'$.

PROOF OF THE CLAIM B. Let $L' \in \mathcal{F}_{t+1}$. By the claim A, it follows that $T_{n_t} \subsetneq T_{n_{t+1}}$ and $L' \notin \mathcal{F}_t$. This means that L' is not a minimal language of T_{n_t} within $\mathcal{L}$. Since $\mathcal{L}$ has M-finite thickness, there exists a language $L'' \in MIN(T_{n_t}, \mathcal{L})$ such that $L'' \subseteq L'$. It implies together with $L \not\subseteq L'$ that $L \not\subseteq L''$, and thus $L'' \in \mathcal{F}_t$.

CLAIM C: There exists an infinite sequence of languages $(L'_t)_{t\in N}$ such that $L'_t \in \mathcal{F}_t$ for every $t \in N$ and $L'_1 \subsetneq L'_2 \subsetneq \cdots$.

PROOF OF THE CLAIM C. By the claim B, for every $t \in N$ and every $L' \in \mathcal{F}_t$, there exists a sequence of languages $L'_1, L'_2, \cdots, L'_t (= L')$ such that $L'_i \in \mathcal{F}_i$ for $i = 1, 2, \cdots, t$

and $L'_1 \subsetneq L'_2 \subsetneq \cdots \subsetneq L'_t$. By the claim A, $\mathcal{F}_1$ is finite. Thus there must exist an infinite sequence $(L'_t)_{t\in N}$ satisfying the conditions in our claim. Otherwise, for every language $L'' \in \mathcal{F}_1$,

$$\max\{t \in N \mid \exists L'_i \in \mathcal{F}_i (i = 1, 2, \cdots, t) \text{ s.t. } L'' = L'_1 \subsetneq L'_2 \subsetneq \cdots \subsetneq L'_t\}$$

is bounded, and is denoted by $t_{L''}$. Since $\mathcal{F}_1$ is finite, $t_{max} = \max\{t_{L''} \mid L'' \in \mathcal{F}_1\}$ is also bounded. This means that for any $L' \in \mathcal{F}_{t_{max}+1}$ there does not exist any language $L'' \in \mathcal{F}_{t_{max}}$ such that $L'' \subsetneq L'$. This contradicts the claim B.

The above claims yield that the sequences $(T_{n_t})_{t\in N}$ and $(L'_t)_{t\in N}$ satisfy the conditions (1)-(5) in our lemma. This completes our proof. □

For a class with M-finite thickness, the following equivalence theorem is valid.

THEOREM 2.15 (SATO&MORIYAMA [10]). *Let $\mathcal{L}$ be a class with M-finite thickness and let $L \in \mathcal{L}$. Then the following two statements are equivalent:*

(i) *L has ftt within $\mathcal{L}$.* (ii) *L has pftt within $\mathcal{L}$.*

By Theorem 2.7, the above statements are also equivalent to that L has finite cross property within $\mathcal{L}_L$, that is, $\langle L, \mathcal{L}_L\rangle$.

The following result is important to establish a characterization theorem for inductive refutable inferability considered in §3.2.

THEOREM 2.16. *Let $\mathcal{L}$ be a class with M-finite thickness and let $L \notin \mathcal{L}$. Then the following three statements are equivalent:*

(i) *L has ftt within $\mathcal{L}$.* (ii) *L has pftt within $\mathcal{L}$.* (iii) *L has pdftt within $\mathcal{L}$.*

PROOF. The proof of (i) $\Longrightarrow$ (ii) can be done similarly to the proof of Theorem 2.15 given in [10].

((ii) $\Longrightarrow$ (iii)) Let $L \notin \mathcal{L}$. Assume that L has *pftt* within $\mathcal{L}$, say $I = (T, F)$. Since $\mathcal{L}$ has M-finite thickness, $MIN(T, \mathcal{L})$ is finite. If $MIN(T, \mathcal{L}) = \phi$, clearly $CON(I, \mathcal{L}) = \phi$. It means that I is *pdftt* of L within $\mathcal{L}$. Otherwise, we put $MIN(T, \mathcal{L}) = \{L_1, L_2, \cdots, L_n\}$ for some $n \in N$. Since $L \notin \mathcal{L}$ and T is *ftt* within $\mathcal{L}$, $L_i \not\subseteq L$ for $i = 1, 2, \cdots, n$. Thus there is a string $w_i \in L_i \setminus L$ for $i = 1, \cdots, n$. Let $F' = F \cup \{w_1, w_2, \cdots, w_n\}$ and $I' = (T, F')$. Clearly $L \in CON(I')$. We will prove $CON(I', \mathcal{L}) = \phi$. If there exists a language $L' \in CON(I', \mathcal{L})$, then for some language $L_i \in MIN(T, \mathcal{L})$, $T \subseteq L_i \subseteq L'$, because $\mathcal{L}$ has M-finite thickness. It implies $w_i \in L'$. This contradicts $L' \in CON(I', \mathcal{L})$.

Clearly (iii) implies (i). □

Next we consider various operations for language classes, and discuss closure properties of M-finite thickness under such operations.

Given classes $\mathcal{L}_1$ and $\mathcal{L}_2$, let us define the following *union*, *intersection*, *concatenation* and *shuffle* operations:

$$\begin{aligned}
\mathcal{L}_1 \tilde{\cup} \mathcal{L}_2 &= \{L^{(1)} \cup L^{(2)} \mid L^{(1)} \in \mathcal{L}_1, L^{(2)} \in \mathcal{L}_2\},\\
\mathcal{L}_1 \tilde{\cap} \mathcal{L}_2 &= \{L^{(1)} \cap L^{(2)} \mid L^{(1)} \in \mathcal{L}_1, L^{(2)} \in \mathcal{L}_2\},\\
\mathcal{L}_1 \tilde{\cdot} \mathcal{L}_2 &= \{L^{(1)} \cdot L^{(2)} \mid L^{(1)} \in \mathcal{L}_1, L^{(2)} \in \mathcal{L}_2\},\\
\mathcal{L}_1 \tilde{\diamond} \mathcal{L}_2 &= \{L^{(1)} \diamond L^{(2)} \mid L^{(1)} \in \mathcal{L}_1, L^{(2)} \in \mathcal{L}_2\},
\end{aligned}$$

where the operations in the right hand sides of the above equations are union, intersection, concatenation and shuffle operation for languages, respectively, in the usual fashion.

Throughout this paper, the set theoretic union operation is called *usual union* in order to distinguish *union* defined above, and denoted by $\cup$. For a given class $\mathcal{L}$, let us define the following operations:

$$\mathcal{L}^{\widetilde{m}} = \{L^m \mid L \in \mathcal{L}\} \quad (m \in N), \qquad \mathcal{L}^{\widetilde{+}} = \{L^+ \mid L \in \mathcal{L}\},$$
$$\mathcal{L}^{\widetilde{*}} = \{L^* \mid L \in \mathcal{L}\}, \qquad \mathcal{L}^C = \{L^C \mid L \in \mathcal{L}\}.$$

Note that $\mathcal{L} \,\widetilde{\cdot}\, \mathcal{L}$ is different from $\mathcal{L}^{\widetilde{2}}$.

THEOREM 2.17. *The property of M-finite thickness is preserved under the operations* $\cup, \widetilde{\cap}, \widetilde{\cdot}, \widetilde{\Diamond}, \widetilde{^m}, \widetilde{^+}$ *and* $\widetilde{^*}$, *respectively.*

PROOF. We only give the proof for the operation $\widetilde{\cdot}$. The proofs for the other operations can be done analogously.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be classes with M-finite thickness and let $\mathcal{L} = \mathcal{L}_1 \,\widetilde{\cdot}\, \mathcal{L}_2$. We show that $\mathcal{L}$ has M-finite thickness. Let $S = \{w_1, w_2, \cdots, w_n\}$ be nonempty finite set of strings for $n \in N$. We call a pair of finite sets (S_1, S_2) of strings a division of S (w.r.t. $\cdot$) if $S_j = \{w_1^{(j)}, w_2^{(j)}, \cdots, w_n^{(j)}\}$ for $j = 1, 2$ and $w_i = w_i^{(1)} \cdot w_i^{(2)}$ for $i = 1, 2, \cdots, n$. By $DIV(S, \cdot)$, let us denote the set of possible divisions of S. Since the set S is finite, $DIV(S, \cdot)$ is finite. Let us define the following subset of $\mathcal{L}$:

$$\mathcal{F} = \{L^{(1)} \cdot L^{(2)} \mid \exists (S_1, S_2) \in DIV(S, \cdot) \text{ s.t. } L^{(j)} \in MIN(S_j, \mathcal{L}_j),\ j = 1, 2\}.$$

Since $\mathcal{L}_j$ has M-finite thickness for $j = 1, 2$ and $DIV(S, \cdot)$ is finite, $\mathcal{F}$ is a finite subset of $\mathcal{L}$, and thus $MIN(S, \mathcal{F}$ is finite. In order to prove our theorem, it is suffice to show the following Claim A and Claim B:

CLAIM A: $MIN(S, \mathcal{L}) = MIN(S, \mathcal{F})$.

PROOF OF THE CLAIM A. Assume first that there exists a language $L \in MIN(S, \mathcal{L}) \setminus MIN(S, \mathcal{F})$. Then there exist languages $L^{(j)} \in \mathcal{L}_j$, $j = 1, 2$ and a division $(S_1, S_2) \in DIV(S, \cdot)$ such that $L = L^{(1)} \cdot L^{(2)}$ and $S_j \subseteq L^{(j)}$ for $j = 1, 2$. Since $\mathcal{L}_j$ has M-finite thickness for $j = 1, 2$, there exist languages $L^{(j)'} \in MIN(S_j, \mathcal{L}_j)$ such that $S_j \subseteq L^{(j)'} \subseteq L^{(j)}$. Let $L' = L^{(1)'} \cdot L^{(2)'}$. Then $L' \in \mathcal{F}$ and $S \subseteq L' \subseteq L$. Since $\mathcal{F}$ is finite, for some $L'' \in MIN(S, \mathcal{F})$, $S \subseteq L'' \subseteq L' \subseteq L$. However, L is a minimal language of S. Thus $L^{(1)'} \cdot L^{(2)'} = L$, which contradicts $L \notin \mathcal{F}$. Hence $MIN(S, \mathcal{L}) \subseteq MIN(S, \mathcal{F})$. Similarly, we can prove the converse inclusion.

Note that Claim A implies $MIN(S, \mathcal{L})$ is finite.

CLAIM B: For a language $L \in \mathcal{L}$ where $S \subseteq L$, there exists a language $L' \in MIN(S, \mathcal{L})$ such that $L' \subseteq L$.

The proof of Claim B is obvious from the Claim A.

By Claim A and Claim B, the class $\mathcal{L}$ has M-finite thickness. $\square$

By the above theorem, it follows immediately that:

THEOREM 2.18. *Given language classes with M-finite thickness, a class obtained by finitely many applying the operations* $\cup, \widetilde{\cap}, \widetilde{\cdot}, \widetilde{\Diamond}, \widetilde{^m}, \widetilde{^+}$ *and* * *to them has M-finite thickness.*

THEOREM 2.19. *The property of M-finite thickness is preserved under neither union* $\widetilde{\cup}$ *nor complement* ^{C}.

PROOF. Let $\Sigma = \{a\}$. Let us consider the following language classes $\mathcal{L}_j = \{L_i^{(j)} \mid i \in N\}$ for $j = 1, 2$:

$$L_1^{(1)} = L_1^{(2)} = \{a\}^+, \quad L_i^{(1)} = \{a^i\}, \quad L_i^{(2)} = \{a\}, \quad i \geq 2.$$

As easily seen, $\mathcal{L}_1$ and $\mathcal{L}_2$ have both M-finite thickness.

First consider the class $\mathcal{L}_1 \widetilde{\cup} \mathcal{L}_2$. Let $S = \{a\}$. Then infinitely many languages $L_i^{(1)} \cup L_2^{(2)} (= \{a, a^i\})$ $(i \geq 2)$ are all minimal languages of S. Hence the class $\mathcal{L}_1 \widetilde{\cup} \mathcal{L}_2$ does not have M-finite thickness.

Next consider the complement language class of $\mathcal{L}_1$. Then clearly infinitely many languages $(L_2^{(1)})^C, (L_3^{(1)})^C, \cdots$ are minimal languages of S defined above. It means that the class $\mathcal{L}_1^C$ does not have M-finite thickness.

Therefore our theorem is valid. □

THEOREM 2.20. *Let* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *be a language classes with M-finite thickness. Then the following class* $\mathcal{L}$ *has M-finite thickness:*

$$\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup (\mathcal{L}_1 \widetilde{\cup} \mathcal{L}_2).$$

PROOF. The proof can be done similarly to that of Theorem 2.17. Let S be a nonempty finite set. We call a pair of finite sets (S_1, S_2) a division of S (w.r.t. $\cup$) when S_1 and S_2 are both nonempty sets and $S = S_1 \cup S_2$. Since the set S is finite, the set of possible divisions of S, denoted by $DIV(S, \cup)$, is finite. Let us put

$$\begin{aligned}\mathcal{F} = \; & MIN(S, \mathcal{L}_1) \cup MIN(S, \mathcal{L}_2) \\ & \cup \{L^{(1)} \cup L^{(2)} \mid \exists (S_1, S_2) \in DIV(S, \cup) \text{ s.t. } L^{(j)} \in MIN(S_j, \mathcal{L}_j), j = 1, 2\}.\end{aligned}$$

Since $\mathcal{L}_j$ has M-finite thickness for $j = 1, 2$ and $DIV(S)$ is finite, $\mathcal{F}$ is finite subset of $\mathcal{L}$. Similar to the proof of Theorem 2.17, we can show the following Claim A and Claim B:

CLAIM A: $MIN(S, \mathcal{L}) = MIN(S, \mathcal{F})$.

CLAIM B: For a language $L \in \mathcal{L}$ where $S \subseteq L$, there exists a language $L' \in MIN(S, \mathcal{L})$ such that $L' \subseteq L$.

By Claim A and Claim B, the class $\mathcal{L}$ has M-finite thickness. □

Let $\mathcal{OPE} = \{\cup, \cap, \cdot, \Diamond, ^+, ^*\}$. Given a language class $\mathcal{L}$, a nonempty set $O \subseteq \mathcal{OPE}$ and $n \in N$, let us denote by $\mathcal{L}(n, O)$ the class obtained by at most n times applying operations in O to languages in $\mathcal{L}$. Then by Theorem 2.17 and Theorem 2.20, the following corollary may be derived:

COROLLARY 2.21. *Let* $\mathcal{L}$ *be a language class with M-finite thickness. Then for* $n \geq 0$ *and a nonempty set* $O \subseteq \mathcal{OPE}$*, the class* $\mathcal{L}(n, O)$ *has M-finite thickness, where* $\mathcal{L}(0, O) = \mathcal{L}$.

PROOF. The proof can be done easily by a mathematical induction on n. □

Let $w_1, w_2, \cdots$ be a recursive enumeration of Σ^* and $\mathcal{L} = L_1, L_2, \cdots$, where $L_i = \{w_i\}$ for $i \in N$. Then for $n \in N$ we denote $\mathcal{L}_\Sigma(n) = \mathcal{L}(n, \{\cup, \cdot, ^*\})$. Then it follows immediately that:

COROLLARY 2.22. *For $n \geq 0$, the class $\mathcal{L}_\Sigma(n)$ has M-finite thickness.*

Note that $\bigcup_{n=0}^{\infty} \mathcal{L}_\Sigma(n)$ is equal to the class of regular languages.

2.3. A Language Class with Finite Elasticity

In this subsection, we are concerned with a special type of language classes, called finite elasticity, introduced by Wright [14] as a sufficient condition for inferability from positive data, defined as follows:

DEFINITION 2.23. A class $\mathcal{L}$ has *infinite elasticity* if there exists an infinite sequence of strings $(w_n)_{n\geq 0}$ and an infinite sequence of languages $(L_n)_{n\in N}$, each in $\mathcal{L}$, such that

$$(1)\ \{w_0, \cdots, w_{n-1}\} \subseteq L_n \qquad \text{and} \qquad (2)\ w_n \notin L_n$$

for all $n \in N$. A class $\mathcal{L}$ has *finite elasticity* if $\mathcal{L}$ does not have infinite elasticity.

Note that Motoki et al.[7] showed that finite elasticity defined by Wright [14] is not a sufficient condition for inferability from positive data, and corrected the definition as the above. Wright's results given below are valid for the above corrected definition of finite elasticity.

We first present a theorem characterizing finite elasticity in terms of finite cross property as follows:

THEOREM 2.24. *$\mathcal{L}$ has finite elasticity if and only if $\langle L, \mathcal{L}\rangle$ for every language L.*

PROOF. ($\Longrightarrow$) Assume that $\mathcal{L}$ has finite elasticity and there exists a language L which has infinite cross property within $\mathcal{L}$. By Lemma 2.2, there exist two infinite sequences $(T_n)_{n\in N}$ and $(L_n)_{n\in N}$ satisfying the conditions (1)-(3) in Lemma 2.2. Without loss of generality, we can assume that $T_1 \neq \phi$. Let $w_0 \in T_1$. Due to the the condition (2), $T_{n+1} \setminus L_n \neq \phi$ for $n \in N$, and thus there is a string $w_n \in T_{n+1} \setminus L_n$. It can be easily shown that two infinite sequences $(w_n)_{n\geq 0}$ and $(L_n)_{n\in N}$ satisfy the conditions of infinite elasticity. This contradicts our assumption.

($\Longleftarrow$) Assume that $\mathcal{L}$ has infinite elasticity. Let $(w_n)_{n\geq 0}$ and $(L_n)_{n\in N}$ be two infinite sequences satisfying the conditions in Definition 2.23. Let us put $L = \{w_n \mid n \geq 0\}$ and $T_n = \{w_0, w_1, \cdots, w_{n-1}\}$ for $n \in N$. As easily seen, two infinite sequences $(T_n)_{n\in N}$ and $(L_n)_{n\in N}$ satisfy the conditions (1)-(3) in Lemma 2.2. Hence L has infinite cross property within $\mathcal{L}$. □

By Theorem 2.9 and Theorem 2.24, it follows immediately that if $\mathcal{L}$ has finite elasticity, then every language has *pftt* within $\mathcal{L}$.

Next we consider the operations for classes introduced in the previous subsection. In [6] and [14], it has been shown that the property of finite elasticity is preserved under such operations as described below.

THEOREM 2.25 (WRIGHT [14]). *The property of finite elasticity is preserved under union operation* $\widetilde{\cup}$.

THEOREM 2.26 (MORIYAMA&SATO [6]). *The property of finite elasticity is preserved under the operations* $\cup, \widetilde{\cap}, \widetilde{\cdot}, \widetilde{^m}, \widetilde{^+}$ *and* $\widetilde{^*}$, *but is not true for the complement operation.*

THEOREM 2.27. *The property of finite elasticity is preserved under the shuffle operation* $\widetilde{\Diamond}$.

PROOF. The proof can be given similarly to that of Theorem 2.26 given in [6]. □

By Theorem 2.25, Theorem 2.26 and Theorem 2.27, the following two theorems may be derived:

THEOREM 2.28. *Given language classes with finite elasticity, a class obtained by finitely many applying the operations* $\cup, \widetilde{\cup}, \widetilde{\cap}, \widetilde{\cdot}, \widetilde{\Diamond}, \widetilde{^m}, \widetilde{^+}$ *and* $\widetilde{^*}$ *to them has also finite elasticity.*

By the above theorem, the following two corollaries may be easily derived:

COROLLARY 2.29. *Let* $\mathcal{L}$ *be a class with finite elasticity. Then for* $n \geq 0$ *and a nonempty set* $O \subseteq \mathcal{OPE}$, *the class* $\mathcal{L}(n, O)$ *has finite elasticity, where* $\mathcal{OPE}$ *and* $\mathcal{L}(n, O)$ *are defined in §2.2.*

COROLLARY 2.30. *For* $n \geq 0$, *the class* $\mathcal{L}_\Sigma(n)$ *has finite elasticity, where* $\mathcal{L}_\Sigma(n)$ *is defined in §2.2.*

PROOF. It is trivial since the class $\mathcal{L} = \{w_1\}, \{w_2\}, \cdots$ has finite elasticity, where $w_1, w_2, \cdots$ is a recursive enumeration of Σ^*. □

3. Inference Machines and Inferability

In this section, we consider inductive inference of formal languages. At the beginning, we define briefly basic notions on inductive inference. For more details about inference machine, we refer to [1].

An *inductive inference machine* is an effective procedure that requests inputs from time to time and produce outputs from time to time. An output produced by an inference machine is called a hypothesis or a guess. The set of hypothesis is called the *hypothesis space*.

Hereafter, we assume that a hypothesis space is an indexed family of recursive languages defined as follows:

DEFINITION 3.1. An infinite sequence of languages $\mathcal{L} = L_1, L_2, \cdots$ is *an indexed family of recursive languages* if there exists a recursive function $f : N \times \Sigma^* \longrightarrow \{0,1\}$ such that

$$f(i,w) = \begin{cases} 1, & \text{if } w \in L_i, \\ 0, & \text{if } w \notin L_i. \end{cases}$$

A *positive presentation* of a nonempty language L is an infinite sequence of strings $w_1, w_2, \cdots$ such that $\{w_n \mid n \in N\} = L$. A *complete presentation* of a language L is an infinite sequence of pairs $(w_1, t_1), (w_2, t_2), \cdots$ such that $\{w_n \mid t_n = 1,\ n \in N\} = L$ and $\{w_n \mid t_n = 0,\ n \in N\} = L^C$.

DEFINITION 3.2. A class of languages $\mathcal{L} = L_1, L_2, \cdots$ is *inferable from positive data* (resp., *complete data*) if there is an inductive inference machine M such that the sequence of outputs produced by M converges to j with $L_j = L_i$ for any index $i \in N$, where $L_i \neq \phi$, and any positive presentation (resp., *complete presentation*) of L_i.

It is well known that every indexed family of recursive languages is inferable from *complete data* (cf. [3]). In the next subsection, we deal with inductive inference from *positive data.*

When the target language does not belong to the hypothesis space, how does the machine work? It is impossible to identify the target language in the limit. Recently, Mukouchi&Arikawa [9] have presented the following inductive refutable inference as a framework of machine discovery:

DEFINITION 3.3. An *inductive inference machine that can refute hypothesis spaces* is an effective procedure that requests inputs from time to time and either (i) produces hypotheses from time to time or (ii) refutes the class and stops after producing some hypothesis.

A language class $\mathcal{L} = L_1, L_2, \cdots$ is *refutably inferable from positive data* (resp., *complete data*) if there is an inductive inference machine M that can refute hypothesis space such that for any language L and any positive presentation (resp., complete presentation) of L, (i) if $L \in \mathcal{L}$, then M infers L from the presentation, (ii) otherwise M refutes the class $\mathcal{L}$ from the presentation.

Refutable inferability from *positive data* is known to be of small power (cf. [9]). In §3.2, we consider inductive refutable inference from *complete data.*

Before going into the detailed discussions on inductive inference, let us give the closure properties of indexed families of recursive languages under the operations considered in §2.2.

By $\mathcal{IFR}$ we denote the collection of indexed families of recursive languages.

LEMMA 3.4. *$\mathcal{IFR}$ is closed under the operations* $\cup, \widetilde{\cup}, \widetilde{\cap}, \widetilde{\ }, \widetilde{\diamondsuit}, \widetilde{\ }^{m}, \widetilde{\ }^{+}$ *and* $\widetilde{\ }^{*}$ *defined in §2.2, respectively.*

PROOF. It is trivial from Definition 3.1. $\square$

3.1. Inductive Inference from Positive Data

In this subsection, we consider inductive inference of languages from positive data.

Angluin [1] presented the following characterization theorem which is of fundamental importance in the study of inductive inference from positive data:

THEOREM 3.5 (ANGLUIN [1]). *Let $\mathcal{L} = L_1, L_2, \cdots$ be a class of $\mathcal{IFR}$. Then $\mathcal{L}$ is inferable from positive data if and only if there is an effective procedure that enumerates all strings of ftt of L_i for any i with $L_i \neq \phi$.*

By the above theorem, the condition that a language of a class has *ftt* within the class is necessary for the inferability from positive data.

The following two sufficient conditions for inferability have to be presented:

THEOREM 3.6 (WRIGHT [14]). *Let $\mathcal{L} \in \mathcal{IFR}$. If $\mathcal{L}$ has finite elasticity, the class is inferable from positive data.*

THEOREM 3.7 (KAPUR [4], SATO&UMAYAHARA [11]). *Let $\mathcal{L} \in \mathcal{IFR}$. If any language of $\mathcal{L}$ has pftt within $\mathcal{L}$, then $\mathcal{L}$ is inferable from positive data.*

Note that the class has *uniformly* inferable from positive data under the above condition (cf. [4] and [5]). Theorem 2.9 and Theorem 2.24 imply that finite elasticity is a stronger sufficient condition that the above condition of *pff* for inferability.

Theorem 2.15 and the above theorem imply the following equivalences:

THEOREM 3.8 (SATO&MORIYAMA [10]). *Let $\mathcal{L} \in \mathcal{IFR}$. If $\mathcal{L}$ has M-finite thickness, then the following three statements are equivalent:*

(i) *$\mathcal{L}$ is inferable from positive data.*
(ii) *Every language of $\mathcal{L}$ has ftt within $\mathcal{L}$.*
(iii) *Every language of $\mathcal{L}$ has pftt within $\mathcal{L}$.*

By Theorem 2.28, Lemma 3.4 and Theorem 3.6, it follows immediately that:

THEOREM 3.9. *Given language classes in $\mathcal{IFR}$ each of which has finite elasticity, a class obtained by finitely many applying the operations $\cup, \widetilde{\cup}, \widetilde{\cap}, \widetilde{\cdot}, \widetilde{\Diamond}, \widetilde{{}^m}, \widetilde{{}^+}$ and * to them is inferable from positive data.*

By the above theorem, the following two useful corollaries are given immediately:

COROLLARY 3.10. *Let $\mathcal{L} \in \mathcal{IFR}$. If $\mathcal{L}$ has finite elasticity, then for $n \in N$ and a nonempty set $O \subseteq \mathcal{OPE}$, the class $\mathcal{L}(n, O)$ is inferable from positive data, where $\mathcal{OPE}$ and $\mathcal{L}(n, O)$ are defined in §2.2.*

COROLLARY 3.11. *For $n \geq 0$, the class $\mathcal{L}_\Sigma(n)$ is inferable from positive data, where $\mathcal{L}_\Sigma(n)$ is defined in §2.2.*

3.2. Inductive Refutable Inference from Complete Data

In this subsection, we consider inductive refutable inference of a language class from complete data introduced by Mukouchi&Arikawa [9].

For a language class $\mathcal{L}$ and a pair of finite sets $I = (T, F)$ of strings, let

$$econ_{\mathcal{L}}(I) = \begin{cases} 1, & \text{if } CON(I, \mathcal{L}) \neq \phi, \\ 0, & \text{o.w.} \end{cases}$$

Let $\mathcal{ECN} = \{\mathcal{L} \mid econ_{\mathcal{L}} \text{ is recursive}\}$.

A characterizing theorem of refutable inferability from complete data has been obtained:

THEOREM 3.12 (MUKOUCHI&ARIKAWA [9]). *Let $\mathcal{L}$ be a class of $\mathcal{IFR}$. Then $\mathcal{L}$ is refutably inferable from complete data if and only if $\mathcal{L} \in \mathcal{ECN}$ and every language $L \notin \mathcal{L}$ has pdftt within $\mathcal{L}$.*

By Theorem 2.16 and the above theorem, a characterization theorem for a class with M-finite thickness is established as follows:

THEOREM 3.13. *Let $\mathcal{L} \in \mathcal{IFR} \cap \mathcal{ECN}$. If $\mathcal{L}$ has M-finite thickness, then the following four statements are equivalent:*

(i) *$\mathcal{L}$ is refutably inferable from complete data.*
(ii) *Every language $L \notin \mathcal{L}$ has ftt within $\mathcal{L}$.*
(iii) *Every language $L \notin \mathcal{L}$ has pftt within $\mathcal{L}$.*
(iv) *Every language $L \notin \mathcal{L}$ has pdftt within $\mathcal{L}$.*

Furthermore, the following sufficient condition may be derived:

THEOREM 3.14. *Let $\mathcal{L} \in \mathcal{IFR} \cap \mathcal{ECN}$. If $\mathcal{L}$ has M-finite thickness and finite elasticity, then $\mathcal{L}$ is refutably inferable from complete data.*

PROOF. As mentioned in §3.1, if $\mathcal{L}$ has finite elasticity, then every language has *pftt* within $\mathcal{L}$. Thus Theorem 3.13 implies our theorem. □

LEMMA 3.15. *$\mathcal{ECN}$ is closed under the operations $\cup, \widetilde{\cup}, \widetilde{\cap}, \tilde{\cdot}, \widetilde{\Diamond}, \widetilde{^m}, \widetilde{^+}$ and $\widetilde{^*}$, defined in §2.2, respectively.*

PROOF. We only prove for the operation $\tilde{\cdot}$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be classes in $\mathcal{ECN}$ and $I = (T, F)$ be a pair of finite sets of strings. Then $DIV(T, \cdot)$ and $DIV(F, \cdot)$ defined in the proof of Theorem 2.17 of §2.2 are both finite and moreover, recursively generable (cf. [9]). Besides, as easily seen, $econ_{\mathcal{L}_1 \tilde{\cdot} \mathcal{L}_2}(T, F) = 1$ *iff* there exists a division $(T_1, T_2) \in DIV(T, \cdot)$ and a division $(F_1, F_2) \in \widetilde{DIV}(F, \cdot)$ such that $econ_{\mathcal{L}_j}(T_j, F_j) = 1$ for $j = 1, 2$. Since $econ_{\mathcal{L}_j}$ is recursive for $j = 1, 2$, $econ_{\mathcal{L}_1 \tilde{\cdot} \mathcal{L}_2}$ is also recursive. This completes our lemma. □

By Theorem 2.18, Theorem 2.28, Lemma 3.4, Theorem 3.14 and Lemma 3.15, the following closedness theorem on inductive refutable inference is established:

THEOREM 3.16. *Given language classes in $\mathcal{IFR} \cap \mathcal{ECN}$ each of which has M-finite thickness and finite elasticity, a class obtained by finitely applying the operations $\cup, \widetilde{\cap}, \tilde{\cdot}, \widetilde{\Diamond}, \widetilde{\ }^{m}, \widetilde{\ }^{+}$ and $\widetilde{\ }^{*}$ to them is refutably inferable from complete data.*

Note that the operation $\widetilde{\cup}$ is not included in the above theorem, because the property of M-finite thickness is not always preserved under the operation $\widetilde{\cup}$ as illustrated in the proof of Theorem 2.19. However, Theorem 2.20 implies the following closedness for union operation $\widetilde{\cup}$:

THEOREM 3.17. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be classes in $\mathcal{IFR} \cap \mathcal{ECN}$. If these classes have M-finite thickness and finite elasticity, then $\mathcal{L}_1 \cup \mathcal{L}_2 \cup (\mathcal{L}_1 \mathbin{\widetilde{\cup}} \mathcal{L}_2)$ is refutably inferable from complete data.*

By the above theorem, the following two corollaries may be derived as well as Corollary 3.10 and Corollary 3.11.

COROLLARY 3.18. *Let $\mathcal{L} \in \mathcal{IFR} \cap \mathcal{ECN}$. If $\mathcal{L}$ has M-finite thickness and finite elasticity, then for $n \in N$ and a nonempty set $O \subseteq \mathcal{OPE}$, $\mathcal{L}(n, O)$ is refutably inferable from complete data, where $\mathcal{OPE}$ and $\mathcal{L}(n, O)$ are defined in §2.2.*

COROLLARY 3.19. *For $n \in N$, the class $\mathcal{L}_\Sigma(n)$ is refutably inferable from complete data, where $\mathcal{L}_\Sigma(n)$ is defined in §2.2.*

PROOF. It is clear since $\mathcal{L}_\Sigma(n) \in \mathcal{ECN}$. □

3.3. Relationships

In this section, we have investigate inductive (refutable) inferability for classes with M-finite thickness. The following Figure 1 shows the relationships obtained so far.

4. Inductive Inference of Elementary Formal Systems

In this section, we deal with a special type of elementary formal system's (EFS's, for short), called length-bounded as a framework defining target languages. That is, a hypothesis space of an inductive inference machine is assumed to be a class of length-bounded EFS's.

Let Σ, Π and X be mutually disjoint sets. We assume that Σ is finite and Π is finite or a countable set. Elements of Σ, Π and X are called *constant* symbols, *predicate* symbols and *variables*, respectively. Each predicate symbol is associated with a positive integer termed *arity*. Let *HB* be the set of all ground atoms. For detailed definitions and results on EFS's, we refer to [2] and [15].

DEFINITION 4.1. A clause $A \leftarrow B_1, \cdots, B_n$ is *length-bounded* if

$$|A\theta| \geq |B_1\theta| + \cdots + |B_n\theta|$$

for any nonempty substitution θ. An EFS Γ is *length-bounded* if any clause in Γ is length-bounded.

$$
\begin{array}{lll}
 & \nearrow\ \forall L \in \mathcal{L},\ \langle L, \mathcal{L}\rangle \longrightarrow & \left\{\begin{array}{l}
\text{(i) } \forall L \in \mathcal{L},\ \exists F \subseteq L^C \text{ s.t. } \langle L, \mathcal{L}_{FC}\rangle \\
\qquad\qquad \text{(i.e., } L \text{ has pftt within } \mathcal{L}) \\
\text{(ii) } \mathcal{L} \text{ is inferable from positive data} \\
\text{(iii) } \forall L \in \mathcal{L},\ \langle L, \mathcal{L}_L\rangle \\
\qquad\qquad \text{(i.e., } L \text{ has ftt within } \mathcal{L})
\end{array}\right. \\
\begin{array}{l}\forall L,\ \langle L, \mathcal{L}\rangle \\ \text{(finite elasticity)}\end{array} & & \\
 & \searrow^{\mathcal{L} \in \mathcal{ECN}}\ \forall L \notin \mathcal{L},\ \langle L, \mathcal{L}\rangle \longrightarrow & \left\{\begin{array}{l}
\text{(i) } \forall L \notin \mathcal{L},\ \langle L^C, \mathcal{L}^C_{LC}\rangle \text{ and} \\
\qquad\qquad \exists F \subseteq L^C \text{ s.t. } \langle L, \mathcal{L}_{FC}\rangle \\
\qquad\qquad \text{(i.e., } L \text{ has pdftt within } \mathcal{L}.\ ) \\
\text{(ii) } \mathcal{L} \text{ is refutably inferable} \\
\qquad\qquad \text{from complete data} \\
\text{(iii) } \forall L \notin \mathcal{L},\ \exists F \subseteq L^C \text{ s.t. } \langle L, \mathcal{L}_{FC}\rangle \\
\qquad\qquad \text{(i.e., } L \text{ has ftt within } \mathcal{L}) \\
\text{(iv) } \forall L \notin \mathcal{L},\ \langle L, \mathcal{L}_L\rangle \\
\qquad\qquad \text{(i.e., } L \text{ has ftt within } \mathcal{L})
\end{array}\right.
\end{array}
$$

Figure 1: Relationships, where $\mathcal{L} \in \mathcal{IFR}$ has M-finite thickness.

Let us denote by $\mathcal{LB}$ the class of all length-bounded EFS's.

We first give the following result on EFS model shown by Arikawa et al.[2].

THEOREM 4.2 (ARIKAWA ET AL.[2]). *For a length-bounded EFS Γ, the least Herbrand model $M(\Gamma)$ is recursive.*

For an EFS Γ and a fixed unary predicate symbol $p \in \Pi$, we denote by $L(\Gamma, p)$ the language over Σ defined by Γ and p.

The next result shows the power of length-bounded EFS's.

THEOREM 4.3 (ARIKAWA ET AL.[2]). *A language $L \subseteq \Sigma^+$ is definable by a length-bounded EFS if and only if L is context-sensitive.*

For a class $\mathcal{G} = \Gamma_1, \Gamma_2, \cdots$ of EFS's, we put

$$L(\mathcal{G}, p) = L(\Gamma_1, p), L(\Gamma_2, p), \cdots .$$

By Theorem 4.2, if $\mathcal{G}$ is a recursive enumeration of length-bounded EFS's, then $L(\mathcal{G}, p)$ is an indexed family of recursive languages, that is, $L(\mathcal{G}, p) \in \mathcal{IFR}$.

Hereafter, we confine ourselves to recursive enumerable classes of length-bounded EFS's.

DEFINITION 4.4. A class $\mathcal{G}$ is *closed under subset operation* if for any $\Gamma \in \mathcal{G}$ and any $\Gamma' \subseteq \Gamma$, Γ' belongs to $\mathcal{G}$.

The following notion has been introduced by Shinohara [12] and [13].

DEFINITION 4.5. Let S be a nonempty subset of HB. An EFS Γ is *reduced w.r.t.* S if $S \subseteq M(\Gamma)$ and $S \not\subseteq M(\Gamma')$ for any $\Gamma' \subsetneq \Gamma$.

For a nonempty set T of strings over Σ and a class $\mathcal{G}$ of EFS's, let us put

$$RED(p(T), \mathcal{G}) = \{\Gamma \in \mathcal{G} \mid \Gamma \text{ is reduced w.r.t. } p(T).\ \},$$

where $p(T) = \{p(w) \mid w \in T\}$.

DEFINITION 4.6. An EFS Γ is *equivalent* to Γ' w.r.t. p, denoted by $\Gamma \equiv_p \Gamma'$, if we can identify them by renaming predicate and variable symbols, except p.

Clearly, the relation $\equiv_p$ is an equivalence on any class $\mathcal{G}$ of EFS's. As easily seen, given $\Gamma, \Gamma' \in \mathcal{G}$, if $\Gamma \equiv_p \Gamma'$, then $L(\Gamma, p) = L(\Gamma', p)$ and moreover

$$\Gamma \in RED(p(T), \mathcal{G}) \text{ if and only if } \Gamma' \in RED(p(T), \mathcal{G}).$$

for every nonempty finite set T. This means that $RED(p(T), \mathcal{G})$ consists of some equivalence classes of $\mathcal{G}$. Let $RED(p(T), \mathcal{G})/\equiv_p$ be the set of equivalence classes of $RED(p(T), \mathcal{G})$.

The following notion introduced in [10] plays an essential role related to inferability for classes with M-finite thickness discussed in the previous section.

DEFINITION 4.7. A class $\mathcal{G}$ is *R-finite* w.r.t. p if $RED(p(T), \mathcal{G})/\equiv_p$ is finite for any finite nonempty set $T \subseteq \Sigma^+$.

LEMMA 4.8 (SATO&MORIYAMA [10]). *Let $\mathcal{G}$ be a class of length-bounded EFS's closed under subset operation. If $\mathcal{G}$ is R-finite w.r.t. p, then $L(\mathcal{G}, p)$ has M-finite thickness.*

We consider what classes are R-finite w.r.t. p below.

LEMMA 4.9 (SHINOHARA [12], [13]).

(i) *If Π is finite, $\mathcal{LB}$ is R-finite w.r.t. p.*
(ii) *For n, $\mathcal{LB}^{\leq n}$ is R-finite w.r.t. p, where*

$$\mathcal{LB}^{\leq n} = \{\Gamma \in \mathcal{LB} \mid \#\Gamma \leq n\}.$$

DEFINITION 4.10. A clause $A \leftarrow B_1, \cdots, B_n$ is *strongly length-bounded* if the clause is length-bounded and in case $n = 1$, $|A\theta| > |B_1\theta|$ for any substitution θ. An EFS Γ is *strongly length-bounded* if any clause in Γ is strongly length-bounded.

THEOREM 4.11. *A language $L \subseteq \Sigma^+$ is context-free, then L is definable by a strongly length-bounded.*

LEMMA 4.12 (SATO&MORIYAMA [10]). *Let $\mathcal{G}$ be a class of strongly length-bounded EFS's closed under subset operation. Then $L(\mathcal{G}, p)$ is R-finite w.r.t. p.*

The above three lemmas mean that the collection of classes with M-finite thickness is sufficiently large.

As shown in Theorem 3.8 or Theorem 3.13, the set theoretic aspect of inferability can be characterized by *ftt* in the criteria considered in this paper. The following notions correspond to *infinite cross property* and *finite cross property* introduced in order to clarify the notion of *ftt* in §2.1.

DEFINITION 4.13. A language L is of *infinite hierarchy w.r.t. p within $L(\mathcal{G},p)$* if there exists an infinite sequence of finite sets $(T_n)_{n\in N}$ and an infinite sequence of EFS's $(\Gamma_n)_{n\in N}$, each in $\mathcal{G}$, such that

(1) $T_1 \subsetneq T_2 \subsetneq \cdots$, (2) $\bigcup_{n=1}^{\infty} T_n = L$, (3) $\Gamma_1 \subsetneq \Gamma_2 \subsetneq \cdots$,

(4) for $n \in N$, Γ_n is reduced w.r.t. $p(T_n)$ within $\mathcal{G}$,

(5) for $n \in N$, $L(\Gamma_n, p) \subsetneq L$.

A language L is of *finite hierarchy w.r.t. p within $L(\mathcal{G},p)$* if L is not of infinite hierarchy w.r.t. p within $L(\mathcal{G},p)$.

Note that if L is of infinite hierarchy w.r.t. p, then the sequence of $L(\Gamma_1,p), L(\Gamma_2,p)$, $\cdots$ in the above definition is strictly monotone increasing, and the class $L(\mathcal{G},p)$ has infinite elasticity.

In terms of the above notion, a characterization of *ftt* in the framework considered is given as follows:

THEOREM 4.14. *Let $\mathcal{G}$ be an R-finite class of length-bounded EFS's w.r.t. p closed under subset operation. A language L has ftt within $L(\mathcal{G},p)$ if and only if L is of finite hierarchy w.r.t. p within $L(\mathcal{G},p)$.*

PROOF. The proof can be done similarly to that of Theorem 4.4 in [10]. □

A characterization of inferability for an R-finite class considered from positive data has been given in [10] as follows:

THEOREM 4.15 (SATO&MORIYAMA [10]). *Let $\mathcal{G}$ be an R-finite class of length-bounded EFS's w.r.t. p closed under subset operation. Then the following two statements are equivalent:*

(i) *$L(\mathcal{G},p)$ is inferable from positive data.*

(ii) *Every language $L \in L(\mathcal{G},p)$ is of finite hierarchy w.r.t. p within $L(\mathcal{G},p)$.*

Concerning inductive refutable inferability, the following characterization can be derived immediately from Lemma 4.8, Theorem 3.13 and Theorem 4.14:

THEOREM 4.16. *Let $\mathcal{G}$ be an R-finite and recursive class of length-bounded EFS's w.r.t. p closed under subset operation and $L(\mathcal{G},p) \in \mathcal{ECN}$. Then the following two statements are equivalent:*

(i) *$L(\mathcal{G},p)$ is refutably inferable from complete data.*

(ii) *Every language $L \notin L(\mathcal{G},p)$ is of finite hierarchy w.r.t. p within $L(\mathcal{G},p)$.*

LEMMA 4.17 (MUKOUCHI& ARIKAWA [9]). *For any $n \in N$, $econ_{L(\mathcal{LB}^{\leq n},p)}$ is recursive, that is, $L(\mathcal{LB}^{\leq n},p) \in \mathcal{ECN}$.*

By Lemma 4.9, Lemma 4.17 and Theorem 4.16, it follows immediately that:

COROLLARY 4.18 (MUKOUCHI&ARIKAWA [9]). *For $n \in N$, the class $L(\mathcal{LB}^{\leq n},p)$ is refutably inferable from complete data.*

Note that Shinohara has proved that $L(\mathcal{LB}^{\leq n},p)$ has finite elasticity, and thus it is inferable from positive data (cf. [12] and [13]). On the other hand, the class $\mathcal{L}$ consisting of all finite languages has infinite elasticity, but is inferable from positive data. This means that $\mathcal{L}$ is not contained in $L(\mathcal{LB}^{\leq n},p)$ for any $n \in N$. As shown in [9], $\mathcal{L}$ is not refutably inferable from complete data. In fact, as easily seen, any language $L \notin \mathcal{L}$ does not have *ftt* within $\mathcal{L}$. As mentioned in §2.2, every class containing $\mathcal{L}$ considered has M-finite thickness. However, such a class is not refutable inferable from complete data.

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