$$\begin{cases} u_{tt}=a^2(u_{xx}+u_{yy}+u_{zz}) \\ u|_{t=0}=\varphi(x,y,z) \\ u_t|_{t=0}=\phi(x,y,z) \end{cases}$$

$$u(M,t)=\frac{1}{4\pi a}\left[\frac{\partial}{\partial t}\iint_{S_M^{at}}\frac{\varphi(M')}{at}ds'+\iint_{S_M^{at}}\frac{\phi(M')}{at}ds'\right]$$

$S_M^{at}: (x'-x)^2+(y'-y)^2+(z'-z)^2=(at)^2$

$$u(x,t)=\frac{1}{2}(\varphi(x+at)+\varphi(x-at))+\frac{1}{2a}\int_{x-at}^{x+at}\phi(s)ds$$
$$=\frac{1}{2a}\left[\frac{\partial}{\partial t}\int_{x-at}^{x+at}\varphi(s)ds+\int_{x-at}^{x+at}\phi(s)ds\right]$$

$$\begin{cases} u_{tt}=a^2(u_{xx}+u_{yy}) \\ u|_{t=0}=\varphi(x,y) \\ u_t|_{t=0}=\phi(x,y) \end{cases}$$

利用降维法，$u(x,y,z)\to u(x,y,0)$，做2次投影

上半球面：$(x'-x)^2+(y'-y)^2\leq(at)^2$ $\quad z'-z=+\sqrt{(at)^2-(x'-x)^2-(y'-y)^2}$

$$\iint_{C_M^{at}}\frac{\phi(x',y',0)}{at}\sqrt{1+z_x^2+z_y^2}\,dx'dy'$$
$$=\iint_{C_M^{at}}\frac{\phi'(x',y',0)}{at}\cdot\frac{at}{\sqrt{(at)^2-(x'-x)^2-(y'-y)^2}}dx'dy'$$

下半球面同理

$$u(x,y,t)=\frac{1}{2\pi a}\left[\frac{\partial}{\partial t}\iint_{C_M^{at}}\frac{\varphi(x',y')}{\sqrt{(at)^2-(x'-x)^2-(y'-y)^2}}dx'dy'\right.$$
$$\left.+\iint_{C_M^{at}}\frac{\phi(x',y')}{\sqrt{(at)^2-(x'-x)^2-(y'-y)^2}}dx'dy'\right]$$

柱面波

齐次化原理 or 冲量原理

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + t\sin x, & -\infty < x < +\infty,\ t > 0 \\ u|_{t=0} = 0 & -\infty < x < +\infty \\ \frac{\partial u}{\partial t}\Big|_{t=0} = 0 \end{cases}$$

先求 $$\begin{cases} \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2}, & t > \tau,\ -\infty < x < +\infty \\ w|_{t=\tau} = 0 & -\infty < x < +\infty \\ \frac{\partial w}{\partial t}\Big|_{t=\tau} = \tau\sin x \end{cases}$$

$$u(x,t) = \int_0^t w(x,t;\tau)\,d\tau$$

$$F(x) = \int_a^x f(x,y)\,dy$$

$$\frac{dF}{dx} = f(x,y)\big|_{y=x} + \int_a^x \frac{\partial f}{\partial x}(x,y)\,dy$$

$$\frac{dF}{dx} = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(x+\Delta x, y)\,dy - \int_a^x f(x,y)\,dy}{\Delta x}$$

Flourier变换

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \qquad e^{i\theta} = \cos\theta + i\sin\theta$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx \qquad n = 0, 1, 2, \cdots$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \qquad n = 1, 2, \cdots$$

$$C_n = \frac{a_n - ib_n}{2} = \frac{1}{l}\int_0^l f(x) e^{-\frac{n\pi x}{l} i}\,dx$$

$$C_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{l}\int_0^l f(x) e^{\frac{n\pi}{l} x i}\,dx = \overline{C_n}$$

记 $\frac{\pi}{l} = w$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n \frac{e^{inwx} + e^{-inwx}}{2} + b_n \frac{e^{inwx} - e^{-inwx}}{2i}\right)$$

$$=\sum_{n=0}^{+\infty}\left(\frac{a_n-ib_n}{2}e^{inwx}+\frac{a_n+ib_n}{2}e^{-inwx}\right)-\frac{a_0}{2}$$

$$=\sum_{n=0}^{+\infty}C_n e^{inwx}+\sum_{n=1}^{+\infty}C_{-n}e^{-inwx}$$

$$=\sum_{n=-\infty}^{+\infty}C_n e^{inwx}$$

$$f(x)=\sum_{n=-\infty}^{+\infty}C_n e^{inwx}\qquad C_n=\frac{1}{2l}\int_{-l}^{l}f(x)e^{-inwx}dx$$

$$f(x)=\sum_{n=-\infty}^{+\infty}\left[\frac{1}{2l}\int_{-l}^{l}f(x)e^{-iwx}dx\right]e^{iw_nx}\qquad \Delta w=w_n-w_{n-1}=\frac{\pi}{l}=w$$

要求 $l\to\infty \Longleftrightarrow f(x)$绝对收敛

$$=\int_{-\infty}^{+\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}f(x)e^{-i\tilde{w}x}dx\right)e^{i\tilde{w}x}d\tilde{w}$$

像函数

$$\hat{f}(w)=\int_{-\infty}^{+\infty}f(x)e^{-i\tilde{w}x}dx$$

像元函数

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(w)e^{iwx}dw$$

复变函数积分实虚分开

$$\int_{-\infty}^{+\infty}|f(x)|dx<+\infty$$

$$\int_{-\infty}^{+\infty}u_{tt}e^{-ixw}dx=\left(\int_{-\infty}^{+\infty}ue^{-iwx}dx\right)_{tt}$$

①线性性:

$$[C_1f+C_2g]^\wedge=C_1\hat{f}+C_2\hat{g}$$

②导数:

$$(f^{(n)})^\wedge=\int_{-\infty}^{+\infty}f^{(n)}(x)e^{iwx}dx$$

$$= \int_{-\infty}^{+\infty} e^{-iwx} df^{(n-1)}$$

$$= -\int_{-\infty}^{+\infty} f^{(n-1)}(x) e^{-iwx}(-iw)dx$$

$$= (iw)^{(n)} \hat{f}(w)$$

③ $$[\hat{f}(w)]_w^{(n)} = \left(\int_{-\infty}^{+\infty} f(x) e^{-iwx} dx\right)_w^{(n)}$$

$$= \int_{-\infty}^{+\infty} (-ix)^n f(x) e^{-iwx} dx$$

$$[x^n f(x)]^\wedge = [i \hat{f}(w)]^{(n)}$$

④卷积： $$f_1(x) * f_2(x) = \int_{-\infty}^{+\infty} f_1(t) f_2(x-t) dt$$

$$= \int_{-\infty}^{+\infty} f_1(x-u) f_2(u) du$$ $x-t=u$

$$(f_1 * f_2)^\wedge = \hat{f_1} \cdot \hat{f_2}$$

$$(f_1 \cdot f_2)^\wedge = \hat{f_1} * \hat{f_2}$$

$$\begin{cases} u_t = a u_{xx} + f(x,t) & -\infty < x < +\infty,\ t > 0 \\ u|_{t=0} = \varphi(x) \end{cases}$$

① $$\int_{-\infty}^{+\infty} u_t e^{-iwx} dx = a^2 \int_{-\infty}^{+\infty} u_{xx} e^{-iwx} dx + \int_{-\infty}^{+\infty} f(x,t) e^{-iwx} dx$$

$$\hat{u}_t(w,t) = -a^2 w^2 \hat{u}(w,t) + \hat{f}(w,t)$$

$$\hat{u}(w,0) = \hat{\varphi}(w)$$

$$\hat{u}(w,t)$$

$$(\hat{u} e^{a^2 w^2 t})' = \hat{f} \cdot e^{a^2 w^2 t}$$

$$\hat{u} = \left(\int_0^t \hat{f} e^{a^2 w^2 t} dt + \hat{\varphi}(w)\right) e^{-a^2 w^2 t}$$

$$\hat{u}(w,t)=\int_0^t \hat{f}(w,\tau)e^{-a^2w^2(t-\tau)}d\tau+\hat{\varphi}(w)e^{-a^2w^2t}$$

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\int_0^t \hat{f}(w,\tau)e^{-a^2w^2(t-\tau)}d\tau\right]e^{iwx}dw$$

$$+\varphi(x)*F^{-1}(e^{-a^2wt})$$

$$F^{-1}(e^{-a^2w^2t})=\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-a^2w^2t}e^{iwx}dw$$

$$=\frac{1}{2a\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}}$$ 热核

三、(2)(3)

四、