数学物理方法第11次作业

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4.1 证明：假设 $u,v$ 在 $D$ 内二次连续可微

在平面上，有

$$\iint_D \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dxdy=\int_C (P\cos\alpha+Q\cos\beta)ds$$

令 $P=u\frac{\partial v}{\partial x}$，$Q=u\frac{\partial v}{\partial y}$

则有 $$\iint_D \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+u\frac{\partial^2 v}{\partial x^2}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+u\frac{\partial^2 v}{\partial y^2}\,dxdy=\int_C \left(u\frac{\partial v}{\partial x}\cos\alpha+u\frac{\partial v}{\partial y}\cos\beta\right)ds$$

$$\iint_D u\Delta v\,dxdy+\iint_D \mathrm{grad}u\cdot\mathrm{grad}v\,dxdy=\int_C u\frac{\partial v}{\partial n}ds$$

同理，令 $P=v\frac{\partial u}{\partial x}$，$Q=v\frac{\partial u}{\partial y}$

得 $$\iint_D v\Delta u\,dxdy+\iint_D \mathrm{grad}u\cdot\mathrm{grad}v\,dxdy=\int_C v\frac{\partial u}{\partial n}ds$$

两式相减，则有

$$\iint_D (v\Delta u-u\Delta v)\,dxdy=\int_C \left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)ds$$

从而得证

4.2 证明：$u=\ln\frac{1}{\rho}$，$\rho=\sqrt{(x-x_0)^2+(y-y_0)^2}$

$$\frac{\partial \rho}{\partial x}=\frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}}\qquad \frac{\partial \rho}{\partial y}=\frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

$$\frac{\partial^2 \rho}{\partial x^2}=\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}-(x-x_0)\cdot\frac{\partial \rho}{\partial x}}{(x-x_0)^2+(y-y_0)^2}\qquad \frac{\partial^2 \rho}{\partial y^2}=\frac{\sqrt{(x-x_0)^2+(y-y_0)^2}-(y-y_0)\cdot\frac{\partial \rho}{\partial y}}{(x-x_0)^2+(y-y_0)^2}$$

从而 $\frac{\partial \rho}{\partial x}=\frac{1}{\rho}(x-x_0)$，$\frac{\partial \rho}{\partial y}=\frac{1}{\rho}(y-y_0)$

$$\frac{\partial^2 \rho}{\partial x^2}=\frac{1}{\rho}-\frac{1}{\rho^3}(x-x_0)^2\qquad \frac{\partial^2 \rho}{\partial y^2}=\frac{1}{\rho}-\frac{1}{\rho^3}(y-y_0)^2$$

而 $u_x=-\frac{1}{\rho}\cdot\frac{\partial \rho}{\partial x}$，$u_y=-\frac{1}{\rho}\frac{\partial \rho}{\partial y}$，

$$u_{xx}=\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\right)^2-\frac{1}{\rho}\frac{\partial^2 \rho}{\partial x^2},\quad u_{yy}=\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial y}\right)^2-\frac{1}{\rho}\frac{\partial^2 \rho}{\partial y^2}$$

$$u_{xx}+u_{yy}=\frac{1}{\rho^2}\left[\left(\frac{\partial \rho}{\partial x}\right)^2+\left(\frac{\partial \rho}{\partial y}\right)^2\right]-\frac{1}{\rho}\left(\frac{\partial^2 \rho}{\partial x^2}+\frac{\partial^2 \rho}{\partial y^2}\right)$$

$$=\frac{1}{\rho^2}-\frac{1}{\rho}\cdot\frac{1}{\rho}=0$$，即 $\Delta u=0$，从而 $u=\ln\frac{1}{\rho}$ 满足二维Laplace方程，得证