$u_{tt}=a^2u_{xx}$

$$\begin{cases} u|_{x=0}=u|_{x=l}=0 \\ u|_{t=0}=\varphi(x) \quad u_t|_{t=0}=\phi(x) \end{cases}$$

; $u_x|_{x=0}=0 \quad u_x|_{x=l}=0$

$u_t=a^2u_{xx}$

$$\begin{cases} u|_{x=0}=0,\ u|_{x=l}=0 \\ u|_{t=0}=\varphi(x) \end{cases}$$

$u_{xx}+u_{yy}=f(x,y)$ Laplace方程

$u_{xx}+u_{yy}=0$ 泊松方程

两个算符: $\frac{df}{dx}=Df$

$$\frac{\partial^2 u}{\partial t^2}-a^2\frac{\partial^2 u}{\partial x^2}$$

$$=\left(\frac{\partial^2}{\partial t^2}-a^2\frac{\partial^2}{\partial x^2}\right)u=Lu$$

L为线性运算符(容易证明: $\begin{cases} L(u+v)=Lu+Lv \\ L(cu)=cL(u) \end{cases}$

叠加原理: 只有级数收敛,才能满足能求导

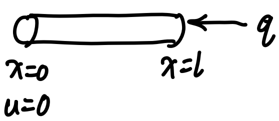
$$f(x)=\sum_{n=1}^{\infty}b_n\sin nx$$

$$b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx$$

$$\Rightarrow \int_{-\pi}^{\pi}f(x)^2\,dx=\int_{-\pi}^{\pi}\left(\sum b_n\sin nx\right)^2dx$$

$$=\pi\sum_{n=1}^{+\infty}b_n^2 \quad \begin{cases} b_n\to 0 \\ n\to+\infty \end{cases}$$

课后习题1:



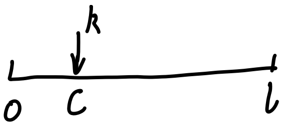
$$u_t = a^2 u_{xx} \quad 0<x<l,\ t>0$$

$$\begin{cases} u|_{x=0}=0 \\ ku_x|_{x=l}=q \\ u|_{t=0}=\frac{x(l-x)}{2},\ 0<x<l \end{cases}$$

温度—热量流动公式

$dQ=q$

习题2:



$$u_{tt}=a^2u_{xx}$$

$$\begin{cases} u|_{x=0}=0,\ u|_{x=l}=0,\ t>0 \\ u|_{t=0}=0 \qquad u_t|_{t=0}=\begin{cases}0, & |x-c|>\frac{\delta}{2} \\ \frac{k}{\rho\delta}, & |x-c|\le\frac{\delta}{2}\end{cases} \end{cases}$$

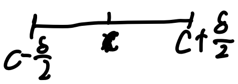
$\delta\to0^+$

$mv=ft=k$

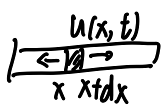
$\downarrow$

微元 $v=\frac{k}{\rho\delta}$

$|x-c|<\frac{\delta}{2}$



习题3:

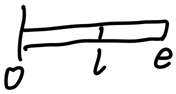


$$Eu_x(x+dx)-Eu_x(x)=mu_{tt}=\rho dx\, u_{tt}$$

$$Eu_{xx}(x+dx)dx=\rho dx\, u_{tt}$$

$$\frac{E}{\rho}u_{xx}=u_{tt} \qquad u_{tt}=a^2u_{xx}$$

习题4:



$$u_{tt} = a^2 u_{xx}$$

$$\begin{cases} u|_{x=0} = 0, \quad u_x|_{x=l} = 0 \\ u|_{t=0} = \frac{e}{l}x, \quad u_t|_{t=0} = 0 \end{cases}$$

分离变量法: $u_{tt} = a^2 u_{xx}, \ t > 0, \ 0 < x < l$

$$\begin{cases} u|_{x=0} = 0, \ u|_{x=l} = 0, \ t > 0 \\ u|_{t=0} = \varphi(x), \ u_t|_{t=0} = \psi(x), \ 0 < x < l \end{cases}$$

(1) $u(x,t) = Z(x)T(t)$ (驻波) 非0且不恒为0的解

(2) $Z T'' = a^2 Z'' T$ $\qquad \dfrac{Z''(x)}{Z(x)} = \dfrac{T''(t)}{a^2 T(t)} \ (= -\lambda)$

(3) $\begin{cases} Z'' + \lambda Z = 0, \ 0 < x < l \\ Z(0) = Z(l) = 0 \end{cases}$ $\qquad T'' + a^2 \lambda T = 0$

$\lambda = 0$, $Z''(x) = 0$, $Z(x) = C_1 x + C_2 \Rightarrow C_1 = C_2 = 0$ ✗

$\lambda < 0$, $Z'' = -\lambda Z = s^2 Z \Rightarrow C_1 = C_2 = 0$ ✗

$\lambda = -s^2$

$Z(x) = C_1 e^{sx} + C_2 e^{-sx}$

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 e^{sl} + C_2 e^{-sl} = 0 \end{cases}$$

$\lambda > 0$, $Z''(x) = -s^2 Z(x)$ $s = \sqrt{\lambda}$

$\lambda = s^2$

$Z(x) = C_1 \cos(sx) + C_2 \sin(sx)$

eigenequation $\begin{cases} Z(0) = 0, \Rightarrow C_1 = 0 \\ Z(l) = 0, \Rightarrow \sqrt{\lambda}\, l = k\pi,\ k \in Z^+ \end{cases}$

$\lambda_k = \frac{1}{l}(k\pi)^2$

$Z_k(x) = C \sin \frac{k\pi x}{l}$

$T'' + a^2 \lambda T = 0$

$T_k''(t) + \left(\frac{ak\pi}{l}\right)^2 T_k(t) = 0$

$T_k(t) = C_k \cos \frac{ak\pi t}{l} + D_k \sin \frac{ak\pi t}{l}$

(4) $u_k(x,t) = \left(C_k \cos \frac{ak\pi t}{l} + D_k \sin \frac{ak\pi t}{l}\right) \sin \frac{n\pi x}{l}$

(5) $\sum_{k=1}^{\infty} u_k(x,t) = u(x,t)$ $\varphi(x) = \sum_{k=1}^{\infty} C_k \sin \frac{k\pi x}{l}$

$\phi(x) = \sum_{k=1}^{\infty} \left(D_k \frac{ak\pi t}{l}\right) \sin \frac{k\pi x}{l}$

$C_k = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{k\pi x}{l}\, dx$

$D_k \frac{ak\pi}{l} = \frac{2}{l} \int_0^l \phi(x) \sin \frac{k\pi x}{l}\, dx$

(1) $u(x,t)=Z(x)T(t)$ 找单频波

(2) $\begin{cases} Z''(x)+\lambda Z(x)=0 \\ Z(0)=Z(l)=0 \end{cases}$

(3) $\lambda=\lambda_k=\left(\frac{k\pi}{l}\right)^2,\ k=1,2,\cdots$ $\quad Z_k(x)=\sin\frac{k\pi x}{l}$

$T_k(t)=C_k\cos\frac{ak\pi t}{l}+D_k\sin\frac{ak\pi t}{l}$

(4) $u_k=Z_kT_k$

(5) $u=\sum_{k=1}^{\infty}u_k=\sum_{k=1}^{\infty}\left(C_k\cos\frac{ak\pi t}{l}+D_k\sin\frac{ak\pi t}{l}\right)\sin\frac{k\pi x}{l}$

$C_k=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{k\pi x}{l}dx$

$D_k=\frac{2}{ak\pi}\int_0^l\varphi(x)\sin\frac{n\pi x}{l}dx$

Flourier级数:

$u_{tt}=a^2u_{xx},\ 0<x<10,\ t>0$

$\begin{cases} u|_{x=0}=u|_{x=10}=0 \\ u|_{t=0}=\frac{x(10-x)}{1000},\ u_t|_{t=0}=0 \end{cases}$

$A_k=\int_0^l x\sin\frac{k\pi x}{l}dx$ $\quad B=\int_0^l x^2\sin\frac{k\pi x}{l}dx$

$=\int_0^l x\,d\cos\left(\frac{k\pi x}{l}\right)\left(-\frac{l}{k\pi}\right)$

$$= -\frac{l}{k\pi}\left[x\cos\left(\frac{k\pi x}{l}\right)\Big|_0^l - \int_0^l \cos\left(\frac{k\pi x}{l}\right)dx \right]$$

$$= -\frac{l^2}{k\pi}(-1)^k = \frac{l^2}{k\pi}(-1)^{k-1}$$

$$B_k = \frac{l^3}{k\pi}(-1)^{k-1} - \frac{2l^2}{(k\pi)^2}\int_0^l \sin\frac{k\pi x}{l}\,dx$$

作业:
2.(1)

$$= \frac{l^3}{k\pi}(-1)^{k-1} + \frac{2l^3}{(k\pi)^3}\left((-1)^k - 1\right)$$