数学物理方法第12次作业

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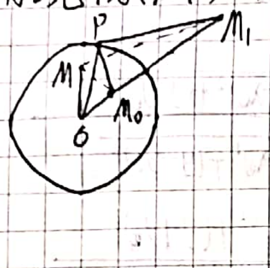
5. 证明：在圆域内，格林函数通式为 $G(M,M_0)=\frac{1}{2\pi}\ln\frac{1}{r_{MM_0}}-v$

其中，v应满足 $\begin{cases}\Delta v=0 & 在\Omega内 \\ v|_\Gamma=\frac{1}{2\pi}\ln\frac{1}{r_{MM_0}} & M\in\Gamma\end{cases}$ ①

利用镜像法，作反演点M_1，令M_0处电荷为+1



则有 $\frac{1}{M_0P}=\frac{q}{M_1P}$ $q=\frac{M_1P}{M_0P}$

又因为在圆内 $OM_0\cdot OM_1=R^2$

$$\frac{OM_0}{R}=\frac{R}{OM_1}$$

于是有 $\triangle OM_0P\sim\triangle OPM_1$ (边角边)

从而 $q=\frac{OP}{OM_0}=\frac{R}{OM_0}$

于是在M_1处电荷为$+\frac{R}{OM_0}$，$v=\frac{1}{2\pi}\ln\frac{R}{\rho_0}\frac{1}{r_{MM_1}}$，满足①式

从而其格林函数为 $G(M,M_0)=\frac{1}{2\pi}\ln\frac{1}{r_{MM_0}}-\frac{1}{2\pi}\ln\frac{R}{\rho_0}\frac{1}{r_{MM_1}}$

记 $OM_0=\rho_1$，$OM_1=\rho_1$，$OM=\rho$，$\angle MOM_0=\gamma$

由余弦定理得 $r_{M_0M}=\sqrt{\rho_0^2+\rho^2-2\rho_0\rho\cos\gamma}$

$$r_{M_1M}=\sqrt{\rho_1^2+\rho^2-2\rho_1\rho\cos\gamma}$$

且 $\rho_0\rho_1=R^2$

那么 $G(M,M_0)=\frac{1}{2\pi}\ln\frac{1}{\sqrt{\rho_0^2+\rho^2+2\rho_0\rho\cos\gamma}}-\frac{1}{2\pi}\ln\frac{R}{\rho_0}\frac{1}{\sqrt{\left(\frac{R^2}{\rho_0}\right)^2+\rho^2-2\frac{R^2}{\rho_0}\rho\cos\gamma}}$

$$=\frac{1}{2\pi}\left(\ln\frac{1}{\sqrt{\rho_0^2+\rho^2+2\rho_0\rho\cos\gamma}}-\ln\frac{R}{\sqrt{R^4+\rho^2\rho_0^2-2R^2\rho_0\rho\cos\gamma}}\right)$$

$$\frac{\partial G}{\partial n}\Big|_\Gamma=\frac{\partial G}{\partial\rho}\Big|_{\rho=R}=\frac{1}{2\pi R}\cdot\frac{\rho_0^2-R^2}{\rho_0^2+R^2-2\rho_0R\cos\gamma}$$

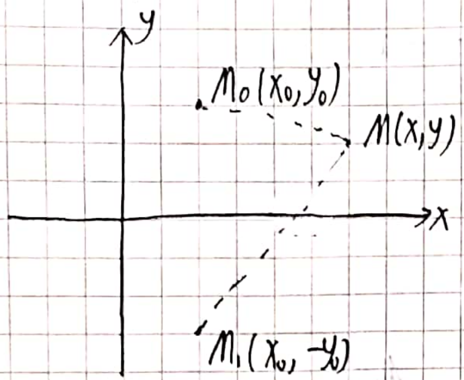
而 $\gamma=\angle MOx-\angle M_0Ox$，可记为 $\gamma=\theta-\theta_0$

从而狄利克雷问题的解为

$$u(\rho_0,\theta_0)=\frac{1}{2\pi}\int_0^{2\pi}\varphi(\theta)\frac{\rho_0^2-R^2}{R^2+\rho_0^2-2\rho_0R\cos\gamma}d\theta$$

$\varphi(\theta)$为边界条件

6. 解:
$$\begin{cases}\Delta u=0 & -\infty<x<+\infty,\ y>0\\ u|_{y=0}=\varphi(x) & -\infty<x<+\infty\end{cases}$$



$$G(M,M_0)=\frac{1}{2\pi}\ln\frac{1}{r_{MM_0}}-\frac{1}{2\pi}\ln\frac{1}{r_{MM_1}}$$

$$r_{MM_0}=\sqrt{(x-x_0)^2+(y-y_0)^2}$$

$$r_{MM_1}=\sqrt{(x-x_0)^2+(y+y_0)^2}$$

$$G(x,y)=\frac{1}{2\pi}\ln\frac{\sqrt{(x-x_0)^2+(y+y_0)^2}}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$$

$$=\frac{1}{4\pi}\ln\frac{(x-x_0)^2+(y+y_0)^2}{(x-x_0)^2+(y-y_0)^2}$$

$$\frac{\partial G}{\partial \vec{n}}\Big|_T=\frac{\partial G}{\partial y}\Big|_{y=0}=\left[\frac{2(y+y_0)}{(x-x_0)^2+(y+y_0)^2}-\frac{2(y-y_0)}{(x-x_0)^2+(y-y_0)^2}\right]\frac{1}{4\pi}\Big|_{y=0}$$

$$=\frac{1}{\pi}\frac{y_0}{(x-x_0)^2+y_0^2}$$

从而其解为

$$u(x_0,y_0)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{y_0}{(x-x_0)^2+y_0^2}\varphi(x)\,dx$$

即 $$u(x,y)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{y}{(x-\xi)^2+y^2}\varphi(\xi)\,d\xi$$