$$u(M_0)=-\oint_{\partial\Omega}\left(u(M)\frac{\partial}{\partial n}\left(\frac{1}{4\pi r_{M_0M}}\right)-\frac{1}{4\pi r_{M_0M}}\frac{\partial}{\partial n}u\right)dS_M$$

双层位势　　单层位势

调和　　调和

$$\begin{cases}\Delta u=0\\ u|_{\partial\Omega}=f\end{cases}\qquad u=-\oint_{\partial\Omega}\frac{\partial G}{\partial n}f\,dS$$

$$\begin{cases}\Delta u=0\\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega}=g\end{cases}\qquad u=\oint_{\partial\Omega}G\cdot g\,dS$$

$$\begin{cases}\Delta u=f\\ u|_{\partial\Omega}=0\end{cases}\qquad u=-\int_{\Omega}G\cdot f\,dV$$

$$\begin{cases}u_{xx}+u_{yy}+u_{zz}=0, & x^2+y^2+z^2<R^2\\ u(x,y,z)=f(x,y,z) & x^2+y^2+z^2=R^2\end{cases}$$

$$G=\frac{1}{4\pi r_{M_0M}}-\frac{R/\rho_0}{4\pi r_{M_1M}}$$

$$u=-\oiint_{S_R}\frac{\partial G}{\partial n}f\,dS\qquad \frac{\partial G}{\partial r}\Big|_{r=R}$$

不好积，球坐标分离变量法

$$\begin{cases}x=r\cos\varphi\sin\theta & 0\leqslant r<R\\ y=r\sin\varphi\sin\theta & 0\leqslant\varphi\leqslant2\pi\\ z=r\cos\theta & 0\leqslant\theta\leqslant\pi\end{cases}$$

$$u_{xx}+u_{yy}+u_{zz}$$

$$x=\rho\cos\varphi,\ y=\rho\sin\varphi$$

$$u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\varphi\varphi}+u_{zz}=0$$

$$\rho=r\sin\theta,\quad z=r\cos\theta$$

$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}+\frac{1}{r^2\sin^2\theta}u_{\varphi\varphi}+\frac{1}{r\sin\theta}u_\rho$$

$$u_\rho \longrightarrow u_r + u_\theta$$

$$u_r \cdot \frac{\rho}{r} \qquad u_\theta \cdot \frac{\frac{1}{z}}{1+(\frac{\rho}{z})^2}$$

(1) $\Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + \frac{\cos\theta}{r^2\sin\theta}u_\theta + \frac{1}{r^2\sin^2\theta}u_{\varphi\varphi} = 0$

$$= \frac{1}{r^2}\left(r^2 u_{rr} + 2r u_r + u_{\theta\theta} + \frac{\cos\theta}{\sin\theta}u_\theta + \frac{1}{\sin^2\theta}u_{\varphi\varphi}\right)$$

$$= \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}u_{\varphi\varphi}$$

(2) $u(r,\varphi,\theta) = R(r)\Phi(\varphi)\Theta(\theta)$

$$\Phi\Theta(r^2R')' + \frac{1}{\sin\theta}R\Phi(\sin\theta\,\Theta')' + \frac{1}{\sin^2\theta}R\Theta\Phi'' = 0$$

$$\frac{1}{R}[r^2R']' = -\frac{1}{\sin\theta}\cdot\frac{1}{\Theta}[\sin\theta\,\Theta']' - \frac{1}{\sin^2\theta}\frac{\Phi''}{\Phi} = \lambda$$

(3) $r^2R'' + 2R' - \lambda R = 0 \qquad \lambda = l(l+1) \qquad$ 欧拉方程 l不一定为整数

$$R_l(r) = C_1 r^l + C_2 r^{-l-1}$$

(4) $\frac{\Phi''}{\Phi} = -\frac{\sin\theta}{\Theta}[\sin\theta\,\Theta']' - \lambda\sin^2\theta = -\mu$

$\Phi'' + \mu\Phi = 0 \qquad \mu_m = m^2 \qquad \Phi_m(\varphi) = A_m\cos m\varphi + B_m\sin m\varphi$

$$-\sin\theta[\sin\theta\,\Theta']' - \lambda\Theta\sin^2\theta = -m^2\Theta$$

$$\sin\theta\,\Theta'' + \cos\theta\,\Theta'$$

$$\sin^2\theta\,\Theta'' + \sin\theta\cos\theta\,\Theta' + \lambda\Theta\sin^2\theta - m^2\Theta = 0$$

$$\Theta'' + \cot\theta\,\Theta' + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right)\Theta = 0$$

$\Theta'_\theta = \Theta'_x x_\theta = \Theta'_x\cdot(-\sin\theta)$

$\Theta'' = [\Theta'_x\cdot(-\sin\theta)]'_x\, x'_\theta$

$= [\Theta''_x(-\sin\theta) + \Theta'_x\cdot(-\sin\theta)'_\theta \frac{1}{x'_\theta}]$

$(-\sin\theta)$

设 $\cos\theta = x \qquad \Theta(\theta) \xlongequal{x=\cos\theta} P(x) \quad P(\cos\theta)$

$$\Theta''_x \sin^2\theta - 2\cos\theta\,\Theta'_x + \left(l(l+1) - \frac{m^2}{1-\cos^2 x}\right)\Theta = 0$$

$$= \Theta''_x \sin^2\theta - \cos\theta\,\Theta'_x$$

$$(1-x^2)P'' - 2xP' + \left(l(l+1) - \frac{m^2}{1-x^2}\right)P = 0$$

$$u(r,\theta,\varphi) = \sum_{m,l} C_l r^l [A_m \cos m\varphi + B_m \sin m\varphi] P_l^m(\cos\theta)$$

考虑 $m=0$ $\quad (1-x^2)P'' - 2xP' + l(l+1)P = 0 \quad -1 \leq x \leq 1$

$Lp = \lambda p$

勒让德多项式

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$$ 罗德里格表示

$$\longrightarrow \quad (1-x^2)^{\frac{m}{2}}\frac{d^m}{dx^m}P(x)$$ legendre方程

$$(1-x^2)P''(x) - 2xP'(x) + n(n+1)P(x) = 0$$

(1) $|x|<1$

(2) n 不一定是整数

设 $P(x) = x^c[a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k + \cdots]$

$$= \sum_{k=0}^{+\infty} a_k x^{k+c} \qquad a_0 \neq 0$$

$$-x^2 P'' - 2xP' + n(n+1)P + P'' = 0$$

$$[-(k+c)(k+c-1) - 2(k+c) + n(n+1)]a_k + (k+c+2)(k+c+1)a_{k+2} = 0$$

$k = 0, 1, 2, \cdots$

$x^{c-2} \quad a_0 c(c-1) = 0 \quad c=0 \quad c=1 \quad a_0 \neq 0$

$x^{c-1} \quad a_1 c(c-1) = 0 \quad c=0 \quad c=-1 \quad a_1 \neq 0$

$c=0$ $\quad a_{k+2} = \dfrac{k(k+1)-n(n+1)}{(k+2)(k+1)} a_k$ $\quad$ n整，有限项多项式

$(k-n)(k+n-1)$

a_0, a_1给定，则全系数可解

$$a_2 = \frac{-n(n-1)}{1\cdot 2} a_0$$

$$a_4 = \frac{(2-n)(n+1)}{3\cdot 4} a_2 = \frac{n(n-2)(n-1)(n+1)}{1\cdot 2\cdot 3\cdot 4} a_0$$

$$a_{2k} = (-1)^k \frac{n(n-2)\cdots(n-2k+2)(n+1)(n+3)\cdots(n+2k-1)}{(2k)!} a_0$$

$$a_3 = \frac{-n(n-1)}{2\cdot 3} a_1$$

$$a_5 = \frac{-(n-3)(n+2)}{4\cdot 5} a_3 = \frac{(n-1)(n-3)n(n+2)}{5!}$$

$$a_{2k+1} = (-1)^k \frac{(n-1)(n-3)\cdots(n-2k+1)n(n+2)\cdots(n+2k)}{(2k+1)!} a_1$$

$$y_1 = \sum_{k=0}^{+\infty} a_{2k} x^{2k} \qquad y_2 = \sum_{k=0}^{+\infty} a_{2k+1} x^{2k+1}$$

方程+$\begin{cases} y(0)=a_0 \\ y'(0)=0 \end{cases}$ $\qquad$ 方程+$\begin{cases} y(0)=0 \\ y'(0)=a_1 \end{cases}$

$c=0$ $\quad P(x) = a_0 y_1 + a_1 y_2 \qquad (c=\pm 1)$

$$y_1 = \sum_{k=0}^{\infty} (-1)^k \frac{n(n-2)\cdots(n-2k+2)(n+1)\cdots(n+2k-1)}{(2k)!} x^{2k}$$

$$y_2 = \sum_{k=0}^{\infty} (-1)^k \frac{(n+1)(n-1)(n-2k+1)\cdot(n+2)\cdots(n+2k)}{(2k+1)!} x^{2k+1}$$

$W(y_1, y_2) = 1 \qquad$ 收敛半径：$\left|\dfrac{a_{k+2}}{a_k}\right| \xrightarrow{k\to\infty} 1$

$|x|<1$

$c=1 \quad a_{k+2} = \frac{(k+1)(k+2)-n(n+1)}{(k+2)(k+3)} a_k \quad a_1=0 \quad y_1$

$c=-1 \quad a_0=0 \quad y_2$

假设n是正整数 $k=0,1,2,\cdots,n-1$

$$a_{n-2} = \frac{-n(n-1)}{2n-1} a_n$$

$$a_k = -\frac{(k+1)(k+2)}{(n-k)(n+k+1)} a_{k+2}$$

取$k+2=n$

$$a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n$$

$$a_{n-4} = \cdots \frac{(n-2)(n-3)}{4(2n-3)} \cdot \cdots \cdot \frac{n(n-1)}{2(2n-1)} a_n$$

$$a_n = \frac{1}{2^n n!}$$

$$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}$$

$$a_{n-2} = (-1)\frac{(2n-2)!}{2^n (n-1)!(n-2)!} = (-1)\frac{1}{2^n n!}\frac{(2n-2)!}{(n-2)!}$$

$$y_1 = \sum_{m=0}^{\frac{n}{2}} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \text{ (偶)}$$

只能得到1个多项式

$$y_2 = \sum_{m=0}^{\frac{n}{2}} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m} \text{ (奇)}$$

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{x^{n-2m}(2n-2m)!}{2^n m!(n-m)!(n-2m)!}$$

$$= \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \qquad P_n(1)=1,\ P_n(-1)=(-1)^n$$

作业:六:1.