数学物理方法14

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2. 证明：(1) $x^2=\frac{2}{3}P_2(x)+\frac{1}{3}P_0(x)$

由前述推导，可得 $P_2(x)=\sum_{m=0}^{1}(-1)^m\frac{(4-2m)!\,x^{2(1-m)}}{2^2 m!(2-m)!(2-2m)!}$

$$=\frac{24\cdot x^2}{4\times2\times2}-\frac{2!\,x^0}{4\times1!\cdot1!\cdot0!}$$

$$=\frac{3}{2}x^2-\frac{1}{2}$$

$$P_0(x)=\frac{1}{2^0\,0!}=1$$

提 $x^2=\frac{2}{3}(\frac{3}{2}x^2-\frac{1}{2})+\frac{1}{3}=x^2$

左边=右边，得证

(2) $x^3=\frac{2}{5}P_3(x)+\frac{3}{5}P_1(x)$

同理 $P_3(x)=\frac{1}{4}(10x^3-3x-1)$

$P_1(x)=\frac{1}{2}x+\frac{2}{3}$

从而 $\frac{2}{5}P_3(x)+\frac{3}{5}P_1(x)=x^3-\frac{3}{10}x-\frac{2}{5}+\frac{3}{10}x+\frac{2}{5}$

$$=x^3$$

左边=右边，得证

3. 解：(1) $\int_0^1 xP_5(x)=\int_0^1 x\cdot\frac{1}{2^5 5!}\frac{d^5}{dx^5}(x^2-1)^5dx$

$$=\frac{1}{2^5 5!}\int_0^1 x\frac{d}{dx}\left[\frac{d^4}{dx^4}(x^2-1)^5\right]dx$$

$$=\frac{1}{2^5 5!}\left[x\frac{d^4}{dx^4}(x^2-1)^5\Big|_0^1-\int_0^1\frac{d^4}{x^4}(x^2-1)^5dx\right]$$

$$=0$$

(2) $\int_{-1}^1[P_2(x)]^2dx=\left(\frac{1}{2^2 2!}\right)^2\int_{-1}^1\frac{d^2}{dx^2}(x^2-1)^2\cdot\frac{d^2}{dx^2}(x^2-1)^2dx$

$$=\left(\frac{1}{2^2 2!}\right)^2\left[\frac{d}{dx}(x^2-1)^2\cdot\frac{d^2}{dx^2}(x^2-1)^2\Big|_{-1}^1-\int_{-1}^1\frac{d}{dx}(x^2-1)^2\cdot\frac{d^3}{dx^3}(x^2-1)^2dx\right]$$

$$=\left(\frac{1}{2^2 2!}\right)^2\left[-(x^2-1)^2\frac{d^3}{dx^3}(x^2-1)^2\Big|_{-1}^1+\int_{-1}^1(x^2-1)^2\frac{d^4}{dx^4}(x^2-1)^2dx\right]$$

$$=\frac{1}{64}\int_{-1}^1 4!\cdot(x^2-1)^2dx=\frac{2}{5}$$

(3) $\int_{-1}^{1} P_2(x)P_4(x)dx = \int_{-1}^{1} P_2(x)\cdot\frac{1}{2^4 4!}\frac{d^4}{dx^4}(x^2-1)^4dx$

$$= \frac{1}{2^4 4!}\left[P_2(x)\frac{d^3}{dx^3}(x^2-1)^4\Big|_{-1}^{1} - \int_{-1}^{1}\frac{d}{dx}P_2(x)\cdot\frac{d^3}{dx^3}(x^2-1)^4dx\right]$$

$$= \frac{1}{2^4 4!}\left[-\frac{d}{dx}P_2(x)\cdot\frac{d^2}{dx^2}(x^2-1)^4\Big|_{-1}^{1} + \int_{-1}^{1}\frac{d^2P_2(x)}{dx}\cdot\frac{d^2}{dx^2}(x^2-1)^4dx\right]$$

$$= \frac{1}{2^4\cdot 4!}\left[\frac{d^2P_2(x)}{dx^2}\cdot\frac{d}{dx}(x^2-1)^4\Big|_{-1}^{1} - \int_{-1}^{1}\frac{d^3P_2(x)}{dx^3}\cdot\frac{d(x^2-1)^4}{dx}dx\right.$$

$$=0$$

8. 解: 在球面坐标系中,

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}$$

$$\begin{cases}\Delta u=0,\ 0<r<1\\ u|_{r=1}=3\cos2\theta+1,\ 0\le\theta\le\pi\\ |u|\,|_{r=0}<\infty \qquad 0\le\varphi\le2\pi\end{cases}$$

令 $u(r,\theta)=R(r)\Theta(\theta)$

代入方程 $$\frac{\frac{\partial}{\partial r}(r^2R')}{R} = \frac{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\Theta')}{\Theta} = \lambda$$

即 $$\begin{cases}r^2R''+2rR'-\lambda R=0\\ \Theta''+\cot\theta\,\Theta''+\lambda\Theta=0\end{cases}$$

令 $\lambda=n(n+1)$, $x=\cos\theta$, $\Theta(\arccos x)=P(x)$,

则得 $(1-x^2)\frac{d^2P}{dx^2}-2x\frac{dP}{dx}+n(n+1)P=0$

u 有界, 则在 $[0,\pi]$ 内 $\Theta(\theta)$ 也应有界, 即 $P(x)$ 在 $[-1,1]$ 上应有界

不妨取 n 为非负整数, 故 $P(x)=C_nP_n(x)+D_nQ_n(x)$

$$\Theta_n(\theta)=C_nP_n(\cos\theta)+D_nQ_n(\cos\theta)$$

由有界性 $D_n=0$, 即: $\Theta_n(\theta)=C_nP_n(\cos\theta)$

从而 $R_n(r)=A_nr^n+B_nr^{-(n+1)}$

当 $r\to0$ 时, $|R_n|$ 应保持有界, 故 $B_n=0$, 即 $R_n(r)=A_nr^n$

利用叠加原理, $u(r,\theta)=\sum_{n=0}^{\infty}C_nr^nP_n(\cos\theta)$

C_n 满足 $3\cos2\theta+1=\sum_{n=0}^{\infty}C_nP_n(\cos\theta)$

令 $t=\cos\theta$, $3\cos 2\theta+1=6t^2-2$

$$6t^2-2=\sum_{n=0}^{\infty}C_nP_n(t)$$

比较系数 $6t^2-2=C_0+C_2P_2(x)=C_0+C_2\cdot\frac{1}{2}(3x^2-1)$

$$C_0=0,\ C_2=4$$

从而，$u(r,\theta)=4r^2P_2(\cos\theta)=2r^2(3\cos^2\theta-1)$