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5. $u_t = a^2 u_{xx}$

$$\begin{cases} u|_{t=0} = x(l-x), \ 0 \le x \le l \\ u|_{x=0} = u|_{x=l} = 0, \ t>0 \end{cases}$$

(1) $u = Z(x)T(t)$

(2) $ZT' = a^2 Z'' T$ $\qquad Z(0)T(t) = 0, \ Z(l)T(t) = 0$

$$\frac{Z''}{Z} = \frac{T'}{a^2 T} = -\lambda \qquad Z(0) = 0, \ Z(l) = 0$$

(3) $\begin{cases} Z'' + \lambda Z = 0 \\ Z(0) = 0, \ Z(l) = 0 \end{cases}$

$\lambda = 0, \ Z = C_1 x + C_2 = 0 \Rightarrow C_1 = C_2 = 0$，舍去

$\lambda < 0, \ \lambda = -s^2, \ Z = C_1 e^{sx} + C_2 e^{-sx}$

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 e^{sl} + C_2 e^{-sl} = 0 \end{cases} \Rightarrow C_1 = C_2 = 0，舍去$$

$\lambda > 0, \ \lambda = s^2, \ Z = C_1 \cos sx + C_2 \sin sx$

$Z(0) = 0, \ C_1 = 0$

$Z(l) = 0, \ C_2 \sin\sqrt{\lambda}\, l = 0$

$\sqrt{\lambda}\, l = k\pi$

$\lambda_k = \left(\frac{k\pi}{l}\right)^2 \quad k = 1, 2, \cdots$

(4) $T' + a^2 \lambda T = 0$

$T = C_1 e^{-a^2 \lambda t}$

(5) $u_k = Z_k T_k = C_k e^{-a^2 \left(\frac{k\pi}{l}\right)^2 t} \sin\frac{k\pi}{l} x$

(6) $u(x,t)=\sum_{k=1}^{\infty} C_k e^{-a^2\left(\frac{k\pi}{l}\right)^2 t}\sin\frac{k\pi}{l}x$

(7) $\sum_{k=1}^{\infty} C_k \sin\frac{k\pi}{l}x = x(l-x)$

从而 $C_k=\frac{2}{l}\int_0^l x(l-x)\sin\frac{k\pi}{l}x\,dx$

$=\frac{2}{l}\int_0^l \left(lx\sin\frac{k\pi}{l}x - x^2\sin\frac{k\pi}{l}x\right)dx$

$=\frac{8l^2}{(2k-1)^3\pi^3}$, $k=1,2,\cdots$

(8) 从而 $u(x,t)=\sum_{k=1}^{\infty}\frac{8l^2}{(2k-1)^3\pi^3}e^{-a^2\left(\frac{(2k-1)\pi}{l}\right)^2 t}\sin\frac{(2k-1)\pi}{l}x$

6. $u_t = a^2 u_{xx}$

$$\begin{cases} u|_{t=0} = x \\ \dfrac{\partial u}{\partial x}\Big|_{x=0} = 0 \\ \dfrac{\partial u}{\partial x}\Big|_{x=l} = 0 \end{cases}$$

(1) $u = Z(x)T(t)$

(2) $ZT' = a^2 Z'' T$ $\qquad Z'(0)T(t)=0,\ Z'(l)T(t)=0$

$$\frac{T'}{a^2 T} = \frac{Z''}{Z} = -\lambda \qquad Z'(0)=0,\ Z'(l)=0$$

(3) $$\begin{cases} Z'' + \lambda Z = 0 \\ Z'(0)=0,\ Z'(l)=0 \end{cases}$$

$\lambda = 0$, $Z = C_1 x + C_2 \Rightarrow C_1 = 0$, C_2自由项，Z为常数 $\Rightarrow T$为常数 舍去

$\lambda < 0$, $\lambda = -s^2$, $Z = C_1 e^{sx} + C_2 e^{-sx}$

$$\begin{cases} C_1 s + C_2(-s) = 0 \\ C_1 s e^{sl} + C_2(-s)e^{-sl} = 0 \end{cases} \Rightarrow C_1 = C_2；\lambda自由，舍去$$

$\lambda > 0$, $\lambda = s^2$, $Z = C_1 \cos sx + C_2 \sin sx$

$Z'(0) = 0,\ C_2 = 0$

$Z'(l) = 0,\ -sC_1 \sin sl = 0,$

$\sqrt{\lambda}\, l = k\pi,\ k = 1, 2, \cdots$

$\lambda_k = \left(\frac{k\pi}{l}\right)^2$

$Z(x) = C_1 \cos \frac{k\pi}{l} x$

(4) $T' + a^2 \lambda T = 0$

$T = C_1 e^{-a^2 \lambda t}$

(5) $u_k = Z_k T_k = C_k e^{-a^2\left(\frac{k\pi}{l}\right)^2 t} \cos \frac{k\pi}{l} x$

(6) $u(x,t)=\sum_{k=1}^{\infty} u_k + \frac{C_0}{2}$

(7) $\sum_{k=1}^{+\infty} C_k \cos\frac{k\pi}{l}x = x$

$C_0=\frac{2}{l}\int_0^l x\,dx = l$

$$C_k=\frac{2}{l}\int_0^l x\cos\frac{k\pi}{l}x\,dx$$

$$=\frac{2}{l}\int_0^l x\,d\sin\frac{k\pi}{l}x\cdot\frac{l}{k\pi}$$

$$=\frac{2}{k\pi}\left(x\sin\frac{k\pi}{l}x\Big|_0^l-\int_0^l\sin\frac{k\pi}{l}x\,dx\right)$$

$$=\frac{2}{k\pi}\int_0^l d\cos\frac{k\pi}{l}x\cdot\frac{l}{k\pi}$$

$$=\frac{2l}{(k\pi)^2}\left((-1)^k-1\right)$$

$$=-\frac{4l}{(2k-1)^2\pi^2},\ k=1,2,\cdots$$

(8) 从而 $u(x,t)=\sum_{k=1}^{\infty}-\frac{4l}{(2k-1)^2\pi^2}e^{-a^2\left(\frac{(2k-1)\pi}{l}\right)^2t}\cos\frac{(2k-1)\pi}{l}x+\frac{l}{2}$