2019302130113 数学物理方法第8次作业

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1. 解:
$$\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = x^2 y, \quad x>1, \ y>0 \\ u|_{y=0} = x^2, \quad u|_{x=1} = \cos y \end{cases}$$

先求通解，为 $u(x,y) = \frac{1}{6}x^3y^2 + C_1(y) + C_2(x)$

$$u|_{y=0} = C_1(0) + C_2(x) = x^2$$

$$u|_{x=1} = \frac{1}{6}y^2 + C_1(y) + C_2(1) = \cos y$$

从而，有 $C_2(x) = x^2 - C_1(0)$

$$C_2(1) = 1 - C_1(0)$$

$$C_1(y) = \cos y - \frac{1}{6}y^2 - 1 + C_1(0)$$

$$u(x,y) = C_1(y) + C_2(x)$$

$$= x^2 + \cos y - \frac{1}{6}y^2 - 1$$

2. 解:
$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} + t\sin x, & -\infty < x < +\infty, t>0 \\ u|_{t=0} = 0, & -\infty < x < +\infty \\ \dfrac{\partial u}{\partial t}\Big|_{t=0} = \sin x, & -\infty < x < +\infty \end{cases}$$

对问题进行分解:

$$(\mathrm{I})\begin{cases} \dfrac{\partial^2 \alpha}{\partial t^2} = \dfrac{\partial^2 \alpha}{\partial x^2}, & -\infty < x < +\infty, t>0 \\ \alpha|_{t=0} = 0, & \\ \dfrac{\partial \alpha}{\partial t}\Big|_{t=0} = \sin x, & -\infty < x < +\infty \end{cases}
\qquad
(\mathrm{II})\begin{cases} \dfrac{\partial^2 \beta}{\partial t^2} = \dfrac{\partial^2 \beta}{\partial x^2} + t\sin x, & -\infty < x < +\infty, t>0 \\ \beta|_{t=0} = 0 & -\infty < x < +\infty \\ \dfrac{\partial \beta}{\partial t}\Big|_{t=0} = \sin x & \end{cases}$$

对于问题(I)，利用一维达朗贝尔公式，有

$$\alpha(x,t) = \frac{1}{2}\int_{x-t}^{x+t} \sin\xi \, d\xi$$

$$= \frac{1}{2}\left(\cos(x-t) - \cos(x+t)\right)$$

对于问题(II)，借助齐次化原理，

先求 $\begin{cases} \dfrac{\partial^2 w}{\partial t^2} = \dfrac{\partial^2 w}{\partial x^2}, & t>\tau, \ -\infty<x<+\infty \\ W|_{t=\tau} = 0 & -\infty<x<+\infty \\ \dfrac{\partial w}{\partial t}\Big|_{t=\tau} = \tau\sin x \end{cases}$

$$W(x,t;\tau) = \frac{1}{2}\int_{x-(t-\tau)}^{x+(t-\tau)} \tau\sin\eta \, d\eta$$

$$\beta(x,t) = \int_0^t W(x,t;\tau)\,d\tau$$

$$= \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} \tau\sin\xi \, d\xi \, d\tau$$

$$= \frac{1}{2}\int_0^t \tau\left(\cos(x-(t-\tau)) - \cos(x+t-\tau)\right) d\tau$$

$$= \frac{1}{2}\left[\tau\sin(x-(t-\tau))\Big|_0^t - \tau\sin(\tau-t-x)\Big|_0^t + \cos(x-(t-\tau))\Big|_0^t - \cos(\tau-t-x)\Big|_0^t\right]$$

$$= \frac{1}{2}\left(t\sin x - t\sin(-x) + \cos x - \cos(x-t) - \cos(-x) + \cos(x+t)\right)$$

$$= t\sin x + \frac{1}{2}\left(\cos(x+t) - \cos(x-t)\right)$$

$$u(x,t) = \alpha(x,t) + \beta(x,t) = t\sin x$$