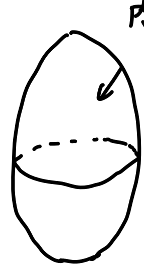
$$\begin{cases}\Delta u = f \quad (f=0) \quad in\ \Omega \in R^3 \\ u|_{\partial\Omega} = g\end{cases}$$



① 高斯公式：

$$\oiint_{\partial\Omega} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\,ds = \iiint_{\Omega} (P_x + Q_y + R_z)\,dV$$

$\vec{F} = (P, Q, R)$

$\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$

$div\,\vec{F} = P_x + Q_y + R_z$

$$\oiint_{\partial\Omega} \vec{F}\cdot\vec{n}\,ds = \iiint_{\Omega} div\,\vec{F}\,dV$$

$P = \dfrac{x}{\sqrt{x^2+y^2+z^2}} \qquad Q = \dfrac{y}{\sqrt{x^2+y^2+z^2}} \qquad R = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

$\Omega: x^2+y^2+z^2 \leq a^2$

$P = u_x v \quad Q = u_y v \quad R = u_z v$

$$\oiint_{\partial\Omega} v[u_x\cos\alpha + u_y\cos\beta + u_z\cos\gamma]\,ds = \iiint_{\Omega} [(u_x v)_x + (u_y v)_y + (u_z v)_z]\,dV$$

$(u_x, u_y, u_z)(\cos\alpha, \cos\beta, \cos\gamma)$

$= grad\,u \cdot \vec{n}$

$= \dfrac{\partial u}{\partial l}$

格林第一恒等式 $\displaystyle\oiint_{\partial\Omega} v\frac{\partial u}{\partial \vec{n}}\,ds = \iiint_{\Omega} \underbrace{(u_{xx}+u_{yy}+u_{zz})v}_{v\Delta u} + \underbrace{u_x v_x + u_y v_y + u_z v_z}_{\nabla u\cdot\nabla v}\,dV$

$$\oiint_{\partial\Omega} v\frac{\partial u}{\partial \vec{n}}\,ds = \iiint_{\Omega} v\Delta u + \nabla u\cdot\nabla v\,dV$$

(1) $v\equiv 1 \quad \nabla v=\vec{0}$

$$\oiint_{\partial\Omega}\frac{\partial u}{\partial n}ds=\iiint_{\Omega}(\Delta u)dV$$

$$\begin{cases}\Delta u=0 & \text{in}\ \Omega\quad \text{调和函数}\\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega}=f\end{cases}$$

$$\oiint_{\partial\Omega}\frac{\partial u}{\partial n}ds=\iiint_{\Omega}(\Delta u)dV$$

$$\oiint_{\partial\Omega}f\,ds=0$$

(2) $u=v$

$$\oiint_{\partial\Omega}u\frac{\partial u}{\partial n}ds=\iiint_{\Omega}u\Delta u\,dV+\iiint_{\Omega}|\nabla u|^2dv$$

用于证明解的唯一性

u,v 二阶可导

格林第二恒等式 $\oiint_{\partial\Omega}\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)ds=\iiint_{\Omega}(v\Delta u-u\Delta v)dV$ 换序相减

调和函数

$\Delta u=0$

eg: $u=x^2-y^2$

$u=x^2+y^2-2z^2$

$$r_x=\frac{1}{2}\left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{-\frac{1}{2}}\cdot 2(x-x_0)$$

$$=\frac{x-x_0}{r}$$

$$u_x=\left(-\frac{1}{r}\right)_x'=-r^{-2}r_x=-r^{-2}\frac{x-x_0}{r}$$

$$=-\frac{x-x_0}{r^3}$$

$$u=\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}$$

$$=\frac{1}{r_{M_0M}}$$

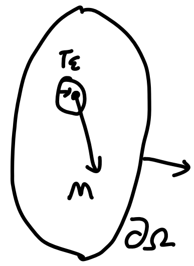
$$u_{xx}=-\frac{1}{r^3}+(x-x_0)3r^{-4}\frac{x-x_0}{r}$$

$$=-\frac{1}{r^3}+\frac{3(x-x_0)^2}{r^5}$$

$u_{xx}+u_{yy}+u_{zz}=0$ (除 (x_0,y_0,z_0))

$$\Delta\left(\frac{1}{4\pi r}\right)=\delta(M-M_0)$$

取 $v=\dfrac{1}{r_{M_0M}}$ 在 $\Omega-B_\varepsilon$ 上用(Ⅱ)式



$$0\equiv\iiint\limits_{\Omega-B_\varepsilon}[(\Delta u)v-(\Delta v)u]\,dV$$

$$=\oiint\limits_{\partial\Omega+\Gamma_\varepsilon}\left(\left(\frac{\partial u}{\partial n}\right)v-\left(\frac{\partial v}{\partial n}\right)u\right)ds$$

$$=\oiint\limits_{\partial\Omega}\left(\frac{\partial u}{\partial n}\frac{1}{r_{M_0M}}-\frac{\partial}{\partial n}\left(\frac{1}{r_{M_0M}}\right)u\right)ds$$

$$+\oiint\limits_{\Gamma_\varepsilon}\left(\frac{\partial u}{\partial n}\frac{1}{r_{M_0M}}-\frac{\partial}{\partial n}\left(\frac{1}{r}\right)u\right)ds_M$$

对于有镜像对称, $\dfrac{\partial u}{\partial n}=\dfrac{\partial u}{\partial r}\Big|_{r=\varepsilon}$

于是 $$\oiint\limits_{\partial\Omega}\left(\frac{\partial u}{\partial n}\frac{1}{r_{M_0M}}-\frac{\partial}{\partial n}\left(\frac{1}{r_{M_0M}}\right)u\right)ds$$ 形变原理

$$=\oiint\limits_{\Gamma_\varepsilon^+}\left(\frac{\partial u}{\partial n}\frac{1}{r_{M_0M}}-\frac{\partial}{\partial n}\left(\frac{1}{r}\right)u\right)ds_M$$ 曲线积分与路径无关原则

条件: u,v 调和

在 Γ_ε 上 $\dfrac{1}{r_{M_0M}}=\dfrac{1}{\varepsilon}$, $\dfrac{\partial(\frac{1}{r})}{\partial n}=\left(\dfrac{1}{r}\right)'_r\Big|_{r=\varepsilon}=-\dfrac{1}{\varepsilon^2}$

从而 $$=\oiint\limits_{\Gamma_\varepsilon}\left(\frac{\partial u}{\partial n}\cdot\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2}u\right)ds$$

$$=\left(\frac{\partial u}{\partial n}\frac{1}{\varepsilon}+\frac{1}{\varepsilon^2}u\right)_{M'}4\pi\varepsilon^2 \qquad \varepsilon\to0^+$$

M' 为球面上一点

$$=4\pi u(M_0)$$

若 $\Delta u = f$ 则有

$$\iiint_{\Omega - B_\varepsilon} \frac{f}{r} dV = \iiint_{\Omega - B_\varepsilon} [(\Delta u)v - (\Delta v)u] dV$$

$$= \oiint_{\Omega + \Gamma_\varepsilon} \left[\left(\frac{\partial u}{\partial n}\right)v - \left(\frac{\partial v}{\partial n}\right)u\right] ds$$

$\forall M_0 \in \Omega$

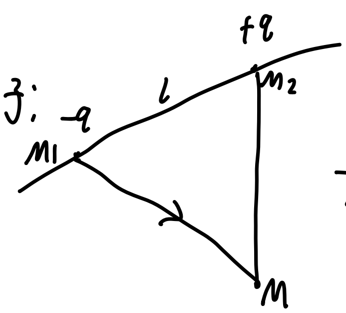
$$u(M_0) = -\oiint_{\partial\Omega} \left(u \frac{\partial}{\partial n}\left(\frac{1}{4\pi r_{M_0M}}\right) - \frac{1}{4\pi r_{M_0M}} \frac{\partial u}{\partial n}\right) dS_M$$

$$-\iiint_{\Omega} \frac{f}{4\pi r_{M_0M}} dV_M$$

物理含义：单层位势 + 偶极位势 + 双层位势

$\Delta\left(\frac{1}{r_{M_0M}}\right) = 0 \quad R^3 \setminus \{M_0\}$

单位偶极子：



$ql = 1$

$$\frac{\frac{1}{l}}{4\pi r_{M_2M}} - \frac{\frac{1}{l}}{4\pi r_{M_1M}} \quad l \to 0^+$$

$$\frac{f(M_2) - f(M_1)}{l} = \frac{\partial f}{\partial l} = \frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right)$$

① $\begin{cases} \Delta u = 0 \\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = f \end{cases} \Leftrightarrow \oiint_{\partial\Omega} f ds = 0$

② $\Delta u = 0 \Leftarrow u(M_0) = \frac{1}{4\pi R^2} \oiint_{S_{M_0}^R} u ds = \frac{1}{\frac{4}{3}\pi R^3} \iiint_{V_M^R} u dV$

平均值等式

$u = x^2 - y^2$ 解析函数的构造

$\begin{cases}\Delta u=0\\ u|_{\partial\Omega}=g\end{cases}$ $\begin{cases}\Delta u=0\\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega}=g\end{cases}$

u_1, u_2是方程的解 $v=u_1-u_2$

$v=u_1-u_2\equiv 0$ 唯一性

$\begin{cases}\Delta v=0\\ v|_{\partial\Omega}=0 \quad \left(\frac{\partial v}{\partial n}\Big|_{\partial\Omega}=0\right)\end{cases}$

$v\equiv 0$ $v\equiv const$

$$\oiint_{\partial\Omega} \underset{\downarrow\,0}{v}\underset{\downarrow\,0}{\frac{\partial v}{\partial n}}ds=\iiint_{\Omega} v\nabla v\,dv+\iiint_{\Omega}|\nabla v|^2dv$$

v是连续可微的调和函数 $(\Delta v=0 \quad v\in C^2)$

$$0=\oiint_{\partial\Omega}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)ds$$

$$u(M_0)=-\oiint_{\partial\Omega}\left[u\frac{\partial}{\partial n}\left(\frac{1}{4\pi r}-v\right)-\left(\frac{1}{4\pi r}-v\right)\frac{\partial u}{\partial n}\right]ds$$

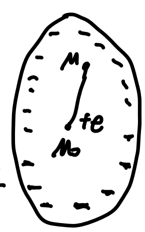
$v:\begin{cases}\Delta v=0 \quad in\ \Omega\\ v|_{\partial\Omega}=\frac{1}{4\pi r_{MM_0}}\end{cases}$

$$u(M_0)=-\oiint_{\partial\Omega} g\frac{\partial}{\partial n}\left(\underbrace{\frac{1}{4\pi r}}_{\text{空间点电荷电势}}-v\right)ds$$

$$=\oiint_{\partial\Omega} G\frac{\partial u}{\partial n}ds$$

$G=\frac{1}{4\pi r}-v$ 格林函数

$\partial\Omega$



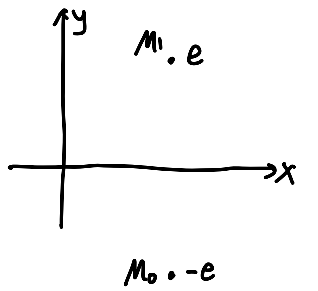
M_1

$-q$

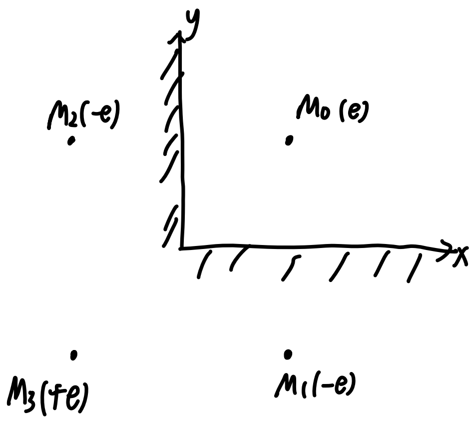
M_1与M_0成镜像点

$\frac{\partial G}{\partial n}\Big|_{\partial\Omega}=\frac{\partial G}{\partial(-z)}\Big|_{z=0}$

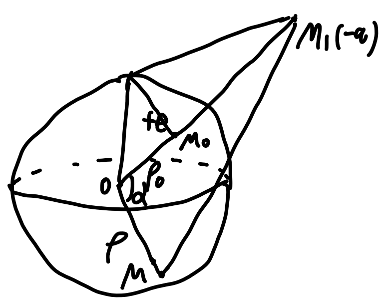
若加一负电荷 $-q$ $G=\frac{1}{4\pi r_{M_0M}}-\frac{q}{4\pi r_{M_1M}}$



$$\frac{1}{2\pi}\ln\frac{1}{r_{M_0M}} \sim \frac{1}{4\pi r_{M_0M}}$$



$$G=\frac{1}{2\pi}\left[\ln\frac{1}{r_{M_0M}}-\ln\frac{1}{r_{M_1M}}-\ln\frac{1}{r_{M_2M}}+\ln\frac{1}{r_{M_3M}}\right]$$



$$OM_0\cdot OM_1=R^2$$

$$\frac{1}{4\pi}\,\frac{1}{r_{M_0M}}=\frac{q}{4\pi\, r_{M_1M}}$$

$$q=\frac{r_{M_1M}}{r_{M_0M}}=\frac{R}{\rho_0}$$

$$\frac{OM}{OM_0}=\frac{OM_1}{OM}=\frac{M_0M}{MM_1}$$

$$G=\frac{1}{4\pi r_{M_0M}}-\frac{\frac{R}{\rho_0}}{4\pi r_{M_1M}}$$

$$\frac{\partial G}{\partial n}\Big|_{\partial S}=\frac{\partial G}{\partial \rho}\Big|_{\rho=R_0}$$

$$\sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos\alpha}$$

$$\sqrt{\rho^2+\left(\frac{R^2}{\rho_0}\right)^2-2\rho\left(\frac{R^2}{\rho_0}\right)\cos\alpha}$$

四 (1) (2)