2019302130113

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第四次作业

13. $\Delta u=0$， $0\le\rho\le a$， $0\le\theta\le\pi$

$$\begin{cases} u(a,\theta)=T\theta(\pi-\theta) \\ u(\rho,0)=u(\rho,\pi)=0 \end{cases}$$

$|u(0,\theta)|<+\infty$ ， $0\le\theta\le\pi$

$$u=R(\rho)\Phi(\theta)$$

$$R''\Phi+\frac{1}{\rho}R'\Phi+\frac{1}{\rho^2}R\Phi''=0$$

令 $\frac{\rho^2 R''}{R}+\frac{\rho R'}{R}=-\frac{\Phi''}{\Phi}=\lambda$

从而 $\begin{cases} \Phi''+\lambda\Phi=0 \\ \Phi(0)=0,\ \Phi(\pi)=0 \end{cases}$ $\begin{cases} \rho^2R''+\rho R'-\lambda R=0 \\ R(a)\Phi(\theta)=T\theta(\pi-\theta) \end{cases}$

$\lambda=0$， $\Phi=C_1\theta+C_2$， $C_1=C_2=0$，舍去

$\lambda<0$， $\lambda=-s^2$， $\Phi=C_1e^{s\theta}+C_2e^{-s\theta}$

$$\begin{cases} C_1+C_2=0 \\ C_1e^{s\pi}+C_2e^{-s\pi}=0 \end{cases}，C_1=C_2=0，舍去$$

$\lambda>0$， $\lambda=s^2$， $\Phi=C_1\cos\sqrt{\lambda}\theta+C_2\sin\sqrt{\lambda}\theta$

$$\begin{cases} C_1=0 \\ C_2\sin\sqrt{\lambda}\pi=0 \end{cases} \qquad \sqrt{\lambda}\pi=n\pi$$

从而 $\Phi(\theta)=C_2\sin\sqrt{\lambda}\theta$ $\qquad \lambda_n=n^2$

从而 $\rho^2R''+\rho R'-n^2R=0$

$$R(\rho)=C_1\rho^n+C_2\rho^{-n}$$

由于 $|R(\rho)|<+\infty$，从而 $R(\rho)=C_1\rho^n$

提 $u_n(\rho,\theta)=R_n(\rho)\Phi_n(\theta)=D_n\sin n\theta\,\rho^n$，$n=1,2,\cdots$

从而 $u(\rho,\theta)=\sum_{n=1}^{+\infty}D_n\rho^n\sin n\theta$，$n=1,2,\cdots$

由边界条件可知， $u(a,\theta)=\sum_{n=1}^{+\infty}D_n a^n\sin n\theta=T\theta(\pi-\theta)$

从而 $D_n=\frac{1}{a^n}\frac{2}{\pi}\int_0^{\pi}T\theta(\pi-\theta)\sin n\theta\,d\theta=\frac{4T[1-(-1)^n]}{\pi n^3a^n}=\frac{8T}{\pi(2k-1)^3a^{2k-1}}$，$k=1,2\cdots$

提 $u(\rho,\theta)=\sum_{k=1}^{+\infty}\frac{8T}{\pi(2k-1)^3a^{2k-1}}a^n\sin n\theta$

14. $\Delta u=0,\ 0\le\theta\le\pi,\quad r_1\le\rho\le r_2$

$$\begin{cases} u(r_1,\theta)=0 \\ u(r_2,\theta)=1 \\ |u(\rho,\theta)|<+\infty \end{cases}$$

$u=R(\rho)\Phi(\theta)$

$\rho^2R''\Phi+\rho R'\Phi+R\Phi''=0$

从而 $\dfrac{\rho^2R''}{R}+\dfrac{\rho R'}{R}+\dfrac{\Phi''}{\Phi}=0$

令 $\dfrac{\rho^2R''}{R}+\dfrac{\rho R'}{R}=-\dfrac{\Phi''}{\Phi}=\lambda$

从而 $\rho^2R''+\rho R'-\lambda R=0,\ \Phi''+\lambda\Phi=0$

$R(r_1)=0,\ \Phi(\theta+2\pi)=\Phi(\theta)$

$\lambda=0,\ \Phi=C_1x+C_2,\quad$ 令$C_1=0,\ \Phi=C_2$

$\lambda<0,\ \lambda=-s^2,\ \Phi=C_1e^{s\theta}+C_2e^{-s\theta}$, 舍去

$\lambda>0,\ \lambda=s^2,\ \Phi=C_1\cos\sqrt{\lambda}\theta+C_2\sin\sqrt{\lambda}\theta$

$\lambda_n=n^2$

从而 $\lambda_n=n^2,\ n=0,1,2,\cdots$

从而 $\rho^2R''+\rho R'-n^2R=0$

$R_n=C_1\rho^n+C_2\rho^{-n}$

$R_0(r)=C_0+d_0\ln r$

于是 $\Phi(0)=a_0$ (常数)

$\Phi_n(\theta)=a_n\cos n\theta+b_n\sin n\theta,\ n=1,2,\cdots$

$R_0(r)=C_0+d_0\ln r$

$R_n(r)=C_nr^n+d_nr^{-n}\ ,n=1,2,\cdots$

又由 $u(r_1,\theta)=0$, 知 $C_0+d_0\ln r_1=0,\ C_nr_1^n+d_nr_1^{-n}=0$

从而 $C_0=-(\ln r_1)d_0,\ C_n=-r_1^{-2n}d_n,\ n=1,2,\cdots$

由叠加原理,

$$u(r,\theta)=D_0\ln\frac{r}{r_1}+\sum_{n=1}^{+\infty}d_n(r^{-n}-r_1^{-2n}r^n)(a_n\cos n\theta+b_n\sin n\theta)$$

$$=D_0\ln\frac{r}{r_1}+\sum_{n=1}^{+\infty}(r^n-r_1^{2n}r^{-n})(A_n\cos n\theta+B_n\sin n\theta)$$

又由 $u(r_2,\theta)=1$

有 $$1=D_0\ln\frac{r_2}{r_1}+\sum_{n=1}^{\infty}(r_2^n-r_1^{2n}r_2^{-n})(A_n\cos n\theta+B_n\sin n\theta)$$

得 $D_0\ln\frac{r_2}{r_1}=1$, $A_n=B_n=0$, $n=1,2,\cdots$

因此 $$u(r,\theta)=\frac{1}{\ln\frac{r_2}{r_1}}\ln\frac{r}{r_1}$$

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第五次作业

22. $\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} + \sin\dfrac{2\pi}{l}x \sin\dfrac{2a\pi}{l}t \\ u|_{x=0} = u|_{x=l} = 0,\ t>0 \\ u|_{t=0} = u_t|_{t=0} = 0,\ 0 \leq x \leq l \end{cases}$

解：令 $u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\frac{n\pi}{l}x$

由 $f(x,t) = \sin\frac{2\pi}{l}x \sin\frac{2a\pi}{l}t$

代入方程，有 $\sum_{n=1}^{\infty}\left(u_n''(t) + \frac{a^2n^2\pi^2}{l^2}u_n(t)\right)\sin\frac{n\pi}{l}x = \sin\frac{2\pi}{l}x\sin\frac{2a\pi}{l}t$

$u_n(0) = u_n'(0) = 0$

比较式子左右两端得 $u_n''(t) + \frac{an^2\pi^2}{l^2}u_n(t) = 0,\ n \neq 2$

$\left(u_2''(t) + \frac{4a^2\pi^2}{l^2}u_2(t)\right)\sin\frac{2\pi}{l}x = \sin\frac{2\pi}{l}x \sin\frac{2a\pi}{l}t$

$u_n(0) = u_n'(0) = 0,\ n = 1, 2, \cdots$

提 $u_n(t) = 0,\ n \neq 2$

且 $\begin{cases} u_2''(t) + \frac{4a^2\pi^2}{l^2}u_2(t) = \sin\frac{2a\pi}{l}t \\ u_2(0) = u_2'(0) = 0 \end{cases}$

令 $u_2(t) = C_1\cos\frac{2a\pi}{l}t + C_2\sin\frac{2a\pi}{l}t$

利用参数变易法

$\begin{cases} C_1'(t)\cos\frac{2a\pi}{l}t + C_2'(t)\sin\frac{2a\pi}{l}t = 0 \\ C_1'(t)\left(-\frac{2a\pi}{l}\sin\frac{2a\pi}{l}t\right) + C_2'(t)\left(\frac{2a\pi}{l}\cos\frac{2a\pi}{l}t\right) = \sin\frac{2a\pi}{l}t \end{cases}$

解得 $C_1'(t) = -\frac{l}{2a\pi}\sin^2\frac{2a\pi}{l}t$，$C_2'(t) = \frac{l}{2a\pi}\sin\frac{2a\pi}{l}t\cos\frac{2a\pi}{l}t$

从而 $C_1(t) = -\frac{l}{4a\pi}t + \left(\frac{l}{4a\pi}\right)^2\sin\left(\frac{4a\pi}{l}t\right)$

$C_2(t) = -\left(\frac{l}{4a\pi}\right)^2\cos\left(\frac{4a\pi}{l}t\right)$

从而 $u_2(t)=\left[\left(-\frac{l}{4a\pi}t\right)+\left(\frac{l}{4a\pi}\right)^2\sin\left(\frac{4a\pi}{l}t\right)\right]\cos\frac{2a\pi}{l}t-\left(\frac{l}{4a\pi}\right)^2\cos\frac{4a\pi}{l}t\sin\frac{2a\pi}{l}t$

$$=-\frac{l}{4a\pi}t\cos\frac{2a\pi}{l}t+\left(\frac{l}{4a\pi}\right)^2\sin\frac{2a\pi}{l}t$$

而 $u(x,t)=u_2(t)\sin\frac{2\pi}{l}x$

得定解问题的解为 $u(x,t)=\frac{l}{4a\pi}\left(\frac{l}{4a\pi}\sin\frac{2a\pi}{l}t-t\cos\frac{2a\pi}{l}t\right)\sin\frac{2\pi}{l}x$

(2) 在单位圆内，转化为极坐标，有

$$
\begin{cases}
u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\theta\theta} = -\rho^2 \sin\theta\cos\theta, & 0 \le \rho < 1 \\
u|_{\rho=1} = 0 & 0 \le \theta \le 2\pi \\
|u(\rho,\theta)| < +\infty \\
u(\rho,\theta) = u(\rho,\theta+2\pi)
\end{cases}
$$

令 $u(\rho,\theta) = \sum_{n=0}^{\infty} (A_n(\rho)\cos n\theta + B_n(\rho)\sin n\theta)$

代入式子，得

$$\sum_{n=0}^{\infty} \left(A_n''(\rho)\cos n\theta + B_n''(\rho)\sin n\theta + \frac{1}{\rho}A_n'(\rho)\cos n\theta + \frac{1}{\rho}B_n'(\rho)\sin n\theta - \frac{n^2}{\rho^2}A_n(\rho)\cos n\theta - \frac{n^2}{\rho^2}B_n(\rho)\sin n\theta\right)$$

$$= -\rho^2 \sin\theta\cos\theta$$

$$= -\frac{\rho^2}{2}\sin 2\theta$$

对比，得 $A_n''(\rho) + \frac{1}{\rho}A_n'(\rho) - \frac{n^2}{\rho^2}A_n(\rho) = 0, \quad n = 0,1,2,3\cdots$

$$B_n''(\rho) + \frac{1}{\rho}B_n'(\rho) - \frac{n^2}{\rho^2}B_n(\rho) = 0, \quad n \ne 2$$

$$B_2''(\rho) + \frac{1}{\rho}B_2'(\rho) - \frac{4}{\rho^2}B_2(\rho) = -\frac{\rho^2}{2}$$

$$A_n(1) = B_n(1) = 0, \quad n = 0,1,2,\cdots$$

$$|A_n(0)| < +\infty, \quad |B_n(0)| < +\infty, \quad n = 0,1,2,\cdots$$

求解 A_n, B_n，得

$$A_n = 0, \quad n = 0,1,2,3\cdots$$

$$B_n = 0, \quad n \ne 2$$

$$B_2 = -\frac{1}{24}\rho^4 + \frac{1}{24}\rho^2$$

故，解为 $u(\rho,\theta) = -\frac{\rho^2}{24}(\rho^2 - 1)\sin 2\theta$

即 $u(x,y) = -\frac{1}{12}xy(x^2+y^2-1)$

23. 在单位圆内，转化为极坐标，有

$$\begin{cases} u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\theta\theta}=-\rho^2\sin\theta\cos\theta, & 0\le\rho<1 \\ u|_{\rho=1}=0, & 0\le\theta\le2\pi \\ |u(\rho,\theta)|<+\infty \\ u(\rho,\theta)=u(\rho,\theta+2\pi) \end{cases}$$

令 $u(\rho,\theta)=\sum_{n=0}^{+\infty}(A_n(\rho)\cos n\theta+B_n(\rho)\sin n\theta)$

代入式子，得

$$\sum_{n=1}^{+\infty}\left(A_n''(\rho)\cos n\theta+B_n''(\rho)\sin n\theta+\frac{1}{\rho}A_n'(\rho)\cos n\theta+\frac{1}{\rho}B_n'(\rho)\sin n\theta-\frac{n^2}{\rho^2}A_n(\rho)\cos n\theta-\frac{n^2}{\rho^2}B_n(\rho)\sin n\theta\right)$$

$$=-\rho^2\sin\theta\cos\theta$$

$$=-\frac{\rho^2}{2}\sin2\theta$$

对比，得 $A_n''(\rho)+\frac{1}{\rho}A_n'(\rho)-\frac{n^2}{\rho^2}A_n(\rho)=0,\ n=0,1,2,3\cdots$

$$B_n''(\rho)+\frac{1}{\rho}B_n'(\rho)-\frac{n^2}{\rho^2}B_n(\rho)=0,\ n\ne2$$

$$B_2''(\rho)+\frac{1}{\rho}B_2'(\rho)-\frac{4}{\rho^2}B_2(\rho)=-\frac{\rho^2}{2}$$

$$A_n(1)=B_n(1)=0,\ n=0,1,2,\cdots$$

$$|A_n(0)|<+\infty,\ |B_n(0)|<+\infty,\ n=0,1,2,\cdots$$

求解 A_n, B_n，得

$$A_n=0,\ n=0,1,2,3\cdots$$

$$B_n=0,\ n\ne2$$

$$B_2=-\frac{1}{24}\rho^4+\frac{1}{24}\rho^2$$

提，解为 $u(\rho,\theta)=-\frac{\rho^2}{24}(\rho^2-1)\sin2\theta$

即 $u(x,y)=-\frac{1}{12}xy(x^2+y^2-1)$