(1) $f(x)$, $T=2\pi$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin(nx))$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad n=0,1,2\cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad n=1,2,\cdots$$

(2) $f(x)$, $T=2l$　Flouier级数

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l} dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l} dx$$

(3) 若 $0 \le x \le l$, $f(x)$　① 延拓至 $[-l, l]$

余弦展开作偶，正弦展开作奇

② 周期延拓

③ $b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} dx$

$f(x)=1$, $0<x<l$

$$f(x) = \sum_{n=1}^{+\infty} b_n \sin\frac{n\pi x}{l} = \sum_{k=0}^{+\infty} \frac{4}{(2k+1)\pi} \sin\frac{(2k+1)\pi x}{l}$$

① $f(x)=x,\ 0<x<l$

$$f(x)=\sum_{n=1}^{+\infty} b_n \sin\frac{n\pi x}{l}=\sum_{n=1}^{+\infty}\frac{2l}{n\pi}(-1)^{n-1}\sin\frac{n\pi}{l}x$$

$$b_n=\frac{2}{l}\int_0^l x\sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\int_0^l x\, d\cos\frac{n\pi x}{l}\cdot\left(\frac{-l}{n\pi}\right)$$

$$=\frac{-2}{n\pi}\left(x\cos\frac{n\pi x}{l}\Big|_0^l-\underbrace{\int_0^l\cos\frac{n\pi}{l}x\,dx}_{0}\right)$$

$$=\frac{-2l}{n\pi}(-1)^n$$

$$=\frac{2l}{n\pi}(-1)^{n-1}$$

② $f(x)=x^2,\quad 0<x<l$

$$f(x)=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$$

$$b_n=\frac{2}{l}\int_0^l x^2\sin\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\int_0^l x^2\, d\cos\frac{n\pi x}{l}\left(-\frac{l}{n\pi}\right)$$

$$=\frac{-2}{n\pi}x^2\cos\frac{n\pi x}{l}\Big|_0^l+\frac{2}{n\pi}\int_0^l\cos\frac{n\pi x}{l}dx^2$$

$$=\frac{2l^2}{n\pi}(-1)^{n-1}+\frac{4}{n\pi}\int_0^l x\, d\sin\frac{n\pi x}{l}\cdot\left(\frac{l}{n\pi}\right)$$

$$=\frac{2l^2}{n\pi}(-1)^{n-1}+\frac{4}{(n\pi)^2}\left(-\int_0^l\sin\frac{n\pi}{l}x\,dx\right)$$

$$=\frac{2l^2}{n\pi}(-1)^{n-1}+\frac{4}{(n\pi)^2}\ \frac{l}{n\pi}\ \cos\frac{n\pi}{l}x\Big|_0^l$$

$$=\frac{2l^2}{n\pi}(-1)^{n-1}+\frac{4l}{(n\pi)^3}((-1)^n-1)$$

① ② $\Rightarrow$ $lx-x^2=\sum_{k=1}^{\infty}\frac{8l^2}{(2k-1)^3\pi^3}\sin\frac{2k-1}{l}\pi x$

更高次可以用逐项积分法

单簧管: $u_{tt} = a^2 u_{xx}$

$$\begin{cases} u|_{x=0} = 0,\ u_x|_{x=l} = 0 \\ u|_{t=0} = x^2 - 2lx,\ u_t|_{t=0} = 0 \end{cases}$$

$\longrightarrow$ 一端不受力

(1) $u = ZT$

(2) $ZT'' = a^2 Z'' T$ $\qquad \dfrac{Z''(x)}{Z} = \dfrac{T''(t)}{a^2 T} = -\lambda$

$Z(0)\,T(t) = 0,\quad Z'(l)\,T(t) = 0$

(3) $\begin{cases} Z'' + \lambda Z = 0 \\ Z(0) = Z'(l) = 0 \end{cases}$ $\qquad T'' + a^2\lambda = 0$

$\lambda = 0,\quad Z = C_1 x + C_2 \Rightarrow C_1 = C_2 = 0$

$\lambda < 0,\quad \lambda = -s^2,\quad C_1 e^{sx} + C_2 e^{-sx}$

$$\begin{cases} C_1 + C_2 = 0 \\ C_1 s e^{sl} - C_2 s e^{-sl} = 0 \end{cases} \Rightarrow \begin{cases} C_1 = 0 \\ C_2 = 0 \end{cases}$$

$\lambda > 0,\quad \lambda = s^2,\quad Z(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$

$Z(0) = 0 \Rightarrow C_1 = 0$

$Z'(l) = 0 \Rightarrow \sqrt{\lambda}l = k\pi \pm \frac{\pi}{2},\ \lambda = \left(\dfrac{k\pi - \frac{\pi}{2}}{l}\right)^2$

$k = 1, 2, \cdots$

$$Z_k = \sin \frac{k\pi - \frac{\pi}{2}}{l} x$$

$$T'' + a^2 \lambda T = 0$$

$$T_k = C_k \cos a\sqrt{\lambda_k}\, t + D_k \sin a\sqrt{\lambda_k}\, t$$

(4) $U_k = Z_k T_k$

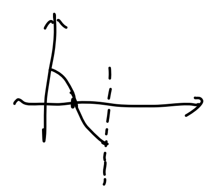
(5) $u(x,t) = \sum U_k$

以 $\left\{ \underline{\sin \frac{(2k-1)\pi x}{2l}} \right\}_{k=1}^{+\infty}$ 为基

$$x = \sum_{k=1}^{+\infty} b_k \sin \frac{2k-1}{2l} \pi x \qquad f(x) = x$$

$$b_k = \frac{2}{l}\int_0^l x\left(\sin \frac{2k-1}{2l}\pi x\right) dx$$

$$= \frac{8l}{(2k-1)^2 \pi^2} (-1)^{k-1}$$



$$x^2 = \sum_{k=1}^{+\infty} b_k \underline{\sin \frac{2k-1}{2l}} \pi x$$

$$b_k = \frac{16l}{[(2k-1)\pi]^2}\left[l(-1)^{k-1} - \frac{2l}{(2k-1)\pi} \right]$$

$$\lambda_k = \left(\frac{k\pi}{l}\right)^2 = k^2 \lambda$$

$$x^2 - 2lx \longrightarrow b_k = -\frac{32l^2}{(2k-1)^3 \pi^3}$$

$$\begin{cases} u_t = a^2 u_{xx} \\ u(0,t) = 0 \\ (u_x + hu)_{x=l} = 0 \\ u(x,0) = \varphi(x), \quad 0<x<l \end{cases}$$

(1) $u(x,t) = Z(x)T(t)$

(2) $ZT' = a^2 Z'' T$

$$\frac{Z''}{Z} = \frac{T'}{a^2 T} = -\lambda$$

(3) $Z(0)T(t) = 0$, $Z'(l)T(t) + hZ(l)T(t) = 0$

$Z(0) = 0$, $Z'(l) + hZ(l) = 0$

(4) $$\begin{cases} Z'' + \lambda Z = 0, \quad 0<x<l \\ Z(0) = 0 \\ Z' + hZ|_{x=l} = 0 \end{cases}$$

$\lambda = 0$, $z = C_1 x + C_2 \Rightarrow C_1 = C_2 = 0$

$\lambda < 0$, $\lambda = -s^2$, $Z = C_1 e^{-sx} + C_2 e^{sx}$

$$\begin{cases} C_1 + C_2 = 0 \\ C_1(-s)e^{-sl} + C_2 s e^{sl} + C_1 h e^{-sl} + C_2 h e^{sl} = 0 \end{cases} \Rightarrow C_1 = C_2 = 0$$

$\lambda > 0$, $Z(x) = C_n \cos\sqrt{\lambda} x + D_n \sin\sqrt{\lambda} x$

$Z(0) = 0 \Rightarrow C_n = 0$

$Z(x) = \sin\sqrt{\lambda} x$

$Z' = \sqrt{\lambda}\cos\sqrt{\lambda} x$

$$Z'+hZ=\sqrt{\lambda}\cos\sqrt{\lambda}l+h\sin\sqrt{\lambda}l=0$$

$$-\frac{\sqrt{\lambda}}{h}=\tan\sqrt{\lambda}l$$

记 $z=\sqrt{\lambda}l$ $\quad \tan z=-\frac{1}{\sqrt{\lambda}l}z=-kz$

$$\lambda_k=\left(\frac{z_k}{l}\right)^2,\ Z_k(x)=\sin\frac{z_k}{l}x$$

(5) $\frac{T_k'}{T_k}=-a^2\lambda_k$ $\qquad T_k(t)=C_k e^{-a^2\lambda_k t}$

(6) $u_k=Z_kT_k=C_k e^{-a^2\lambda_k t}\sin\sqrt{\lambda_k}t$

(7) $u=\sum_{k=1}^{\infty}u_k$

(8) $\sum_{k=1}^{+\infty}C_k\sin\sqrt{\lambda_k}x=\varphi(x)$ $\qquad \int_0^l\sin\sqrt{\lambda_k}x\sin\sqrt{\lambda_m}x\,dx=0$

正交性

$$C_k\int_0^l(\sin\sqrt{\lambda_k}x)^2dx=L_k$$

$$=\int_0^l\varphi(x)\sin\sqrt{\lambda_k}x\,dx$$

关于正交性：不难证明：$\int_0^l Z_mZ_n\,dx=0$ $\qquad$ 2 (5)(6)

$$u_{xx}+u_{yy}\longrightarrow\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta\end{cases}$$

$$u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\theta\theta}=0$$