$$\begin{cases} u_{tt} = a^2 u_{xx} & -\infty < x < +\infty \quad t > 0 \qquad a \neq 0 \\ u|_{t=0} = \varphi(x), & u_t|_{t=0} = \phi(x) \end{cases}$$

$$\left(\frac{\partial^2}{\partial t^2} - a^2 \frac{\partial^2}{\partial x^2}\right) u = 0$$

$$\underbrace{\left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right)}_{\frac{\partial u}{\partial \eta}} \underbrace{\left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)}_{\frac{\partial u}{\partial \xi}} u = 0$$

(1) $\begin{cases} \xi = t + \frac{1}{a}x \sim x + at \\ \eta = t - \frac{1}{a}x \sim x - at \end{cases}$

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(2) $u_{\xi\eta} = 0$

$$\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right) = 0$$

$u_t = u_\xi \xi_t + u_\eta \eta_t$

$= u_\xi + u_\eta$

$u_{tt} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}$

$u_x = \frac{1}{a}u_\xi - \frac{1}{a}u_\eta$

$u_{xx} = \frac{1}{a^2}(u_{\xi\xi} + u_{\eta\eta} - 2u_{\xi\eta})$

代入 $u_{\xi\eta} = 0$

(3) $\frac{\partial u}{\partial \eta} = F(\eta)$

$$u(\xi, \eta) = F(\eta) + G(\xi)$$

$$= F(x-at) + G(x+at)$$

(4) $\begin{cases} \varphi(x) = F(x) + G(x) \\ \phi(x) = -aF'(x) + aG'(x) \end{cases}$ $F'(x-at)\underbrace{(x-at)'}_{-a}$

$$\frac{1}{a}\int_0^x \phi(s)\,ds + C = -F(x) + aG(x)$$

$$F(x) = \frac{\varphi(x)}{2} - \frac{1}{2a}\int_0^x \phi(s)\,ds - \frac{C}{2} \qquad x \sim x - at$$

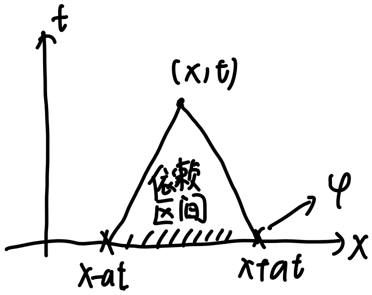
$$G(x) = \frac{\varphi(x)}{2} + \frac{1}{2a}\int_0^x \phi(s)\,ds + \frac{C}{2} \qquad x \sim x + at$$

※: $u(x,t) = \frac{1}{2}\varphi(x-at) - \frac{1}{2a}\int_0^{x-at}\phi(s)\,ds + \frac{1}{2}\varphi(x+at) + \frac{1}{2a}\int_0^{x+at}\phi(s)\,ds$

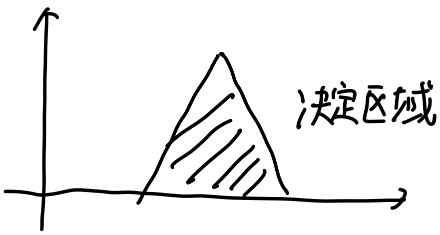
$$= \frac{1}{2}[\varphi(x-at) + \varphi(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at}\phi(s)\,ds \quad \text{平均位移}$$

行波法　$u(x,t)=F(x-at)+G(x+at)$

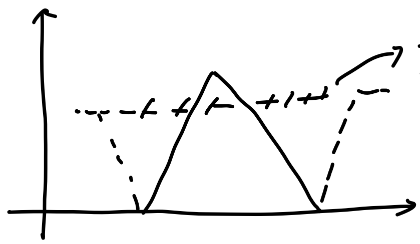
右行波　左行波



$u(x,t)$ 依赖于 $[x-at, x+at]$



三角形内每点确定



影响区域：扩散可影响到的区域

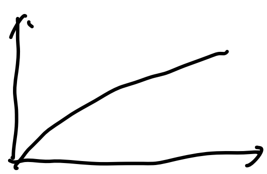
$AU_t+BU_x=F$　守恒矩方程

$|B-\lambda A|=0$　特征

若λ有2个不同实根，则方程为双曲型

1个则为抛物型

共轭根则为椭圆型



特征曲线

$\begin{cases} x=x(\sigma) \\ t=t(\sigma) \end{cases}$　$x=x(t)$

$|Bt'(\sigma)-Ax'(\sigma)|=0$

$|B-A\frac{dx}{dt}|=0$　$\lambda=\frac{dx}{dt}$

$$AU_t + BU_x = 0$$

$$\begin{cases} u_t = a v_x \\ v_t = a u_x \end{cases}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_t - \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x = 0$$

$$|B - \lambda A| = \left| -\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

$$\lambda^2 = a^2, \quad \lambda = \pm a$$

$$\frac{dx}{dt} = \pm a, \quad x = \pm at + C$$

$$\begin{cases} \xi = x + at \\ \eta = x - at \end{cases}$$

$$a u_{xx} + 2b u_{xy} + c u_{yy} = 0 \qquad a(dy)^2 + 2b(dx)(dy) + c(dx)^2 = 0$$

$$(a u_x + b u_y)_x + (b u_x + c u_y)_y = 0$$

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_x \\ u_y \end{pmatrix}_x + \begin{pmatrix} b & c \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u_x \\ u_y \end{pmatrix}_y = 0$$

$$\left| \begin{pmatrix} b & c \\ -1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} b - a\lambda & c - b\lambda \\ -1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda(b - a\lambda) + c - b\lambda = 0$$

$$-b\lambda + a\lambda^2 + c - b\lambda = 0$$

$$a\lambda^2 - 2b\lambda + c = 0$$

$$\frac{dy}{dx} = \lambda = \frac{b \pm \sqrt{b^2 - ac}}{2a}$$

$$\xi = y - \lambda_1 x, \quad \eta = y - \lambda_2 x$$

$b^2 - ac > 0$, hyperbolic $u_{tt} = a^2 u_{xx}$

$b^2 - ac = 0$, parabolic $u_t = a^2 u_{xx}$

$b^2 - ac < 0$, elliptic $u_{xx} + u_{yy} = 0$

e.g: $\begin{cases} \dfrac{\partial^2 u}{\partial x \partial y} = 1, & x>0, y>0 \\ u|_{x=0} = y+1, & y\geq 0 \\ u|_{y=0} = 1, & x\geq 0 \end{cases}$

(1) $u_{xy} = 1$ $u_x = y + \varphi(x)$

$u(x,y) = xy + f(x) + g(y)$

$y+1 = f(0) + g(y)$ $f(0) + g(0) = 1$

$1 = f(x) + g(0)$

$u(x,y) = xy + 1 - g(0) + y + 1 - f(0)$

$= xy + y + 1$

半无限问题:

$\begin{cases} u_{tt} = a^2 u_{xx} & 0<x<+\infty, t>0 \\ u|_{x=0} = h(t) \\ u|_{t=0} = \varphi(x), & u_t|_{t=0} = \phi(x) \end{cases}$

$u(x,t) = F(x-at) + G(x+at)$

$h(t) - G(at) = F(-at)$

假设 $h(t)=0$ $x>at$ 时, $u(x,t) = \dfrac{\varphi(x-at)+\varphi(x+at)}{2} + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}\phi(s)ds$

$x<at$ 时, $u(x,t) = G(x+at) - G(at-x)$

$= \dfrac{1}{2}[\varphi(x+at) - \varphi(at-x)] + \dfrac{1}{2a}\displaystyle\int_{at-x}^{at+x}\phi(s)ds$

$\overline{u}(x) = \begin{cases} u(x), & x>0 \\ -u(-x), & x<0 \end{cases}$

$h(t)\neq 0$ 时, $v = u - h(t+\frac{x}{a})$ $v|_{x=0} = h(x)$

$v_{tt} = u_{tt} - h''$ $a^2[u_{xx} - \frac{1}{a^2}h'']$

$v_{xx} = u_{xx} - h''\frac{1}{a^2}$ $a^2 v_{xx}$

$$\begin{cases} v_{tt} = a^2 v_{xx} \\ v|_{x=0} = 0 \\ v|_{t=0} = \varphi - h(\frac{x}{a}) \quad \varphi_1 \\ v_t|_{t=0} = \phi(x) - h'(\frac{x}{a}) \quad \phi_1 \end{cases}$$

(1) $u(x,t) = F(x-at) + G(x+at)$

(2) $$\begin{cases} F(-at) + G(at) = h(t) \\ F(x) + G(x) = \varphi(x) \\ F'(x)(-a) + G'(x) \cdot a = \phi(x) \end{cases}$$

$x > at$, $u = \frac{\varphi(x-at) + \varphi(x+at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \phi(s)\, ds$

$x < at$, $\qquad F(-at) = h(t) - G(at)$

$u = F(-(at-x)) + G(x+at) \qquad F(-(at-x)) = h(\frac{at-x}{a}) - G(at-x)$

$= h(\frac{at-x}{a}) - G(at-x) + G(x+at)$

非齐次:

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t), & -\infty < x < +\infty, \ t > 0 \\ u|_{t=0} = \varphi(x), \ u_t|_{t=0} = \phi(x) \end{cases}$$

(I) $$\begin{cases} \bar{u}_{tt} = a^2 \bar{u}_{xx} \\ \bar{u}|_{t=0} = \varphi(x) \\ \bar{u}_t|_{t=0} = \phi(x) \end{cases} \qquad u = \bar{u} + v$$

(II) $$\begin{cases} v_{tt} = a^2 v_{xx} + f \\ v|_{t=0} = 0 = v_t|_{t=0} \end{cases}$$ 齐次化原理

三.2

$$\frac{dF}{dx} = f(x,y)|_{y=x} + \int_a^x f_x(x,y)\,dy$$

$$F(x) = \int_a^x f(x,y)\,dy$$

习题三、13