

$$\begin{cases} u_{tt} = a^2 u_{xx} + A \\ u|_{x=0} = 0, u|_{x=l} = B \\ u|_{t=0} = u_t|_{t=0} = 0 \end{cases}$$

$$(1) u = v + w$$

$$w|_{x=0} = 0 \quad w|_{x=l} = B$$

$$(2) v_{tt} = a^2 v_{xx} + a^2 w_{xx} + A$$

$$a^2 w_{xx} + A = 0$$

$$(3) w_{xx} = -\frac{A}{a^2}$$

$$w_x = -\frac{A}{a^2} x + C_1$$

$$(4)$$

$$(5)$$

$$+k \frac{\partial u}{\partial n} = q_0$$

$$\begin{cases} u_t = a^2 u_{xx} \\ u|_{x=0} = u_0, \quad u_x|_{x=l} = \frac{q_0}{k}, \quad t > 0 \\ u|_{t=0} = u_0, \quad 0 \leq x \leq l \end{cases}$$

$$(1) W(x) = u_0 + \frac{q_0}{k} x$$

$$(2) \begin{cases} v_t = a^2 v_{xx}, \quad 0 < x < l, \quad t > 0 \\ v|_{x=0} = 0, \quad v_x|_{x=l} = 0 \\ v|_{t=0} = -\frac{q_0}{k} x \end{cases}$$

$$(3) v = Z(x) T(t)$$

$$\frac{Z''}{Z} = \frac{T'}{a^2 T} = -\lambda$$

$$\begin{cases} Z'' + \lambda Z = 0 \\ Z(0) = Z'(l) = 0 \end{cases}$$

$$\lambda_k = \left(\frac{k\pi - \frac{\pi}{2}}{l} \right)^2$$

$$Z_k(x) = \sin\left(\frac{k - \frac{1}{2}}{l}\right) \pi x$$

$$(4) \frac{T'}{a^2 T} = -\lambda, \quad T_k = C_k e^{-\lambda_k a^2 t}$$

$$v(x, t) = \sum_{k=1}^{+\infty} C_k e^{-a^2 \left(\frac{k\pi - \frac{\pi}{2}}{l}\right)^2 t} \sin \frac{(k - \frac{1}{2})\pi x}{l}$$

例3:

$$\left\{ \begin{array}{l} u_t = a^2 u_{xx} - b^2 u, 0 < x < l \\ u_x|_{x=0} = 0, u|_{x=l} = u_1 \\ u|_{t=0} = \frac{u_1}{l^2} x^2 \end{array} \right. \rightarrow \left\{ \begin{array}{l} v_t = a^2 v_{xx} \\ v_x|_{x=0} = 0, v|_{x=l} = u_1 e^{-b^2 t} \\ v|_{t=0} = \frac{u_1}{l^2} x^2 \end{array} \right.$$

$$(1) \quad u = v + w \quad v = u - w \quad v = e^{b^2 t} u$$

$$w = \frac{u_1}{l^2} x^2$$

$$(2) \quad \left\{ \begin{array}{l} v_t = a^2 v_{xx} + a^2 \frac{2u_1}{l^2} - b^2 v - b^2 w \\ v_x|_{x=0} = 0, v|_{x=l} = 0 \\ v|_{t=0} = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} v_t = a^2 v_{xx} - b^2 v \\ v_x|_{x=0} = 0, v|_{x=l} = 0 \end{array} \right.$$

$$v = z \tau$$

$$z \tau' = a^2 z'' \tau - b^2 z \tau$$

$$\frac{\tau'}{a^2 \tau} + \frac{b^2}{a^2} = \frac{z''}{z} = -\lambda$$

$$\left\{ \begin{array}{l} z'' + \lambda z = 0 \\ z'(0) = z(l) = 0 \end{array} \right.$$

$$z = C_1 \cos \sqrt{\lambda} x$$

$$\sqrt{\lambda} l = k\pi - \frac{\pi}{2}$$

$$\lambda_k = \left(\frac{2k-1}{2l} \pi \right)^2$$

$$z_k = \cos \left(\frac{2k-1}{2l} \pi x \right)$$

$$-\frac{b^2 u_1}{l^2} x^2 + \frac{a^2 \cdot 2u_1}{l^2} = \sum_{k=1}^{+\infty} a_k \cos \frac{2k-1}{2l} \pi x$$

$$\Delta u = 0$$

$$u|_{x=0} = u|_{x=l} = 0$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2$$

$$z_n(x) = \sin \frac{n\pi x}{l}$$

$$n=1, 2, 3, \dots$$

$$u_x|_{x=0} = u_x|_{x=l} = 0$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2$$

$$z_n = \cos \frac{n\pi x}{l}$$

$$n=0, 1, 2, \dots$$

$$\tan x = x$$

$$x_n \in \left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$$

$$\lambda_n - \lambda_{n-1} \rightarrow \pi^2$$

$$\tan(x_n - x_{n-1}) \rightarrow 0$$

$$\tan x_n - \tan x_{n-1}$$

$$u|_{x=0} = u_x|_{x=0} = 0$$

$$\lambda_n = \left[\frac{(2n-1)\pi}{2l}\right]^2$$

$$z_n(x) = \sin \frac{(2n-1)\pi x}{2l}$$

$$u_x|_{x=0} = u|_{x=l} = 0$$

$$\lambda_n = \left[\frac{(2n-1)\pi}{2l}\right]^2$$

$$z_n(x) = \cos \frac{(2n-1)\pi x}{2l}$$

习题:

$$1. \begin{cases} u_{tt} = a^2 u_{xx} \\ u|_{x=0} = u|_{x=l} = 0 \\ u|_{t=0} = \varphi(x), u_t|_{t=0} = 0 \end{cases}$$

$$\varphi(x) = \begin{cases} \frac{h}{c}x & 0 \leq x \leq c \\ -\frac{h}{b-c}(x-l) & c \leq x \leq l \end{cases}$$

$$2. \begin{cases} u_{tt} = a^2 u_{xx} \\ u(0, t) = u(l, t) = 0 \\ u(x, 0) = 0, u_x(x, 0) = x(l-x) \end{cases}$$

$$3. \begin{cases} u_{tt} = a^2 u_{xx} \\ u|_{x=0} = u|_{x=l} = 0 \\ u|_{t=0} = \begin{cases} x & 0 \leq x \leq \frac{l}{2} \\ l-x & \frac{l}{2} \leq x \leq l \end{cases} \end{cases}$$

$$4. \begin{cases} u_{tt} = a^2 u_{xx} \\ u(0, t) = u(l, t) = 0 \\ u|_{t=0} = 0, u_t = \text{冲量} \end{cases}$$

弦振——分部积分

$$5. \begin{cases} u_t = a^2 u_{xx} \\ u|_{x=0} = u|_{x=l} = 0 \\ u|_{t=0} = x(l-x) \end{cases}$$

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-a^2 \lambda_n^2 t} \sin \frac{n\pi x}{l}$$

$$C_n = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{n\pi x}{l} dx$$

正弦展开

$$6. \begin{cases} u_t = a^2 u_{xx} \\ \underline{u_x|_{x=0} = u_x|_{x=l} = 0} \\ u|_{t=0} = \lambda \end{cases}$$

$\rightarrow n$ 可从 0 取起

余弦展开

$$7. \begin{cases} u_t = a^2 u_{xx} \\ u|_{x=0} = u|_{x=l} = 0 \\ u|_{t=0} = \varphi(x) \end{cases}$$

正弦展开

$$\begin{cases} u_{xx} - a^2 u_t + A e^{-ax} = 0 \\ u|_{x=0} = u|_{x=l} = 0 \\ u|_{t=0} = T \end{cases}$$

$$\begin{cases} u_t = a^2 u_{xx} + A \\ \text{---} \\ \text{---} \end{cases}$$

$$\begin{cases} u_t = a^2 u_{xx} \\ u|_{x=0} = 0, u|_{x=l} = 5 \\ u|_{t=0} = kx \end{cases}$$

$$v_n' + p v_n = f$$

$$(v_n e^{\int p dx})' = f e^{\int p dx}$$

$$v_n e^{\int p dx} = \int e^{\int p dx} f dx + C$$