$$u_{tt} = a^2(u_{xx} + u_{yy} + u_{zz}) \quad (x,y,z)\in R^3$$
$$u|_{t=0} = \varphi(x,y,z)$$
$$u_t|_{t=0} = \phi(x,y,z)$$

① $u(x,y,z,t) = u(\sqrt{x^2+y^2+z^2}, t)$ 球对称性
$= u(r,t)$

$$u_x = u_r r_x = u_r \frac{x}{\sqrt{x^2+y^2+z^2}} = u_r \frac{x}{r}$$
$$u_{xx} = u_{rr}\frac{x^2}{r^2} + u_r \frac{r^2-x^2}{r^3}$$
$$u_{yy} = u_{rr}\frac{y^2}{r^2} + u_r \frac{r^2-y^2}{r^3}$$
$$u_{zz} = u_{rr}\frac{z^2}{r^2} + u_r \frac{r^2-z^2}{r^3}$$
$$u_{xx}+u_{yy}+u_{zz} = u_{rr} + u_r\frac{2}{r}$$
$$\frac{1}{r}(r u_{rr} + 2u_r)$$

$$u_{tt} = (u_{rr} + u_r\frac{2}{r})a^2$$
$$u|_{t=0} = \varphi(r) \quad r>0,\ t>0$$
$$u_t|_{t=0} = \phi(r)$$

$$(ru)_{tt} = a^2(ru)_{rr}$$
$$ru = v(r,t)$$
$$\begin{cases} v_{tt} = a^2 v_{rr} & 0<r<+\infty \\ v|_{t=0} = r\varphi(r) & v|_{r=0}=0 \\ v_t|_{t=0} = r\phi(r) \end{cases}$$

作奇延拓 $v(r,t) = F(r-at) + G(r+at)$

$$\begin{cases} F(r) + G(r) = r\varphi(r) \\ -aF'(r) + aG'(r) = r\phi(r) \end{cases}$$
$$F(-at) + G(at) = 0$$
$$\forall s>0,\ F(-s) = -G(s)$$

$$-F(r) + G(r) = \int_0^r s\phi(s) + C$$
$$G(r) = \frac{1}{2}r\varphi(r) + \frac{1}{2a}\int_0^r s\phi(s)\,ds + \frac{C}{2}$$
$$F(r) = \frac{1}{2}r\varphi(r) - \frac{1}{2a}\int_0^r s\phi(s)\,ds - \frac{C}{2}$$

$$v(r,t) = F(r-at) + G(r+at)$$

$$= \frac{1}{2}\left[(r+at)\varphi(r+at) + (r-at)\varphi(r-at)\right] + \frac{1}{2a}\int_{r-at}^{r+at} s\phi(s)\,ds$$

$(r-at>0)$

$$v(r,t) = F(r-at) + G(r+at)$$

$$= -G(at-r) + G(r+at)$$

$$= \frac{1}{2}\left[(r+at)\varphi(r+at) - (at-r)\varphi(at-r)\right] \quad (r-at<0)$$

$$+ \frac{1}{2a}\int_{at-r}^{at+r} s\phi(s)\,ds$$

$$v(r,t) = \begin{cases} \sim & r>at \\ \sim & r<at \end{cases}$$

$u(r,t) = \frac{v(r)}{r}$ $r=0$，利用 $\lim\limits_{r\to 0}\frac{v(r)}{r}$ 代替 $u(0,t)$

$$= \lim_{r\to 0} v'(0)$$

$$= \varphi(0)$$

若 $v'|_{r=0}=0$，先求通解

球面平均 $\bar{u} = \frac{1}{4\pi r^2} \iint\limits_{S_M^r} u(x, y, z, t) dS$

$= \frac{1}{4\pi r^2} \iint\limits_{C_M^r} u \sqrt{1+z_x^2+z_y^2}\, dx dy$

$M(x_0, y_0, z_0)$

$\begin{cases} x = x_0 + r\cos\theta\sin\varphi & 0 \le \theta \le 2\pi \\ y = y_0 + r\sin\theta\sin\varphi & 0 \le \varphi \le \pi \\ z = z_0 + r\cos\varphi \end{cases}$

$dS = r\sin\varphi d\theta\, r d\varphi$

$= r^2 \sin\varphi d\theta d\varphi$

$\iint\limits_{\Sigma} f(x,y,z) dS \xlongequal[(u,v)\in D']{} \iint\limits_{D'} f(u,v) \sqrt{EG-F^2}\, du dv$

$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$ $E = x_u^2 + y_u^2 + z_u^2$

$G = y_v^2 + x_v^2 + z_v^2$

$F = x_u x_v + y_u y_v + z_u z_v$

$\bar{u} = \frac{1}{4\pi r^2} \iint\limits_{S_0^1} u(x_0 + rx', y_0 + ry', z_0 + rz', t) r^2 dw$

$= \frac{1}{4\pi} \iint\limits_{S_0^1} u(x_0 + rx', y_0 + ry', z_0 + rz', t) dw \sim F(r,t)$

① $\bar{u}(r,t) = \frac{1}{4\pi r^2} \iint\limits_{S_M^r} u(x,y,z,t) dS \xrightarrow{\text{积分中值定理}} \frac{1}{4\pi r^2} u(x', y', z', t) 4\pi r^2$

$= u(x', y', z', t)$

$= \frac{1}{4\pi} \iint\limits_{S_0^1} u(x_0 + r\bar{x}, y_0 + r\bar{y}, z_0 + r\bar{z}, t) d\bar{S}$

② $\lim\limits_{r\to 0} \bar{u}(r,t) = u(M,t)$

③ $\bar{u}(-r,t) = \bar{u}(r,t)$ 偶函数

$\|$

$\frac{1}{4\pi} \iint\limits_{S_0^1} u(x_0 - r\bar{x}, y_0 - r\bar{y}, z_0 - r\bar{z}, t) d\bar{S}$

④
$$\begin{cases} \left(\frac{1}{4\pi r^2}\iint\limits_{S_M^r} u\,ds\right)_{tt} = a^2\iint\limits_{S_M^r} (u_{xx}+u_{yy}+u_{zz})\,ds\,\frac{1}{4\pi r^2} \\ \frac{1}{4\pi r^2}\iint\limits_{S_M^r} (u|_{t=0})\,ds = \frac{1}{4\pi r^2}\iint\limits_{S_M^r} \varphi\,ds \\ \frac{1}{4\pi r^2}\iint\limits_{S_M^r} (u_t|_{t=0})\,ds = \frac{1}{4\pi r^2}\iint\limits_{S_M^r} \phi\,ds \end{cases}$$

⑤
$$\begin{cases} \bar{u}_{tt} = a^2 v_{rr} \quad 0<r<+\infty,\ t>0 \\ \bar{u}|_{t=0} = \bar{\varphi}(r) \\ \bar{u}_t|_{t=0} = \bar{\phi}(r) \end{cases}$$

$$\iiint\limits_{B_M^r} (u_{xx}+u_{yy}+u_{zz})\,dV$$

$$= \oiint\limits_{S_M^r} (u_x\cos\alpha + u_y\cos\beta + u_z\cos\gamma)\,ds\,\frac{1}{4\pi r^2}$$

$$= \oiint\limits_{S_M^r} \frac{\partial u}{\partial \vec{n}}\,ds\,\frac{1}{4\pi r^2} \qquad \left.\frac{\partial u}{\partial \vec{n}}\right|_S = \operatorname{grad}\vec{u}\,\vec{e_n}$$

$$= \frac{1}{4\pi a^2}\oiint\limits_{S_M^r} \left(\frac{\partial u}{\partial r}\right) ds$$

$$= \frac{1}{4\pi}\iint\limits_{S_0^1} \frac{\partial u}{\partial r}\,ds$$

$$= \frac{\partial \bar{u}}{\partial r}$$

$$= \frac{1}{4\pi r^2}\iiint\limits_{B_M^r} \frac{1}{a^2} u_{tt}\,dV$$

$$r^2\bar{u}_r = \int_0^r \frac{1}{a^2} R^2 \bar{u}_{tt}\,dR$$

$$a^2(r^2\bar{u}_r)_r = r^2\bar{u}_{tt}$$

$$a^2[\bar{u}_{rr} r^2 + 2r\bar{u}_r] = r^2\bar{u}_{tt}$$

$$a^2[r\bar{u}]_{rr} = [r\bar{u}]_{tt}$$

$$v = r\bar{u} \qquad v_{tt} = a^2 v_{rr}$$

$$\iiint\limits_{B_M^r} f\,dV = \int_0^r dR \iint\limits_{S_0^1} f R^2\,dw$$

$v(r)$是r的奇函数，$v = r\bar{u}$

$$v(r,t) = \frac{1}{2}[(r+at)\bar{\varphi}(r+at) + (r-at)\bar{\varphi}(r-at)] + \frac{1}{2a}\int_{r-at}^{r+at} s\phi(s)\,ds \quad r>at$$

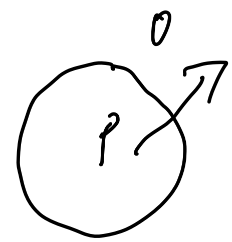
$$= \sim \qquad r<at$$

$$u=\lim_{r\to 0}\bar{u}=\lim_{r\to 0}\frac{v}{r}=\lim_{r\to 0}v'(0,t)$$

$$u(M,t)=\frac{1}{4\pi a}\frac{\partial}{\partial t}\iint_{S_M^{at}}\frac{\varphi}{at}dS+\frac{1}{4\pi a}\iint_{S_M^{at}}\frac{\phi}{at}dS$$ 球面波 半径为at

$$\begin{cases} u_{tt}=a^2(u_{xx}+u_{yy}+u_{zz}) \\ u|_{t=0}=x+y+z=\varphi \\ u_t|_{t=0}=0=\phi \end{cases}$$ 可拆为阶方程分别解

$$u(x,y,z,t)=\frac{1}{4\pi a}\frac{\partial}{\partial t}\iint_{S_M^{at}}\frac{x'+y'+z'}{at}dS'$$



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