1. 振动方程

2. 热传导方程

3. 静电场方程

偏微分方程

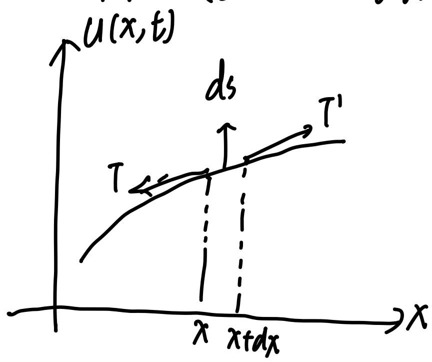
初值条件 initial

边界条件 bounding
- 取值 Drichlet
- 一阶导取值 Neumann
- 一阶导乘积 Robin

$u_{tt}=a^2\Delta u$ (双曲型方程)

一、振动方程(多维)

$u_{tt}=a^2(u_{xx})$ $\qquad$ $u_{tt}=a^2\sum_{i=1}^{n} u_{x_ix_i}$



$$ds=\sqrt{dx^2+du^2}$$

$$\begin{cases} T\cos\alpha = T'\cos\alpha' \\ T'\sin\alpha' - T\sin\alpha - \rho ds g = \rho ds \dfrac{\partial^2 u}{\partial^2 t} \end{cases}$$

$$T'\cos\alpha' \underbrace{\tan\alpha'}_{=\frac{\partial u}{\partial x}(x+dx)} - T\cos\alpha \underbrace{\tan\alpha}_{=\frac{\partial u}{\partial x}(x)} - \rho dx g = \rho dx\, u_{tt}$$

$dx\to 0$

$$T_0\frac{\partial^2 u}{\partial x^2} - \underbrace{\rho g}_{\text{略去}g} = \rho u_{tt}$$

非齐次化齐次

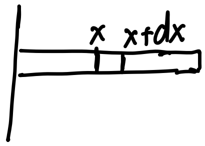
$$\left.\begin{aligned} L(u+v) &= Lu+Lv \\ L(cu) &= cL(u) \end{aligned}\right\} \Rightarrow \text{线性}$$

$\Rightarrow$ 线性偏微分算子; operator

$$Lu = \left(\frac{\partial^2}{\partial t^2} - a^2\frac{\partial^2}{\partial x^2}\right)u$$
$$= \left(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x}\right)u$$

若令 $\xi = x+at$, $\eta = x-at$

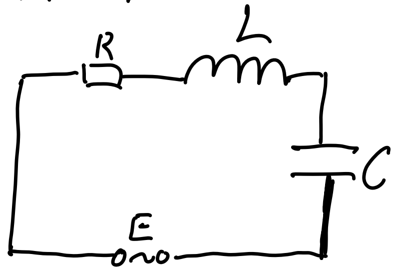
则有 $Lu = u_{\xi\eta}$



$$E u_x(x+dx) - E u_x(x) = \rho\, dx\, u_{tt}$$

$$\frac{E}{\rho} u_{xx} = u_{tt}$$

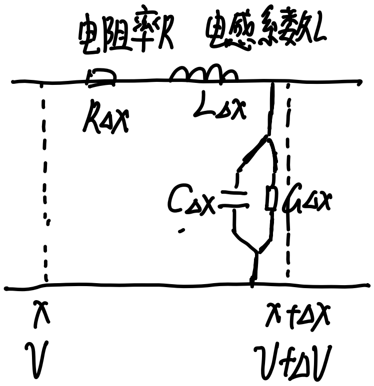
RLC电路



$$E(t) - L\frac{dI}{dt} = RI(t) + \frac{1}{C}\int_0^t I(t)\,dt$$

$$\frac{dE}{dt} - L\frac{d^2I}{dt^2} = R\frac{dI}{dt} + \frac{1}{C}I(t)$$

$$y'' + py' + qy = f(x)$$



$$\begin{cases} V - (V+\Delta V) = R\Delta x\, i(t) + L\Delta x \dfrac{\partial i}{\partial t} \\ i - (i+\Delta i) = C\Delta x \dfrac{\partial v}{\partial t} + vG\Delta x \end{cases}$$

$$\frac{1}{C}\int_0^t i(t)\,dt = v$$

$$\begin{cases} L\dfrac{\partial i}{\partial t} + Ri(t) + \dfrac{\partial v}{\partial x} = 0 & ① \\ C\dfrac{\partial v}{\partial t} + Gv + \dfrac{\partial i}{\partial x} = 0 & ② \end{cases}$$

消v，$①\dfrac{d}{dt}\times C - ②\dfrac{d}{dx}$　　　消i　$①\dfrac{d}{dx} - ②\dfrac{d}{dt}\times L$

$$\Rightarrow \frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} + (RC+GL)\frac{\partial I}{\partial t} + GRI$$

高频电路中，G, C忽略不计

从而有 $I_{tt} = \left(\frac{1}{\sqrt{LC}}\right)^2 I_{xx}$ $U_{tt} = \left(\frac{1}{\sqrt{LC}}\right)^2 U_{xx}$

$U_{tt} = a^2 U_{xx}$ $u(x,t) = \underbrace{f(x-at)}_{\text{右行波}} + \underbrace{g(x+at)}_{\text{左行波}}$ 行波解

麦克斯韦方程： $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$

$\nabla \cdot \vec{B} = 0$

$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\nabla \times \vec{B} = (\underbrace{\sigma E}_{\text{真空中为0}} + \varepsilon \frac{\partial E}{\partial t})\mu$

梯度算子： $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$\text{grad}\, f = \nabla f$

$\text{div}\,\vec{F} = \nabla \cdot \vec{F}$ 高斯公式

$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ P & Q & R \end{vmatrix} = \text{Rot}\,\vec{F}$

斯托克斯公式

Laplace算子 $\Delta = \nabla^2$ 当$\Delta f = 0$时，f为调和函数 e.g: $z = x^2 - y^2$

$\nabla^2 f = (\nabla \cdot \nabla) f = \nabla \cdot (\nabla f) = \nabla \cdot (\text{grad} f) = \text{div}(\text{grad} f) = \Delta f$

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

易证 ↓ 与 $\vec{b}, \vec{c}$ 共面

$\nabla \times (\nabla \times \vec{B}) = \nabla(\nabla \cdot \vec{B}) - (\nabla \cdot \nabla)\vec{B}$

$= -\Delta \vec{B} = \varepsilon\mu \frac{\partial}{\partial t}(\nabla \times \vec{E})$

$$\Rightarrow -\Delta \vec{B} = \varepsilon\mu(-B_{tt}) \qquad \Delta \vec{B} = \varepsilon\mu B_{tt}$$

$$c = \frac{1}{\sqrt{\varepsilon\mu}} \qquad B_{tt} = \frac{1}{\varepsilon\mu}(\vec{B}_{xx} + \vec{B}_{yy} + \vec{B}_{zz})$$

E同上

$E = -\mathrm{grad}\, u$ 势场做功只与位置有关

势函数

$$-\mathrm{div}(\mathrm{grad}\, u) = \frac{\rho}{\varepsilon}$$

$$-\Delta u = \frac{\rho}{\varepsilon} \longrightarrow \text{泊松方程}$$

$$-\Delta u = 0 \longrightarrow \text{Laplace 方程}$$

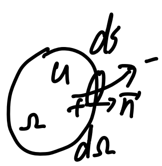
elliptic型方程 hyperbolic型方程

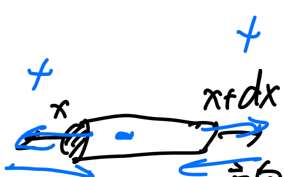
2. 热传导方程

通量：$\Phi = \vec{F} \cdot \vec{n} \cdot ds$

$$dQ = \underbrace{k}_{\text{热导率}}(-\mathrm{grad}\, \vec{u}) \cdot \vec{n} \cdot ds\, dt \ (\text{放热})$$

$\vec{F} = -\mathrm{grad}\, \vec{u}$ $\qquad \to -k\frac{\partial u}{\partial n}$





$$\frac{\partial u}{\partial l} = \mathrm{grad}\, u \ \vec{e_l} \longrightarrow \text{单侧导数}$$

方向导数

吸热 $k\frac{\partial u}{\partial(-x)}\Big|_x ds\,dt + k\frac{\partial u}{\partial x}\Big|_{(x+dx)} ds\,dt = cm\Delta u$ $\quad \Delta u = \underbrace{u(t+dt) - u(t)}$ $\quad \frac{\partial u}{\partial t}dt$

$m = \rho\, ds\, dx$

$$k\,ds\,dt\left(u_x(x+dx)-u_x(x)\right)=c\rho\,ds\,dx\,dt\,u_t$$

parabolic方程 $u_t=\frac{k}{c\rho}u_{xx}=a^2u_{xx}$

$$u_t=a^2\Delta u$$

$u|_r=f_1$ $u_x|_r=f_2$ $u_x+au|_r=f_3$

习题1: 5 6 1-4例