$$F(f)=\int_{-\infty}^{+\infty} f(x)e^{-ix\xi}dx=\hat{f}(\xi)$$

$$F^{-1}(\hat{f}(\xi))=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(\xi)e^{ix\xi}d\xi=f(x)$$

利用阶跃函数

$$\int_{-\infty}^{+\infty} H(x)f(x)e^{-\alpha x}e^{-ix\xi}dx$$

$$=\int_{0}^{+\infty} f(x)e^{-(\alpha+i\xi)x}dx \quad (s=\alpha+i\xi)$$

Laplace变换

$$=\int_{0}^{+\infty} f(x)e^{-sx}dx=F(s)=\mathcal{L}(f)$$

(1) $f(x)=1 \quad (x>0)$

$$\mathcal{L}(1)=\int_0^{+\infty}e^{-sx}dx=-\frac{1}{s}e^{-sx}\Big|_0^{+\infty}$$

$s>0$ 时收敛

$$=\frac{1}{s} \quad (Re[s]>0)$$

关于s求导

(2) $f(x)=x$

$$\mathcal{L}(x)=\int_0^{+\infty}xe^{-sx}dx=\frac{-1}{s}\int_0^{+\infty}x\,de^{-sx}$$

$$=-\frac{1}{s}xe^{-sx}\Big|_0^{+\infty}+\frac{1}{s}\int_0^{+\infty}e^{-sx}dx$$

$$=\frac{1}{s^2} \quad (Re[s]>0)$$

$$\int_0^{+\infty}x^ne^{-sx}dx=\frac{n!}{s^{n+1}} \quad n=0,1,2,\cdots$$

条件: (1) $x\to+\infty \quad |f(x)|<e^{cx}$ $\qquad Res>c$

(2) $\int_0^{+\infty}[C_1f(x)+C_2g(x)]e^{-sx}dx$

$$=C_1\int_0^{+\infty}f(x)dx+C_2\int_0^{+\infty}g(x)dx$$

$$\mathcal{L}(C_1 f(x) + C_2 g(x)) = C_1 \mathcal{L}(f(x)) + C_2 \mathcal{L}(g(x))$$

(3)

$$f(x) = e^{ax}$$

$$\int_{-\infty}^{+\infty} e^{ax} e^{-sx} dx = \int_0^{+\infty} e^{-(s-a)x} dx$$

$$= -\frac{1}{s-a} e^{-(s-a)x} \Big|_0^{+\infty}$$

$$= \frac{1}{s-a} \quad Re(s-a) > 0$$

$$\int_0^{+\infty} (f(x) e^{ax}) e^{-sx} dx = F(s-a) \quad Re(s-a) > c$$

(4) $e^{i w x} = \cos wx + i \sin wx$

$$\cos wx \rightarrow \frac{s}{s^2 + w^2} \quad \sin wx \rightarrow \frac{w}{s^2 + w^2} \quad Re[s] > 0$$

e.g $$\int_0^{+\infty} e^{x} \sin wx \, e^{-sx} dx = \frac{w}{(s-1)^2 + w^2}$$

$$\mathcal{L}[f'] = \int_0^{+\infty} f'(x) e^{-sx} dx = e^{-sx} f(x) \Big|_0^{+\infty} + s \int_0^{+\infty} f(x) e^{-sx} dx$$

$$= s \mathcal{L}[f] - f(0)$$

$$\mathcal{L}[f^{(n)}] = s^n \mathcal{L}[f] - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

$$\begin{cases} y'' - 3y' + 2y = e^{-x} \\ y(0) = 0, \; y'(0) = 0 \end{cases}$$

$$\mathcal{L}(y) = Y(s)$$

$$\mathcal{L}(y') = s \mathcal{L}(y) - y(0) = s Y(s)$$

$$\mathcal{L}(y'') = s^2 Y(s)$$

$$Y = \frac{1}{(s-1)(s-2)} \cdot \frac{1}{(s+1)} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) - \frac{1}{3}\left(\frac{1}{s-2} - \frac{1}{s+1}\right)$$

$$= \frac{1}{2}\frac{1}{s-1} - \frac{1}{6}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s-2}$$

$$y(x) = \frac{1}{2}e^{x} - \frac{1}{3}e^{2x} - \frac{1}{6}e^{-x}$$

Laplace变换解常微分方程

$$\begin{cases} v_n'' + \Big(\underbrace{\frac{an\pi}{l}}^{\uparrow w}\Big)^2 v_n = f_n \\ v_n(0) = v_n'(0) = 0 \end{cases}$$

(1) $\mathcal{L}[v] = V(s)$ $\mathcal{L}[f] = F(s)$

$$s^2 V(s) + w^2 V = F(s)$$

$$\mathcal{L}[f*g] = \mathcal{L}[f]\cdot\mathcal{L}[g]$$

$$V(s) = \frac{1}{w}\left(\frac{w}{s^2+w^2}\right)\cdot F(s)$$

$$v_n(t) = \frac{1}{w}\int_0^t f(\tau)\sin w(t-\tau)\,d\tau$$

$$\mathcal{L}[f] = F(s)$$

$$\mathcal{L}[f'] = sF(s) - f(0)$$

$$\mathcal{L}\left[\int_0^x f(t)dt\right] = \frac{1}{s}\mathcal{L}[f] \qquad \mathcal{L}[g(x)] = \frac{1}{s}\mathcal{L}[g'(x)] + \frac{1}{s}\mathcal{L}[g(0)]$$

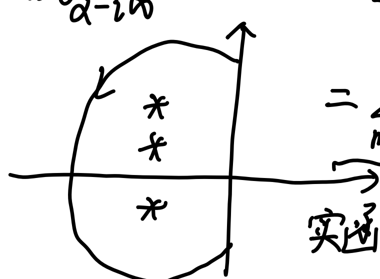
与Fourier变换联系： $H(x) f(x) e^{-\alpha(x)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\alpha+i\xi)\, e^{ix\xi} d\xi \quad x>0$

$$f(x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} F(\alpha+i\xi)\, e^{i(\alpha+i\xi)x} d(\alpha+i\xi)$$

留数定理

$$= \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} F(s) e^{sx} ds = \frac{1}{2\pi i}\oint_C F(z)\, e^{xz} dz$$

复变



$$= \sum_{m=1}^{k} \operatorname{Res}_{z_k}[F(z) e^{xz}]$$

实函数分解

流数

$$F(s) = \frac{1}{s(s-1)^2}$$
$$= \frac{1}{(s-1)^2} - \frac{1}{s-1} + \frac{1}{s}$$
$$f(x) = xe^x - e^x + 1$$
$$\frac{1}{(s-1)^2} \to xe^x$$

$$f(x) = \sum Res\,[F(s)e^{sx}, s_k]$$
$s=0$, $s=1$
$$Res\,[Fe^{sx}, 0] = \lim_{s\to 0} \frac{1}{(s-1)^2} e^{sx} = 1$$
$$Res\,[Fe^{sx}, 1] = \lim_{s\to 1} \left(\frac{e^{sx}}{s}\right)' = xe^x - e^x$$

$$F(s) = \frac{1}{s} \frac{1}{(s-1)^2}$$
$$f(x) = \int_0^x u(\tau) w(x-\tau) d\tau$$
$$= \int_0^x \tau e^\tau d\tau$$
$$= [\tau e^\tau - e^\tau]\Big|_0^x$$
$$= xe^x - e^x + 1$$

$H(x)$, $\delta(x)$
$H'(x) = \delta(x)$

$$\forall \varphi(x) \in k \quad \langle H'(x)\ \varphi(x)\rangle = \langle \delta(x)\ \varphi(x)\rangle$$
$$\hat{\delta} = \int_{-\infty}^{+\infty} \delta(x) e^{-ix\xi} dx = 1$$
$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} d\xi$$
$$\mathcal{L}[\delta(x)] = \int_0^{+\infty} H'(x) e^{-sx} dx = 1$$
$$s\int_0^{+\infty} e^{-sx} dx$$

$$\begin{cases} u_t = a^2 u_{xx} & -\infty < x < +\infty,\ t > 0 \\ u|_{t=0} = \varphi(x) \end{cases}$$
$$F(u) = \int_{-\infty}^{+\infty} u(x,t) e^{ixw} dx = U(w,t)$$

$$F(u_t) = \int_{-\infty}^{+\infty} u_t e^{-ixw} dx = \left(\int_{-\infty}^{+\infty} u e^{-ixw} dx\right)_t$$
$$= U_t$$

$$F(u_{xx}) = \int_{-\infty}^{+\infty} u_{xx} e^{-ixw} dx = -w^2 U$$
$$= (-iw)^2 \left(\int_{-\infty}^{+\infty} u e^{-ixw} dx\right)$$

$$\begin{cases} U_t = -a^2 w^2 U, & t>0 \\ U|_{t=0} = \hat{\varphi}(w) \end{cases}$$

$$U(w,t) = \hat{\varphi}(w) e^{-a^2w^2t}$$

$$u(x,t) = \varphi(x) \, F^{-1}(e^{-a^2w^2t}) = \varphi(x) * k(x,t) = \int_{-\infty}^{+\infty} \varphi(s) \frac{1}{2a\sqrt{\pi t}} e^{-\frac{(x-s)^2}{4a^2t}} ds$$

$$F^{-1}(e^{-a^2w^2t}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-a^2w^2t} e^{-ixw} dw$$ 作一步实虚互换
$$= \frac{1}{2a\sqrt{\pi t}} e^{-\frac{x^2}{4a^2t}}$$

非齐次 $u_t = a^2 u_{xx} + f(x,t) \Rightarrow u = k*f$

$f(x), g(x)$ 定义在 $[0,+\infty)$ 上 $f(x)*g(x) = \int_{-\infty}^{+\infty} f(t) g(x-t) dt$
$$= \int_{-\infty}^{0} + \int_{0}^{x} + \int_{x}^{+\infty}$$
$$= \int_{0}^{x} f(t) g(x-t) dt$$

$\mathcal{L}[u(x,t)] = U(x,s)$ $U(x,s) = \int_0^{+\infty} u(x,t) e^{-st} dt$

$$\begin{cases} sU(s) = a^2 U_{xx} \\ U|_{x=0} = F(s) \quad U|_{x=+\infty} = 0 \end{cases}$$

$$U(x,s) = c_1 e^{\frac{\sqrt{s}}{a}x} + c_2 e^{-\frac{\sqrt{s}}{a}x}$$ $$u(x,t) = f(t) * \mathcal{L}^{-1}\left[e^{-\frac{x}{a}\sqrt{s}}\right]$$

$c_1 = 0$ $c_2 = F(s)$

$$\mathcal{L}^{-1} = \frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{+\infty} e^{-y^2}dy$$

$$\mathcal{L}^{-1}[e^{-\frac{x}{a}\sqrt{s}}] = \mathcal{L}^{-1}[s\cdot\frac{1}{s}e^{-\frac{x}{a}\sqrt{s}}]$$

$$= \left(\frac{2}{\sqrt{\pi}}\int_{\frac{x}{2a\sqrt{t}}}^{+\infty} e^{-y^2}dy\right)_t$$

$$= \frac{x}{2a\sqrt{\pi t}\,t}e^{-\frac{x^2}{4a^2t}}$$

$U_t = U_{xx}$

$\uparrow$

$u(x,t)$

$u(\lambda x, \lambda^2 t)$ $\lambda = \frac{1}{x}$

$\searrow u(1, \frac{t}{x^2})$ 非各向同性的伸缩变换法

$U_{tt} = a^2 U_{xx}$, $0<x<l$, $t>0$

$u|_{x=0} = 0$ $U_x|_{x=l} = \frac{A}{E}\sin wt$

$u|_{t=0} = 0$ $U_t|_{t=0} = 0$

$\mathcal{L}[u(x,t)] = U(x,s)$

$$\begin{cases} s^2U = a^2U_{xx} \\ U|_{x=0} = 0, \quad U_x|_{x=l} = \frac{A}{E}\frac{w}{s^2+w^2} \end{cases}$$

$U_{xx} = \frac{s^2}{a^2}U$ $U(x,s) = C_1e^{\frac{s}{a}x} - C_1e^{-\frac{s}{a}x}$

$$\frac{s}{a}C_1[e^{\frac{s}{a}l} + e^{-\frac{s}{a}l}] = \frac{A}{E}\frac{w}{s^2+w^2}$$

$$U(x,s) = \frac{A}{E}\frac{w}{s^2+w^2}\frac{a}{s}\cdot\frac{1}{e^{\frac{s}{a}l}+e^{-\frac{s}{a}l}}[e^{\frac{s}{a}x} - e^{-\frac{s}{a}x}]$$

$$u(x,t) = \sum \mathrm{Res}[U(x,s)e^{st}, s_k]$$

s_k: $s(s^2+w^2)\,\mathrm{ch}(\frac{s}{a}l) = 0$

$0 \quad \pm \omega i \quad \pm \frac{ia}{l}(k-\frac{1}{2})\pi \, (k=1,2,\cdots)$

(3.81) (3.94)

5. (2) (5)
7. (3) (4)
8.