勒让德多项式的奇偶项不能同时为无限项
应用：代替泰勒级数 —— 广义级数

① 勒让德多项式的正交性

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0 & m \neq n \\ \dfrac{2}{2n+1} & m = n \end{cases}$$

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \sum \int_{-1}^{1} x^c \frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$\int_{-1}^{1} x^c \frac{d^n}{dx^n}(x^2-1)^n\,dx$ 分部积分至x的多低次

$$I_n^2 = \frac{2\cdot 2\cdot 4\cdot 4\cdots 2n\cdot 2n}{3\cdot 3\cdot 5\cdot 5\cdots(2n+1)\cdot(2n+1)}$$

沃利公式

$$f(x) = \sum_{k=0}^{+\infty} C_k P_k(x)$$

$$\int_{-1}^{1} f(x)P_m(x)\,dx = \int_{-1}^{1}\left(\sum_{k=0}^{+\infty} C_k P_k(x)\right)P_m(x)\,dx$$

$$= C_m\int_{-1}^{1} P_m^2(x)\,dx$$

$$= C_m\frac{2}{2n+1}$$

$$C_k = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_m(x)\,dx$$

$P_{2n}(x)$ 偶函数　　$P_{2n-1}(x)$ 奇函数

以 |x| 为例

$$C_{2n} = \frac{4n+1}{2}\int_{-1}^{1} f(x)\, P_{2n}(x)\, dx$$

$$= \frac{4n+1}{2}\int_{-1}^{0} (-x)\, P_{2n}(x)\, dx + \frac{4n+1}{2}\int_{0}^{1} x\, P_{2n}(x)\, dx$$

$$= \frac{4n+1}{2}\int_{-1}^{0} (-x)\frac{1}{2^{2n}(2n)!}\frac{d^{2n}}{dx^{2n}}(x^2-1)^{2n} dx$$

$$+ \frac{4n+1}{2}\int_{0}^{1} x\frac{1}{2^{2n}(2n)!}\frac{d^{2n}}{dx^{2n}}(x^2-1)^{2n} dx$$

$$= \frac{4n+1}{2^{2n+1}(2n)!}\left[-x\frac{d^{2n-1}}{dx^{2n-1}}(x^2-1)^{2n}\Big|_{-1}^{0} - \right.$$

$y = x^2$

$$x^2 = C_0 P_0 + C_2 P_2 + \cdots + C_{2n} P_{2n}$$

$$C_0 = \int_{-1}^{1} x^2 dx = \frac{2}{3}$$

$$C_2 = \frac{k}{2}\Big|_{-1}^{1}$$

$$x^2 = C_0 + P_2$$

$$\frac{1}{3} + \frac{2}{3}P(x)$$

$$(n+1)P_{n+1}(x)+nP_{n-1}(x)=(2n+1)xP_n(x)$$

$$(1-x)^2P''(x)-2xP'(x)+l(l+1)P(x)=0$$

$l=n$时，方程为有限项多项式

$$-((1-x^2)P')'=n(n+1)P(x)$$

$$xP_n=\frac{n+1}{2n+1}P_{n+1}(x)+\frac{n}{2n+1}P_{n-1}(x)$$

$$xP_n=\sum_{k=1}^{+\infty}C_kP_k(x)$$

$$=C_{n-1}P_{n-1}(x)+C_{n+1}P_{n+1}(x)$$

勒让德多项式的递推式

$$\begin{cases}u_{xx}+u_{yy}+u_{zz}=0\\ u|_{x=0}=\cos^2\theta=z^2\end{cases}$$

$\lambda=l(l+1)$ $\quad r=C_1r^l+\underline{C_2r^{-l-1}}$

$$r=C_lr^l$$

$\cos\theta=x$， $\theta=p(x)=P(\cos\theta)$

$$(1-x^2)P''-2xP'+l(l+1)P=0$$

$$P_l(x)$$

$$\sum (C_l r^l P_l(\cos\theta))\big|_{r=1} = \cos^2\theta$$

$$x^2 = \sum C_l P_l(x) = \frac{1}{3} + \frac{2}{3} P_2(x)$$

$$u(r,\theta) = \frac{1}{3} + \frac{2}{3} r^2 P_2(\cos\theta)$$

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