考试：分离变量法(*) 16周考试　书上例题、课后题

$$\begin{cases}\Delta u = F \text{ in } \Omega \\ u|_{\partial\Omega} = f\end{cases} \qquad u(M_0) = -\oint_{\partial\Omega} f \frac{\partial G}{\partial n} dS_M - \int_{\Omega} G F \, dV$$

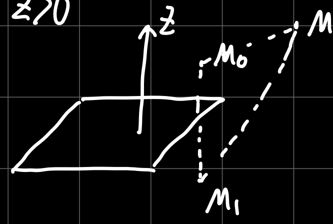
$\Omega \in R^3$，（Δ的基本解为 $\frac{1}{4\pi r_{M_0M}}$）　　其格林函数：

G：是M_0及其镜像点上的电位代数和

$\Omega \in R^2$　　$\frac{1}{2\pi} \ln \frac{1}{r_{M_0M}}$

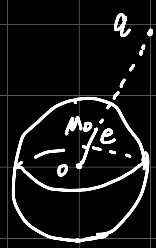
$M_0 \in \Omega$

(1) 对于 $z>0$



$$G = \frac{1}{4\pi r_{M_0M}} - \frac{1}{4\pi r_{M_1M}}$$

(2)



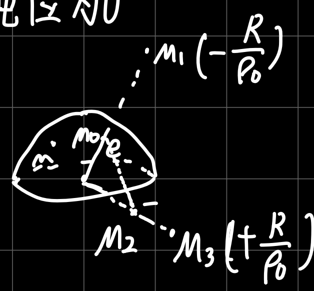
$OM_0 \cdot OM_1 = R^2$　　$OM_0 = \rho_0$

$$q = \frac{R}{\rho_0} e$$

电荷的选择：
使边界电位为0

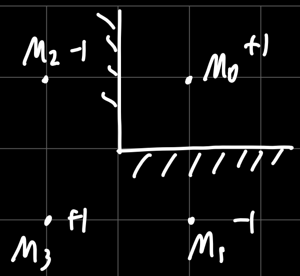
$$G = \frac{1}{4\pi r_{M_0M}} - \frac{Re}{\rho_0} \frac{1}{4\pi r_{M_1M}}$$

(3)



$$G = \frac{1}{4\pi}\left[\frac{1}{r_{M_0M}} - \frac{R}{\rho_0}\frac{1}{r_{M_1M}} - \frac{1}{r_{M_2M}} + \frac{R}{\rho_0}\frac{1}{r_{M_3M}}\right]$$

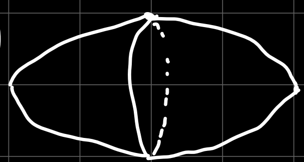
(4)



$\cdot M$

$$G=\frac{1}{2\pi}\left[\ln\frac{1}{r_{M_0M}}-\ln\frac{1}{r_{M_1M}}+\ln\frac{1}{r_{M_3M}}-\ln\frac{1}{r_{M_2M}}\right]$$

(5)

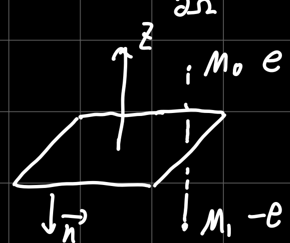


参考黎曼的文献

e.g: 91:

$$\begin{cases}\Delta u=0 & z>0\\ u(x,y,0)=f(x,y)\end{cases}$$

$$u(M_0)=-\oint_{\partial\Omega} f\frac{\partial G}{\partial n}ds$$



$$G(M,M_0)=\frac{1}{4\pi}\left(\frac{1}{r_{M_0M}}-\frac{1}{r_{M_1M}}\right)$$

$$=\frac{1}{4\pi}\left(\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}-\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}\right)$$

边界上为0； 除M_0外$\Delta G=0$也即调和

$$\frac{\partial G}{\partial n}\Big|_{\partial\Omega}=-\frac{\partial G}{\partial z}\Big|_{z=0}=\frac{-1}{4\pi}\left(-\frac{1}{2}(\ )^{-\frac{3}{2}}2(z-z_0)-\left(-\frac{1}{2}(\ )^{-\frac{3}{2}}2(z+z_0)\right)\right)_{z=0}$$

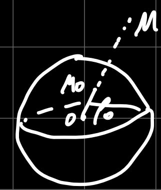
$$=\frac{-1}{4\pi}\left[\frac{1}{\left(\sqrt{(x-x_0)^2+(y-y_0)^2+z_0^2}\right)^3}\cdot 2z_0\right]$$

$$u(M_0)=\iint_{R^2}\frac{z_0 f(x,y)\,dx\,dy}{2\pi\left(\sqrt{(x-x_0)^2+(y-y_0)^2+z_0^2}\right)^3}$$

9.2:

$$\begin{cases}\Delta u=0 & x^2+y^2+z^2<R^2\\ u(x,y,z)=f(x,y,z) & x^2+y^2+z^2=R^2\end{cases}$$

采用分离变量法
较繁琐



$$u(M_0) = -\oint_{\partial\Omega} f\frac{\partial G}{\partial n}\,dS$$

$$G = \frac{1}{4\pi}\left[\frac{1}{r_{M_0M}} - \frac{R}{\rho_0}\frac{1}{r_{M_1M}}\right]$$

$OM=\rho$

$\angle MOM_0=\theta_0$

$OM_0\cdot OM_1=R^2$

$$r_{M_0M} = \sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos\theta_0}$$

$$r_{M_1M} = \sqrt{\rho^2+\left(\frac{R^2}{\rho_0}\right)^2-2\rho\left(\frac{R^2}{\rho_0}\right)\cos\theta}$$

$$\left.\frac{\partial G}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial G}{\partial \rho}\right|_{\rho=R}$$

$$= \frac{1}{4\pi}\Big[-\frac{1}{2}(\ )^{-\frac{3}{2}}(2\rho-2\rho_0\cos\theta) - \frac{R}{\rho_0}\left(-\frac{1}{2}\right)(\ )^{-\frac{3}{2}}$$

$$\left(2\rho-2\left(\frac{R}{\rho_0}\right)\cos\theta\right)\Big]_{\rho=R}$$

$$= \frac{1}{4\pi}\Big[\frac{R}{\rho_0}\left(R-\frac{R^2}{\rho_0}\cos\theta\right)\Big(R^2+\left(\frac{R^2}{\rho_0}\right)^2$$

$$-2R\left(\frac{R^2}{\rho_0}\right)\cos\theta\Big)^{-\frac{3}{2}}$$

$$-(R^2+\rho_0^2-2R\rho_0\cos\theta)^{-\frac{3}{2}}(R-\rho_0\cos\theta)\Big]$$

$$= -\frac{R^2-\rho_0^2}{4\pi(R^2+\rho_0^2-2R\rho_0\cos\theta)^{\frac{3}{2}}}\cdot\frac{1}{R}$$

从而，$u(M_0)=\frac{1}{4\pi R}\oint\limits_{\partial\Omega}\frac{(R^2-\rho_0^2)f}{(R^2+\rho_0^2+2R\rho_0\cos\theta_0)^{\frac{3}{2}}}ds$

$$ds=R^2\sin\varphi\, d\theta d\varphi$$

$$=\frac{R}{4\pi}\int_0^{2\pi}d\theta\int_0^{\pi}\frac{(R^2-\rho_0^2)f(\theta,\varphi)}{(R^2+\rho_0^2+2R\rho_0\cos\theta_0)^{\frac{3}{2}}}\sin\varphi d\varphi$$

$$M_0(x_0,y_0,z_0)\sim\rho_0(\cos\theta_0\sin\varphi_0,\ \sin\theta_0\sin\varphi_0,\ \cos\varphi_0)$$

$$M(x,y,z)\sim\rho(\cos\theta\sin\varphi,\ \sin\theta\sin\varphi,\ \cos\varphi)$$

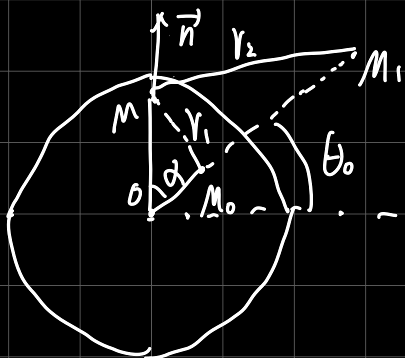
$\theta_0\to\alpha$ $\qquad \alpha=\langle\overrightarrow{OM},\overrightarrow{OM_0}\rangle$

$$\cos\alpha=\frac{\overrightarrow{OM_0}\cdot\overrightarrow{OM}}{|\overrightarrow{OM_0}|\cdot|\overrightarrow{OM}|}=\cos(\theta-\theta_0)\sin\varphi\sin\varphi_0+\cos\varphi\cos\varphi_0$$

93:

$$\begin{cases}\Delta u = 0 & x^2+y^2<R^2\\ u(x,y)=f(x,y) & x^2+y^2=R^2\end{cases}$$

$$u(M_0) = -\oint_{\partial\Omega} f \frac{\partial G}{\partial n}\, ds$$



$$G = \frac{1}{2\pi}\ln\frac{1}{r_{M_0M}} - \frac{1}{2\pi}\ln\frac{\frac{R}{\rho_0}}{r_{M_1M}}$$

$$= \frac{1}{2\pi}\left(\ln r_{M_1M} - \ln r_{M_0M} - \ln\frac{R}{\rho_0}\right)$$

$$r_{M_0M} = \sqrt{\rho^2+\rho_0^2-2\rho\rho_0\cos\alpha}$$

$$r_{M_1M} = \sqrt{\left(\frac{R^2}{\rho_0}\right)^2+\rho^2-2\rho\left(\frac{R^2}{\rho_0}\right)\cos\alpha}$$

$$\left.\frac{\partial G}{\partial \rho}\right|_{\rho=R} = \frac{1}{4\pi}\cdot\left(\frac{2\rho-2\frac{R^2}{\rho_0^2}\cos\alpha}{\frac{R^4}{\rho_0^2}+\rho^2-\frac{2\rho R^2}{\rho_0}\cos\alpha} - \frac{1}{4\pi}\frac{2\rho-2\rho_0\cos\alpha}{\rho^2+\rho_0^2-2\rho\rho_0\cos\alpha}\right)\Bigg|_{\rho=R}$$

$$= \frac{1}{2\pi}\,\frac{\frac{\rho_0^2}{R}-R}{R^2+\rho_0^2-2R\rho_0\cos\alpha}$$

$$u(M_0) = \frac{1}{2\pi}\oint_{\partial\Omega}\frac{R-\frac{\rho_0^2}{R}}{R^2+\rho_0^2-2R\rho_0\cos\alpha}f(\theta)\,ds$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-\rho_0^2}{R^2+\rho_0^2-2R\rho_0\cos\alpha}f(\theta)\,d\theta$$

$$\alpha = \theta-\theta_0$$

(1) $u=\ln\frac{1}{\rho}=\ln\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$ 调和

$$=-\frac{1}{2}\ln\left[(x-x_0)^2+(y-y_0)^2\right]$$

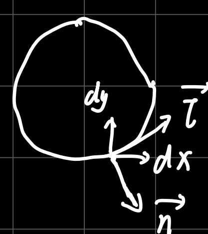
$$u_x=-\frac{1}{2}\frac{2(x-x_0)}{(x-x_0)^2+(y-y_0)^2}$$

$$u_{xx}=-\frac{1}{(x-x_0)^2+(y-y_0)^2}+(x-x_0)(\ )(\ )^{-2}\cdot 2(x-x_0)$$

$$=\frac{2(x-x_0)^2}{\left[(x-x_0)^2+(y-y_0)^2\right]^2}-\frac{1}{\left[(x-x_0)^2+(y-y_0)^2\right]}$$

从而 $u_{xx}+u_{yy}=0$

(2) $\oint_{\partial\Omega} P\,dx+Q\,dy=\iint_{\Omega}(Q_x-P_y)\,d\sigma$



$$\oint_{\partial\Omega}\left[P\cos(x,\vec{\tau})+Q\cos(y,\vec{\tau})\right]ds$$

$$(x,\vec{\tau})=\pi-(\vec{n},y)$$
$$(y,\vec{\tau})=(\vec{n},x)$$

$$=\oint_{\partial\Omega}-P\cos(\vec{n},y)+Q\cos(n,x)\,ds$$ 去掉负号仍成立

$$=\iint_{\Omega}(-P_y+Q_x)\,d\sigma$$

在上式中取 $P=u_x v \quad Q=u_y v$

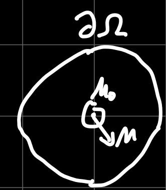
$$\oint_{\partial\Omega}\frac{\partial u}{\partial n}v\,ds=\iint_{\Omega}(\Delta u)v\,d\sigma+\iint_{\Omega}(\nabla u\nabla v)\,d\sigma \quad (I)$$

$u\leftrightarrow v$，相减

$$\oint_{\partial\Omega}\left(\frac{\partial u}{\partial n}v-\frac{\partial v}{\partial n}u\right)ds=\iint_{\Omega}\left[(\Delta u)v-(\Delta v)u\right]d\sigma$$

$$\begin{cases}\Delta u=0 \\ u|_{\partial\Omega}=f\end{cases}$$

$\Omega\in R^2$ $\qquad$ $\ln\frac{1}{r_{M_0M}}$ 是调和函数



(1) $\Delta u=0$

$\Delta v=0 \quad \text{in}\ \Omega\setminus\{M_0\}$

(2) $$\iint_{\Omega-B_\varepsilon}\left[(\Delta u)v-(\Delta v)u\right]d\sigma=0$$

$$=\oint_{\partial\Omega^{+}+\Gamma_\varepsilon}\left(\frac{\partial u}{\partial n}v-\frac{\partial v}{\partial n}u\right)ds$$

(3) $$\oint_{\partial\Omega}\left(\frac{\partial u}{\partial n}v-\frac{\partial v}{\partial n}u\right)ds=\oint_{\Gamma_\varepsilon}\left(\frac{\partial u}{\partial n}v-\frac{\partial v}{\partial n}u\right)ds$$

在 Γ_ε 上，$v=\ln\frac{1}{r_{M_0M}}=\ln\frac{1}{\varepsilon}$

$$\left.\frac{\partial v}{\partial n}\right|_{\Gamma_\varepsilon}=(-\ln r)'_{r=\varepsilon}=-\left.\frac{1}{r}\right|_{r=\varepsilon}=-\frac{1}{\varepsilon}$$

$$=\oint_{\Gamma_\varepsilon}\left(\frac{\partial u}{\partial n}\cdot(-\ln\varepsilon)+\frac{1}{\varepsilon}u\right)ds=(-\ln\varepsilon)\left.\frac{\partial u}{\partial n}\right|_{\bar{M}}2\pi\varepsilon+u|_{\bar{M}}\frac{1}{\varepsilon}2\pi\varepsilon$$

$\varepsilon\to0^{+}$

从而有 $u(M_0)=\frac{1}{2\pi}\oint_{\partial\Omega}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)ds$

$$u(M_0)=-\oint\left(u\frac{\partial}{\partial n}\left(\frac{1}{2\pi}\ln\frac{1}{r}\right)-\left(\frac{1}{2\pi}\ln\frac{1}{r}\right)\frac{\partial u}{\partial n}\right)ds$$

$$0=\oint_{\partial\Omega}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)ds$$

$$u(M_0)=-\oint_{\partial\Omega}\left[u\frac{\partial}{\partial n}\left(\frac{1}{2\pi}\ln\frac{1}{r}-v\right)-\frac{\partial u}{\partial n}\left(\frac{1}{2\pi}\ln\frac{1}{r}-v\right)\right]ds$$

选取 v: $\begin{cases}\Delta v=0 & \text{in}\,\Omega\\ v|_{\partial\Omega}=\frac{1}{2\pi}\ln\frac{1}{r_{M_0M}}\end{cases}$ $\qquad G=\frac{1}{2\pi}\ln\frac{1}{r_{M_0M}}-v$

$$u(M_0)=-\oint_{\partial\Omega}f\frac{\partial G}{\partial n}ds$$

R^n $\Delta v=0$ $v=\frac{C}{r^{n-2}}$

C 为单位球面积

n 维球可用极坐标计算

G: M_0 及其镜像点 $M_1\sim M_n$ 产生的电势之和

镜像点相当于使平面接地

$\begin{cases}\Delta u+a(x)\frac{\partial u}{\partial x}+b(y)\frac{\partial u}{\partial y}=0\\ u=f\end{cases}$ 对于更一般的形式

G 仍存在

四、5、6 说明 G 的表达式