$\Delta u=0$, $u_{xx}+u_{yy}=0$

在圆域内，$u_{xx}+u_{yy}=0$, $x^2+y^2<\rho_0^2$

$$\begin{cases} u(x,y)=f(x,y), & x^2+y^2=\rho_0 \\ \end{cases}$$

$$\rho=\sqrt{x^2+y^2},\quad \theta=\arctan\left(\frac{y}{x}\right)$$

$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases} \quad \begin{matrix} (0\leq\rho\leq\rho_0) \\ 0\leq\theta\leq 2\pi \end{matrix}$$

$$u_x = u_\rho \rho_x + u_\theta \theta_x = u_\rho \frac{x}{\rho} + u_\theta\cdot\left(\frac{-y}{\rho^2}\right)$$

$$\rho_x = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\cdot(2x) = \frac{x}{\rho} \qquad \theta_x = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} = \frac{-y}{x^2+y^2} = \frac{-y}{\rho^2}$$

$$\rho_y = \frac{y}{\rho}$$

$$\theta_y = \frac{x}{\rho}$$

$$u_y = u_\rho \frac{y}{\rho} + u_\theta \frac{x}{\rho^2}$$

$$u_{xx} = (u_\rho)_x \frac{x}{\rho} + u_\rho\left(\frac{x}{\rho}\right)_x + (u_\theta)_x \frac{-y}{\rho^2} + \left(-\frac{y}{\rho^2}\right)_x u_\theta$$

$$= \left(u_{\rho\rho}\frac{x}{\rho} + u_{\rho\theta}\frac{-y}{\rho^2}\right)\frac{x}{\rho} + u_\rho\left(\frac{\rho^2-x^2}{\rho^3}\right) + \left(u_{\theta\rho}\frac{x}{\rho} + u_{\theta\theta}\frac{-y}{\rho^2}\right)\frac{-y}{\rho^2} + u_{\theta\theta}(-y)(-2\rho^{-3})\frac{x}{\rho}$$

$$= u_{\rho\rho}\frac{x^2}{\rho^2} + 2u_{\rho\theta}\left(-\frac{xy}{\rho^3}\right) + u_{\theta\theta}\frac{y^2}{\rho^4} + u_\rho\frac{\rho^2-x^2}{\rho^3} + u_\theta\frac{2xy}{\rho^4}$$

$$u_{yy} = u_{\rho\rho}\frac{y^2}{\rho^2} + 2u_{\rho\theta}\left(\frac{xy}{\rho^3}\right) + u_{\theta\theta}\frac{x^2}{\rho^4} + u_\rho\frac{\rho^2-y^2}{\rho^3} + u_\theta\frac{-2xy}{\rho^4}$$

$u_{xx}+u_{yy} = u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\theta\theta}=0 \rightarrow$ 当化为欧拉公式时，可推 $u(\rho)$

在三维情形下，道理相同 $0\le\rho\le\rho_0$ ，$0\le\theta\le2\pi$ ， $0\le\varphi\le\pi$

$$\begin{cases}\rho=\sqrt{x^2+y^2+z^2}\\ \theta=\arctan\frac{y}{x}\\ \varphi=\arctan\frac{\sqrt{x^2+y^2}}{z}\end{cases}$$

迭代法： ① $(x,y,z)\to(\rho,\theta,z)$

$$u_{xx}+u_{yy}+u_{zz}=u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\theta\theta}+u_{zz}$$

② $(\rho,\theta,z)\to(r,\theta,\varphi)$

$\rho=r\sin\varphi$

$z=r\cos\varphi$

$$\text{原式}=u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\varphi\varphi}+\frac{1}{r^2\sin^2\varphi}u_{\theta\theta}+\frac{1}{r\sin\varphi}(u_r r_\rho+u_\varphi\varphi_r)$$

$$=\underbrace{u_{rr}+\frac{2}{r}u_r}_{\frac{1}{r^2}[r^2u_r]_r}+\underbrace{\frac{1}{r^2}u_{\varphi\varphi}+\frac{1}{r^2\tan\varphi}u_\varphi}_{\frac{1}{r^2\sin\varphi}(u_\varphi\sin\varphi)_\varphi}+\frac{1}{r^2\sin^2\varphi}u_{\theta\theta}$$

u若球对称，则 $u_{\theta\theta}=0$，则 $u=\frac{C}{r}$ ，C一般取 $\frac{1}{4\pi}$

① $\begin{cases}u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\theta\theta}=0, & \rho<\rho_0\\ u=f(\theta), & \rho=\rho_0\end{cases}$

f 为 θ 的 2π 为T的周期函数

② $u(\rho,\theta)=R(\rho)\Phi(\theta)$

$R''\Phi+\frac{1}{\rho}R'\Phi+\frac{1}{\rho^2}R\Phi''=0$

③ $\rho^2 R''\Phi + \rho R'\Phi + R\Phi'' = 0 \Rightarrow \frac{\rho^2 R''}{R} + \frac{\rho R'}{R} = -\frac{\Phi''(\theta)}{\Phi} = \lambda$

④ $\Phi'' + \lambda\Phi = 0, \quad \rho^2 R'' + \rho R' - \lambda R = 0$

Φ 以 2π 为周期 $\quad |R(\rho)|$ 处处有界 圆盘温度有限

$\lambda = 0, \ \Phi = C_2$ ✓

$\lambda < 0, \ \Phi'' = s^2\Phi, \quad \Phi = e^{s\theta} + e^{-s\theta}$ ✗

$\lambda > 0, \ \Phi(\theta) = C_1 \cos\sqrt{\lambda}\theta + C_2 \sin\sqrt{\lambda}\theta \qquad \lambda = n^2$

⑤ $\lambda_n = n^2, \ n = 0, 1, 2, \cdots \quad \Phi_n(\theta) = C_n \cos n\theta + D_n \sin n\theta$

⑥ $\lambda = 0, \ \rho^2 R_0'' + \rho R_0' = 0 \quad \rho R_0'' + R_0' = 0, \quad (\rho R_0')' = 0, \ R_0 = C_1 \ln\rho + C_2$

$\Rightarrow R_0(\rho) = 1$

$\lambda = n^2, \ \rho^2 R_n'' + \rho R_n' - n^2 R_n = 0$

$R_n(\rho) = \rho^k \quad k = \pm n$

$R_n(\rho) = C_n \rho^n + D_n \rho^{-n} \quad \rightarrow R_n(\rho) = C_n \rho^n$

(有界, ✗)

⑦ $u_n = R_n(\rho)\Phi_n(\theta)$

⑧ $\sum_{n=0}^{+\infty} u_n = C_0 + \sum_{n=1}^{+\infty} [C_n \cos n\theta + D_n \sin n\theta]\rho^n$

$f(\theta) = C_0 + \sum_{n=1}^{+\infty} [(C_n \rho_0^n)\cos n\theta + (D_n \rho_0^n)\sin n\theta]$

$C_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$

$$\begin{cases} a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta \quad n=0,1,2,\cdots \quad \underline{C_n \rho_0^n} \\ b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta \, d\theta, \quad n=1,2,\cdots \quad \underline{D_n \rho_0^n} \end{cases}$$

⑨ $$u(\rho,\theta) = \frac{C_0}{2} + \sum_{n=1}^{+\infty}\left(\frac{\rho}{\rho_0}\right)^n (C_n\cos n\theta + D_n \sin n\theta)$$

$$\begin{aligned} u(\rho,\theta) &= \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}f(t)\,dt + \sum_{n=1}^{+\infty}\frac{1}{\pi}\frac{\rho^n}{\rho_0^n}\int_0^{2\pi} f(t)[\cos n(\theta-t)]\,dt \\ &= \frac{1}{\pi}\sum_{n=0}^{+\infty}\int_0^{2\pi} f(t)\left(\frac{1}{2} + a^n \cos n(\theta-t)\right)dt \qquad a=\frac{\rho}{\rho_0} \\ &= \frac{1}{\pi}\int_0^{2\pi} f(t)\left(\frac{1}{2} + \sum_{n=1}^{+\infty}(a^n\cos n(\theta-t))\right)dt \end{aligned}$$

$$q = \theta - t$$

$$\sum_{n=1}^{+\infty} = \frac{1}{2}\frac{ae^{iq}}{1-ae^{iq}} + \frac{1}{2}\frac{ae^{-iq}}{1-ae^{-iq}}$$

$$\frac{1}{2} + \sum = \frac{1}{2} + \frac{a\cos q - a^2}{1+a^2-2a\cos q}$$

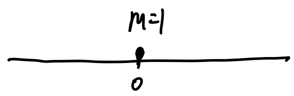
$$= \frac{\frac{1}{2}-\frac{1}{2}a^2}{1+a^2-2a\cos q}$$

求此积分确能代换

从而，原式 $= \frac{1}{2\pi}\int_0^{2\pi} f(t)\frac{1-a^2}{1+a^2-2a\cos(\theta-t)}\,dt \qquad a=\frac{\rho}{\rho_0}$

当 $\rho=\rho_0$ 时，注意积分不为0

$\delta(\theta-t)$



$$\int_{-\infty}^{+\infty} \rho(x)\,dx = 1 \qquad \int_{-\infty}^{+\infty} \delta(x) f(x)\,dt = f(0)$$

$$\int_{-\infty}^{+\infty} \delta(x-t) f(t)\,dt = f(x)$$

$x \neq 0,\ \rho(x) = 0$

$x = 0,\ \rho(x) = +\infty$

卷积： $f * g(x) = \int_{-\infty}^{+\infty} f(t)\, g(x-t)\, dt$

$$= \int_{-\infty}^{+\infty} g(t) f(x-t)\, dt$$

$$\delta * f = f$$

$$\int_{-\infty}^{+\infty} \delta(x) f(x)\, dx = \int_{-\infty}^{-\varepsilon} \delta f + \int_{-\varepsilon}^{\varepsilon} \delta f + \int_{\varepsilon}^{+\infty} \delta f$$

$$\parallel$$

$$f(\varepsilon') \int_{-\varepsilon}^{\varepsilon} \delta\, dx \qquad \varepsilon \to 0,\ \varepsilon' \to 0$$

$$= f(\varepsilon')$$

$$u_{xx} + u_{yy} = 0$$

$u|_{x=0} = 0,\ u|_{x=a} = Ay$ $\qquad u = ZY$

$u_y|_{y=0} = u_y|_{y=b} = 0$ $\qquad Z''Y + ZY'' = 0$

$$\frac{Z''}{Z} = -\frac{Y''}{Y} = \lambda$$

$Z'' = \lambda Z$ $\qquad Y'' + \lambda Y = 0$

$Z_0 = C_0 x + d_0$ $\qquad Y'(0) = Y'(b) = 0$

$Z_n = C_n e^{\frac{n\pi x}{b}} + D_n e^{-\frac{n\pi x}{b}}$ $\qquad Y_0 = 1$

$Y_n = \cos\left(\frac{n\pi y}{b}\right)$

$\lambda_n = \left(\frac{n\pi}{b}\right)^2,\ n = 0, 1, 2, \ldots$

$\lambda_n = \left(\frac{n\pi}{b}\right)^2 \quad n = 0, 1, 2, \cdots$

$$u = \sum Z_n Y_n$$

$$= (C_0 x + d_0) + \sum \left(C_n e^{\frac{n\pi x}{b}} + D_n e^{-\frac{n\pi x}{b}} \right) \cos\left(\frac{n\pi y}{b}\right)$$

$= C_n$

$$\Rightarrow d_0 = 0$$

2 (13) (14)