非齐次线性方程的求解

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t) \\ u|_{x=0} = u|_{x=l} = 0 \\ u|_{t=0} = \varphi(x),\ u_t|_{t=0} = \phi(x) \end{cases}$$

$$\longrightarrow \text{I)} \begin{cases} w_{tt} = a^2 w_{xx} \\ w|_{x=0} = w|_{x=l} = 0 \\ w|_{t=0} = \varphi(x),\ w_t|_{t=0} = \phi(x) \end{cases} + \text{II)} \begin{cases} v_{tt} = a^2 v_{xx} + f \\ v|_{x=0} = v|_{x=l} = 0 \\ v|_{t=0} = 0,\ v_t|_{t=0} = 0 \end{cases}$$

$$w(x,t) = X(x)T(t)$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{l}$$

$$v(x,t) = \sum_{n=1}^{+\infty} v_n(t) \sin\frac{n\pi x}{l}$$

$$f(x,t) = \sum_{n=1}^{+\infty} f_n(t) \sin\frac{n\pi x}{l}$$

$$f_n(t) = \frac{2}{l}\int_0^l f(x,t)\sin\frac{n\pi x}{l}\,dx$$

按特征函数展开法

① $u = w + v$

② $W \to \lambda_n,\ X_n$

③ $v(x,t) = \sum_{n=1}^{+\infty} v_n(t)\sin\frac{n\pi x}{l}$

$$f(x,t) = \sum_{n=1}^{+\infty} f_n(t)\sin\frac{n\pi x}{l}$$

代入方程，有

$$\sum_{n=1}^{+\infty} v_n''(t)\sin\frac{n\pi x}{l} = \sum_{n=1}^{+\infty} -\left(\frac{an\pi}{l}\right)^2 \sin\left(\frac{n\pi x}{l}\right) v_n(t) + \sum_{n=1}^{+\infty} f_n(t)\sin\frac{n\pi x}{l}$$

$$\sum_{n=1}^{+\infty}\left[v_n'' + \left(\frac{an\pi}{l}\right)^2 v_n - f_n(t)\right]\sin\frac{n\pi x}{l} = 0$$

$$\begin{cases} v_n'' + \left(\frac{an\pi}{l}\right)^2 v_n - f_n(t) = 0 \\ v_n(0) = v_n'(0) = 0 \end{cases}$$

(1) $v_n'' + \left(\frac{an\pi}{l}\right)^2 v_n = 0$

$$v_n(t) = C_1 \cos\frac{an\pi}{l}t + C_2 \sin\frac{an\pi}{l}t$$

(2) $v_n'' + \left(\frac{an\pi}{l}\right)^2 v_n = f_n(t)$ 特征

$$v_n^*(t) = C_1(t)\cos\frac{an\pi t}{l} + C_2(t)\sin\frac{an\pi t}{l}$$

$$\begin{cases} C_1'\cos\frac{an\pi t}{l} + C_2'\sin\frac{an\pi t}{l} = 0 \\ -C_1'\frac{an\pi}{l}\sin\frac{an\pi t}{l} + C_2'\frac{an\pi}{l}\cos\frac{an\pi t}{l} = f_n(t) \end{cases}$$

$$\begin{pmatrix} \cos\frac{an\pi}{l}t & \sin\frac{an\pi}{l}t \\ -\sin\frac{an\pi}{l}t & \cos\frac{an\pi}{l}t \end{pmatrix}\begin{pmatrix} C_1' \\ C_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{l}{an\pi}f_n \end{pmatrix}$$

从而 $C_1' = -\frac{l}{an\pi}f_n(t)\sin\frac{an\pi}{l}t$

$$C_2' = \frac{l}{an\pi}f_n(t)\cos\frac{an\pi}{l}t$$

$$v^* = \int_0^t \frac{l}{an\pi}f_n(s)\sin\frac{an\pi}{l}(t-s)\,ds$$

从而 $v_n(t) = C_1\cos\frac{an\pi}{l}t + C_2\sin\frac{an\pi t}{l} + \frac{l}{an\pi}\int_0^t f_n(s)\sin\frac{an\pi}{l}(t-s)\,ds$

$V_n(0) = 0 \Rightarrow C_1 = 0$

$F(x) = \int_a^x f(x,t)\,dt$

含参变量的变上限积分

$F'(x) = f(x,x) + \int_a^x f_x'(x,t)\,dt$

导函数定X

$v_n'(0) = 0 \Rightarrow C_2 = 0$

从而 $v_n(t) = \frac{1}{an\pi}\int_0^t f_n(s)\sin\frac{an\pi}{l}(t-s)\,ds$

从而,

$$u=\sum_{n=1}^{+\infty}\left[C_n\cos\frac{an\pi t}{l}+D_n\sin\frac{an\pi t}{l}\right]\sin\frac{n\pi x}{l}+\sum_{n=1}^{+\infty}v_n(t)\sin\frac{n\pi x}{l}$$

$$C_n=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{n\pi x}{l}dx$$

$$D_n=\frac{2}{an\pi}\int_0^l\phi(x)\sin\frac{n\pi x}{l}dx$$

$$v_n(t)=\frac{l}{an\pi}\int_0^l f_n(s)\sin\frac{an\pi(t-s)}{l}ds$$

$$f_n(s)=\frac{2}{l}\int_0^l f(x,s)\sin\frac{n\pi x}{l}dx$$

Laplace变换

e.g $\int_0^{+\infty}\frac{\sin x}{x}dx$

$$=\int_0^{+\infty}\sin x\,dx\int_0^{+\infty}e^{-xy}dy$$

$$=\int_0^{+\infty}dy\int_0^{+\infty}e^{-xy}\sin x\,dx$$

$$=\int_0^{+\infty}\frac{1}{1+y^2}dy$$

$$=\frac{\pi}{2}$$

$$\begin{cases} u_{xx}+u_{yy}=12(x^2-y^2) & a^2<x^2+y^2<b^2 \\ u|_{R=a}=0,\ \frac{\partial u}{\partial \vec{n}}\Big|_{R=b}=0 \end{cases}$$

$$\frac{\partial u}{\partial \vec{n}} = \mathrm{grad}\,u\cdot\vec{e_n}$$

$$\frac{\partial u}{\partial \vec{n}}\Big|_{\partial\Omega} = \mathrm{grad}\,u\cdot\vec{n}$$

$$=\mathrm{grad}\,u\Big|_R \frac{\vec{R}}{R}$$

$$=u_\rho|_{\rho=R}$$

① $u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\theta\theta}=12\rho^2\cos 2\theta$

$u|_{\rho=a}=0 \qquad u_\rho|_{\rho=b}=0$

②找特征函数

$$\begin{cases} u_{\rho\rho}+\frac{1}{\rho}u_\rho+\frac{1}{\rho^2}u_{\theta\theta}=0 \\ u|_{\rho=a}=0,\ u_\rho|_{\rho=b}=0 \end{cases}$$

$u=R(\rho)\Phi(\theta) \qquad \Phi$ 周期 2π

$R''\Phi+\frac{1}{\rho}R'\Phi+\frac{1}{\rho^2}R\Phi''=0$

$$\frac{\rho^2R''+\rho R'}{R}=-\frac{\Phi''}{\Phi}=\lambda$$

$\rho^2R''+\rho R'-\lambda R=0$

$$\begin{cases} \Phi''+\lambda\Phi=0 \\ \Phi(\theta)=\Phi(\theta+2\pi) \end{cases}$$

$\Phi_n=C_n\cos n\theta+D_n\sin n\theta$

$\lambda_n=n^2,\ n=0,1,2,\cdots$

③ $\lambda_n=n^2,\ n=0,1,2\cdots$

$\{\cos n\theta,\ \sin n\theta\}_{n=0}^{+\infty}$

④ $u(\rho,\theta)=\sum_{n=0}^{+\infty} A_n(\rho)\cos n\theta+B_n(\rho)\sin n\theta$

$=A_0(\rho)+\sum_{n=1}^{+\infty} A_n(\rho)\cos n\theta+B_n(\rho)\sin n\theta$

⑤ $A_0''+\sum A_n''\cos n\theta+B_n''\sin n\theta+\frac{1}{\rho}[A_0'+\sum A_n'\cos n\theta+B_n'\sin n\theta]$

$+\frac{1}{\rho^2}[-\sum(A_n n^2\cos n\theta+B_n n^2\sin n\theta)]=12\rho^2\cos 2\theta$

$A_2''+\frac{1}{\rho}A_2'-\frac{4}{\rho^2}A_2=12\rho^2$ $\quad A_n(a)=0,\ B_n(a)=0$

$A_n''+\frac{1}{\rho}A_n'-\frac{n^2}{\rho^2}A_n=0$ $\quad A_n'(b)=0,\ B_n'(b)=0$

$A_0''+\frac{1}{\rho}A_0'=0$

$B_n''+\frac{1}{\rho}B_n'-\frac{n^2}{\rho^2}B_n=0$

(1) $\rho A_0''+A_0'=0$, $A_0=0$

(2) $\rho^2A_n''+\rho A_n'-n^2A_n=0,\ A_n(\rho)=\rho^k$

$k(k-1)\rho^k+k\rho^k-n^2\rho^k=0$

(3) $\begin{cases} A_2''+\frac{1}{\rho}A_2'-\frac{4}{\rho^2}A_2=12\rho^2 \\ A_2(a)=A_2'(b)=0 \end{cases}$ $\quad A_2(\rho)=C_1\rho^2+C_2\rho^{-2}$

$A_2^*=C_1(\rho)\rho^2+C_2(\rho)\rho^{-2}$

$\begin{cases} C_1'\rho^2+C_2'\rho^{-2}=0 \\ C_1'(2\rho)+C_2'(-2\rho^{-3})=12\rho^2 \end{cases}$ $\quad C_2'=-C_1'\rho^4$

$C_1'\rho+C_1'\rho=6\rho^2$

$$C_1' = 3\rho, \quad C_2' = -3\rho^5$$

$$A_2^* = \frac{3}{2}\rho^4 + \left(-\frac{1}{2}\rho^6\right)\rho^{-2}$$

$$= \rho^4$$

$$A_2(\rho) = C_1\rho^2 + C_2\rho^{-2} + \rho^4$$

$$u(\rho,\theta) = A_2(\rho)\cos 2\theta$$

$$\begin{cases} C_1 a^2 + C_2 a^{-2} + a^4 = 0 \\ 2C_1 b - 2C_2 b^{-3} + 4b^3 = 0 \end{cases}$$

$$C_1 = -\frac{a^4+b^4}{a^6+2b^6}$$

$$C_2 = -a^6 + \frac{(a^4+b^4)a^4}{a^6+2b^6}$$

多变量微分方程的求解

$$\sum C_{mn}\sin\frac{n\pi x}{a} = x(a-x)\,\frac{2}{b}\int_0^b y(b-y)\sin\frac{n\pi y}{b}\,dy$$

$$C_{mn} = \frac{2}{a}\int_0^a x(a-x)\sin\frac{n\pi x}{a}\,dx\ \frac{2}{b}\int_0^b y(b-y)\sin\frac{n\pi y}{b}\,dy$$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad \lambda_{mn} = \left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)\pi^2$$

$$\begin{cases} u_{tt} = a^2 u_{xx} + f(x,t) \\ u|_{x=0} = u_1(t),\ u|_{x=l} = u_2(t) \\ u|_{t=0} = \varphi(x),\ u_t|_{t=0} = \psi(x) \end{cases}$$

(1) $u = v + w$ $\quad$ w是过$(0,u_1)(l,u_2)$的直线

$$w(x) = \frac{u_2-u_1}{l}(x) + u_1$$

$$u = v + \left(\frac{x}{l}u_2 + \frac{l-x}{l}u_1\right)$$

(2) $(v+w)_{tt} = a^2(v+w)_{xx} + f(x,t)$

$u|_{x=0} = u_1(t)$, $u|_{x=l} = u_2(t)$

$$w = A(t)x^2 + B(t)x + C(t) \quad \begin{cases} v_{tt} = a^2 v_{xx} + \underbrace{f(x,t) - \left(\frac{x}{l}u_2'' + \frac{l-x}{l}u_1''\right)}_{f_1(x,t)} \\ v|_{x=0} = v|_{x=l} = 0 \\ v|_{t=0} = \varphi_1(x) \quad v_t|_{t=0} = \phi_1(x) \end{cases}$$

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