$$H(\underline{R}(t))|\psi(t)\rangle = i\hbar \frac{d}{dt}|\psi(t)\rangle$$

$$E_n(\underline{R}(t))\,|n(\underline{R}(t))\rangle = \hbar\left(\frac{d}{dt}\theta(t)\right)|n(\underline{R}(t))\rangle + i\hbar\frac{d}{dt}|n(\underline{R}(t))\rangle$$

$$\Rightarrow \theta(t) = \underbrace{\frac{1}{\hbar}\int_0^t E_n(\underline{R}(t'))\,dt'}_{\text{dynamical phase}} \underbrace{- i\int_0^t \langle n(\underline{R}(t'))|\frac{d}{dt'}|n(\underline{R}(t'))\rangle\,dt'}_{\text{Berry phase } \gamma_n}$$

use $\frac{d}{dt'}|n(\underline{R}(t'))\rangle = \underline{\nabla}_{\underline{R}}|n(\underline{R})\rangle \frac{d\underline{R}}{dt'}$

$$\gamma_n = i\int_C \langle n(\underline{R})|\underline{\nabla}_{\underline{R}}|n(\underline{R})\rangle\, d\underline{R}$$

$$= \int_C \underline{A}_n \cdot d\underline{R} \qquad \text{with } \underline{A}_n(\underline{R}) = i\langle n(\underline{R})|\underline{\nabla}_{\underline{R}}|n(\underline{R})\rangle \in \mathbb{R}$$

... Berry connection

$$= -\operatorname{Im}\left\{\int_C \langle n(\underline{R})|\underline{\nabla}_{\underline{R}}|n(\underline{R})\rangle\, d\underline{R}\right\}$$

- consider closed path with $\underline{R}(0) = \underline{R}(T)$
- assume $\underline{R} \in \mathbb{R}^3$

Stokes theorem:

$$\gamma_n = -\operatorname{Im}\iint d\underline{S}\cdot\underline{\nabla}_{\underline{R}}\times\underline{A}_n(\underline{R})$$

$$= -\mathrm{Im} \iint d\underline{S} \cdot \left(\langle \nabla n(\underline{R}) | \times | \nabla n(\underline{R}) \rangle \right)$$

$$= -\mathrm{Im} \iint d\underline{S} \cdot \underline{\Omega}_n(\underline{R})$$

with $\underline{\Omega}_n(\underline{R}) = \underline{\nabla} \times \underline{A}_n(\underline{R})$ ... Berry curvature

continue:

$$\langle \nabla n | \times | \nabla n \rangle = \sum_{m \neq n} \langle \nabla n | m \rangle \times \langle m | \nabla n \rangle$$

use $\langle m | \nabla n \rangle = \dfrac{\langle m | \nabla H | n \rangle}{E_n - E_m}$

$$= \sum_{m \neq n} \frac{\langle n | \nabla H | m \rangle \times \langle m | \nabla H | n \rangle}{(E_m - E_n)^2}$$