LIFTING LINE THEORY (LLT) TUTORIAL & EXAMPLE

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This document presents a step-by-step methodology for the application of the Lifting Line Theory (LLT) to subsonic straight wings. This material is a complementary material to the aerodynamics lecture.

LLT assumes that even though the flow around an aircraft's wing is 3-D, it may be satisfactorily approximated by a linear summation of flows around the elemental aerofoils, which makes up the overall wing, where the flow around each aerofoil is assumed to be 2-D. This approach gives a reasonable result provided that the model flow considers the effect of the vortex sheet, which is shed at the trailing edge of the wing. The trailing vortex sheet induces a downwash velocity, which varies along the span wise direction.

The wing is assumed to be a flat plate lying on the *x-y* plane. Therefore, the theory does not consider the wing's thickness distribution. It is also unable to handle any dihedral or sweepback angle. However, it can model a tapered wing with geometrical and aerodynamic twists ($\Lambda_{c/4} = 0$ [*deg*]).

For the analysis of a wing with LLT the following wing information is need it:

Wing geometry:

- Wing span (b_w)
- Wing taper ratio (λ_w)
- Wing reference area (S_w)
- Wing twist angle (β_{tip})
- Wing geometric angle of attack (α_{geo})

Aerofoil aerodynamic characteristics:

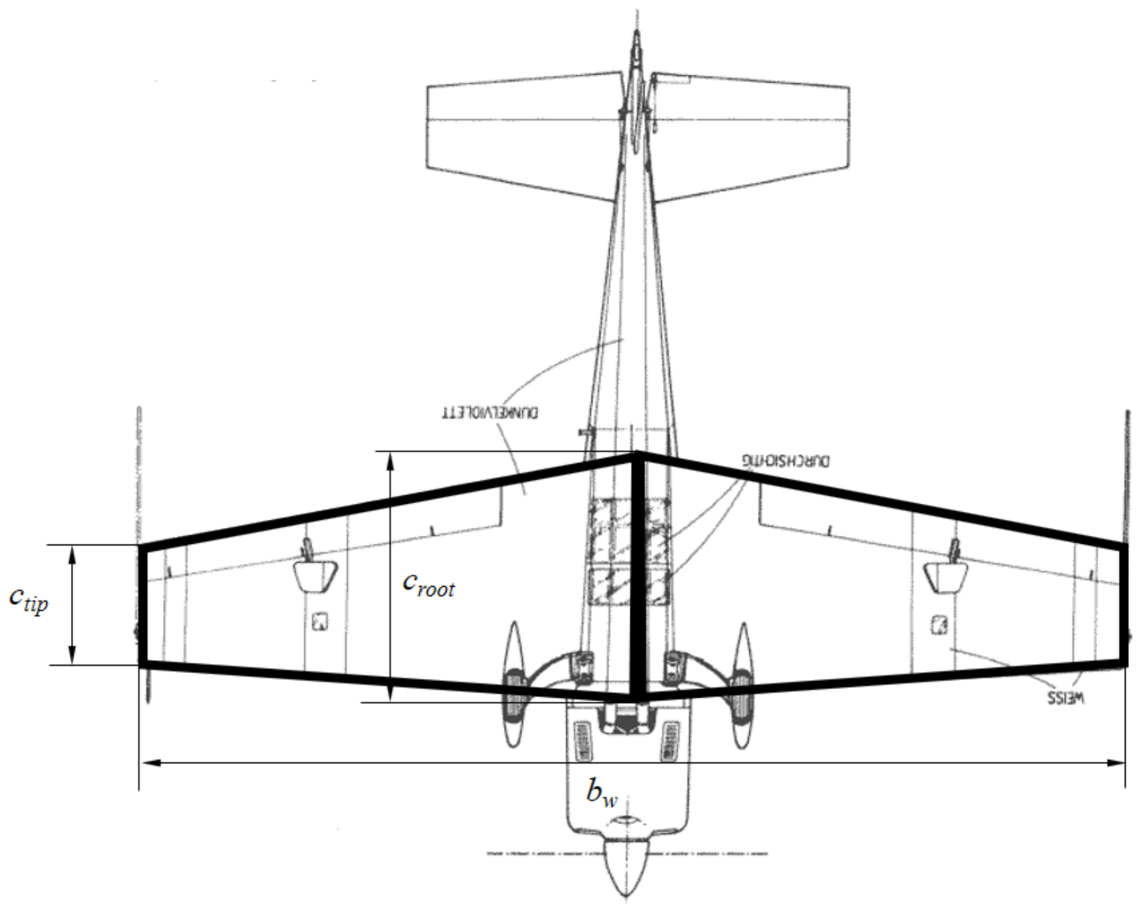
- Aerofoil reference – name or reference of the tip and root aerofoils
- Lift slope (a_0) [*1/rad*] – tip and root aerofoils
- Angle of attack for $L = 0$ ($\alpha_{L=0}$) [*deg*]
- Profile drag (C_{d_0})

Flying (airflow) characteristics:

- Air density (ρ_∞)
- Air speed (V_∞)
- Air dynamic viscosity (μ_∞)

The *LLT* procedure is described in the following steps:

Identify the wing geometry to be analysed



Calculate the wing aspect ratio

$$AR = \frac{b_w^2}{S}$$

Calculate the wing average chord

$$c_{av} = \frac{S}{b_w} = \frac{1}{2} c_{root} \left(1 + \lambda_w\right) = \frac{1}{2\lambda_w} c_{tip} \left(1 + \lambda_w\right)$$

Calculate the root aerofoil chord distance

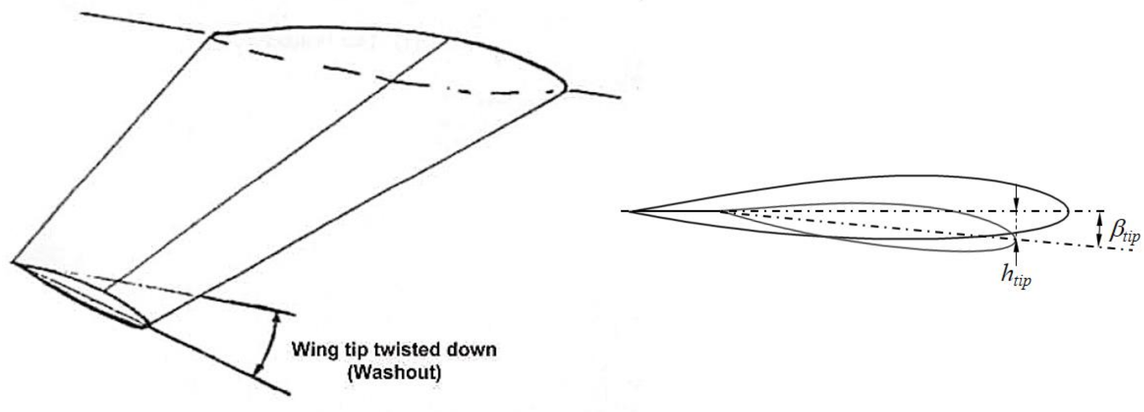
$$c_{root} = \frac{2 \cdot S}{b_w \left(1 + \lambda_w\right)}$$

Calculate the tip aerofoil chord distance

$$c_{tip} = \lambda_w \cdot c_{root}$$

Calculate the root to tip leading edge height difference (geometrical twist)

$$\sin \beta_{tip} = \frac{h_{tip}}{c_{tip}}$$



Calculate the wing mean aerodynamic chord

$$\bar{c} = \frac{2}{3} c_{root} \frac{\left(1 + \lambda_w + \lambda_w^2\right)}{\left(1 + \lambda_w\right)}$$

Calculate the location of the mean aerodynamic chord

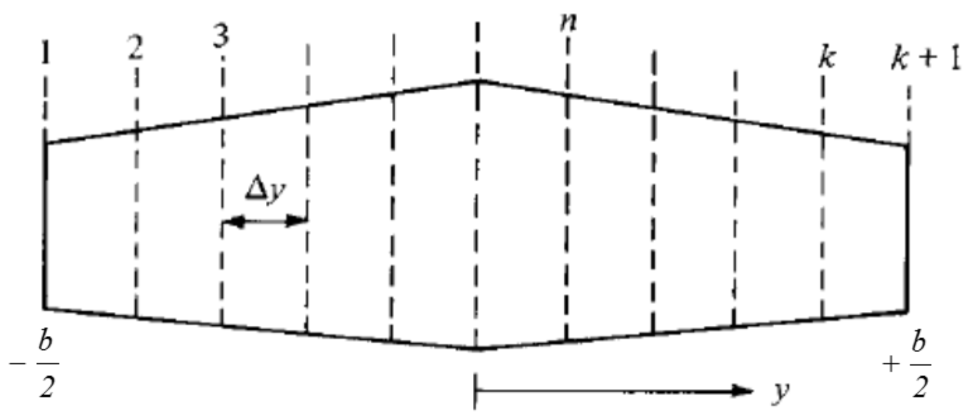
$$\bar{Y} = \frac{b_w}{6} \left[\frac{\left(1 + 2\lambda_w\right)}{\left(1 + \lambda_w\right)} \right]$$

Calculate the Reynolds number

$$\text{Re} = \frac{\rho_\infty V_\infty \bar{c}}{\mu_\infty}$$

With the previous information, the computational procedure can begin:

Divide the wing into a define number of spanwise stations. Here *k + 1* stations are shown, with *n* designating any specific station



For $k = 1, 2, 3, .., N$ points, calculate the following: The N points along the span are chosen so that they are equally spaced. In other words, the wing span is divided up into N equal intervals, and the midpoint of each interval is chosen to be a control point. The port (left) wing tip is located at $y = -b_w/2$ whereas the starboard (right) wing tip is located at $y = b_w/2$.

The coordinates of the control points are then given as follows, for each value of k, from 1 to N, we have:

$$y(k) = -\frac{b_w}{2}\left(1 - \frac{2k-1}{2N}\right)$$

Transforming the points:

$$\theta(k) = \cos^{-1}\left(-\frac{2y(k)}{b_w}\right)$$

The chord length at y is then given by the following equation:

$$c(k) = c_{root}\left(1 - 2\frac{\lambda - 1}{b_w}y(k)\right)$$

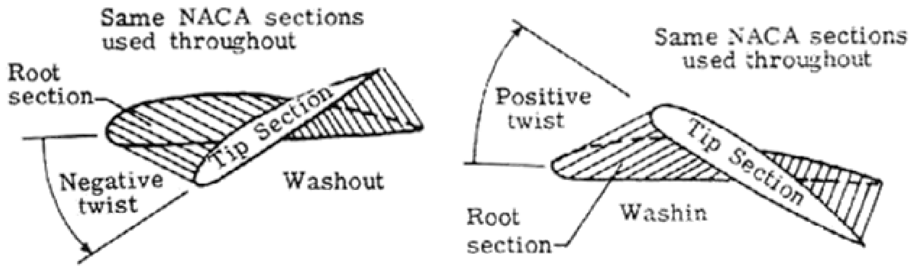
The specification for <u>aerodynamic twist</u> is quite complicated, since it requires knowledge of the shape of the aerofoil section at each station y along the span. In the absence of such information, this problem can simplify the problem somewhat by requiring that the tip aerofoil differs only slightly from the root aerofoil such that the lift coefficient and zero angle of attack at y are given by the following linear relationships

$$a_0(k) = a_{0\_root} - 2\frac{a_{0\_tip} - a_{0\_root}}{b_w}y(k)$$

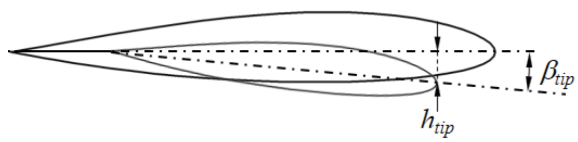
$$\alpha_{L=0}(k) = \alpha_{L=0\_root} - 2\frac{\alpha_{L=0\_tip} - \alpha_{L=0\_root}}{b_w}y(k)$$

The geometric angle of attack (α_{geo}) may vary as a function of y if the wing is given a geometric twist. A wing without twist is one where the α_{geo} is constant for all values of y, such that the leading edge and the trailing edge of the wing are straight lines, which lie on the same horizontal plane when $\alpha_{geo} = 0$.

A wing may be given a wash-out, where the wing is twisted such that the leading edge of the wing tip aerofoil is now lower than the leading edge of the root aerofoil (the root aerofoil is the aerofoil located at the plane of symmetry if it is imagined that the fuselage is not there and the two halves of the wing meet at the plane of symmetry). A wing with wash-in is one where the leading edge of the tip aerofoil is now higher than the leading edge of the root aerofoil, whereas the trailing edge of the wing remains on the horizontal plane. It follows, therefore, that the chord of the aerofoil at y may have negative or positive geometric angle of attack values when $\alpha_{geo} = 0$ at the wing root, depending on whether the wing has a wash-out or a wash-in.



Let the height difference between the leading edge of the wing tip aerofoil from the leading edge of the root aerofoil is h_{tip}, which is negative for wash-out and positive for wash-in.



It should be noted that the leading edge of the wing is required to remain as a straight line. Therefore, the twist angle or the geometric angle of attack at y relative to the geometric angle of attack at the wing root can be calculated as follows:

$$\beta(k) = \sin^{-1}\left(-\frac{2y(k)}{b_w \cdot c(k)} \cdot h_{tip}\right); \quad h_{tip} = c_{tip} \sin \beta_{tip}$$

Let´s apply now *LLT* to find the solution in the form of span wise wing load or lift per unit span length distribution, the overall wing's lift coefficient (C_{L_w}) and the induced drag coefficient of the wing (C_{Di_w}). From the *Kutta-Joukowski* Lift Theorem it is known that lift is directly proportional to circulation or vortex strength. Therefore, the theory must be capable of predicting the span-wise bound vortex strength per unit length distribution. The unknown vortex strength distribution, $\Gamma(y)$, is approximated by a *Fourier* series:

$$\Gamma(\theta) = 2b_w V_\infty \sum_{n=1}^{N} A_n \sin(n\theta)$$

The basic problem is how to calculate the unknown *Fourier* series coefficients or amplitudes, A_n. The approximation using a Fourier series becomes more accurate as the number of terms, N, increases. The lifting line equation that needs to be solved is:

$$\frac{4b_w}{a(y)c(y)} \sum_{n=1}^{N} A_n \sin(n\theta(y)) + \sum_{n=1}^{N} nA_n \frac{\sin(n\theta(y))}{\sin(\theta(y))} = \alpha_{geo}(y) - \alpha_{L=0}(y)$$

Rearranging the equation:

$$\sum_{n=1}^{N}\left[\left(\frac{4b_w}{a(y)c(y)} + \frac{n}{\sin(\theta(y))}\right)\sin(n\theta(y))\right]A_n == \alpha_{geo}(y) - \alpha_{L=0}(y)$$

Defining quantities:

$$C(y,n) = \left[\left(\frac{4b_w}{a(y)c(y)} + \frac{n}{\sin(\theta(y))}\right)\sin(n\theta(y))\right]$$

$$A(n) = A_n$$

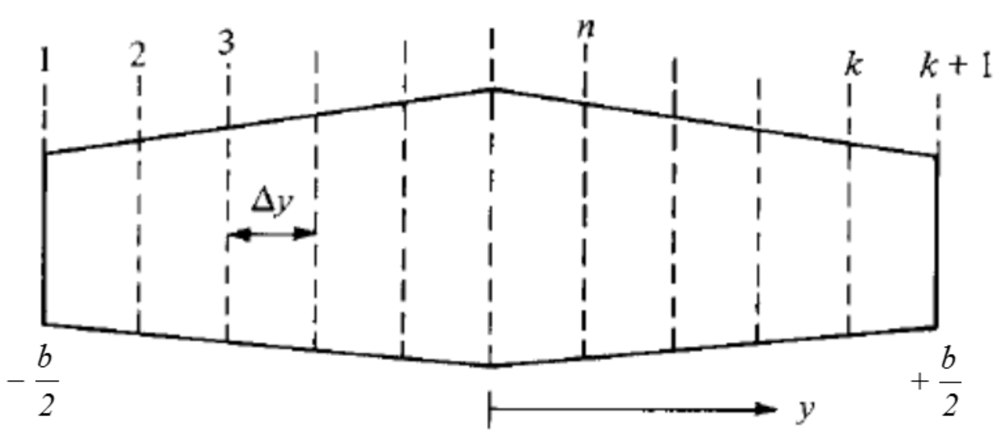
$$D(y) = \alpha_{geo}(y) - \alpha_{L=0}(y)$$

With this the lifting line equation can be written as:

$$\sum_{n=1}^{N} C(y,n)A(n) = D(y)$$

This equation contains N unknowns, namely $A(n)$ for $n = 1$ to N. It is therefore necessary to apply the last equation at N different control points or values of distance along the span, y, so that we have a system of equations that can be solved simultaneously to calculate the values of $A(n)$. The points chosen should not include the wing tips, since regardless of the values of the Fourier coefficients, the vortex strength distribution ($\Gamma(\theta)$) is always satisfied at those points. Selecting those 2 points will not provide any new information regarding the values of the Fourier amplitudes. It is also recommended that the midpoint ($y = 0$) should also not be selected as a control point, for similar reasons. To get the most accurate result for a given number of control points, the following method for selecting the control points is recommended. Firstly, N should be chosen to be an even integer, such that for example $N = 2M$, say.

The N points along the span are chosen so that they are equally spaced. In other words, the span is divided up into N equal intervals, and the midpoint of each interval is chosen to be a control point. The port (left) wing tip is located at $y = -b_w/2$ whereas the starboard (right) wing tip is located at $y = b_w/2$.



The equation of $\Gamma(\theta)$ can be rewritten in a way, which separates the even terms from the odd terms as follows:

$$\Gamma(\theta) = \sum_{n=1}^{N} A_n \sin(n\theta) = 2b_w V_\infty \left[\sum_{n=1}^{N} A_n \sin((2n-1)\theta) + \sum_{n=1}^{N} A_n \sin(2n\theta) \right]$$

Therefore, if the load distribution is symmetrical then the last equation is simplified to:

$$\Gamma(\theta) = \sum_{n=1}^{N} A_{2n-1} \sin((2n-1)\theta) = 2b_w V_\infty \left[\sum_{n=1}^{N} A_n (2n-1) \sin((2n-1)\theta) \right]$$

Keep in mind that is always assumed that the load distribution is symmetrical.

The lifting line equation, which is a system of equations, that must be solved to calculate the *Fourier sine coefficients*, can now be written as follows

$$\sum_{n=1}^{N} C(k, 2n-1) A_n (2n-1) = D(k)$$

where the values of y are given by equation of $y(k)$ and

$$\theta_k = \theta(k) = \theta(y_k) = \cos^{-1}\left(1 - \frac{2k-1}{2N}\right)$$

$$C(k, 2n-1) = C(\theta_k, 2n-1) = \left(\frac{4b_w}{a(k)c(k)} + \frac{2n-1}{\sin(\theta_k)} \right) \sin((2n-1)\theta_k)$$

$$D(k) = \alpha_{geo}(k) - \alpha_{L=0}(k) = \alpha_{geo}(y_k) - \alpha_{L=0}(y_k)$$

It should be noted that $c(k)$ is the aerofoil's chord length at the station $y(k)$, or $\theta(k)$, whereas α_{geo} and $\alpha_{L=0}(k)$ are the geometric and zero lift angle of attacks at $y(k)$. The geometric angle of attack may vary as a function of y if the wing is given a geometric twist. A wing without twist is one where the geometric angle of attack is constant for all values of y, such that the leading edge and the trailing edge of the wing are straight lines, which lie on the same horizontal plane when $\alpha_{geo} = 0$.

A wing may be given a wash-out, where the wing is twisted such that the leading edge of the wing tip aerofoil is now lower than the leading edge of the root aerofoil (the root aerofoil is the aerofoil located at the plane of symmetry if it is imagined that the fuselage is not there and the two halves of the wing meet at the plane of symmetry).

A wing with wash-in is one where the leading edge of the tip aerofoil is now higher than the leading edge of the root aerofoil, whereas the trailing edge of the wing remains on the horizontal plane. It follows, therefore, that the chord of the aerofoil at y may have negative or positive geometric angle of attack values when $\alpha_{geo} = 0$ at the wing root, depending on whether the wing has a wash-out or a wash-in.

The height difference between the leading edge of the wing tip aerofoil from the leading edge of the root aerofoil is h_{tip}, which is negative for wash-out and positive for wash-in. It should be noted that the leading edge of the wing is required to remain as a straight line. Therefore, the twist angle or the geometric angle of attack at y relative to the geometric angle of attack at the wing root can be calculated as follows

$$\beta(k) = \sin^{-1}\left(\frac{h_{tip}}{c_{tip}}\right) = \sin^{-1}\left(-\frac{2y(k)}{b_w \cdot c(k)} \cdot h_{tip}\right) = \sin^{-1}\left(\left(1 - \frac{2k-1}{N}\right)\frac{h_{tip}}{c_{tip}}\right)$$

The angle of attack of the wing or the aircraft be denoted by the angle of attack at the wing root, and is given the symbol of α_{geo}. This angle obviously can be varied and represents the attitude of the aircraft (when the aircraft is at a level cruising flight this angle may have a small positive value of not more than *3* degrees). Using this definition, it can now calculate the geometric angle of attack at y as follows

$$\alpha_{geo}(k) = \alpha_{geo} + \beta(k)$$

Equation of $D(k)$ can now be rewritten as follows:

$$D(k) = \alpha_{geo} - \alpha_{L=0}(k) + \beta(k)$$

A wing may be given an aerodynamic twist as well as a geometric twist. This means that the aerofoil shape at the wing root is different from that at the wing tip. The shape of the aerofoil in between the two limiting stations is then determined by insisting that the wing cross-section should have a smoothly varying shape along the span wise direction. Since the aerofoil shape at the wing tip is different from that at the root, therefore the value of the sectional lift coefficient as well as its zero-lift angle of attack may also vary along the span wise direction. Provided the variation of $a_0(y)$ and $\alpha_{L=0}(y)$ are given, equation of $C(k, 2n\text{-}1)$ can still be used to compute the matrix coefficients, thus the lifting line theory can handle such a problem.

The theory can also handle the problem involving a variation in the chord length of the sections as a function of y, if the functional form of $c(y)$ is given. This means that the theory is also applicable for analysing tapered wing shape, so long as the quarter chord line is normal or almost normal to the aircraft's longitudinal axis. Obviously, the theory is not valid for a highly swept wing. For swept wings, a more elaborated methodology like Vortex Lattice Method (VLM) should be used.

For each value of $n = 1, 2, 3, \ldots, N$ calculate the matrix coefficients:

$$C(k,n) = \left[\left(\frac{4b_w}{a_0(k)c(y)} + \frac{2n-1}{\sin(\theta(k))}\right)\sin((2n-1)\theta(k))\right]$$

Next step in the *LLT* methodology is to solve the following system of simultaneous equations:

$$\sum_{n=1}^{N} C(k,n)A(n) = D(k), \text{ for } k = 1, 2, 3, \ldots, N$$

A simple direct method for computing the solutions of $A(n)$, is the *Gaussian Elimination Method*. Other methods, such as the *Jacobi* or *Gauss-Seidel* iterative methods may also be used.

After the *Fourier* coefficients, $A(n)$, have been calculated it can now be compute the non-dimensional wing load distribution as follows:

$$\Gamma_{ND}(k) = \frac{\Gamma(y)}{2b_w V_\infty} = \sum_{1}^{N} A(n)\sin((2n-1)\theta)$$

$$\theta(k) = \cos^{-1}\left(-\frac{2y}{b_w}\right)$$

The wing´s lift coefficient can be calculated now:

$$C_L = \pi \cdot AR \cdot A(1)$$

The Oswald efficiency factor, e, is

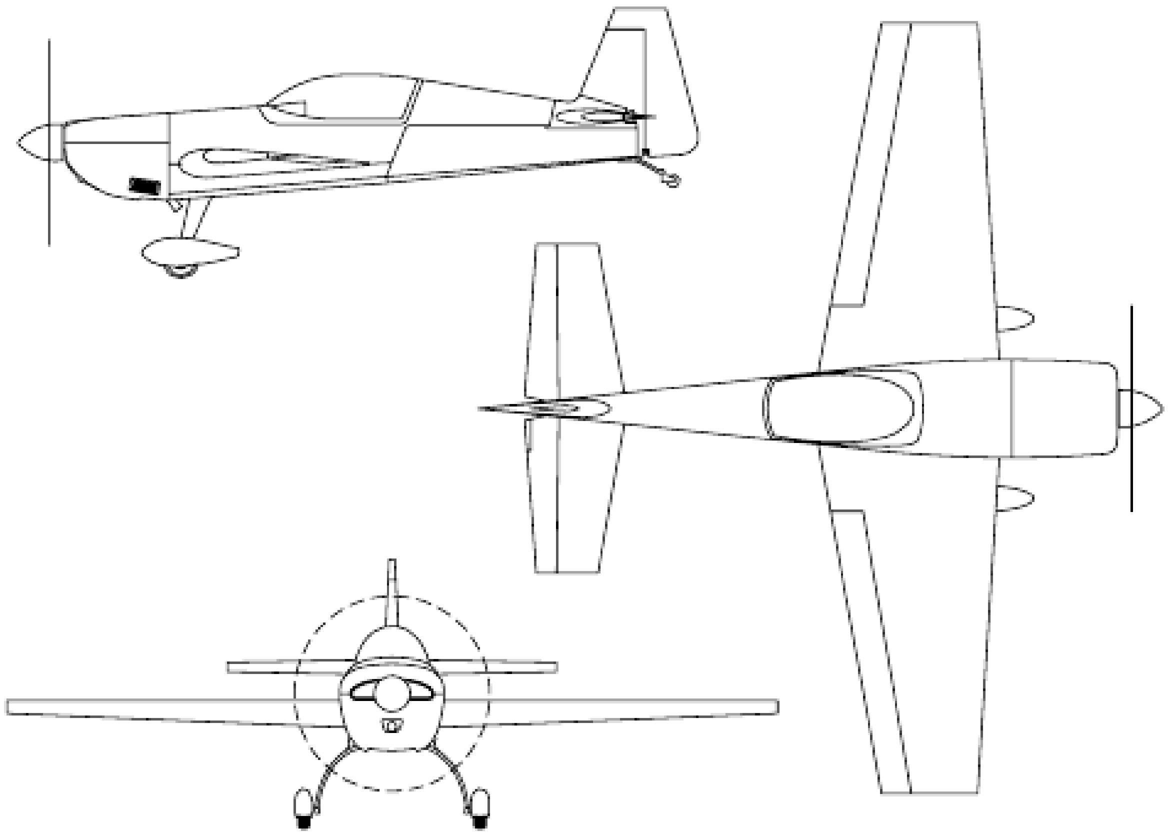
$$e = (1+\delta)^{-1}, \text{ where: } \delta = \sum_{n=2}^{N} n\left(\frac{A(n)}{A(1)}\right)^2$$

The induced drag coefficient C_{Di} is obtained by:

$$C_{Di} = \frac{C_L^2}{\pi \cdot e \cdot AR} = k \cdot C_L^2, \text{ where: } k = \frac{1}{\pi \cdot e \cdot AR}$$

Example:

For the acrobatic monoplane *Extra EA-300*, calculate (1) the aerodynamic characteristics of flight to see if the actual geometry can counteract the Maximum Take-Off Weight (*MTOW*) for a normal cruise condition with a wing incidence angle of *2* [*deg*]; and (2) find the distributed aerodynamic load (Lift force) over the wing. An airplane 3-view picture is shown:



Specifications:

- $MTOW = 950$ [kg]
- Wing span (b_w) = 8.0 [m]
- Taper ratio (λ_w) = 0.45
- Wing reference area (S_w) = 10.7 [m^2]
- Aerofoil information:
    - Root – *NACA 0015*
    - Tip – *NACA 0012*

Performance:

- $V_{cruise} = 317.0$ [km/h]
- $V_{stall} = 102.0$ [km/h]
- $V_{NE} = 408.0$ [km/h]

Solution:

First, the geometric characteristics of the wing must be determined in order to begin with the LLT methodology.

The Aspect Ratio of the wing is:

$$AR = \frac{b_w^2}{S} = \frac{8.0^2}{10.7} = 5.98$$

The average wing chord:

$$c_{av} = \frac{S}{b_w} = \frac{10.7}{8.0} = 1.34[m]$$

Chord length at the root:

$$c_{root} = \frac{2 \cdot S}{b_w\ (1+\lambda_w)} = \frac{2c_{av}}{(1+\lambda)} = \frac{2 \cdot (1.34)}{(1+0.45)} = 1.84[m]$$

Chord length at the tip:

$$c_{tip} = \lambda_w c_{root} = 0.45 \cdot 1.84 = 0.83[m]$$

Mean aerodynamic chord:

$$\bar{c} = \frac{2}{3} c_{root} \frac{(1+\lambda_w+\lambda_w^2)}{(1+\lambda_w)} = \frac{2}{3}(1.84)\left[\frac{(1+0.45+0.45^2)}{(1+0.45)}\right] = 1.40[m]$$

Mean aerodynamic chord distance:

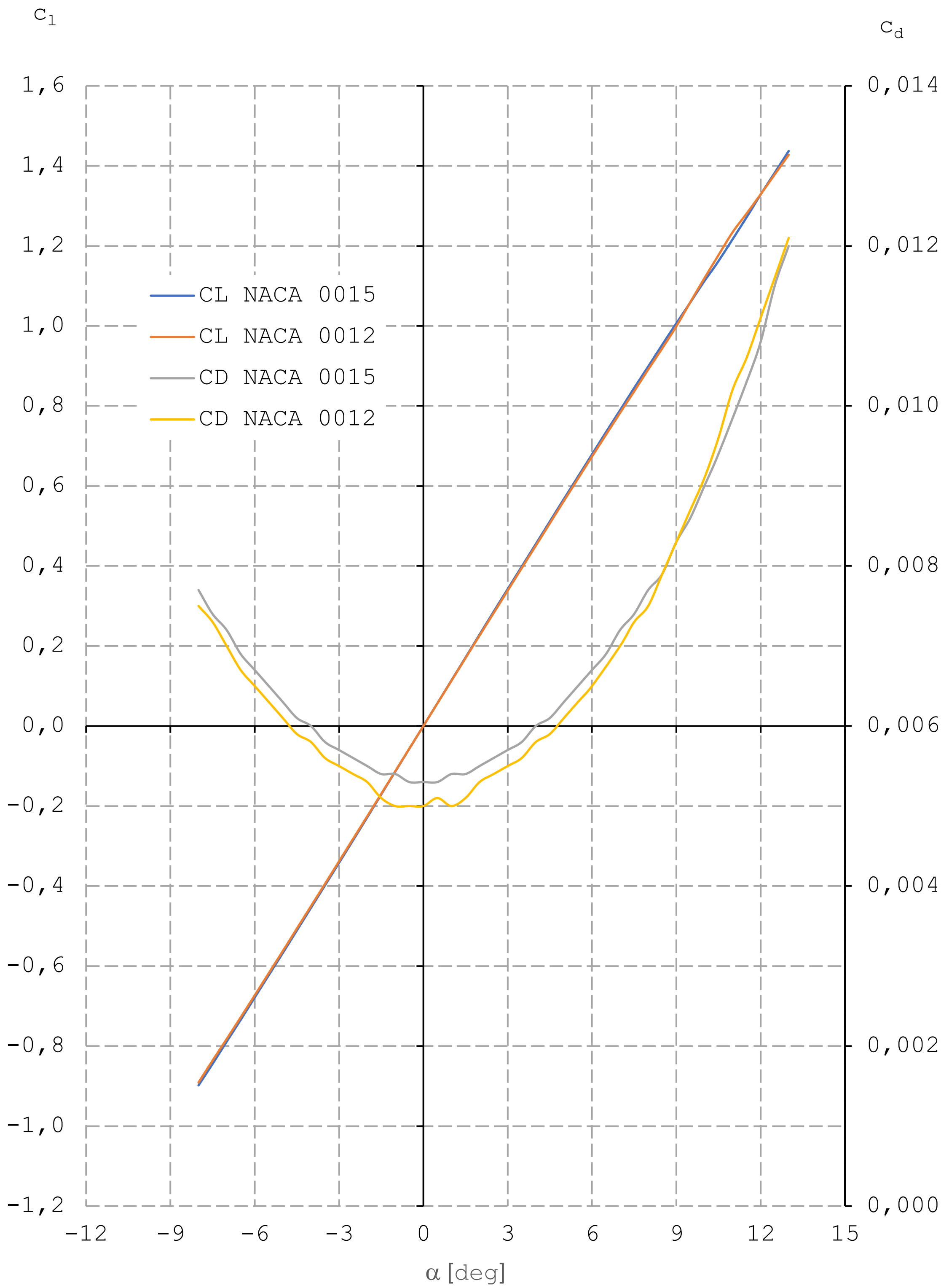
$$\bar{Y} = \frac{b_w}{6}\left[\frac{(1+2\lambda_w)}{(1+\lambda_w)}\right] = \frac{8.0}{6}\left[\frac{(1+2(0.45))}{(1+0.45)}\right] = 1.75[m]$$

Assuming sea-level conditions ($\rho_\infty = 1.225$ [kg/m^3]; $\mu_\infty = 1.789$ x $10\text{-}5$ [$kg/(m\ sec)$]), and cruise flight conditions, then:

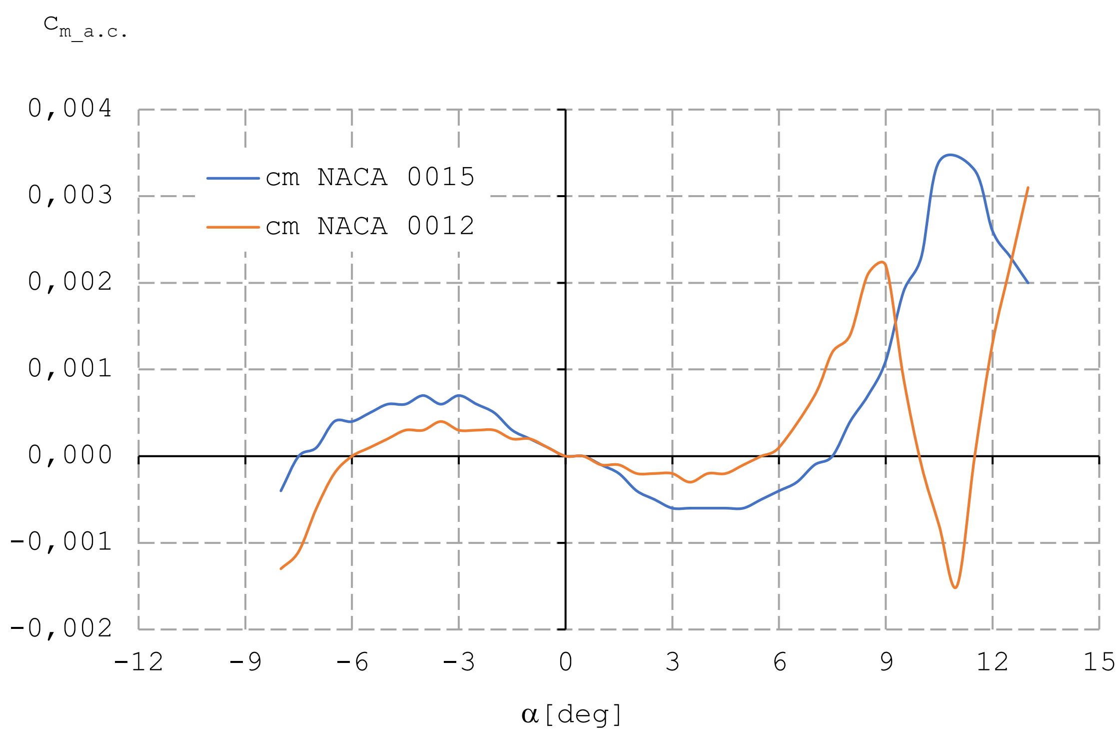
$$\mathrm{Re} = \frac{\rho_\infty V_\infty \bar{c}}{\mu_\infty} = \frac{1.225 \times \left(317 \times \frac{1000}{3600}\right) \times 1.40}{1.789 \times 10^{-5}} = 8451241$$

With this *Reynolds* number, the aerofoil characteristics can be obtained. The characteristic curves of the two wing aerofoils aerodynamic characteristics are presented in the following graphs:

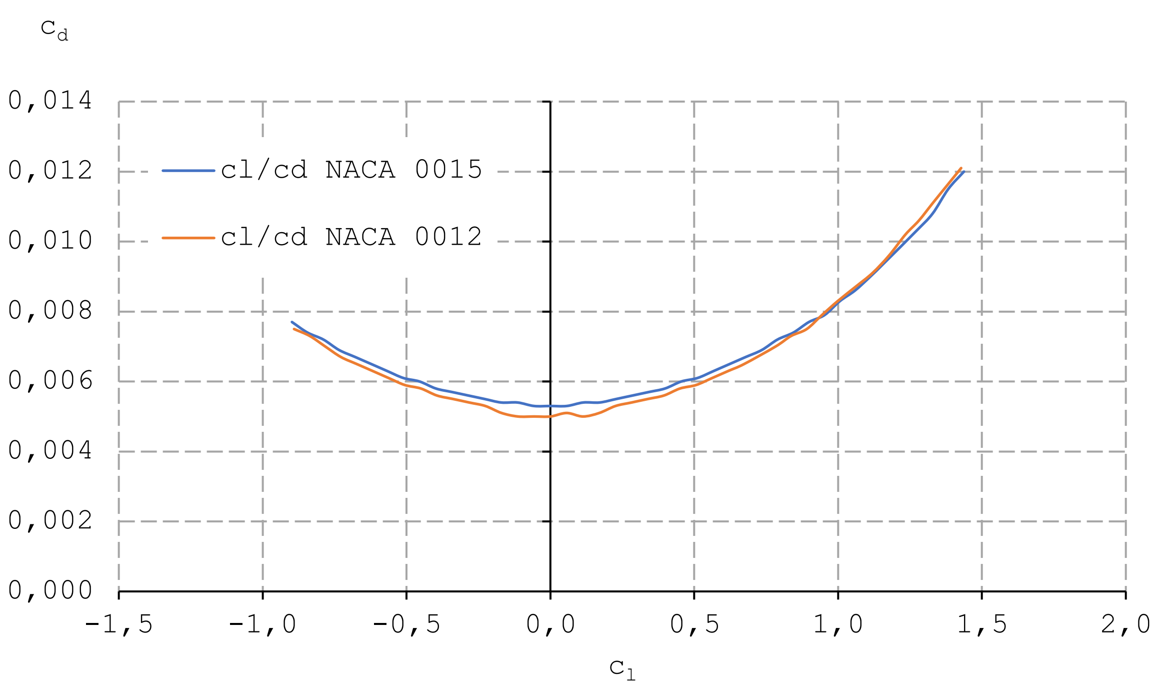
c_l vs. α [*deg*] and c_d vs. α [*deg*]



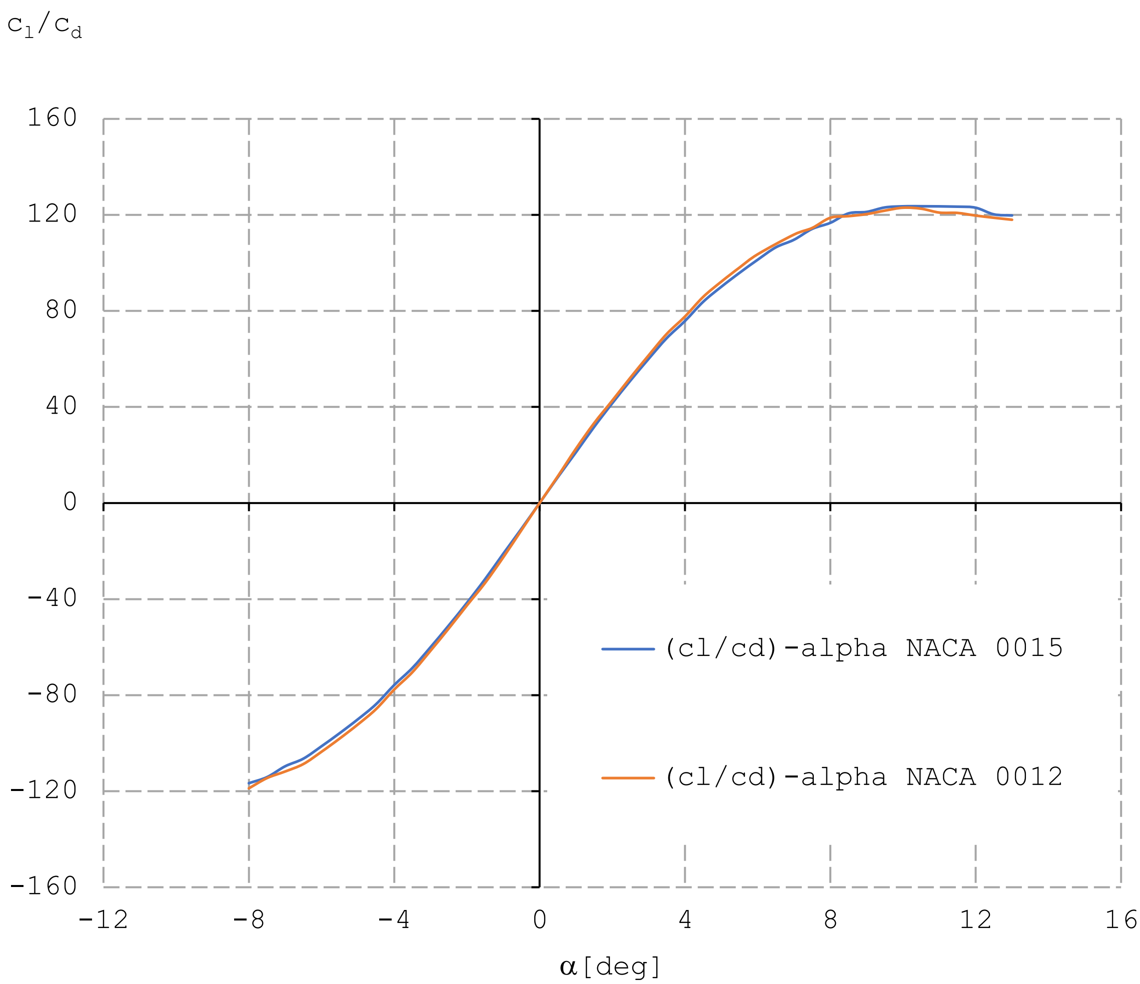
$c_{m\_a.c.}$ vs. α [deg]



c_l vs. c_d



(c_l/c_d) vs. α [*deg*]



This curves were obtain through a software that uses *Vortex Panel Method* to give the characteristics of c_l, c_d and $c_{m\_a.c.}$ at different angles of attack [*deg*] for the two aerofoils used in this wing. These curves were analysed at a Reynolds number of 8×10^6. From the curves, the following information is obtained:

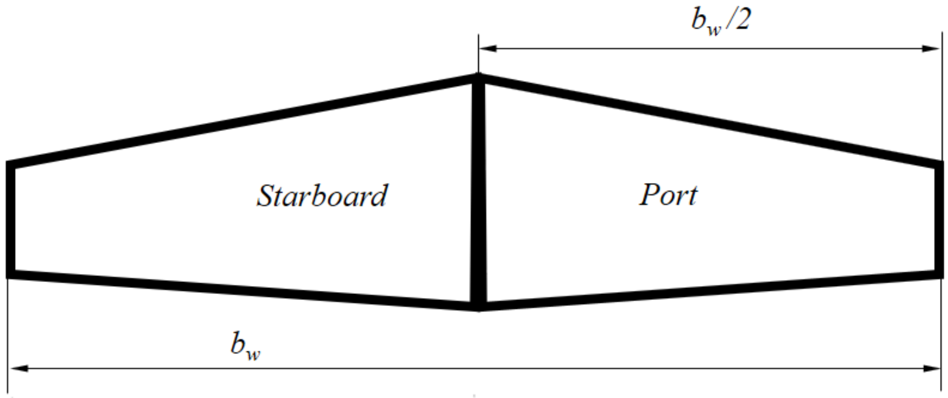
Wing position	**Aerofoil**	a_0 **[*1/rad*]**	$\alpha_{L=0}$ **[deg]**
Root	*NACA 0015*	6,436	0
Tip	*NACA 0012*	6,363	0

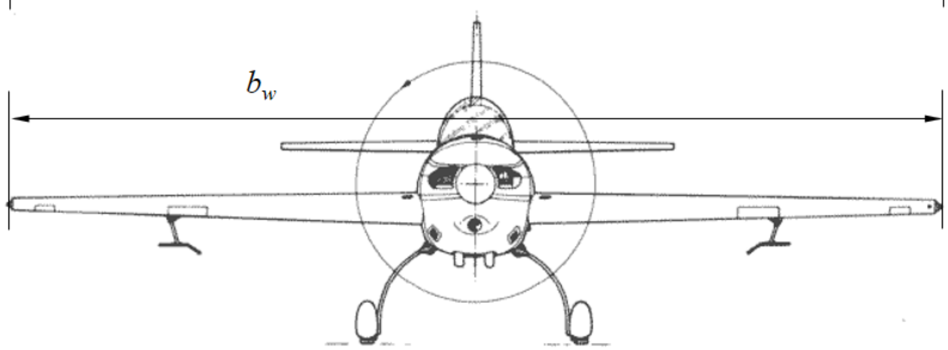
As this wing has aerodynamic twist it can be assume that there is no need for a geometrical twist, therefore $\beta_{tip} = 0$.

$$\sin \beta_{tip} = \frac{h_{tip}}{c_{tip}} \Rightarrow h_{tip} = c_{tip} \sin \beta_{tip}$$

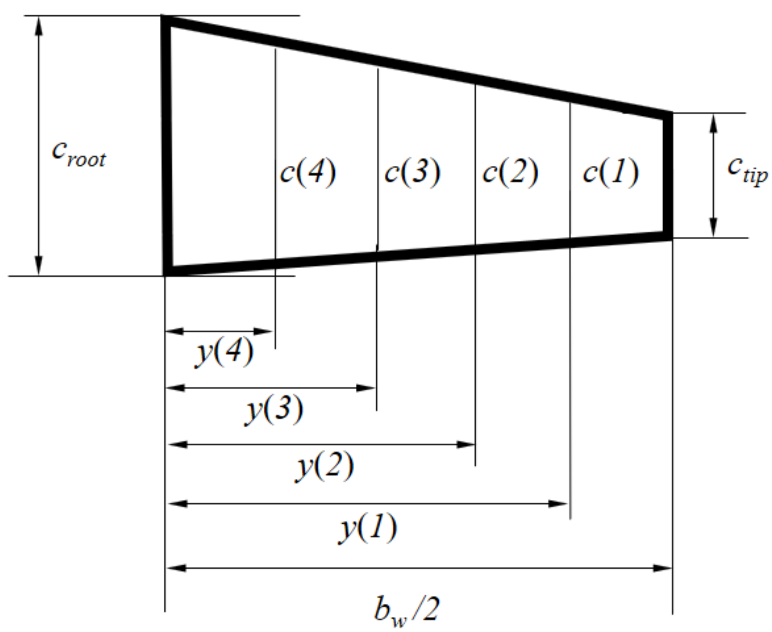
If no geometrical twist ($\beta_{tip} = 0$) is need it for the wing, then the length $h_{tip} = 0$.

First, the wing is divided into two symmetrical sections, the left wing (port side) and the right wing (starboard side):





Then we can stablish an amount of control points (k) that will be calculated for the wing, In this case we are going to put eight control points, which means that it is going to be four per half-wing.



Stablishing the above, the *y*-stations on the wing span (half in this case) can be found as follows:

$$y(k) = -\frac{b_w}{2}\left(1 - \frac{2k-1}{2N}\right)$$

Results are shown in the following table:

k	**y(k) [m]**
1	-3.5
2	-2.5
3	-1.5
4	-0.5

The local chord length values at each station (control point) are found by:

$$c(k) = c_{root}\left(1 - 2\frac{\lambda - 1}{b_w}y(k)\right)$$

Results are shown in the following table:

k	**c(k) [m]**
1	0.957
2	1.211
3	1.464
4	1.718

Next, the twist angle distribution is calculated using the relations:

$$\beta(k) = \sin^{-1}\left(-\frac{2y(k)}{b_w \cdot c(k)} \cdot h_{tip}\right); \quad h_{tip} = c_{tip}\sin\beta_{tip}$$

In this airplane the wing has an aerodynamical twist instead of a geometric one, therefor the results are:

k	**β (k) [deg]**
1	0
2	0
3	0
4	0

The variation of the lift slope along the wing will vary because the root and tip aerofoils have different references. This also happens with the zero lift angle of attack for the different points of control. The variations are calculated using:

$$a_0(k) = a_{0\_root} - 2\frac{a_{0\_tip} - a_{0\_root}}{b_w} y(k)$$

$$\alpha_{L=0}(k) = \alpha_{L=0\_root} - 2\frac{\alpha_{L=0\_tip} - \alpha_{L=0\_root}}{b_w} y(k)$$

The results are shown in the following table:

k	$a_0(k)$	$\alpha_{L=0}(k)$
1	6.372	0
2	6.390	0
3	6.408	0
4	6.427	0

Note that for this wing, as booth aerofoils are symmetrical the zero lift angle of attack is equal to zero for both cases, therefor the variation will also have this same value along the span.

Now, let's calculate the transformation $\theta(k)$ using:

$$\theta(k) = \cos^{-1}\left(-\frac{2y}{b_w}\right)$$

Results of this relation are shown as follows,

k	$\theta(k)$ [*deg*]
1	28.955
2	51.318
3	67.976
4	82.819

With these values now the $D(k)$ variation can be calculated:

$$D(k) = \alpha_{geo} - \alpha_{L=0}(k) + \beta(k)$$

Results for this equation are:

k	$D(k)$ [*rad*]
1	0.035
2	0.035
3	0.035
4	0.035

Next, we want to create the *4*x*4* sized matrix coefficients with the following formula:

$$C(k,n)=\left[\left(\frac{4b_w}{a_0(k)c(k)}+\frac{2n-1}{\sin(\theta(k))}\right)\sin((2n-1)\theta(k))\right]$$

Where $k = n = 4$, this is because we are taking advantage of the symmetry of the wing loading distribution we have $N = 4$ and we can use $n = 1$, 2, 3 and 4 (port wing only) or $n = 5$, 6, 7 and 8 (starboard wing only). If port wing control points only are chosen, then we can calculate the variation for each point. For this example, we present only 4 points as reference:

For $k = 1$ and $n = 1$

$$C(1,1)=\left[\left(\frac{4(8.0)}{a_0(1)c(1)}+\frac{2(1)-1}{\sin(\theta(1))}\right)\sin((2(1)-1)\theta(1))\right]$$

$$C(1,1)=\left[\left(\frac{4(8.0)}{(6.372)(0.957)}+\frac{2(1)-1}{\sin(28.955)}\right)\sin((2(1)-1)(28.955))\right]$$

$$C(1,1)=3.541$$

For $k = 2$ and $n = 1$

$$C(2,1)=\left[\left(\frac{4(8.0)}{a_0(2)c(2)}+\frac{2(1)-1}{\sin(\theta(2))}\right)\sin((2(1)-1)\theta(2))\right]$$

$$C(2,1)=\left[\left(\frac{4(8.0)}{(6.39)(1.211)}+\frac{2(1)-1}{\sin(51.318)}\right)\sin((2(1)-1)(51.318))\right]$$

$$C(2,1)=4.229$$

For $k = 1$ and $n = 2$

$$C(1,2)=\left[\left(\frac{4(8.0)}{a_0(1)c(1)}+\frac{2(2)-1}{\sin(\theta(1))}\right)\sin((2(2)-1)\theta(1))\right]$$

$$C(1,2)=\left[\left(\frac{4(8.0)}{(6.372)(0.957)}+\frac{2(2)-1}{\sin(28.955)}\right)\sin((2(2)-1)(28.955))\right]$$

$$C(1,1)=11.428$$

For $k = 2$ and $n = 2$

$$C(2,2)=\left[\left(\frac{4(8.0)}{a_0(2)c(1)}+\frac{2(2)-1}{\sin(\theta(2))}\right)\sin((2(2)-1)\theta(2))\right]$$

$$C(2,2)=\left[\left(\frac{4(8.0)}{(6.39)(1.211)}+\frac{2(2)-1}{\sin(51.318)}\right)\sin((2(2)-1)(51.318))\right]$$

$$C(2,2)=3.504$$

Results for this equation are given in the following matrix:

$$C(k=4,n=4)=\begin{vmatrix} 3.541 & 11.428 & 8.984 & -7.6 \\ 4.229 & 3.504 & -10.254 & -0.177 \\ 4.161 & -2.696 & -3.029 & 9.866 \\ 3.876 & -5.508 & 6.43 & -6.363 \end{vmatrix}$$

Next step in the *LLT* methodology is to solve the following system of simultaneous equations:

$$\sum_{n=1}^{N} C(k,n)A(n)=D(k)\text{, for } k = 1,\ 2,\ 3,\ \ldots,\ N$$

For this case the equation system gives:

$$\sum_{n=1}^{N} C(k,n)A(n)=D(k)\Rightarrow\begin{pmatrix} C(1,1) & C(1,2) & C(1,3) & C(1,4) \\ C(2,1) & C(2,2) & C(2,3) & C(2,4) \\ C(3,1) & C(3,2) & C(3,3) & C(3,4) \\ C(4,1) & C(4,2) & C(4,3) & C(4,4) \end{pmatrix}\cdot\begin{pmatrix} A(1) \\ A(2) \\ A(3) \\ A(4) \end{pmatrix}=\begin{pmatrix} D(1) \\ D(2) \\ D(3) \\ D(4) \end{pmatrix}$$

$$\sum_{n=1}^{N} C(k,n)A(n)=D(k)\Rightarrow\begin{pmatrix} 3.541 & 11.428 & 8.984 & -7.6 \\ 4.229 & 3.504 & -10.254 & -0.177 \\ 4.161 & -2.696 & -3.029 & 9.866 \\ 3.876 & -5.508 & 6.43 & -6.363 \end{pmatrix}\cdot\begin{pmatrix} A(1) \\ A(2) \\ A(3) \\ A(4) \end{pmatrix}=\begin{pmatrix} 0.035 \\ 0.035 \\ 0.035 \\ 0.035 \end{pmatrix}$$

This equation system can be solve by the *Gaussian Elimination Method*. Other methods, such as the *Jacobi* or *Gauss-Seidel* iterative methods may also be used.

The solution of an equation system (matrices) by the *Gaussian Elimination* is:

$$C(k,n)A(n)=D(k)\Rightarrow A(n)=[C(k,n)]^{-1}\cdot D(k)$$

Meaning that the inverse of the matrix $C(k,n)$ must be find first, in order to solve the system, this is done in this case by the *Gauss-Jordan* method:

$$[C(k,n)]^{-1} = \left(\begin{array}{cccc|cccc} 3.541 & 11.428 & 8.984 & -7.6 & 1 & 0 & 0 & 0 \\ 4.229 & 3.504 & -10.254 & -0.177 & 0 & 1 & 0 & 0 \\ 4.161 & -2.696 & -3.029 & 9.866 & 0 & 0 & 1 & 0 \\ 3.876 & -5.508 & 6.43 & -6.363 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$[C(k,n)]^{-1} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0.038 & 0.057 & 0.079 & 0.076 \\ 0 & 1 & 0 & 0 & 0.055 & 0.009 & 0.002 & -0.063 \\ 0 & 0 & 1 & 0 & 0.034 & -0.070 & 0.032 & 0.011 \\ 0 & 0 & 0 & 1 & 0.010 & -0.043 & 0.079 & -0.046 \end{array}\right)$$

With the matrix inverse result, the system can be solve for $A(n)$ by multiplying $[C(k,n)]^{-1}$ by $D(k)$:

$$A(n) = [C(k,n)]^{-1} \cdot D(k) = \begin{pmatrix} A(1) \\ A(2) \\ A(3) \\ A(4) \end{pmatrix} = \begin{pmatrix} 0.038 & 0.057 & 0.079 & 0.076 \\ 0.055 & 0.009 & 0.002 & -0.063 \\ 0.034 & -0.070 & 0.032 & 0.011 \\ 0.010 & -0.043 & 0.079 & -0.046 \end{pmatrix} \cdot \begin{pmatrix} 0.035 \\ 0.035 \\ 0.035 \\ 0.035 \end{pmatrix}$$

$$\begin{pmatrix} A(1) \\ A(2) \\ A(3) \\ A(4) \end{pmatrix} = \begin{pmatrix} 0.008734 \\ 0.000133 \\ 0.000244 \\ -0.000034 \end{pmatrix}$$

With these results, the wing lift coefficient can now be found by the relation:

$$C_{L\_w} = \pi \cdot AR \cdot A(1) = \pi \cdot (5.98) \cdot (0.008734) = 0.164$$

The total lift force exerted by the wing is:

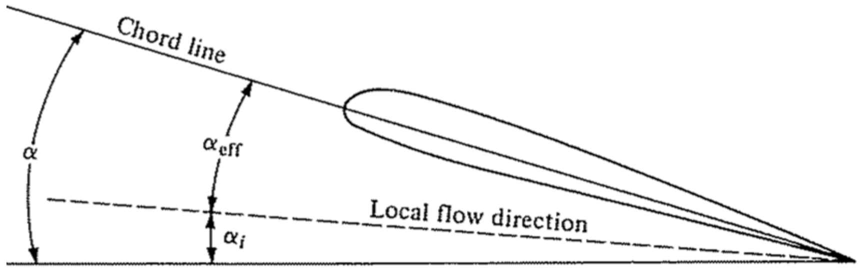
$$L_w = q_\infty S_w C_{L\_w} = \frac{1}{2}\rho_\infty V_\infty^2 S_w C_{L\_w} = \frac{1}{2}(1.225)(88.056)^2(10.7)(0.164) = 8340.38[N]$$

$$L_w = 8340.38[N] = 850.192[kg]$$

Analysing this result and comparing it with the opposing force (airplane´s *MTOW*) the geometric angle of attack of two degrees it´s not enough to lift-off the airplane from the ground; $L_w < MTOW$.

It means that for a geometric angle of attack (incidence angle) of two degrees the airplane still needs more lift force to counteract the *MTOW*. Some considerations must be considered to assure the wing with its geometric and aerodynamic characteristics can achieve enough lift force, then a study of different incidence angles must be made to study the possibilities of wing attitude with respect to the horizontal plane of the aircraft.

The following figure illustrates the differences between the geometric (incidence) angle of attack, the effective angle of attack and the induced angle of attack:



Examining the profile drag curves of both aerofoils, it is decided that the total profile drag coefficient for the wing is $c_{d\_0} = 0.0054$ for a geometric angle of two degrees.

Now, let´s calculate the total wing drag force at the established conditions:

The *Oswald* efficiency planform factor is crucial for the calculation of the wing induced drag at the flight conditions given. The equation to find it is:

$$e = (1+\delta)^{-1}, \text{ where: } \delta = \sum_{n=2}^{N} n\left(\frac{A(n)}{A(1)}\right)^2$$

The delta coefficient depends on the *Fourier* series coefficients, which were calculated previously.

$$\delta = \sum_{n=2}^{N} n\left(\frac{A(n)}{A(1)}\right)^2 = 2\cdot\left(\frac{0.000133}{0.008734}\right)^2 + 3\cdot\left(\frac{0.000244}{0.008734}\right)^2 + 4\cdot\left(\frac{-0.000034}{0.008734}\right)^2 = 0.002876$$

$$e = (1+\delta)^{-1} = (1+0.002876)^{-1} = 0.9971$$

As the result shows a high value, it is recommended to check it with some other methodology that predicts the wing efficiency.

The induced drag coefficient C_{Di} is obtained by:

$$C_{Di} = \frac{C_L^2}{\pi\cdot e\cdot AR} = k\cdot C_L^2, \text{ where: } k = \frac{1}{\pi\cdot e\cdot AR} = \frac{1}{\pi(0.9971)(5.98)} = 0.0534$$

$$C_{Di} = \frac{C_L^2}{\pi\cdot e\cdot AR} = k\cdot C_L^2 = (0.0534)(0.164)^2 = 0.00144$$

The total wing drag coefficient will be:

$$C_{D\_w} = c_{d\_prof} + C_{Di} = c_{d\_prof} + \frac{C_L^2}{\pi\cdot e\cdot AR} = 0.00684$$

So, for this flight condition the total wing drag will be equal to:

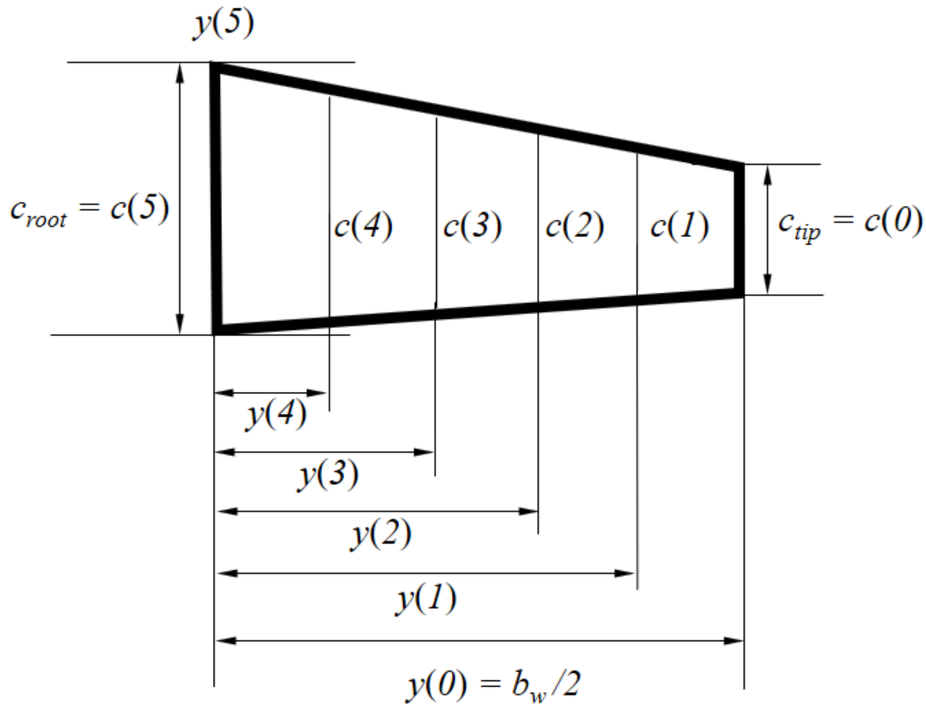
$$D_w = q_\infty S_w C_{D\_w} = \frac{1}{2}\rho_\infty V_\infty^2 S_w C_{D\_w} = \frac{1}{2}(1.225)(88.056)^2(10.7)0.00684 = 347.466[N]$$

$$D_w = 347.466[N] = 35.42[kg]$$

Analysing the wing load distribution:

The main idea with this analysis is to establish how the lift load distribution over the wing is going to be. For this it is expected to obtain a distribution that describe an elliptical shape to make it as aerodynamically efficient as possible. First define the stations in which you want to study the local lift load, for this help yourself with already y stations already stablished. In this case we are analysing half span beginning at the root of the wing:

Station $y(k)$	**Points of control + c_{tip} and $c_{root} - y$**	**Location [m]**
$y(0) = b_w/2$	0	4.0
$y(1)$	1	3.5
$y(2)$	2	2.5
$y(3)$	3	1.5
$y(4)$	4	0.5
$y(5)$	5	0



Recalling the transformation $\theta(k)$ using:

$$\theta(k) = \cos^{-1}\left(-\frac{2y}{b_w}\right)$$

This time the results include the values at the root and the tip of the wing,

Points of control + c_{tip} and c_{root} – y	**y [m]**	**$\theta(y)$ [deg]**
0	*4.0*	*0*
1	*3.5*	*28.955*
2	*2.5*	*51.318*
3	*1.5*	*67.976*
4	*0.5*	*82.819*
5	*0*	*90.0*

Keeping in mind that the load distribution is symmetrical then circulation on each station over the wing can be calculated by the equation:

$$\Gamma(y) = \Gamma(\theta(y)) = 2b_w V_\infty \left[A(1)\sin\theta(y) + (A(2))\sin(3\cdot\theta(y))\right]$$

Results are presented in the following table:

Points of control + c_{tip} and c_{root} – y	**y [m]**	**$\Gamma(y)$ [m^2/sec]**
0	*4.0*	*0*
1	*3.5*	*5.958*
2	*2.5*	*9.606*
3	*1.5*	*11.408*
4	*0.5*	*12.209*
5	*0*	*12.306*

Non-dimensionlazing the circulation it is obtained (Γ_0 – circulation at the root of the wing):

$$\Gamma_{ND} = \frac{\Gamma(y)}{\Gamma_0}$$

Points of control + c_{tip} and c_{root} – y	**y [m]**	**Γ_{ND}**
0	*4.0*	*0*
1	*3.5*	*0.484*
2	*2.5*	*0.781*
3	*1.5*	*0.927*
4	*0.5*	*0.992*
5	*0*	*1.0*

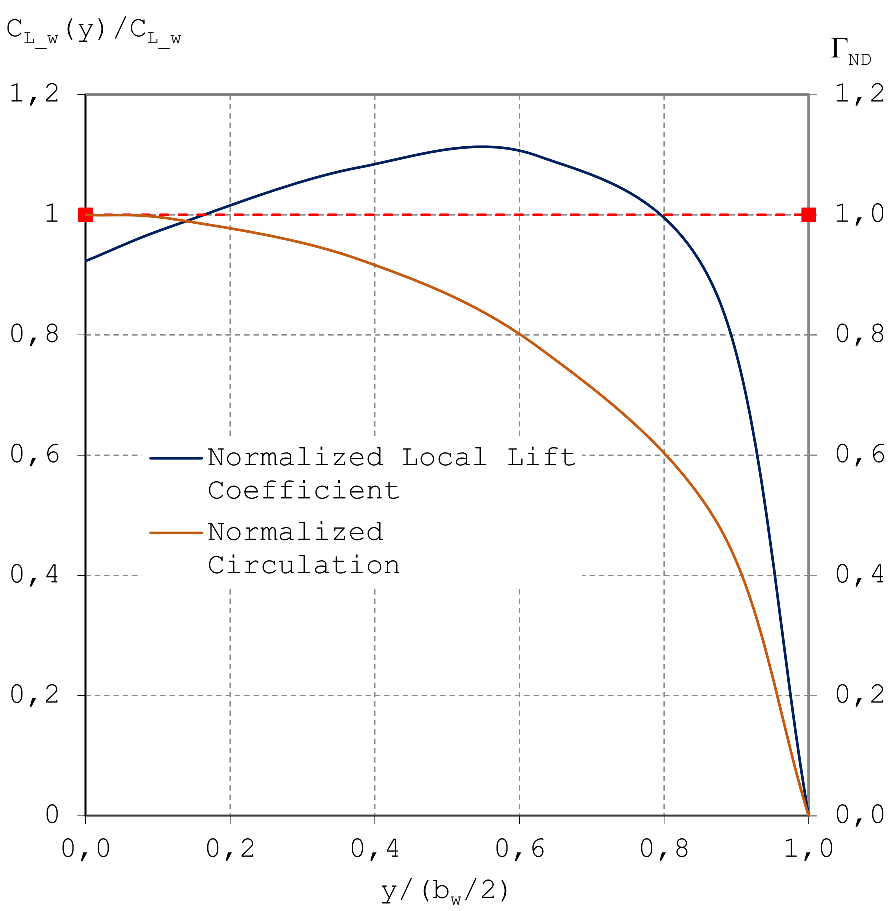
Now, let´s compute the wing lift coefficient and the force in each station of the half-span. The local wing lift coefficient and the correspondent force both can be calculated by the equations:

$$C_{L\_w}(y) = \frac{2\Gamma(y)}{V_\infty c(y)} \Rightarrow \frac{C_{L\_w}(y)}{C_{L\_w}} = \frac{1}{C_{L\_w}}\left(\frac{2\Gamma(y)}{V_\infty c(y)}\right)$$

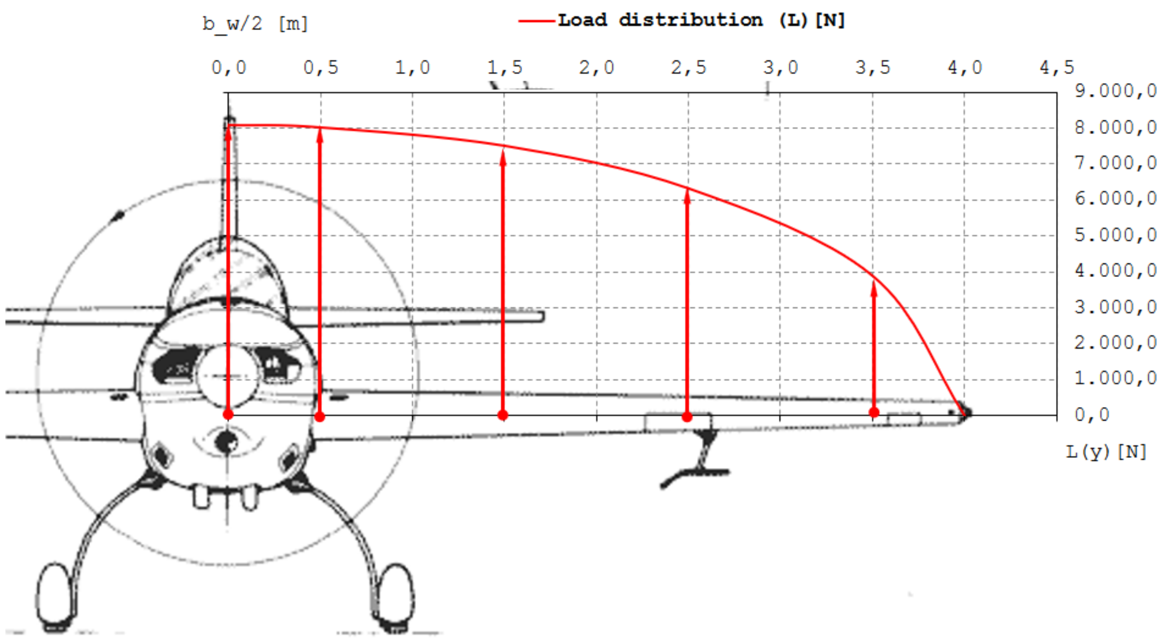
$$L(y) = \frac{1}{2}\rho_\infty V_\infty^2 c(y)\frac{C_{L\_w}(y)}{C_{L\_w}}$$

Points of control + c_{tip} and c_{root} – y	**y [m]**	**$y/(b_w/2)$**	**$C_{L\_w}(y)/C_{L\_w}$**	**$L(y)$ [N]**
0	*4.0*	*1.0*	*0*	*0*
1	*3.5*	*0.875*	*0.862*	*3915.515*
2	*2.5*	*0.625*	*1.098*	*6313.475*
3	*1.5*	*0.375*	*1.078*	*7497.437*
4	*0.5*	*0.125*	*0.983*	*8024.157*
5	*0*	*0*	*0.923*	*8087.584*

Using this information, the normalized local lift coefficient and the local circulation are plotted as shown in the following graph:



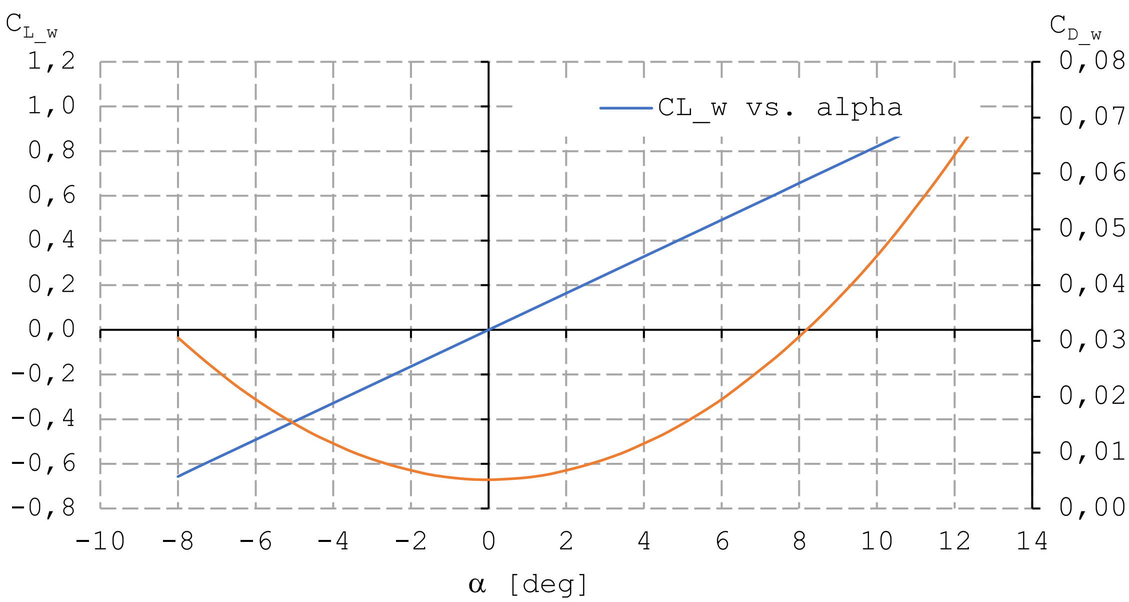
In addition, the lift distribution is plotted along the half wingspan in order to get the values of the distributed load over the wing:



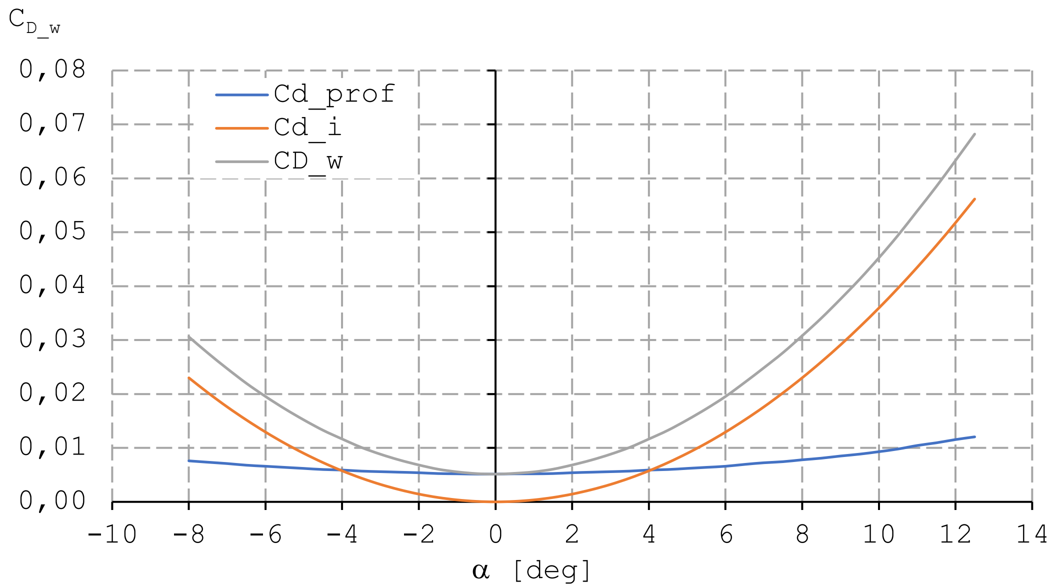
As stablished before, the net lift produced by this wing is not enough to counteract the MTOW, a further study should be done to assure proper values of the wing incidence angle of attack.

Using the *LLT* mathematical model, we can find different values of the net wing lift coefficient at different angles of attack. The following graphs show the results of this procedure:

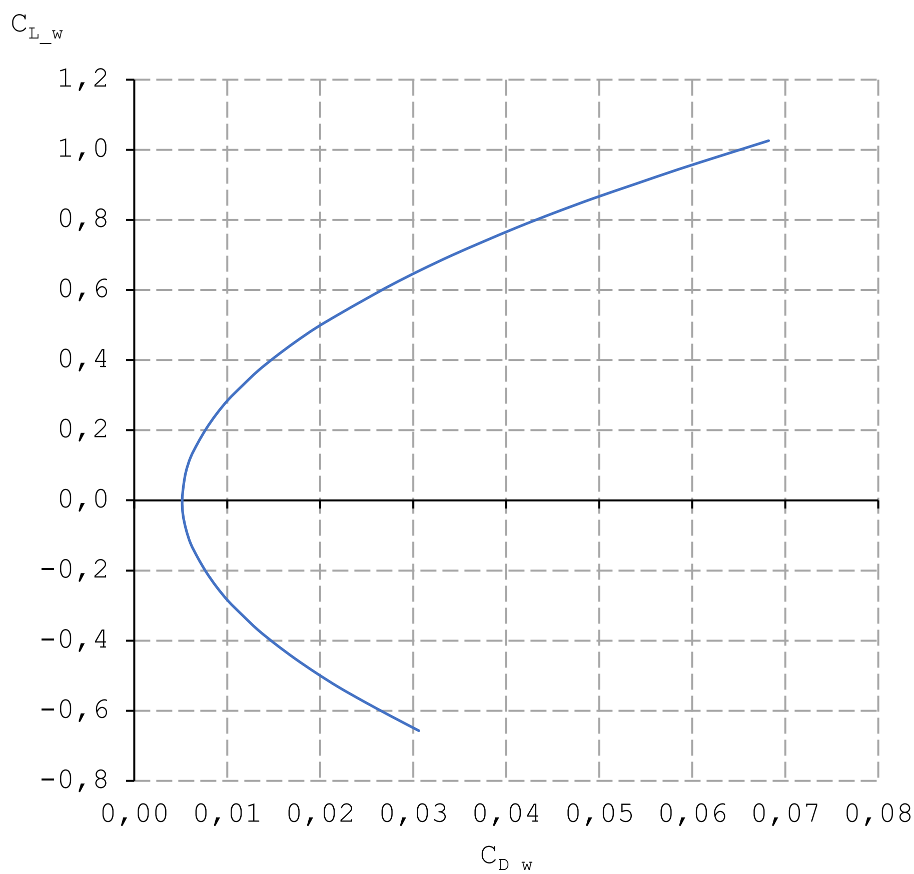
C_{L_w} vs. α [*deg*] and C_{D_w} vs. α [*deg*]



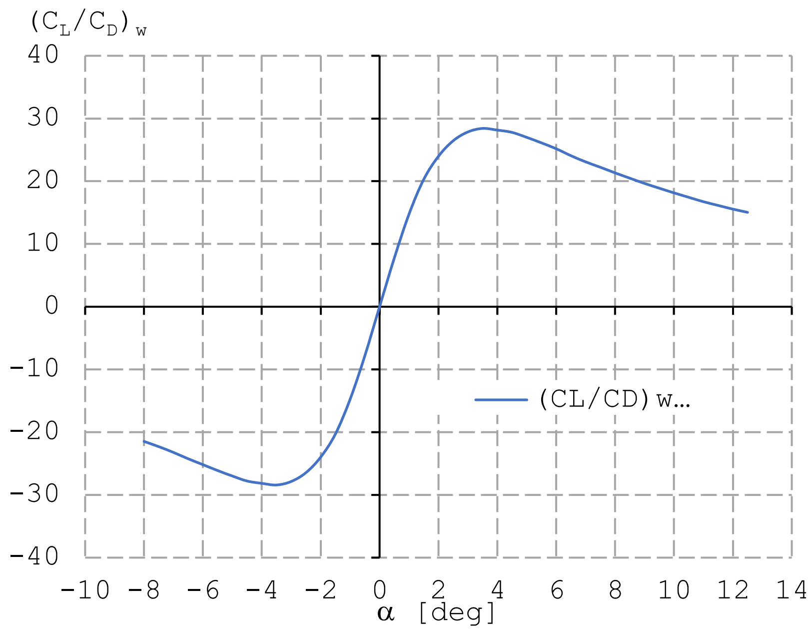
$C_{D\_w.} = c_{d\_prof} + C_{D\_w}$ vs. α [*deg*]



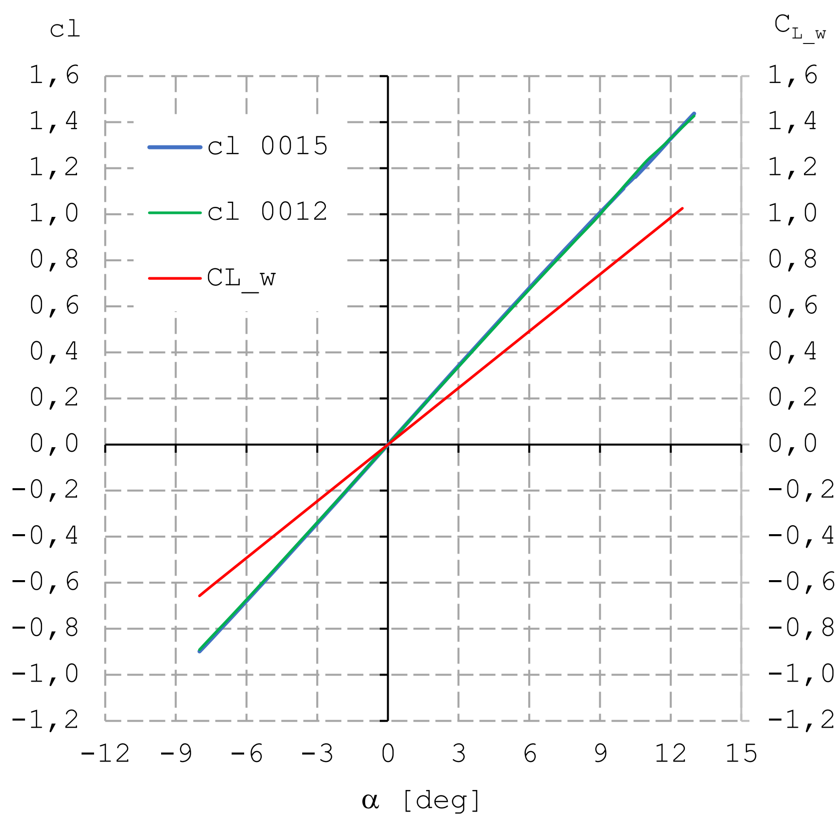
Polar curve: C_{L_w} vs. C_{D_w}



$(C_L/C_D)_w$ vs. α [deg]



The following graph shows the comparison between the aerofoils lift coefficient and the wing lift coefficient showing the changes on the lift slope:



With all this information, a new value of the net wing lift coefficient can be found. We must first be clear that in normal cruise flight at constant acceleration the *MTOW* of the airplane is equal to the total net lift produced by the wing:

$$L_w = MTOW = \frac{1}{2}\rho_\infty V_\infty^2 S_w C_{L\_w} = q_\infty S_w C_{L\_w}$$

From this relation the net wing lift coefficient is obtained:

$$C_{L\_w} = \frac{MTOW}{q_\infty S_w} = 0.1834$$

From the graph C_{L_w} vs. α [*deg*] the wing lift slope is calculated:

$$a = \frac{C_{L\_w}}{\alpha - \alpha_{L=0}} = 0.08213\left[{}^{1}\!/_{\mathrm{deg}}\right] = 4.705\left[{}^{1}\!/_{rad}\right]$$

With this same relation, the net wing lift coefficient is calculated. It is to note that in this graph (C_{L_w} vs. α [*deg*]) it can be seen that $\alpha_{L=0} = 0$ for this particular wing, then

$$C_{L\_w} = a\left(\alpha - \alpha_{L=0}\right) = C_{L\_w} = a \cdot \alpha$$

Equating the last relation, the angle of attack required for the cruise condition is found to be:

$$\alpha = \frac{C_{L\_w}}{a} = \frac{0.1834}{0.08213} \approx 2.24[\mathrm{deg}]$$

With this new value of the geometric (wing incidence) angle of attack, the values of the wing aerodynamic characteristics are:

Wing Aerodynamic Characteristic	**Value**	**Units**
Lift force (L_w)	*952.215*	*kgf*
Induced drag coefficient (C_{D_i})	*0.0018*	-
Total drag force (D_w)	*37.314*	*kgf*

You could study more about the response to change of some variables like the altitude (height), velocity, different aerofoils, applying of a geometric twist angle and even the change in the geometric shape of the wing to check the consequences in the aerodynamic characteristics of this airplane wing.