Analytic characterization of oblique shock waves in flows around wedges

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ANALYTIC CHARACTERIZATION OF OBLIQUE SHOCK WAVES IN FLOWS AROUND WEDGES

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Abstract. We consider the compressible flow around triangular wedges in which oblique shock waves are formed. We report on the novel analytic solution regarding the evaluation of the maximum wedge angle beyond which the shock wave detaches from the wedge to promote the formation of a bow shock. In addition, the limit line at which the flow past the oblique shock becomes sonic is determined whereby an analytic characterization for the corresponding shock angle is presented.
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1. Introduction

The supersonic flow past triangular wedge is discussed in many gas dynamics classics and is considered a renowned problem in the study of oblique shock wave in two-dimensional compressible flows [1–3]. Applying the fundamental conservation laws across the shock wave, the governing equation relating the free stream Mach number M_1, the wedge (deflection) angle θ, and, the shock wave angle β, is formed. This equation is given as [3, 4]

$$\tan\theta = \frac{2}{\tan\beta}\frac{M_1^2\sin^2\beta - 1}{M_1^2(\gamma+\cos 2\beta)+2} \tag{1}$$

γ being the gas specific heat ratio. The resulting Mach number for the flow past the oblique shock wave, M_2, is calculated as follows [1]:

$$M_2^2\sin^2(\beta-\theta) = \frac{2+(\gamma-1)M_1^2\sin^2\beta}{2\gamma M_1^2\sin^2\beta-\gamma+1} \tag{2}$$

While Eq. (1) appears to necessitate a numerical nonlinear solver if β is to be solved in terms of M_1 and θ, a clever substitution of the trigonometric functions in favour of $\tan\beta$ generates a cubic equation solvable in closed form by the method of radicals [4, 5]. It is expressed as

$$\begin{aligned}\left[2+(\gamma-1)M_1^2\right]\tan\theta\tan^3\beta + 2\left(1-M_1^2\right)\tan^2\beta \\ +\left[2+(\gamma+1)M_1^2\right]\tan\theta\tan\beta + 2 = 0\end{aligned} \tag{3}$$

Algebraically, Eq. (3) admits three roots. The negative root is rejected and the remaining positive roots constitute the physical solution. The family $\theta-\beta$ curves for various M_1 are shown in Fig. 1. For any given $M_1 > 1$, the range of β is bounded from below by the Mach angle, $\mu = \arcsin\left(\frac{1}{M_1}\right)$, and its upper bound

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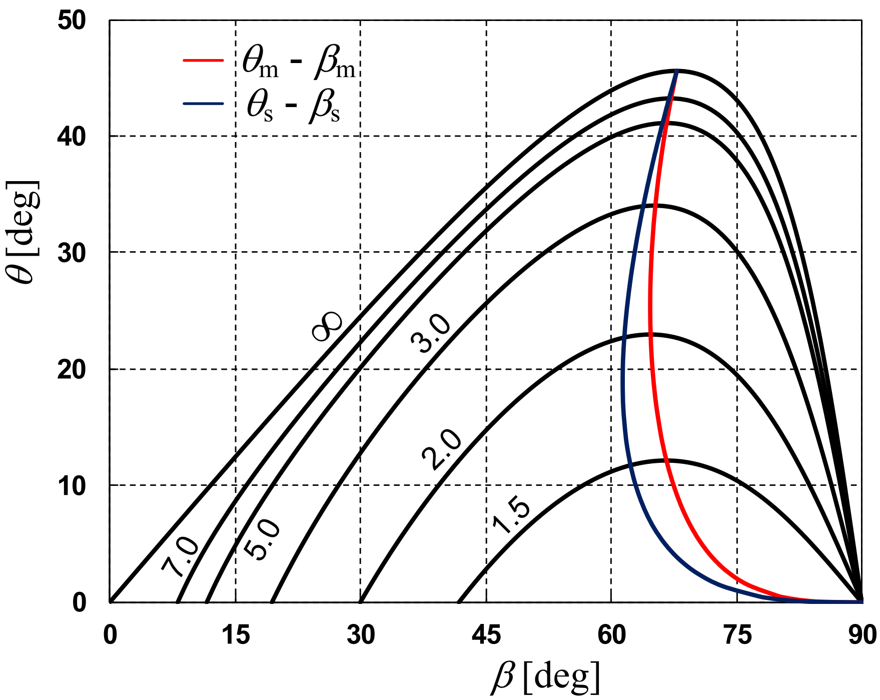

FIGURE 1. $\theta - \beta$ family curves for some special values of M_1. The sonic separatrix ($\theta_s - \beta_s$) and the detachment separatrix ($\theta_m - \beta_m$) are plotted. Note their unique intersection at $M_1 = 1$ and $M_1 \to \infty$.

is always 90 deg corresponding to the case of normal shock wave. The solution curve of β features a local maximum point at $\beta = \beta_m$ with corresponding deflection angle denoted θ_m. For $\theta > \theta_m$, the oblique shock detaches from the wedge and a bow shock is formed. The contour line joining the points (θ_m, β_m) constitutes a *separatrix* splitting the region of the $\theta - \beta$ curves into a zone of strong shocks (situated to the right of separatrix) and a zone of weak shocks (situated to the left of separatrix) [3]. In all previous studies, the (θ_m, β_m) points are determined by solving the nonlinear equation resulting from the straightforward method to determine local extrema points, i.e. set $\frac{d\theta}{d\beta} = 0$ and then solve for β [5]. In our work, we present a different approach to evaluate these maxima points. In fact, the first objective of this work is to formulate the analytic expressions of $\beta_m(M_1, \gamma)$ and $\theta_m(M_1, \gamma)$.

Another line of interest on the $\theta - \beta$ family curves is the sonic limit. On this line, the flow past the oblique shock is sonic, thus $M_2 = 1$. The sonic line intersects the $\theta - \beta$ curves at the particular points (θ_s, β_s). In similarity to the strong-weak shock segregation, the sonic limit separatrix splits the region of the $\theta - \beta$ family curves into a zone of supersonic flow to the left of separatrix and subsonic flow on its right side. The two separatrices intersect at two special points. The first, corresponds to $M_1 = 1$ whereby $\beta_m = \beta_s = 90$ deg and $\theta_m = \theta_s = 0$ deg; in

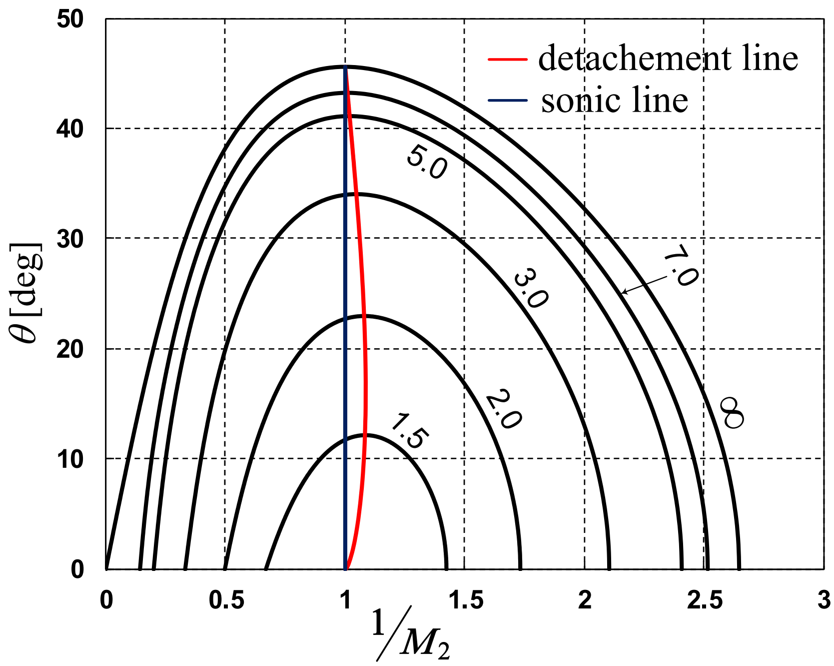

FIGURE 2. $\theta - M_2$ family curves for some special values of M_1. For convenience, $\frac{1}{M_2}$ is plotted. The sonic line corresponding to $M_2 = 1$ and the detachment line which joins all θ_{m} points are also plotted.

fact for the trivial case $M_1 = 1$, the entire $\theta - \beta$ curve collapses to the one point $(0\,\mathrm{deg}, 90\,\mathrm{deg})$. The second point of intersection corresponds to $M_1 \to \infty$, in such a case, we obtain $\beta_{\mathrm{m}} = \beta_{\mathrm{s}} \to \arctan\sqrt{\frac{\gamma+1}{\gamma-1}}$ and $\theta_{\mathrm{m}} = \theta_{\mathrm{s}} \to \arctan\sqrt{\frac{1}{\gamma^2-1}}$, these results are derived in the upcoming sections. We also prove that for all physically meaningful problems, $\beta_{\mathrm{m}} \geq \beta_{\mathrm{s}}$ and $\theta_{\mathrm{m}} \geq \theta_{\mathrm{s}}$; equality occurs only at the two limiting points just introduced. From this inequality, we deduce that strong shocks always result in a subsonic flow, while flows past weak shocks can either be subsonic or supersonic. This conclusion is graphically illustrated in Fig. 2 whereby the variation of θ with respect to $\frac{1}{M_2}$ is plotted. The detachment separatrix is situated in the zone $\frac{1}{M_2} > 1$, corresponding to subsonic downstream flow. The zone between the two separatrices corresponds to weak shocks albeit a subsonic downstream flow.

2. CLOSED FORM EVALUATION OF β_{m} AND θ_{m}

A general cubic equation of the form $ax^3 + bx^2 + cx + d = 0$ admits a double root x_{d} if its discriminant vanishes. The double root also nullifies the first derivative of the polynomial equation. Therefore,

$$27a^2d^2 - b^2c^2 + 4ac^3 - 18abcd + 4b^3d = 0 \tag{4a}$$

$$3ax_{\mathrm{d}}^2 + 2bx_{\mathrm{d}} + c = 0 \tag{4b}$$

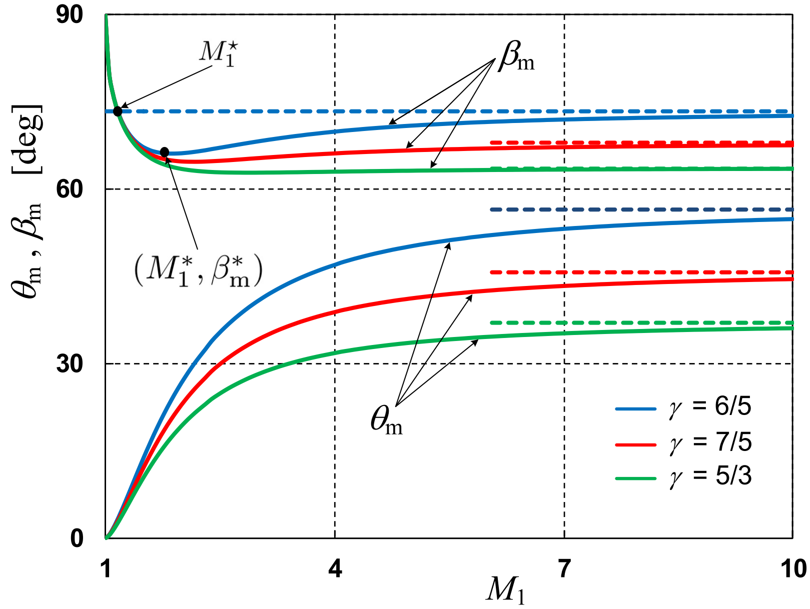

FIGURE 3. The variation of $\beta_{\rm m}$ and $\theta_{\rm m}$ with respect to M_1. The asymptotes for large M_1 are also shown. The local minima points on the $\beta_{\rm m}$ curve $(M_1^*, \beta_{\rm m}^*)$ are indicated.

Applying the formula of Eq. (4a) to the coefficients of Eq. (3), the result is a mathematical characterization of the local maximum point $(\theta_{\rm m}, \beta_{\rm m})$ and consequently, an algebraic equation with $\tan\theta_{\rm m}$ as unknown emerges. Thus

$$
\begin{aligned}
(5)\quad & \left[2+(\gamma-1)\,M_1^2\right]\left[2+(\gamma+1)\,M_1^2\right]^3\tan^4\theta_{\rm m} \\
&\quad + \Big\{27\left[2+(\gamma-1)\,M_1^2\right]^2-\left(1-M_1^2\right)^2\left[2+(\gamma+1)\,M_1^2\right]^2 \\
&\quad - 18\left(1-M_1^2\right)\left[2+(\gamma-1)\,M_1^2\right]\left[2+(\gamma+1)\,M_1^2\right]\Big\}\tan^2\theta_{\rm m} \\
&\quad + 16\left(1-M_1^2\right)^3 = 0
\end{aligned}
$$

Eq. (5) is of bi-quadratic type with only one root being physically meaningful (the product of all four roots is negative). In conjunction with the admissible solution of Eq. (5), a closed form expression for $\cos 2\theta_{\rm m}$ is provided in [5]. In the limiting case $M_1 \to \infty$, the physical solution approaches the asymptotic value $\sqrt{\frac{1}{\gamma^2-1}}$. The solution $\theta_{\rm m}\,(M_1)$ for various values of γ is plotted in Fig. 3; all plots exhibit identical behaviour regarding the consistent increase of $\theta_{\rm m}$ toward its asymptotic value.

Considering Eq. (4b) and Eq. (3) and eliminating $\tan\theta_{\mathrm{m}}$, we obtain the governing equation for $\tan\beta_{\mathrm{m}}$ which takes the following form

$$(6) \quad \left(1 - M_1^2\right) \frac{2 + (\gamma - 1)\, M_1^2}{2 + (\gamma + 1)\, M_1^2} \tan^4 \beta_{\mathrm{m}} + \frac{(\gamma + 1)\, M_1^4 + 2\,(\gamma - 1)\, M_1^2 + 4}{2 + (\gamma + 1)\, M_1^2} \tan^2 \beta_{\mathrm{m}} + 1 = 0$$

This is also a bi-quadratic equation in $\tan\beta_{\mathrm{m}}$, admitting one admissible solution expressed as

$$(7) \quad \frac{\tan^2 \beta_{\mathrm{m}}}{M_1^2} = \frac{\sqrt{(\gamma+1)\left[\left(M_1^2 - 4\right)^2 + M_1^2 \gamma \left(M_1^2 + 8\right)\right]}}{2\left(M_1^2 - 1\right)\left[2 + (\gamma - 1)\, M_1^2\right]} + \frac{(\gamma + 1)\, M_1^2 + 2\,(\gamma - 1) + \dfrac{4}{M_1^2}}{2\left(M_1^2 - 1\right)\left[2 + (\gamma - 1)\, M_1^2\right]}$$

From Eq. (7), it is easy to verify that $\lim\limits_{M_1 \to \infty} \tan\beta_{\mathrm{m}} = \sqrt{\frac{\gamma+1}{\gamma-1}}$. Interestingly, β_{m} crosses this limit at a finite value of M_1 denoted $M_1^{\star}$. From Fig. 3 where β_{m} is plotted for various γ, we notice that β_{m} decreases from 90 deg toward a local minimum point $(M_1^*, \beta_{\mathrm{m}}^*)$ and then approaches its asymptote from below. The intersection of β_{m} with its asymptote corresponds to $M_1^{\star}$. In the following, we evaluate $M_1^{\star}$, M_1^*, and β_{m}^*.

To evaluate $M_1^{\star}$, we consider Eq. (6) and substitute for $\tan^2 \beta_{\mathrm{m}}^{\star} = \frac{\gamma+1}{\gamma-1}$. The result is a bi-quadratic equation in M_1 having a single meaningful root $M_1^{\star}$ given by

$$(8) \quad M_1^{\star} = \sqrt{\frac{2\gamma}{(\gamma + 1)\,(2 - \gamma)}}$$

The evaluation of M_1^*, and β_{m}^* requires the rewriting of Eq. (6) as to make M_1 the unknown and $\tan\beta_{\mathrm{m}}$ the parameter. For simplicity, we substitute $\tan^2\beta_{\mathrm{m}}$ by t and M_1^2 by m. The treated equation becomes

$$(9) \quad \left[-t^2\,(\gamma - 1) + t\,(\gamma + 1)\right] m^2 + \left[(\gamma - 3)\, t^2 + 2\,(\gamma - 1)\, t + \gamma + 1\right] m + 2\,(t + 1)^2 = 0$$

The local minimum point, $(M_1^*, \beta_{\mathrm{m}}^*)$ corresponds to Eq. (9) having a double root in m. Indeed, for $\tan^2 \beta_{\mathrm{m}}^* < t < \frac{\gamma+1}{\gamma-1}$, there exists two distinct roots for m. For Eq. (9) to have a double root, its discriminant must vanish. The discriminant is $(t+1)^2 \left[(\gamma + 1)\, t^2 + 2\,(\gamma - 7)\, t + \gamma + 1\right]$ and the three values of t that nullify it are $t_1 = -1$, $t_2 = \frac{\left(\sqrt{3-\gamma}-2\right)^2}{\gamma+1}$, and $t_3 = \frac{\left(\sqrt{3-\gamma}+2\right)^2}{\gamma+1}$. t_1 being negative, does not correspond to a physical solution. t_2 and t_3 are both positive, nevertheless, t_2 generates a double root $m_{\mathrm{d}} < 1$ and this is unacceptable since $M_1 > 1$. Thus, t_3

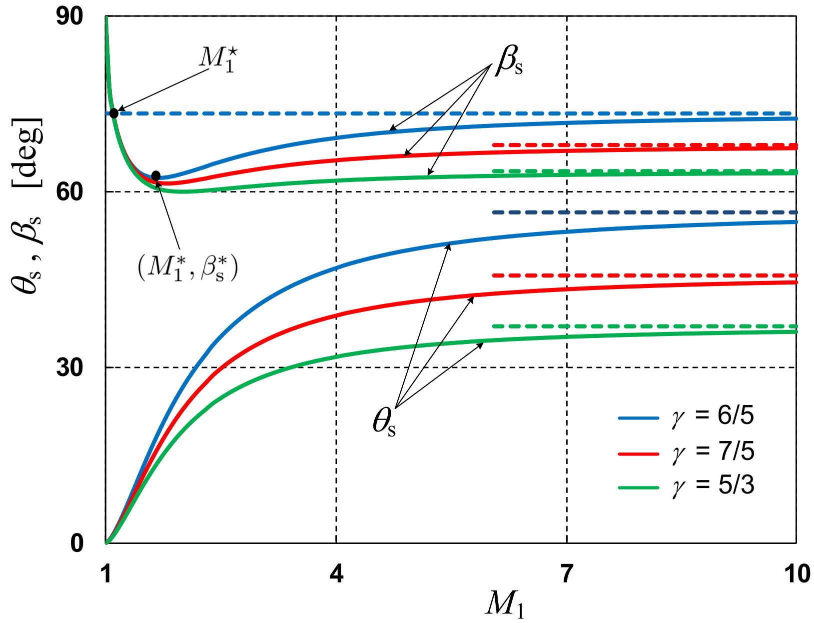

FIGURE 4. The variation of β_s and θ_s with respect to M_1. The asymptotes for large M_1 are also shown. The local minima points on the β_s curve (M_1^*, β_s^*) are indicated.

will be the only acceptable solution corresponding to

$$M_1^* = 2\sqrt{\frac{\gamma - 1 + \sqrt{3-\gamma}}{(\gamma+1)(2-\gamma)}} \tag{10a}$$

$$\tan \beta_m^* = \frac{\sqrt{3-\gamma}+2}{\sqrt{\gamma+1}} \tag{10b}$$

3. Closed form evaluation of β_s

Setting $M_2 = 1$ in Eq. (2), a relation between β_s and θ_s emerges

$$\sin^2(\beta_s - \theta_s) = \frac{2 + (\gamma-1) M_1^2 \sin^2 \beta_s}{2\gamma M_1^2 \sin^2 \beta_s - \gamma + 1} \tag{11}$$

The relation of Eq. (11), along with that of Eq. (1) constitute the incomplete system of nonlinear equations whose unknowns are β_s, θ_s, and M_1. After extensive algebraic work aiming to eliminate θ_s, a bi-quadratic equation with unknown $\tan\beta_s$ and parameter M_1 is obtained. Hence,

$$\begin{aligned}(12)\quad & \left(1 - M_1^2\right)\left[2 + (\gamma-1) M_1^2\right] \tan^4 \beta_s \\ & + \left[M_1^4(\gamma+1) + M_1^2(\gamma-3) + 4\right] \tan^2 \beta_s + 2 = 0\end{aligned}$$

and the physically meaningful root of Eq. (12) is

$$\frac{\tan^2\beta_{\mathrm{s}}}{M_1^2}=\frac{\sqrt{\left[M_1^2(\gamma+1)+\gamma-3\right]^2+16\gamma}+M_1^2(\gamma+1)+\gamma-3+\frac{4}{M_1^2}}{2\left(M_1^2-1\right)\left[2+(\gamma-1)M_1^2\right]} \tag{13}$$

Having solved analytically for $\beta_{\mathrm{s}}(M_1,\gamma)$, the expression for $\theta_{\mathrm{s}}(M_1,\gamma)$ is derived by engaging Eq. (1). The plots of $\theta_{\mathrm{s}}(M_1,\gamma)$ and $\beta_{\mathrm{s}}(M_1,\gamma)$ for three distinct values of γ are shown in Fig. 4. In the limiting case $M_1\to\infty$, the root in Eq. (13), i.e. $\tan\beta_{\mathrm{s}}$, approaches $\sqrt{\frac{\gamma+1}{\gamma-1}}$ while the asymptotic limit for $\tan\theta_{\mathrm{s}}$ is $\sqrt{\frac{1}{\gamma^2-1}}$. This latter limit is obtained by substituting for the appropriate values of M_1 and β_{s} in Eq. (1).

In the plots of Fig. 4, we highlight on two interesting points: the first corresponding to the intersection of the curve with its asymptote, and the second being the local minimum. The first point is evaluated by substituting for the asymptotic value in Eq. (12) to obtain $M_1^\star=2\sqrt{\frac{\gamma}{(\gamma+1)(3-\gamma)}}$. The local minimum is determined by first rewriting Eq. (12) in which the unknown becomes $m=M_1^2$ and the parameter is $t=\tan^2\beta_{\mathrm{s}}$. Hence, the following quadratic equation is produced

$$\left[(\gamma-1)t^2-(\gamma+1)t\right]m^2+(3-\gamma)\left(t^2+t\right)m-2(t+1)^2=0 \tag{14}$$

The local minimum $(M_1^*,\beta_{\mathrm{s}}^*)$ coincides with the point at which Eq. (14) admits a double root. The caveat for double root along with its value correspond to the following results

$$\beta_{\mathrm{s}}^*=\arctan\sqrt{\frac{8}{\gamma+1}} \tag{15a}$$

$$M_1^*=\sqrt{\frac{\gamma+9}{2(3-\gamma)}} \tag{15b}$$

4. Proof of $\beta_{\mathrm{m}}>\beta_{\mathrm{s}}$

The aim of the following work is to prove that $\beta_{\mathrm{m}}>\beta_{\mathrm{s}}$ for all admissible γ and M_1. Considering the two equations that solve for $t_{\mathrm{m}}=\tan^2\beta_{\mathrm{m}}$ and $t_{\mathrm{s}}=\tan^2\beta_{\mathrm{s}}$, mainly Eqs. (6) and (12), we realise that they are both of the form $At^2+Bt+C=0$ sharing a common A term but with different B and C. The expressions for these coefficients along with the roots of interest t_{m} and t_{s}, are given in Eq. (16).

$$A=\left(1-M_1^2\right)\left[2+(\gamma-1)M_1^2\right]<0 \tag{16a}$$

$$B_{\mathrm{m}}-B_{\mathrm{s}}=C_{\mathrm{m}}-C_{\mathrm{s}}=M_1^2(\gamma+1)>0 \tag{16b}$$

$$B_{\mathrm{m}}+B_{\mathrm{s}}=2(\gamma+1)M_1^4+(3\gamma-5)M_1^2+8 \tag{16c}$$

$$t_{\mathrm{m}}=\frac{-B_{\mathrm{m}}-\sqrt{B_{\mathrm{m}}^2-4AC_{\mathrm{m}}}}{2A} \tag{16d}$$

$$t_{\mathrm{s}}=\frac{-B_{\mathrm{s}}-\sqrt{B_{\mathrm{s}}^2-4AC_{\mathrm{s}}}}{2A} \tag{16e}$$

To prove that $\beta_{\mathrm{m}} > \beta_{\mathrm{s}}$ is equivalent to prove $t_{\mathrm{m}} > t_{\mathrm{s}}$. From the expressions of t_{m} and t_{s}, we have

$$
\begin{aligned}
&\frac{-B_{\mathrm{m}} - \sqrt{B_{\mathrm{m}}^2 - 4AC_{\mathrm{m}}}}{2A} > \frac{-B_{\mathrm{s}} - \sqrt{B_{\mathrm{s}}^2 - 4AC_{\mathrm{s}}}}{2A} \\
\Leftrightarrow & - B_{\mathrm{m}} - \sqrt{B_{\mathrm{m}}^2 - 4AC_{\mathrm{m}}} < -B_{\mathrm{s}} - \sqrt{B_{\mathrm{s}}^2 - 4AC_{\mathrm{s}}} \\
\Leftrightarrow & \sqrt{B_{\mathrm{s}}^2 - 4AC_{\mathrm{s}}} - \sqrt{B_{\mathrm{m}}^2 - 4AC_{\mathrm{m}}} < B_{\mathrm{m}} - B_{\mathrm{s}} \\
\Leftrightarrow & \frac{B_{\mathrm{s}}^2 - 4AC_{\mathrm{s}} - B_{\mathrm{m}}^2 + 4AC_{\mathrm{m}}}{\sqrt{B_{\mathrm{s}}^2 - 4AC_{\mathrm{s}}} + \sqrt{B_{\mathrm{m}}^2 - 4AC_{\mathrm{m}}}} < B_{\mathrm{m}} - B_{\mathrm{s}} \\
\Leftrightarrow & \frac{4A\left(C_{\mathrm{m}} - C_{\mathrm{s}}\right) - \left(B_{\mathrm{m}} - B_{\mathrm{s}}\right)\left(B_{\mathrm{m}} + B_{\mathrm{s}}\right)}{B_{\mathrm{m}} - B_{\mathrm{s}}} < \sqrt{B_{\mathrm{s}}^2 - 4AC_{\mathrm{s}}} + \sqrt{B_{\mathrm{m}}^2 - 4AC_{\mathrm{m}}} \\
\Leftrightarrow & 4A - \left(B_{\mathrm{m}} + B_{\mathrm{s}}\right) < \sqrt{B_{\mathrm{s}}^2 - 4AC_{\mathrm{m}}} + \sqrt{B_{\mathrm{m}}^2 - 4AC_{\mathrm{m}}} \\
\Leftrightarrow & M_1^2\left[(\gamma - 7) + 2\left(1 - 3\gamma\right) M_1^2\right] < \sqrt{B_{\mathrm{s}}^2 - 4AC_{\mathrm{m}}} + \sqrt{B_{\mathrm{m}}^2 - 4AC_{\mathrm{m}}}
\end{aligned}
$$

the last inequality is always satisfied since the right hand side is positive and the left hand side is negative. Thus the initial assumption $t_{\mathrm{m}} > t_{\mathrm{s}}$ is true and consequently $\beta_{\mathrm{m}} > \beta_{\mathrm{s}}$ for all $M_1 > 1$. A direct deduction to this result is $\theta_{\mathrm{m}} > \theta_{\mathrm{s}}$ which implies the feasibility of having weak shock with subsonic downstream flow.

5. Conclusion

In this paper, the oblique shock problem is revisited and the closed form expressions for angles of shocks corresponding to sonic limit and detachment limit are formulated. These angles are further characterised whereby their asymptotic limit and local minimum point are evaluated analytically. The followed mathematical procedure is based on seeking the multiple roots of third and fourth order algebraic parametric equations. This approach lead to the composition of a concise proof for the restriction relation $\beta_{\mathrm{m}} > \beta_{\mathrm{s}}$.

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