LIFTING LINE THEORY (LLT) TUTORIAL & EXAMPLE

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This document presents a step-by-step methodology for applying the Lifting Line Theory (LLT) to subsonic straight wings. It complements the aerodynamics lecture.

LLT assumes that even though the flow around an aircraft's wing is 3-D, it may be satisfactorily approximated by a linear summation of flows around the elemental airfoils, which makes up the overall wing, where the flow around each airfoil is assumed to be 2-D. This approach gives a reasonable result, provided that the model flow considers the effect of the vortex sheet, which is shed at the wing's trailing edge. The trailing vortex sheet induces a downwash velocity, which varies along the spanwise direction.

The wing is assumed to be a flat plate on the x-y plane. Therefore, the theory does not consider the wing's thickness distribution. It is also unable to handle any dihedral or sweepback angle. However, it can model a tapered wing with geometrical and aerodynamic twists ($\Lambda_{c/4} = 0$ [deg]).

For the analysis of a wing with LLT, the following wing information is needed:

Wing geometry:

- Wingspan (b_w)
- Wing taper ratio (λ_w)
- Wing reference area (S_w)
- Wing twist angle (β_{tip})
- Wing geometric angle of attack (α_{geo})

Airfoil aerodynamic characteristics:

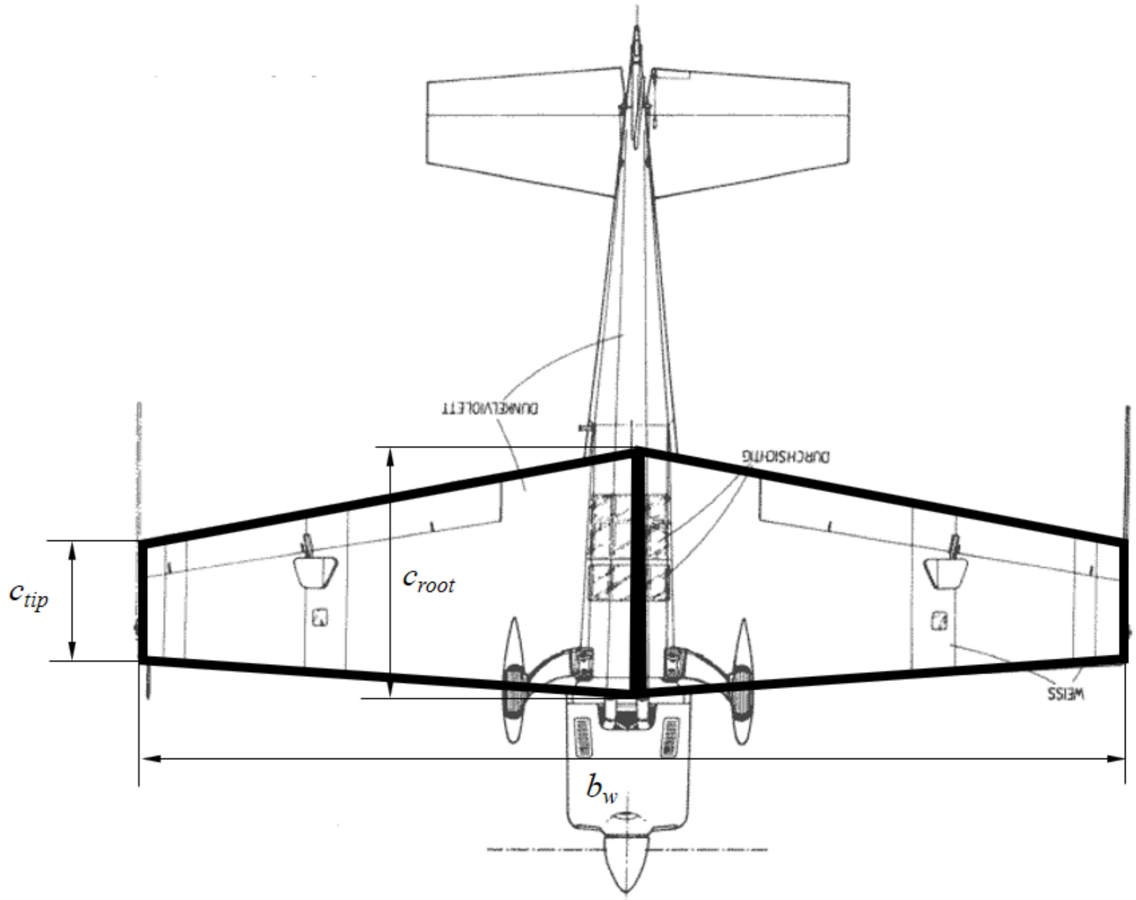
- Airfoil reference – name or reference of the tip and root airfoils
- Lift slope (a_0) [1/rad] – tip and root airfoils
- Angle of attack for L = 0 ($\alpha_{L=0}$) [deg]
- Profile drag (C_{d_0})

Flying (airflow) characteristics:

- Air density (ρ_∞)
- Airspeed (V_∞)
- Air dynamic viscosity (μ_∞)

The LLT procedure is described in the following steps:

Identify the wing geometry to be analyzed



Calculate the wing aspect ratio

$$AR = \frac{b_w^2}{S}$$

Calculate the wing average chord

$$c_{av} = \frac{S}{b_w} = \frac{1}{2} c_{root} \left(1 + \lambda_w\right) = \frac{1}{2\lambda_w} c_{tip} \left(1 + \lambda_w\right)$$

Calculate the root airfoil chord distance

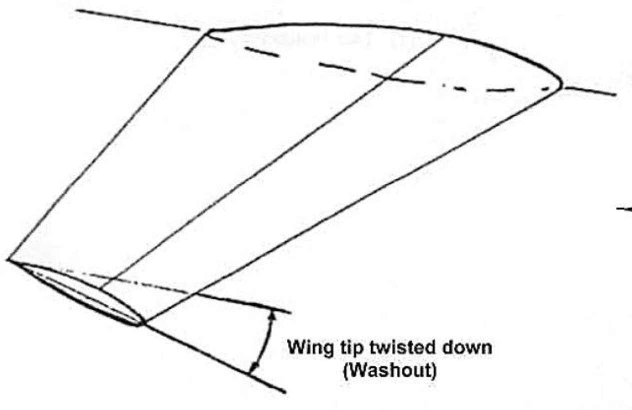
$$c_{root} = \frac{2 \cdot S}{b_w \left(1 + \lambda_w\right)}$$

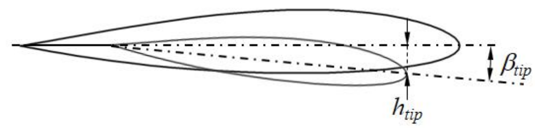
Calculate the tip airfoil chord distance

$$c_{tip} = \lambda_w \cdot c_{root}$$

Calculate the root-to-tip leading edge height difference (geometrical twist)

$$\sin \beta_{tip} = \frac{h_{tip}}{c_{tip}}$$





Calculate the wing mean aerodynamic chord

$$\bar{c} = \frac{2}{3} c_{root} \frac{\left(1 + \lambda_w + \lambda_w^2\right)}{\left(1 + \lambda_w\right)}$$

Calculate the location of the mean aerodynamic chord

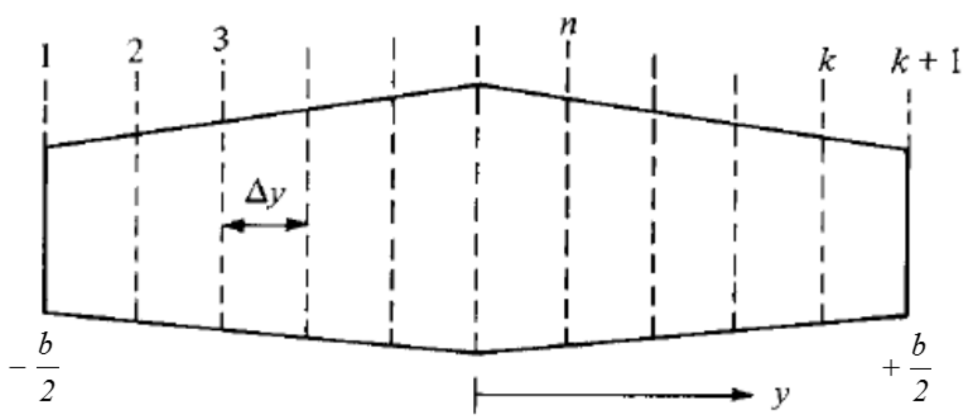
$$\bar{Y} = \frac{b_w}{6}\left[\frac{\left(1 + 2\lambda_w\right)}{\left(1 + \lambda_w\right)}\right]$$

Calculate the Reynolds number

$$\mathrm{Re} = \frac{\rho_\infty V_\infty \bar{c}}{\mu_\infty}$$

With the previous information, the computational procedure can begin:

Divide the wing into a defined number of spanwise stations. Here, k + 1 stations are shown, with n designating any specific station



For k = 1, 2, 3, .., N points, calculate the following: The N points along the span are chosen to be equally spaced. In other words, the wingspan is divided into N equal intervals, and each interval's midpoint is chosen as a control point. The port (left) wing tip is located at y = $-b_w$ /2, whereas the starboard (right) wing tip is located at y = b_w /2.

The coordinates of the control points are then given as follows: for each value of k, from 1 to N, we have:

$$y(k) = -\frac{b_w}{2}\left(1 - \frac{2k-1}{2N}\right)$$

Transforming the points:

$$\theta(k) = \cos^{-1}\left(-\frac{2y(k)}{b_w}\right)$$

The following equation then gives the chord length at y:

$$c(k) = c_{root}\left(1 - 2\frac{\lambda - 1}{b_w}y(k)\right)$$

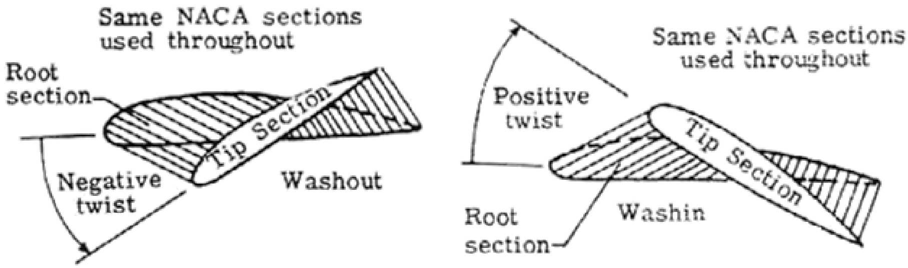
The specification for aerodynamic twist is quite complicated since it requires knowledge of the shape of the airfoil section at each station y along the span. In the absence of such information, this problem can be somewhat by requiring that the tip airfoil differs only slightly from the root airfoil such that the following linear relationships give the lift coefficient and zero angle of attack at y

$$a_0(k) = a_{0\_root} - 2\frac{a_{0\_tip} - a_{0\_root}}{b_w}y(k)$$

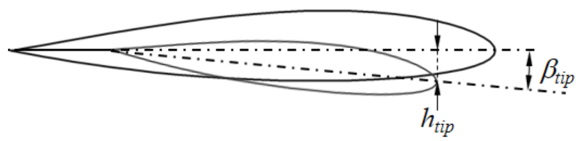
$$\alpha_{L=0}(k) = \alpha_{L=0\_root} - 2\frac{\alpha_{L=0\_tip} - \alpha_{L=0\_root}}{b_w}y(k)$$

The geometric angle of attack (α_{geo}) may vary as a function of y if the wing is given a geometric twist. A wing without twist is one where the α_{geo} is constant for all values of y, such that the leading edge and the trailing edge of the wing are straight lines, which lie on the same horizontal plane when α_{geo} = 0.

A wing may be given a wash-out, where the wing is twisted such that the leading edge of the wing tip airfoil is now lower than the leading edge of the root airfoil (the root airfoil is the airfoil located at the plane of symmetry if it is imagined that the fuselage is not there, and the two halves of the wing meet at the plane of symmetry). A wing with wash-in is one where the leading edge of the tip airfoil is now higher than the root airfoil’s leading edge, whereas the wing's trailing edge remains on the horizontal plane. Therefore, the airfoil chord at y may have negative or positive geometric angle of attack values when α_{geo} = 0 at the wing root, depending on whether the wing has a wash-out or a wash-in.



Let the height difference between the wing tip airfoil's leading edge and the root airfoil's leading edge be h_{tip}, which is negative for wash-out and positive for wash-in.



It should be noted that the wing's leading edge must remain in a straight line. Therefore, the twist angle or the geometric angle of attack at y relative to the geometric angle of attack at the wing root can be calculated as follows:

$$\beta(k) = \sin^{-1}\left(-\frac{2y(k)}{b_w \cdot c(k)} \cdot h_{tip}\right); \quad h_{tip} = c_{tip} \sin \beta_{tip}$$

Let´s now apply LLT to find the solution in the form of span-wise wing load or lift per unit span length distribution, the overall wing's lift coefficient (C_{L_w}), and the induced drag coefficient of the wing (C_{Di_w}). From the *Kutta-Joukowski* Lift Theorem, it is known that lift is directly proportional to circulation or vortex strength. Therefore, the theory must be capable of predicting the span-wise bound vortex strength per unit length distribution. The unknown vortex strength distribution, Γ(y), is approximated by a Fourier series:

$$\Gamma(\theta) = 2b_w V_\infty \sum_{n=1}^{N} A_n \sin(n\theta)$$

The fundamental problem is calculating the unknown Fourier series coefficients or amplitudes, A_n. The Fourier series approximation becomes more accurate as the number of terms, N, increases. The lifting line equation that needs to be solved is:

$$\frac{4b_w}{a(y)c(y)} \sum_{n=1}^{N} A_n \sin(n\theta(y)) + \sum_{n=1}^{N} nA_n \frac{\sin(n\theta(y))}{\sin(\theta(y))} = \alpha_{geo}(y) - \alpha_{L=0}(y)$$

Rearranging the equation:

$$\sum_{n=1}^{N} \left[\left(\frac{4b_w}{a(y)c(y)} + \frac{n}{\sin(\theta(y))} \right) \sin(n\theta(y)) \right] A_n == \alpha_{geo}(y) - \alpha_{L=0}(y)$$

Defining quantities:

$$C(y,n) = \left[\left(\frac{4b_w}{a(y)c(y)} + \frac{n}{\sin(\theta(y))}\right)\sin(n\theta(y))\right]$$

$$A(n) = A_n$$

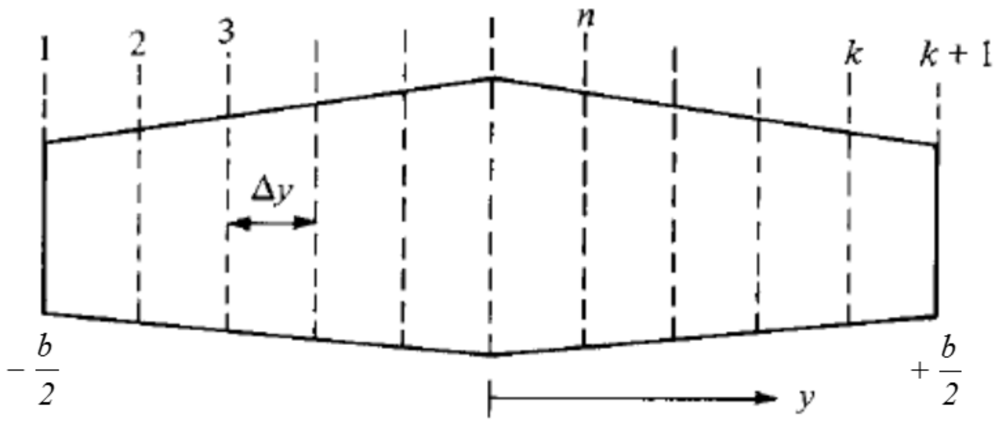
$$D(y) = \alpha_{geo}(y) - \alpha_{L=0}(y)$$

With this, the lifting line equation can be written as:

$$\sum_{n=1}^{N} C(y,n)A(n) = D(y)$$

This equation contains N unknowns, A(n) for n = 1 to N. It is, therefore, necessary to apply the last equation at N different control points or values of distance along the span, y so that we have a system of equations that can be solved simultaneously to calculate the values of A(n). The points chosen should not include the wing tips since, regardless of the values of the Fourier coefficients, the vortex strength distribution ($\Gamma(\theta)$) is always satisfied at those points. Selecting those 2 points will not provide any new information regarding the values of the Fourier amplitudes. It is also recommended that the midpoint (y = 0) should not be selected as a control point for similar reasons. The following method for selecting the control points is recommended to get the most accurate result for a given number of control points. First, N should be chosen as an even integer, such as N = 2M.

The N points along the span are chosen to be equally spaced. In other words, the span is divided into N equal intervals, and each interval's midpoint is chosen as a control point. The port (left) wing tip is at y = -bw /2, whereas the starboard (right) wing tip is at y = $b_w/2$.



The equation $\Gamma(\theta)$ can be rewritten in a way which separates the even terms from the odd terms as follows:

$$\Gamma(\theta) = \sum_{n=1}^{N} A_n \sin(n\theta) = 2b_w V_\infty \left[\sum_{n=1}^{N} A_n \sin((2n-1)\theta) + \sum_{n=1}^{N} A_n \sin(2n\theta)\right]$$

Therefore, if the load distribution is symmetrical, then the last equation is simplified to:

$$\Gamma(\theta) = \sum_{n=1}^{N} A_{2n-1} \sin((2n-1)\theta) = 2b_w V_\infty \left[\sum_{n=1}^{N} A_n (2n-1) \sin((2n-1)\theta) \right]$$

Keep in mind that is always assumed that the load distribution is symmetrical.

The lifting line equation, which is a system of equations that must be solved to calculate the Fourier sine coefficients, can now be written as follows

$$\sum_{n=1}^{N} C(k, 2n-1) A_n (2n-1) = D(k)$$

where the values of y are given by the equation of y(k) and

$$\theta_k = \theta(k) = \theta(y_k) = \cos^{-1}\left(1 - \frac{2k-1}{2N}\right)$$

$$C(k, 2n-1) = C(\theta_k, 2n-1) = \left(\frac{4b_w}{a(k)c(k)} + \frac{2n-1}{\sin(\theta_k)}\right)\sin((2n-1)\theta_k)$$

$$D(k) = \alpha_{geo}(k) - \alpha_{L=0}(k) = \alpha_{geo}(y_k) - \alpha_{L=0}(y_k)$$

It should be noted that c(k) is the airfoil's chord length at the station y(k), or θ (k), whereas α_{geo} and $\alpha_{L=0}$ (k) are the geometric and zero lift angle of attacks at y(k). The geometric angle of attack may vary as a function of y if the wing is given a geometric twist. A wing without a twist is one where the geometric angle of attack is constant for all values of y, such that the wing's leading edge and trailing edge are straight lines, which lie on the same horizontal plane when $\alpha_{geo} = 0$.

A wing may be given a wash-out, where the wing is twisted such that the leading edge of the wing tip airfoil is now lower than the leading edge of the root airfoil (the root airfoil is the airfoil located at the plane of symmetry if it is imagined that the fuselage is not there, and the two halves of the wing meet at the plane of symmetry).

A wing with wash-in is one where the leading edge of the tip airfoil is now higher than the root airfoil's leading edge, whereas the wing's trailing edge remains on the horizontal plane. Therefore, the airfoil chord at y may have negative or positive geometric angle of attack values when $\alpha_{geo} = 0$ at the wing root, depending on whether the wing has a wash-out or a wash-in.

The height difference between the wing tip airfoil's leading edge and the root airfoil's leading edge is h_{tip}, which is negative for wash-out and positive for wash-in. It should be noted that the wing's leading edge must remain in a straight line. Therefore, the twist angle or the geometric angle of attack at y relative to the geometric angle of attack at the wing root can be calculated as follows

$$\beta(k) = \sin^{-1}\left(\frac{h_{tip}}{c_{tip}}\right) = \sin^{-1}\left(-\frac{2y(k)}{b_w \cdot c(k)} \cdot h_{tip}\right) = \sin^{-1}\left(\left(1 - \frac{2k-1}{N}\right)\frac{h_{tip}}{c_{tip}}\right)$$

The angle of attack of the wing or the aircraft is denoted by the angle of attack at the wing root and is given the symbol of α_{geo}. This angle obviously can be varied and represents the aircraft's attitude (when the aircraft is at a level cruising flight this angle may have a small positive value of not more than 3 degrees). Using this definition, it can now calculate the geometric angle of attack at y as follows

$$\alpha_{geo}(k) = \alpha_{geo} + \beta(k)$$

The equation of D(k) can now be rewritten as follows:

$$D(k) = \alpha_{geo} - \alpha_{L=0}(k) + \beta(k)$$

A wing may be given an aerodynamic twist and a geometric twist. This means that the airfoil shape at the wing root is different from that at the wing tip. The shape of the airfoil in between the two limiting stations is then determined by insisting that the wing cross-section should have a smoothly varying shape along the spanwise direction. Since the airfoil shape at the wing tip is different from that at the root, therefore the value of the sectional lift coefficient, as well as its zero-lift angle of attack, may also vary along the span-wise direction. Provided the variation of a_0 (y) and $\alpha_{L=0}$ (y) are given, the equation of C (k, 2n-1) can still be used to compute the matrix coefficients. Thus, the lifting line theory can handle such a problem.

The theory can also handle the problem involving a variation in the chord length of the sections as a function of y if the functional form of c(y) is given. This means the theory is also applicable for analyzing tapered wing shape, so long as the quarter chord line is normal or almost normal to the aircraft’s longitudinal axis. Obviously, the theory is not valid for a highly swept wing. A more elaborated methodology, like the Vortex Lattice Method (VLM), should be used for swept wings.

For each value of n = 1, 2, 3, …, N calculate the matrix coefficients:

$$C(k,n) = \left[\left(\frac{4b_w}{a_0(k)c(y)} + \frac{2n-1}{\sin(\theta(k))}\right)\sin((2n-1)\theta(k))\right]$$

The next step in the LLT methodology is to solve the following system of simultaneous equations:

$$\sum_{n=1}^{N} C(k,n)A(n) = D(k), \text{ for k = 1, 2, 3, …, N}$$

The Gaussian elimination method is a simple direct method for computing the solutions of A(n). Other methods, such as the Jacobi or Gauss-Seidel iterative methods, may also be used.

After the Fourier coefficients, A(n), have been calculated it can now compute the non-dimensional wing load distribution as follows:

$$\Gamma_{ND}(k) = \frac{\Gamma(y)}{2b_w V_\infty} = \sum_{1}^{N} A(n)\sin((2n-1)\theta)$$

$$\theta(k) = \cos^{-1}\left(-\frac{2y}{b_w}\right)$$

The wing´s lift coefficient can be calculated now:

$$C_L = \pi \cdot AR \cdot A(1)$$

The Oswald efficiency factor, *e*, is

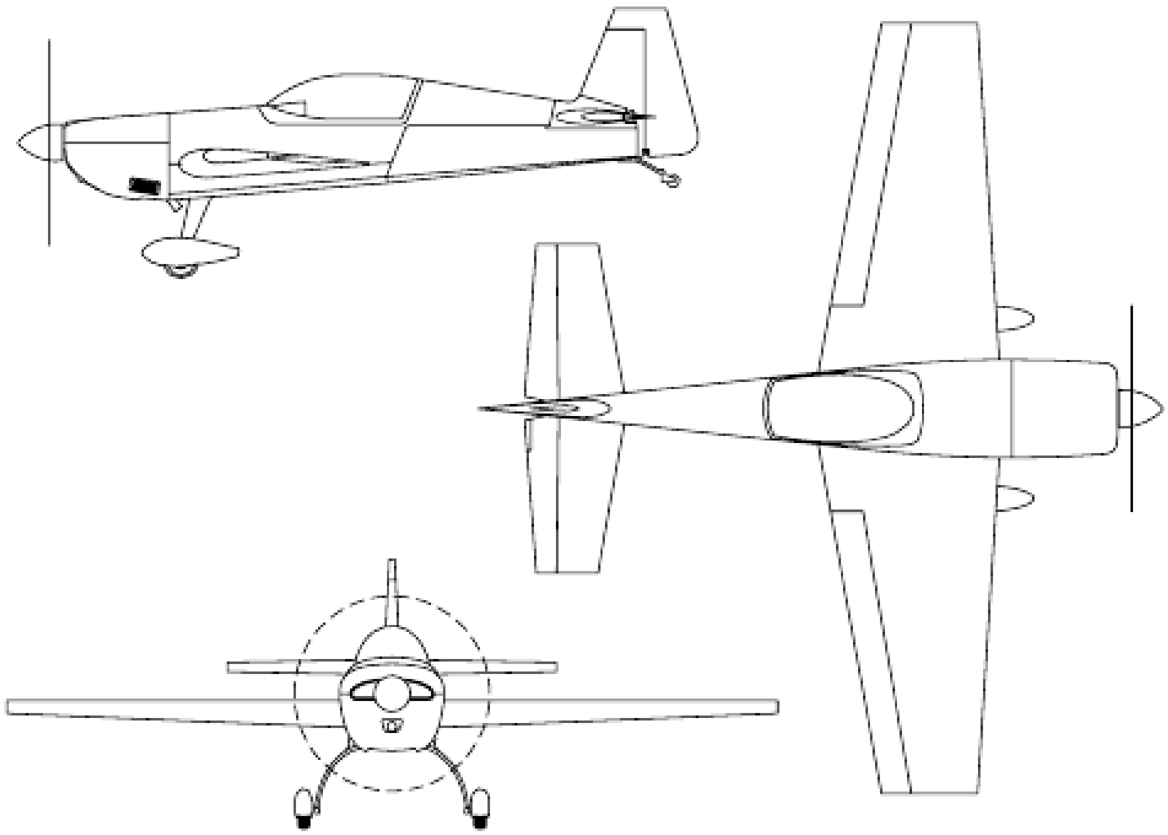
$$e = (1+\delta)^{-1}\text{, where: } \delta = \sum_{n=2}^{N} n\left(\frac{A(n)}{A(1)}\right)^2$$

The induced drag coefficient C_{Di} is obtained by:

$$C_{Di} = \frac{C_L^2}{\pi \cdot e \cdot AR} = k \cdot C_L^2\text{, where: } k = \frac{1}{\pi \cdot e \cdot AR}$$

Example:

For the acrobatic monoplane *Extra EA-300*, calculate (1) the aerodynamic characteristics of flight to see if the actual geometry can counteract the Maximum Take-Off Weight (MTOW) for an average cruise condition with a wing incidence angle of 2 [deg] and (2) find the distributed aerodynamic load (Lift force) over the wing. An airplane 3-view picture is shown:



Specifications:

- MTOW = 950 [kg]
- Wingspan (b_w) = 8.0 [m]
- Taper ratio (λ_w) = 0.45
- Wing reference area (S_w) = 10.7 [m^2]
- Airfoil information:
  - Root – NACA 0015
  - Tip – NACA 0012

Performance:

- V_{cruise} = 317.0 [km/h]
- V_{stall} = 102.0 [km/h]
- V_{NE} = 408.0 [km/h]

Solution:

First, the geometric characteristics of the wing must be determined, beginning with the LLT methodology.

The Aspect Ratio of the wing is:

$$AR = \frac{b_w^2}{S} = \frac{8.0^2}{10.7} = 5.98$$

The average wing chord:

$$c_{av} = \frac{S}{b_w} = \frac{10.7}{8.0} = 1.34[m]$$

Chord length at the root:

$$c_{root} = \frac{2 \cdot S}{b_w(1+\lambda_w)} = \frac{2c_{av}}{(1+\lambda)} = \frac{2 \cdot (1.34)}{(1+0.45)} = 1.84[m]$$

Chord length at the tip:

$$c_{tip} = \lambda_w c_{root} = 0.45 \cdot 1.84 = 0.83[m]$$

Mean aerodynamic chord:

$$\bar{c} = \frac{2}{3} c_{root} \frac{(1+\lambda_w+\lambda_w^2)}{(1+\lambda_w)} = \frac{2}{3}(1.84)\left[\frac{(1+0.45+0.45^2)}{(1+0.45)}\right] = 1.40[m]$$

Mean aerodynamic chord distance:

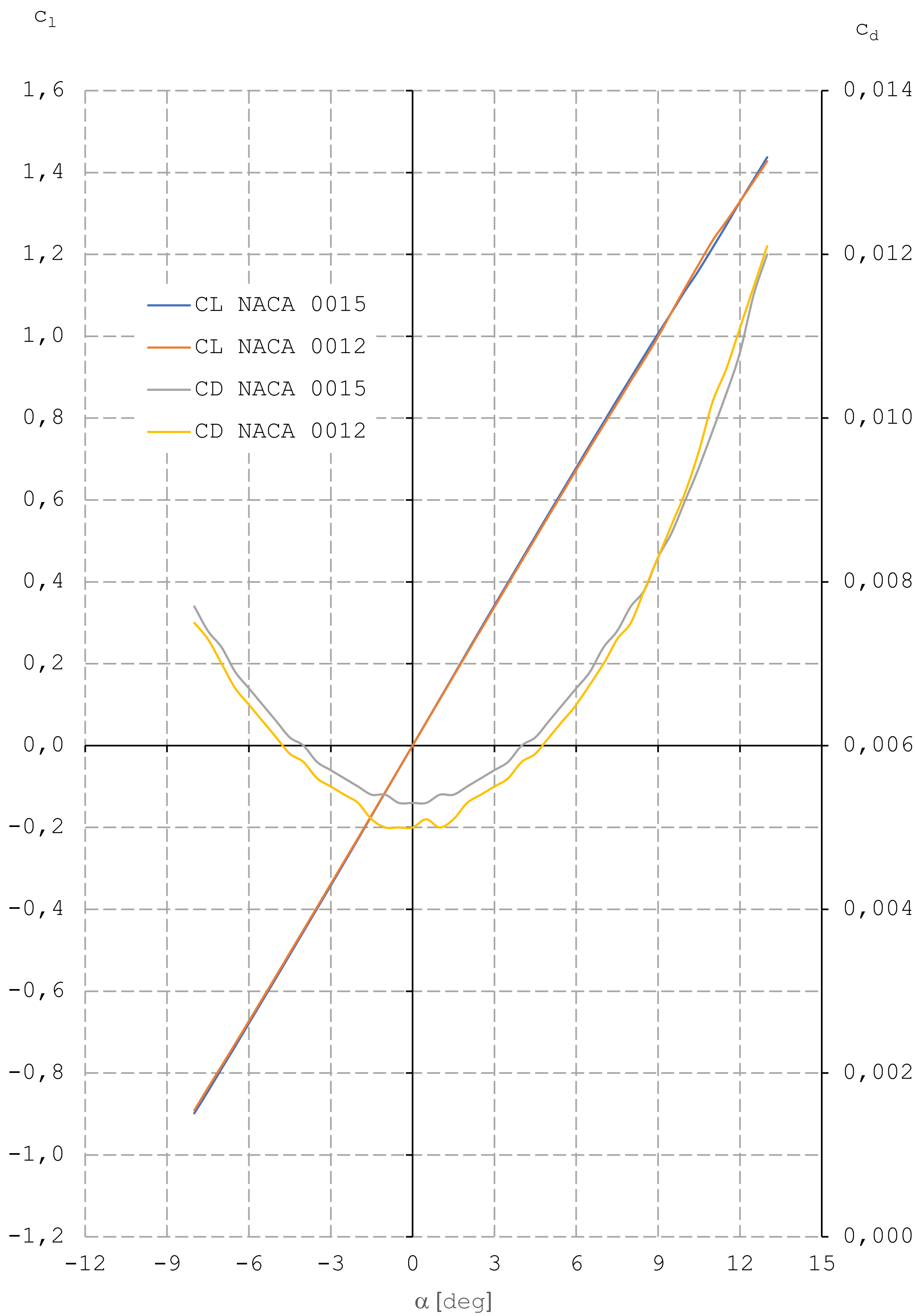
$$\bar{Y} = \frac{b_w}{6}\left[\frac{(1+2\lambda_w)}{(1+\lambda_w)}\right] = \frac{8.0}{6}\left[\frac{\left(1+2(0.45)\right)}{(1+0.45)}\right] = 1.75[m]$$

Assuming sea-level conditions (ρ_∞ = 1.225 [kg/m^3]; μ_∞ = 1.789 x 10-5 [kg/(m sec)]), and cruise flight conditions, then:

$$Re = \frac{\rho_\infty V_\infty \bar{c}}{\mu_\infty} = \frac{1.225 \times \left(317 \times \frac{1000}{3600}\right) \times 1.40}{1.789 \times 10^{-5}} = 8451241$$

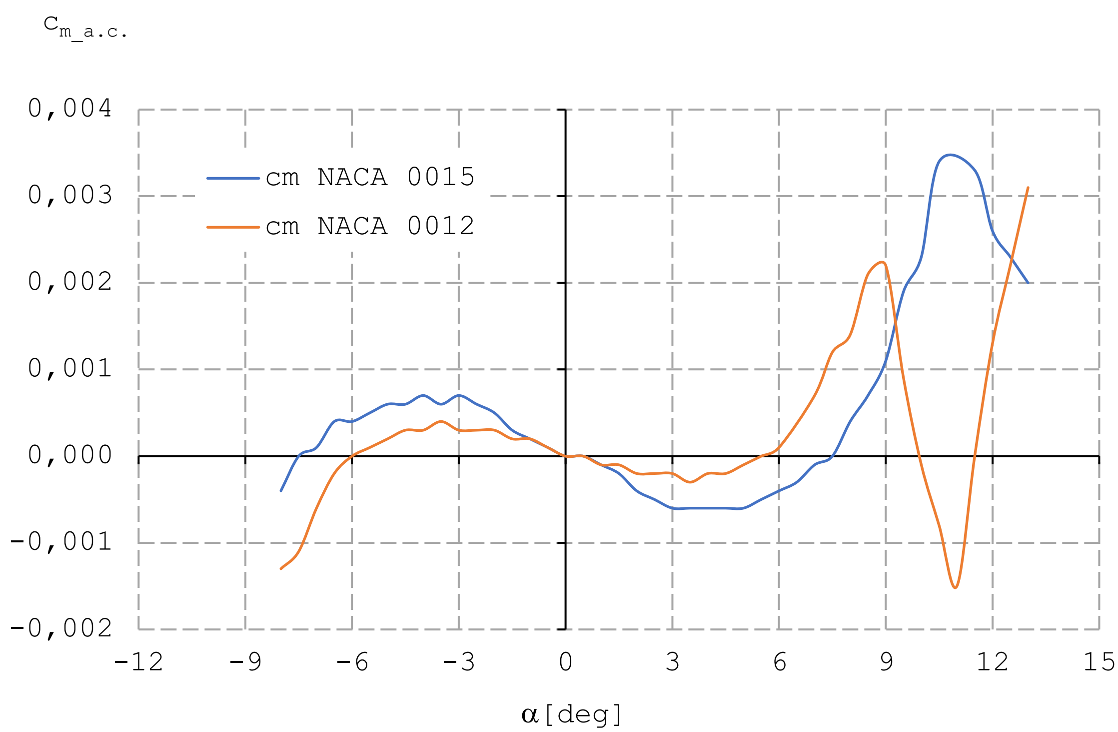
With this Reynolds number, the airfoil characteristics can be obtained. The characteristic curves of the two wing airfoils aerodynamic characteristics are presented in the following graphs:

c_l vs. α [deg] and c_d vs. α [deg]

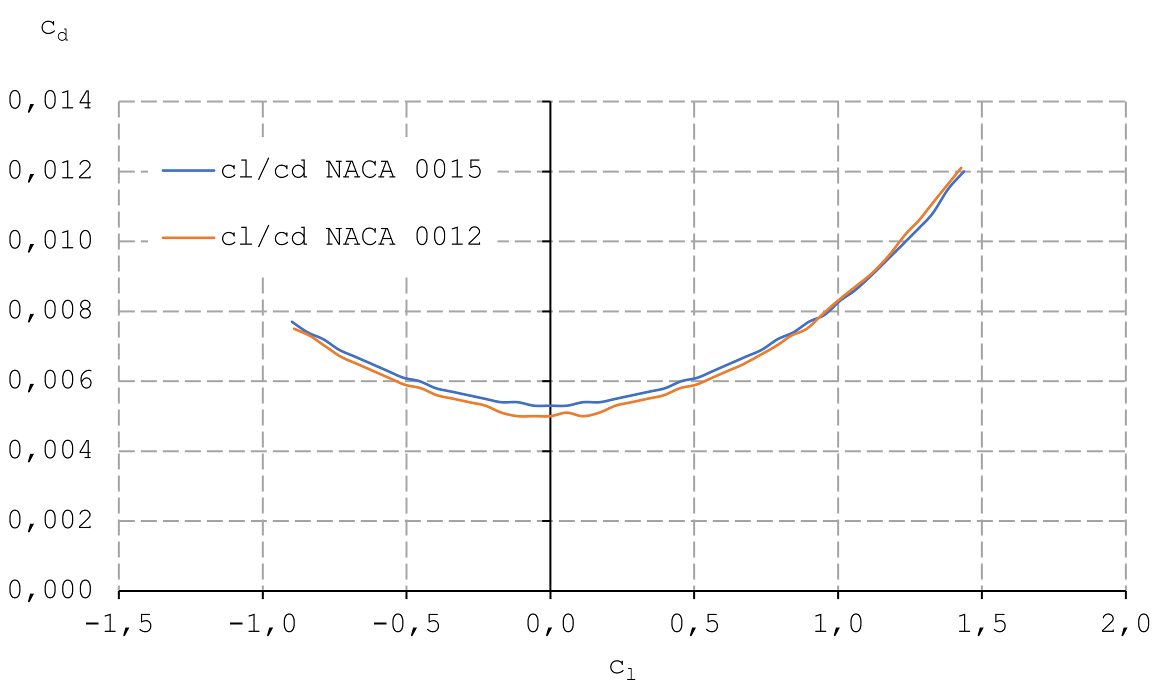


c_l
c_d
1,6
1,4
1,2
1,0
0,8
0,6
0,4
0,2
0,0
-0,2
-0,4
-0,6
-0,8
-1,0
-1,2
0,014
0,012
0,010
0,008
0,006
0,004
0,002
0,000
-12
-9
-6
-3
0
3
6
9
12
15
α [deg]
CL NACA 0015
CL NACA 0012
CD NACA 0015
CD NACA 0012

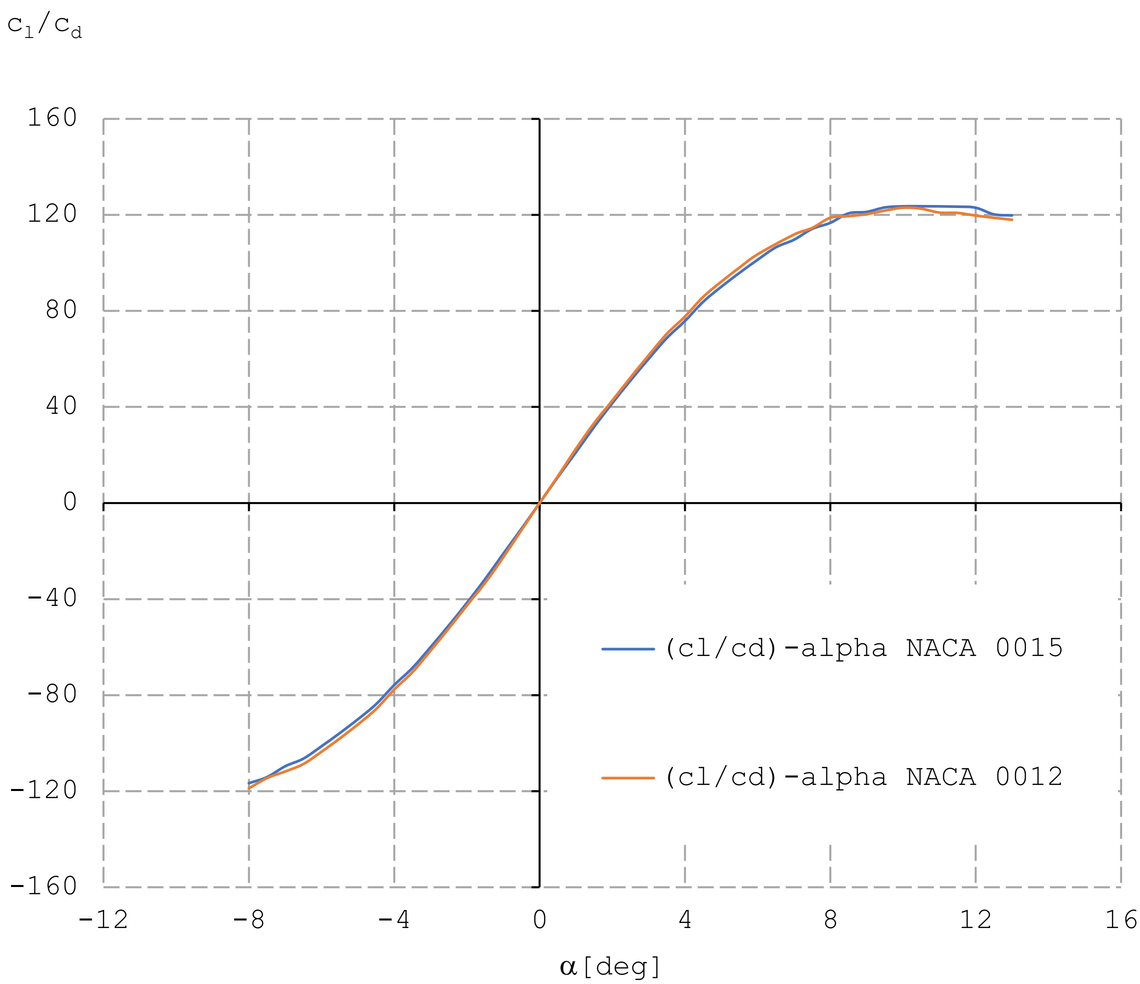
$c_{m\_a.c.}$ vs. α [deg]



c_l vs. c_d



(c_l / c_d) vs. α [deg]



These curves were obtained through software that uses the Vortex Panel Method to give the characteristics of cl, cd, and $c_{m\_a.c.}$ at different angles of attack [deg] for the two airfoils used in this wing. These curves were analyzed at a Reynolds number of $8x10^6$. From the curves, the following information is obtained:

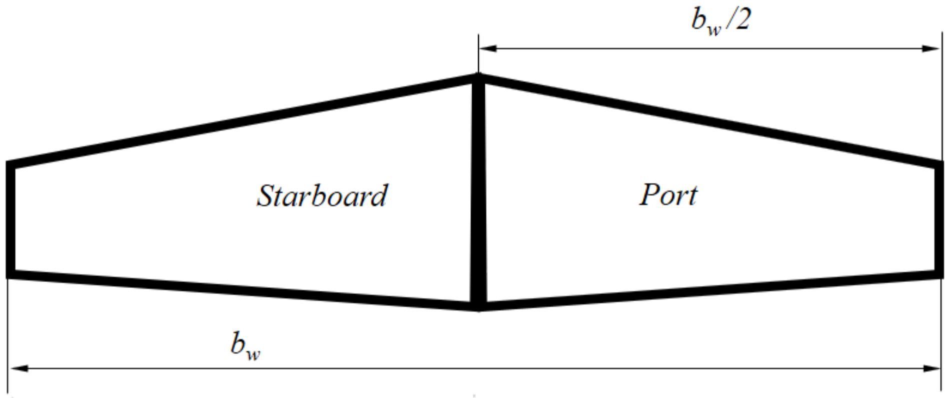
Wing position	**Airfoil**	**a_0 [1/rad]**	**$\alpha_{L=0}$ [deg]**
Root	NACA 0015	6,436	0
Tip	NACA 0012	6,363	0

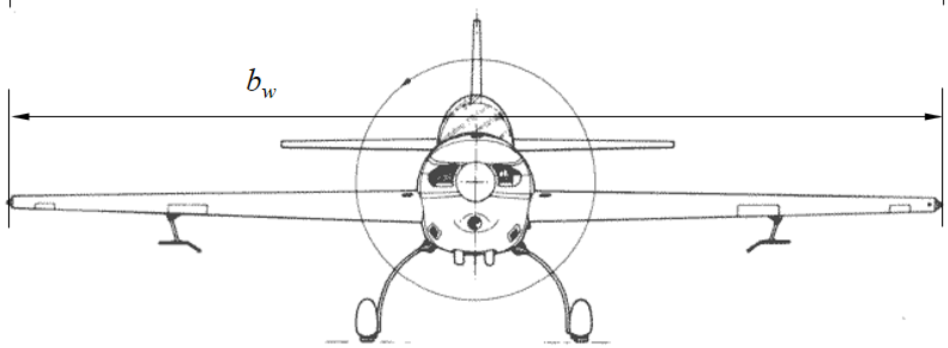
As this wing has an aerodynamic twist, it can be assumed there is no need for a geometric twist; therefore $\beta_{tip} = 0$.

$$\sin \beta_{tip} = \frac{h_{tip}}{c_{tip}} \Rightarrow h_{tip} = c_{tip} \sin \beta_{tip}$$

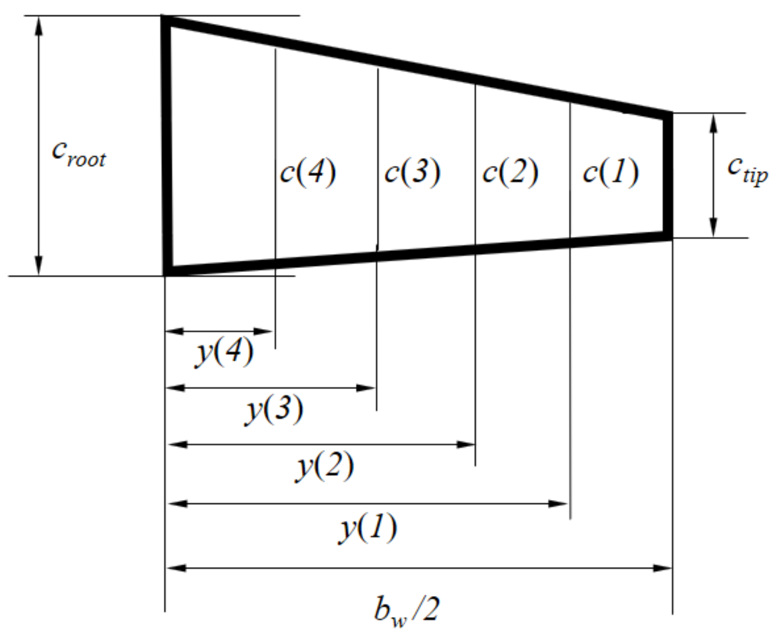
If no geometrical twist ($\beta_{tip} = 0$) is needed for the wing, then the length $h_{tip} = 0$.

First, the wing is divided into two symmetrical sections, the left wing (port side) and the right-wing (starboard side):





Then, we can establish the number of control points (k) calculated for the wing. In this case, we will put eight control points, which means there will be four per half-wing.



Establishing the above, the y-stations on the wingspan (half in this case) can be found as follows:

$$y(k) = -\frac{b_w}{2}\left(1 - \frac{2k-1}{2N}\right)$$

Results are shown in the following table:

k	y(k) [m]
1	-3.5
2	-2.5
3	-1.5
4	-0.5

The local chord length values at each station (control point) are found by:

$$c(k) = c_{root}\left(1 - 2\frac{\lambda - 1}{b_w}y(k)\right)$$

Results are shown in the following table:

k	c(k) [m]
1	0.957
2	1.211
3	1.464
4	1.718

Next, the twist angle distribution is calculated using the relations:

$$\beta(k) = sin^{-1}\left(-\frac{2y(k)}{b_w \cdot c(k)} \cdot h_{tip}\right); h_{tip} = c_{tip}\, sin\, \beta_{tip}$$

In this airplane, the wing has an aerodynamical twist instead of a geometric one therefor the results are:

k	β (k) [deg]
1	0
2	0
3	0
4	0

The variation of the lift slope along the wing will vary because the root and tip airfoils have different references. This also happens with the zero-lift angle of attack for the different points of control. The variations are calculated using:

$$a_0(k) = a_{0\_root} - 2\frac{a_{0\_tip} - a_{0\_root}}{b_w} y(k)$$

$$\alpha_{L=0}(k) = \alpha_{L=0\_root} - 2\frac{\alpha_{L=0\_tip} - \alpha_{L=0\_root}}{b_w} y(k)$$

The results are shown in the following table:

k	**a_0(k)**	**$\alpha_{L=0}$(k)**
1	6.372	0
2	6.390	0
3	6.408	0
4	6.427	0

Note that for this wing, as booth airfoils are symmetrical, the zero-lift angle of attack is equal to zero for both cases; therefore, the variation will also have this same value along the span.

Now, let's calculate the transformation θ(k) using:

$$\theta(k) = cos^{-1}\left(-\frac{2y}{b_w}\right)$$

The results of this relation are shown as follows,

k	**θ (k) [deg]**
1	28.955
2	51.318
3	67.976
4	82.819

With these values now, the D(k) variation can be calculated:

$$D(k) = \alpha_{geo} - \alpha_{L=0}(k) + \beta(k)$$

The results for this equation are:

k	**D(k) [rad]**
1	0.035
2	0.035
3	0.035
4	0.035

Next, we want to create the 4x4-sized matrix coefficients with the following formula:

$$C(k,n) = \left[\left(\frac{4b_w}{a_0(k)c(k)} + \frac{2n-1}{sin\big(\theta(k)\big)}\right) sin\big((2n-1)\theta(k)\big)\right]$$

Where k = n = 4, this is because we are taking advantage of the symmetry of the wing loading distribution, we have N = 4, and we can use n = 1, 2, 3, and 4 (port wing only) or n = 5, 6, 7 and 8 (starboard wing only). We can calculate each point's variation if only port wing control points are chosen. For this example, we present only 4 points as references:

For k = 1 and n =1

$$C(1,1) = \left[\left(\frac{4(8.0)}{a_0(1)c(1)} + \frac{2(1)-1}{sin\big(\theta(1)\big)}\right) sin\big((2(1)-1)\theta(1)\big)\right]$$

$$C(1,1) = \left[\left(\frac{4(8.0)}{(6.372)(0.957)} + \frac{2(1)-1}{sin(28.955)}\right) sin\big((2(1)-1)(28.955)\big)\right]$$

$$C(1,1) = 3.541$$

For k = 2 and n =1

$$C(2,1) = \left[\left(\frac{4(8.0)}{a_0(2)c(2)} + \frac{2(1)-1}{sin\big(\theta(2)\big)}\right) sin\big((2(1)-1)\theta(2)\big)\right]$$

$$C(2,1) = \left[\left(\frac{4(8.0)}{(6.39)(1.211)} + \frac{2(1)-1}{sin(51.318)}\right) sin\big((2(1)-1)(51.318)\big)\right]$$

$$C(2,1) = 4.229$$

For k = 1 and n =2

$$C(1,2) = \left[\left(\frac{4(8.0)}{a_0(1)c(1)} + \frac{2(2)-1}{sin\big(\theta(1)\big)}\right) sin\big((2(2)-1)\theta(1)\big)\right]$$

$$C(1,2) = \left[\left(\frac{4(8.0)}{(6.372)(0.957)} + \frac{2(2)-1}{sin(28.955)}\right) sin\big((2(2)-1)(28.955)\big)\right]$$

$$C(1,1) = 11.428$$

For k = 2 and n =2

$$C(2,2) = \left[\left(\frac{4(8.0)}{a_0(2)c(1)} + \frac{2(2)-1}{sin\big(\theta(2)\big)}\right) sin\big((2(2)-1)\theta(2)\big)\right]$$

$$C(2,2) = \left[\left(\frac{4(8.0)}{(6.39)(1.211)} + \frac{2(2)-1}{sin(51.318)}\right) sin\big((2(2)-1)(51.318)\big)\right]$$

$$C(2,2) = 3.504$$

Results for this equation are given in the following matrix:

$$C(k=4, n=4) = \begin{vmatrix} 3.541 & 11.428 & 8.984 & -7.6 \\ 4.229 & 3.504 & -10.254 & -0.177 \\ 4.161 & -2.696 & -3.029 & 9.866 \\ 3.876 & -5.508 & 6.43 & -6.363 \end{vmatrix}$$

The next step in the LLT methodology is to solve the following system of simultaneous equations:

$\sum_{n=1}^{N} C(k,n)A(n) = D(k)$, for k = 1, 2, 3, …, N

For this case, the equation system gives:

$$\sum_{n=1}^{N} C(k,n)A(n) = D(k) \Rightarrow \begin{pmatrix} C(1,1) & C(1,2) & C(1,3) & C(1,4) \\ C(2,1) & C(2,2) & C(2,3) & C(2,4) \\ C(3,1) & C(3,2) & C(3,3) & C(3,4) \\ C(4,1) & C(4,2) & C(4,3) & C(4,4) \end{pmatrix} \cdot \begin{pmatrix} A(1) \\ A(2) \\ A(3) \\ A(4) \end{pmatrix} = \begin{pmatrix} D(1) \\ D(2) \\ D(3) \\ D(4) \end{pmatrix}$$

$$\sum_{n=1}^{N} C(k,n)A(n) = D(k) \Rightarrow \begin{pmatrix} 3.541 & 11.428 & 8.984 & -7.6 \\ 4.229 & 3.504 & -10.254 & -0.177 \\ 4.161 & -2.696 & -3.029 & 9.866 \\ 3.876 & -5.508 & 6.43 & -6.363 \end{pmatrix} \cdot \begin{pmatrix} A(1) \\ A(2) \\ A(3) \\ A(4) \end{pmatrix} = \begin{pmatrix} 0.035 \\ 0.035 \\ 0.035 \\ 0.035 \end{pmatrix}$$

The Gaussian elimination method can solve this equation system. Other methods, such as the Jacobi or Gauss-Seidel iterative methods, may also be used.

The solution of an equation system (matrices) by the Gaussian Elimination is:

$$C(k,n)A(n) = D(k) \Rightarrow A(n) = [C(k,n)]^{-1} \cdot D(k)$$

This means that the inverse of the matrix C(k,n) must be found first, to solve the system, this is done in this case by the Gauss-Jordan method:

$$[C(k,n)]^{-1} = \left(\begin{array}{cccc|cccc} 3.541 & 11.428 & 8.984 & -7.6 & 1 & 0 & 0 & 0 \\ 4.229 & 3.504 & -10.254 & -0.177 & 0 & 1 & 0 & 0 \\ 4.161 & -2.696 & -3.029 & 9.866 & 0 & 0 & 1 & 0 \\ 3.876 & -5.508 & 6.43 & -6.363 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$[C(k,n)]^{-1} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0.038 & 0.057 & 0.079 & 0.076 \\ 0 & 1 & 0 & 0 & 0.055 & 0.009 & 0.002 & -0.063 \\ 0 & 0 & 1 & 0 & 0.034 & -0.070 & 0.032 & 0.011 \\ 0 & 0 & 0 & 1 & 0.010 & -0.043 & 0.079 & -0.046 \end{array}\right)$$

With the matrix inverse result, the system can be solve for A(n) by multiplying $[C(k,n)]^{-1}$ by D(k):

$$A(n) = [C(k,n)]^{-1} \cdot D(k) = \begin{pmatrix} A(1) \\ A(2) \\ A(3) \\ A(4) \end{pmatrix} = \begin{pmatrix} 0.038 & 0.057 & 0.079 & 0.076 \\ 0.055 & 0.009 & 0.002 & -0.063 \\ 0.034 & -0.070 & 0.032 & 0.011 \\ 0.010 & -0.043 & 0.079 & -0.046 \end{pmatrix} \cdot \begin{pmatrix} 0.035 \\ 0.035 \\ 0.035 \\ 0.035 \end{pmatrix}$$

$$\begin{pmatrix} A(1) \\ A(2) \\ A(3) \\ A(4) \end{pmatrix} = \begin{pmatrix} 0.008734 \\ 0.000133 \\ 0.000244 \\ -0.000034 \end{pmatrix}$$

With these results, the wing lift coefficient can now be found by the relation:

$$C_{L\_w} = \pi \cdot AR \cdot A(1) = \pi \cdot (5.98) \cdot (0.008734) = 0.164$$

The total lift force exerted by the wing is:

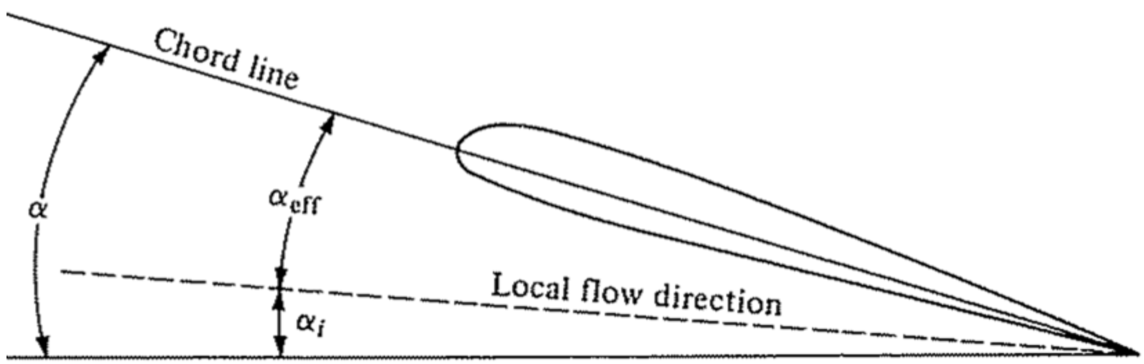
$$L_w = q_\infty S_w C_{L\_w} = \frac{1}{2}\rho_\infty V_\infty^2 S_w C_{L\_w} = \frac{1}{2}(1.225)(88.056)^2(10.7)(0.164) = 8340.38[N]$$

$$L_w = 8340.38[N] = 850.192[kg]$$

Analyzing this result and comparing it with the opposing force (airplane´s MTOW), the geometric angle of attack of two degrees is not enough to lift off the aircraft from the ground; L_w < MTOW.

This means the airplane still needs more lift force to counteract the MTOW for a geometric angle of attack (incidence angle) of two degrees. Some considerations must be made to ensure the wing, with its geometric and aerodynamic characteristics, can achieve enough lift force; then, different incidence angles must be studied to study the possibilities of wing attitude with respect to the horizontal plane of the aircraft.

The following figure illustrates the differences between the geometric (incidence) angle of attack, the effective angle of attack, and the induced angle of attack:



Examining the profile drag curves of both airfoils, it is decided that the wing's total profile drag coefficient is $c_{d\_0} = 0.0054$ for a geometric angle of two degrees.

Now, let´s calculate the total wing drag force at the established conditions:

The Oswald efficiency planform factor is crucial for calculating the wing-induced drag under the given flight conditions. The equation to find it is:

$$e = (1+\delta)^{-1}, \text{ where: } \delta = \sum_{n=2}^{N} n\left(\frac{A(n)}{A(1)}\right)^2$$

The delta coefficient depends on the Fourier series coefficients, which were calculated previously.

$$\delta = \sum_{n=2}^{N} n\left(\frac{A(n)}{A(1)}\right)^2 = 2\cdot\left(\frac{0.000133}{0.008734}\right)^2 + 3\cdot\left(\frac{0.000244}{0.008734}\right)^2 + 4\cdot\left(\frac{-0.000034}{0.008734}\right)^2 = 0.002876$$

$$e = (1+\delta)^{-1} = (1+0.002876)^{-1} = 0.9971$$

As a result, it shows a high value, so it should be checked with another methodology that predicts wing efficiency.

The induced drag coefficient C_{Di} is obtained by:

$$C_{Di} = \frac{C_L^2}{\pi \cdot e \cdot AR} = k \cdot C_L^2, \text{ where: } k = \frac{1}{\pi \cdot e \cdot AR} = \frac{1}{\pi(0.9971)(5.98)} = 0.0534$$

$$C_{Di} = \frac{C_L^2}{\pi \cdot e \cdot AR} = k \cdot C_L^2 = (0.0534)(0.164)^2 = 0.00144$$

The total wing drag coefficient will be:

$$C_{D\_w} = c_{d\_prof} + C_{Di} = c_{d\_prof} + \frac{C_L^2}{\pi \cdot e \cdot AR} = 0.00684$$

So, for this flight condition, the total wing drag will be equal to:

$$D_w = q_\infty S_w C_{D\_w} = \frac{1}{2}\rho_\infty V_\infty^2 S_w C_{D\_w} = \frac{1}{2}(1.225)(88.056)^2(10.7)0.00684 = 347.466[N]$$

$$D_w = 347.466[N] = 35.42[kg]$$

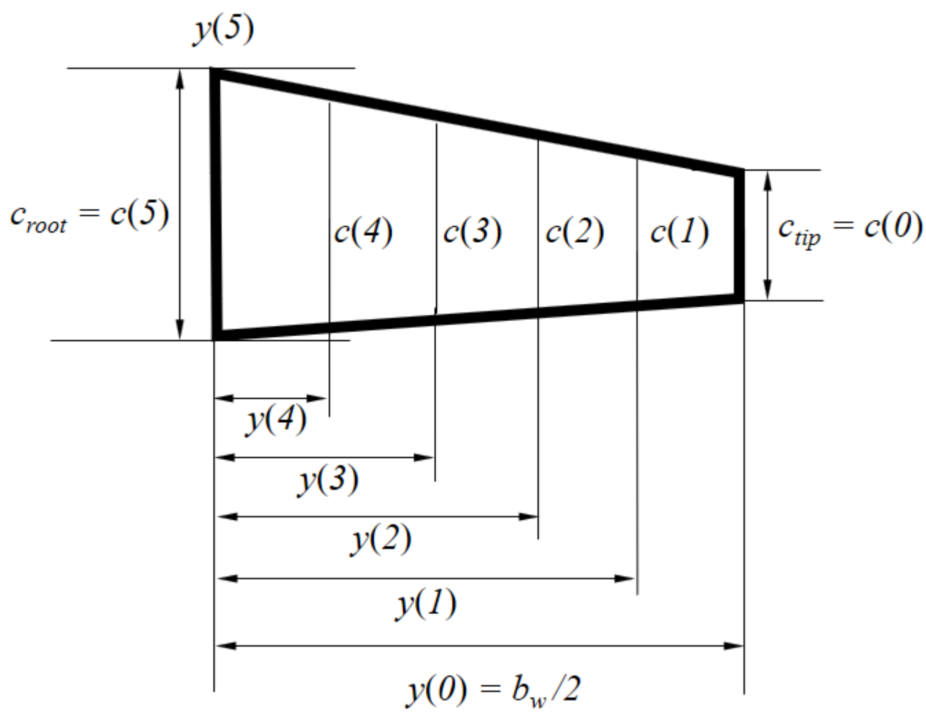
Analyzing the wing load distribution:

The main idea of this analysis is to establish how the lift load distribution over the wing will be. For this, a distribution that describes an elliptical shape is expected to be obtained to make it as aerodynamically efficient as possible. First, define the stations where you want to study the local lift load; this helps you with already established stations. In this case, we are analyzing half span beginning at the root of the wing:

Station y(k)	**Points of control + c_{tip} and c_{root} – y**	**Location [m]**
y(0) = $b_w/2$	0	4.0
y(1)	1	3.5
y(2)	2	2.5
y(3)	3	1.5
y(4)	4	0.5
y(5)	5	0

Recalling the transformation θ(k) using:

$$\theta(k) = cos^{-1}\left(-\frac{2y}{b_w}\right)$$



This time, the results include the values at the root and the tip of the wing,

Points of control + c_{tip} and c_{root} – y	y [m]	θ(y) [deg]
0	4.0	0
1	3.5	28.955
2	2.5	51.318
3	1.5	67.976
4	0.5	82.819
5	0	90.0

Keeping in mind that the load distribution is symmetrical then, the equation can calculate circulation on each station over the wing:

$$\Gamma(y) = \Gamma\big(\theta(y)\big) = 2b_w V_\infty\big[A(1)\sin\theta\,(y) + \big(A(2)\big)\,sin\big(3\cdot\theta(y)\big)\big]$$

Results are presented in the following table:

Points of control + c_{tip} and c_{root} – y	y [m]	Γ (y) [m^2/sec]
0	4.0	0
1	3.5	5.958
2	2.5	9.606
3	1.5	11.408
4	0.5	12.209
5	0	12.306

Non-dimensionalizing the circulation is obtained (Γ_0 – circulation at the root of the wing):

$$\Gamma_{ND} = \frac{\Gamma(y)}{\Gamma_0}$$

Points of control + c_{tip} and c_{root} – y	**y [m]**	**Γ_{ND}**
0	4.0	0
1	3.5	0.484
2	2.5	0.781
3	1.5	0.927
4	0.5	0.992
5	0	1.0

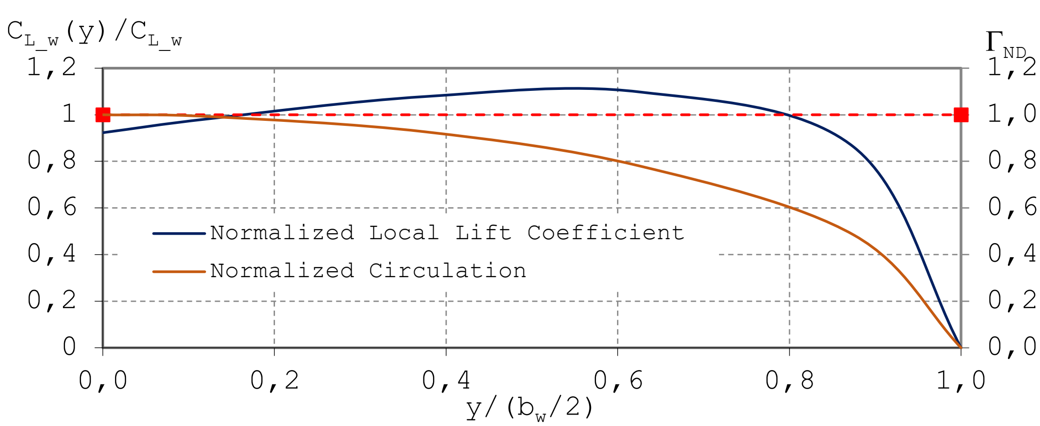
Now, let´s compute the wing lift coefficient and the force in each station of the half-span. The equations can calculate the local wing lift coefficient and the correspondent force:

$$C_{L\_w}(y) = \frac{2\Gamma(y)}{V_\infty c(y)} \Rightarrow \frac{C_{L\_w}(y)}{C_{L\_w}} = \frac{1}{C_{L\_w}}\left(\frac{2\Gamma(y)}{V_\infty c(y)}\right)$$

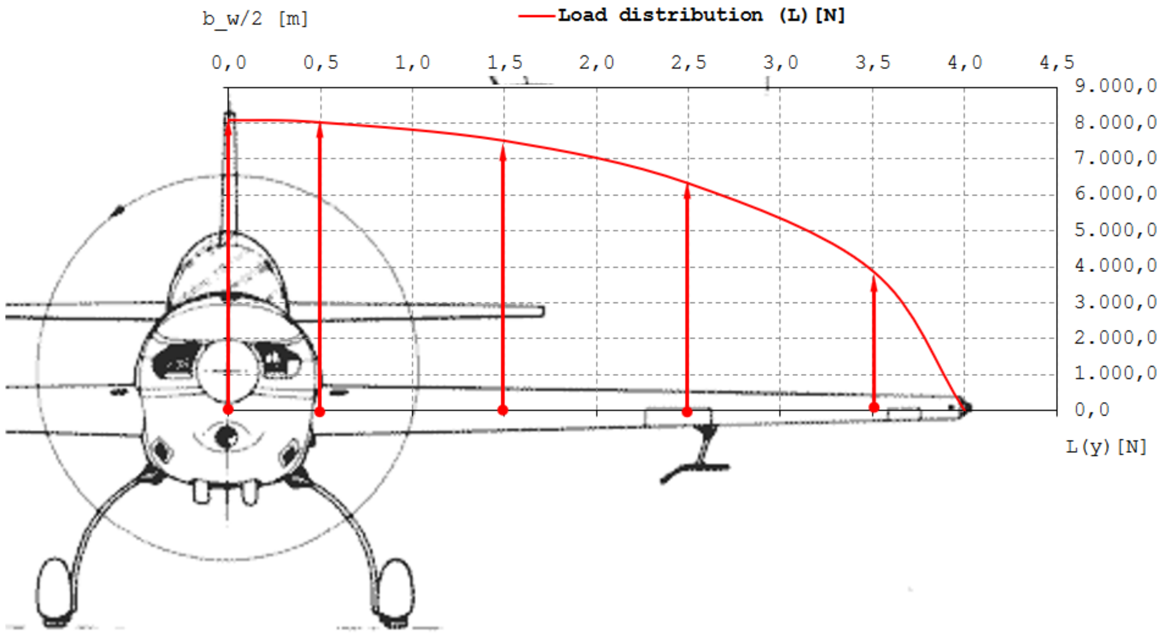
$$L(y) = \frac{1}{2}\rho_\infty V_\infty^2 c(y)\frac{C_{L\_w}(y)}{C_{L\_w}}$$

Points of control + c_{tip} and c_{root} – y	**y [m]**	**$y/(b_w/2)$**	**$C_{L\_w}(y)/C_{L\_w}$**	**L(y) [N]**
0	4.0	1.0	0	0
1	3.5	0.875	0.862	3915.515
2	2.5	0.625	1.098	6313.475
3	1.5	0.375	1.078	7497.437
4	0.5	0.125	0.983	8024.157
5	0	0	0.923	8087.584

Using this information, the normalized local lift coefficient and the local circulation are plotted as shown in the following graph:



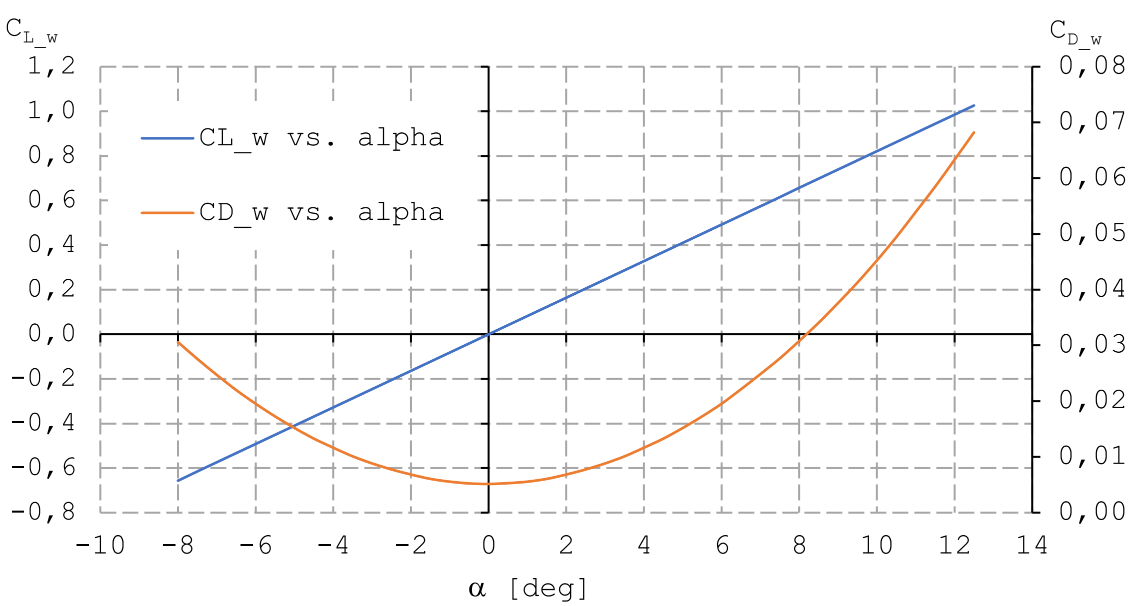
In addition, the lift distribution is plotted along the half wingspan to get the values of the distributed load over the wing:



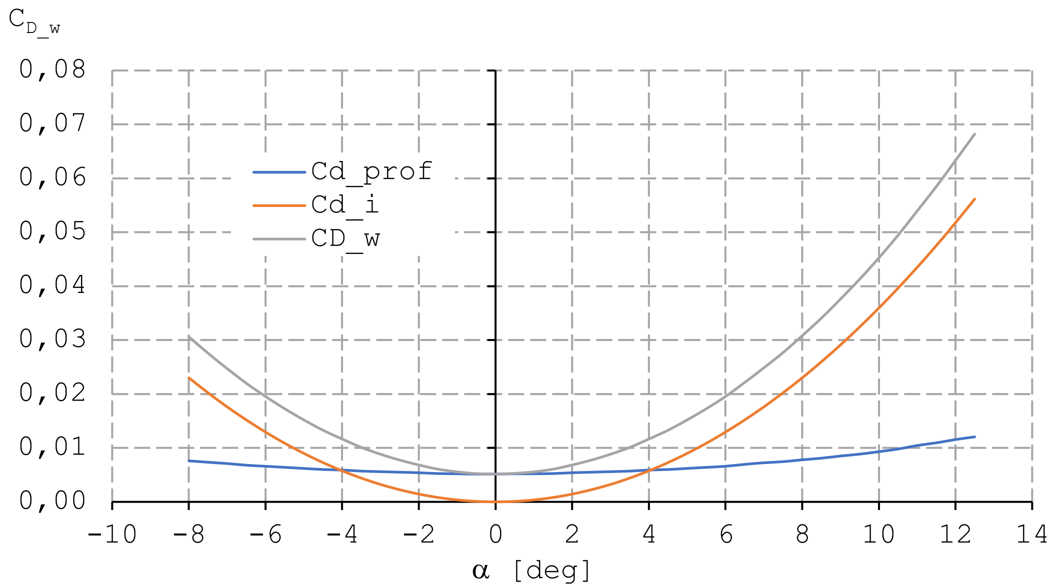
As established before, the net lift produced by this wing is not enough to counteract the MTOW. Further studies should be done to ensure proper values of the wing incidence angle of attack.

Using the LLT mathematical model, we can find different values of the net wing lift coefficient at different angles of attack. The following graphs show the results of this procedure:

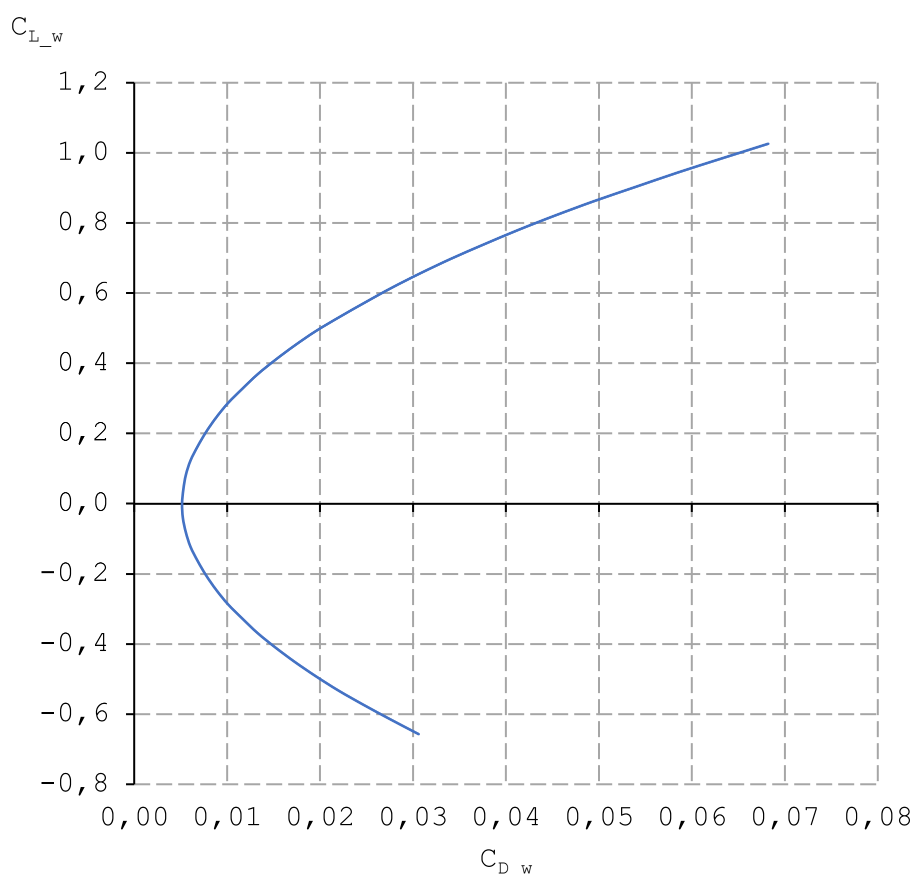
C_{L_w} vs. α [deg] and C_{D_w} vs. α [deg]



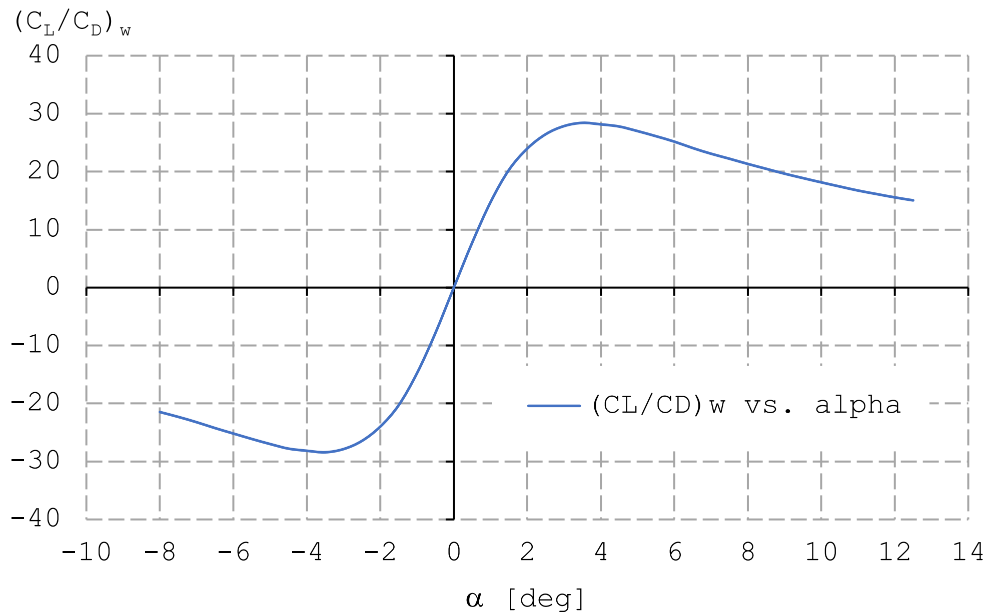
$C_{D\_w.} = c_{d\_prof} + C_{D\_w}$ vs. α [deg]



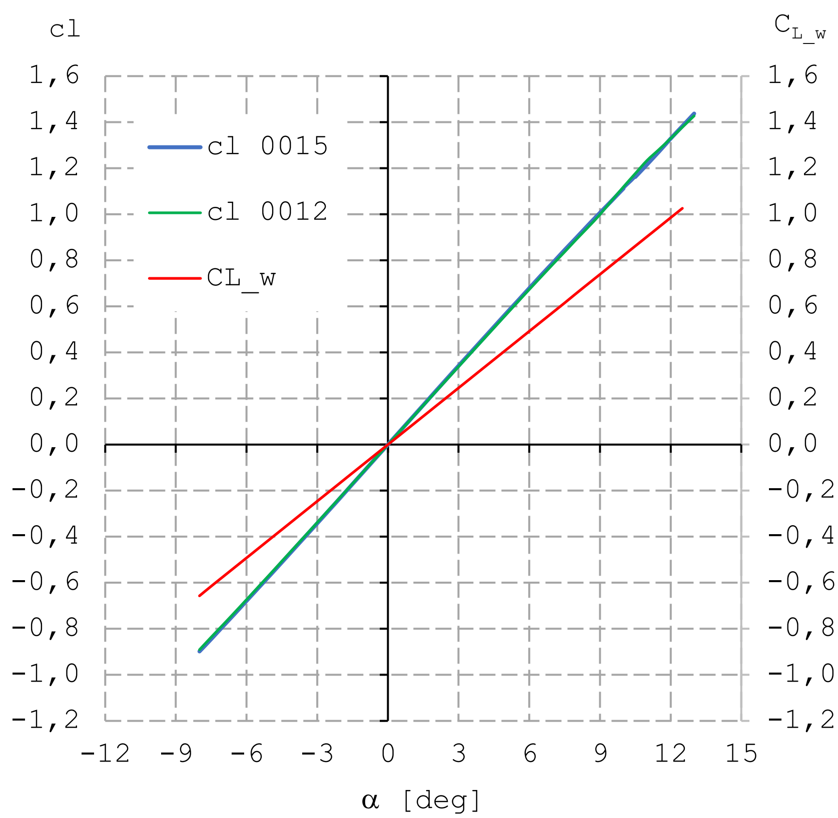
Polar curve: C_{L_w} vs. C_{D_w}



$(C_L/C_D)_w$ vs. α [deg]



The following graph shows the comparison between the airfoil lift coefficient and the wing lift coefficient, showing the changes in the lift slope:



A new net wing lift coefficient value can be found with all this information. We must first be clear that in normal cruise flight at constant acceleration, the MTOW of the airplane is equal to the total net lift produced by the wing:

$$L_w = MTOW = \frac{1}{2}\rho_\infty V_\infty^2 S_w C_{L\_w} = q_\infty S_w C_{L\_w}$$

From this relation, the net wing lift coefficient is obtained:

$$C_{L\_w} = \frac{MTOW}{q_\infty S_w} = 0.1834$$

From the graph C_{L_w} vs. α [deg], the wing lift slope is calculated:

$$a = \frac{C_{L\_w}}{\alpha - \alpha_{L=0}} = 0.08213\left[{}^{1}/_{deg}\right] = 4.715\left[{}^{1}/_{rad}\right]$$

With this same relation, the net wing lift coefficient is calculated. It is to note that in this graph (C_{L_w} vs. α [deg]), it can be seen that $\alpha_{L=0} = 0$ for this particular wing, then

$$C_{L\_w} = a(\alpha - \alpha_{L=0}) = C_{L\_w} = a \cdot \alpha$$

Equating the last relation, the angle of attack required for the cruise condition is found to be:

$$\alpha = \frac{C_{L\_w}}{a} = \frac{0.1834}{0.08213} \approx 2.24[deg]$$

With this new value of the geometric (wing incidence) angle of attack, the values of the wing aerodynamic characteristics are:

Wing Aerodynamic Characteristic	**Value**	**Units**
Lift force (L_w)	952.22	kgf
Induced drag coefficient (C_{D_i})	0.0018	-
Total drag force (D_w)	37.32	kgf

You could study more about the response to changes in variables like altitude (height), velocity, different airfoils, the application of a geometric twist angle, and even the change in the geometric shape of the wing to check the consequences for the aerodynamic characteristics of this airplane wing.