Proceedings of the 2006 IEEE/RSJ
International Conference on Intelligent Robots and Systems
October 9 - 15, 2006, Beijing, China

On the Observability and Observability Analysis of SLAM

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Abstract **– Simultaneous localization and mapping problem for mobile robots has received considerable attention over the last decade. The widely used formulation of the SLAM problem has been the augmented state approach in an estimation theoretic framework. Although, many related issues of SLAM such as computational complexity, loop closing and data association have received much attention, the observability issue has largely remained ignored. System observability is an important aspect in any state estimation problem. Observability analysis provides for understanding of the fundamental limits of the solution obtainable, regardless of process and measurement noises. The standard *world-centric* SLAM formulation is a highly non-linear system. Thus the direct use of linear observability tools and criteria in the analysis of its observability yields incorrect and inconsistent results. In this paper an appropriate method of analysis of the observability of non-linear systems is applied to investigate the properties of the standard SLAM formulation. Contrary to popular belief, it is shown through theoretical analysis that the standard 2D planar *world-centric* SLAM formulation involving odometry inputs for robot speed and heading, and range/bearing measurements to features in the environment is unobservable. It is also shown that for the system to be observable, it requires at least two absolutely known feature point positions, thus questioning the very meaning implied by SLAM. The analytical results thus established are verified through simulations.**

Index Terms – Localization, SLAM & Mobile Robot.

I. INTRODUCTION

Synthesis of autonomous localization and mapping algorithms are fundamental to the successful application of mobile robotics. Of late the robotics community has been engaged in the development and implementation of algorithms and related issues for simultaneous localization and mapping (SLAM) [1], [2].

Many successful SLAM algorithms have been proposed and implemented as evidenced by the plenitude of literature on SLAM [2]. However, the theoretical issues relating to fundamental properties of the SLAM problem formulation had not received much attention. Almost every research has dedicated itself to solving the computational complexity [3], loop closing [8], scaling [9] and data association [4] problems related to SLAM. In all of these, SLAM is posed essentially as a state estimation problem. In the formulation and its solution observability issues have been neglected and various methods of state estimation have been applied, tweaked and modified to squeeze out performance. A serious flaw in these approaches is the presumption that SLAM system is essentially observable.

In this paper, the observability issues of a typical *world-centric* planar SLAM problem involving proprioceptive sensor measurements as inputs and extereoceptive sensor measurements as observations investigated in detail using non-linear observability analysis tools. It is shown that to obtain distinctive solutions in a *world-centric* model of SLAM, absolute information of some states is required. As a consequence, an observable SLAM formulation is proposed and a solution is presented drawing on existing works.

II. RELATED WORK

In principle, SLAM problem involves incrementally building a feature map of the robot's environment while simultaneously using this map to localize itself. In the standard SLAM formulation, known as *world-centric* SLAM an attempt is made to estimate the pose of robot and features of the map with respect to a known global reference frame. The starting pose of the robot may or may not be known. Andrade-Cetto *et. al.* [7] showed for the first time that the *world-centric* SLAM problem is unobservable. They investigated observability by applying observability tools on a linearized *world-centric* SLAM system model. Although, their conclusion that conventional *world-centric* 2D SLAM problem was unobservable, the analysis lacked theoretical correctness. The latter has contributed in part to some incorrect results of observability for 2D *world-centric* SLAM. *World-centric* SLAM is an inherently highly non-linear and coupled problem. The control inputs to such systems play a pivot role in observability of the system, and methods based on linearized system models completely ignore the effects of inputs. Although, inputs do not affect observability analysis in linear systems, this is not quite the case for highly non-linear and coupled systems, rendering observability analysis based on linearized models inappropriate. In this paper, we correctly address the effects of inputs, coupling and non-linear effects of the SLAM system on its observability.

The paper provides a theoretically more correct non-linear analysis of observability of *world-centric* SLAM. Observability conditions for 2D *world-centric* SLAM is thus established and *per-case* stimulations are presented to verify the claims.

The remainder of the paper is organized as follows. Section III introduces the concepts of observability as applied to non-linear systems. Section IV summarizes the typical *world-centric* SLAM problem and its representation. In Section V the observability of the typical *world-centric* SLAM problem is theoretically analyzed. The results of analysis and the insights hence gained are used to establish requirements and conditions of observability for *world centric* SLAM. Section VI validates the theoretical results through simulation case studies. Section VII concludes the paper.

III. NONLINEAR SYSTEM OBSERVABILITY

In general, the *world-centric* SLAM formulation is of higher dimension, highly non-linear and coupled. In this section we summarize important concepts involving observability of non-linear, high dimensional coupled systems and their observability analysis.

A. Non-linear observability concept

For non-linear systems, Hermann and Krener [5] related observability to the concept of "indistinguishability" of states with respect to the inputs. In other words they emphasise the dependence of observability of non-linear systems on the control inputs unlike linear systems where it is not so. This renders methods of observability analysis based on linearized system models incorrect.

Observability of a non-linear system can be characterised from a differential geometric point of view. Consider the system Σ :

$$\Sigma \begin{cases} \dot{X} = \mathbf{f}(X,u) = \begin{bmatrix} f_1(X,u) \\ f_2(X,u) \\ \vdots \\ f_A(X,u) \end{bmatrix} & X = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_A \end{bmatrix} \in R^A \\ Z = \mathbf{h}(X) = \begin{bmatrix} h_1(X) \\ h_2(X) \\ \vdots \\ h_B(X) \end{bmatrix} & Z \in R^B \end{cases} \tag{1}$$

where, $\mathbf{X} \in R^A$ is the state vector that is an element of an A-dimensional manifold Ξ (ie., $\mathbf{X} \in \Xi$), $\mathbf{u} \in R^E$ is the input vector and $\mathbf{Z} \in R^B$ is the measurable output of the system Σ . This system is said to be observable at $\mathbf{X}(0)$ if the state vector $\mathbf{X}(t_0)$ can be determined from the observation, $\mathbf{Z}(t)$ over a finite time interval, $t_0 \le t \le t_1$. Hemann and Krener proposed a rank condition test for what they termed "local weak observability" of a non-linear system [5]. If the system Σ is locally weakly observable, then "one can instantaneously distinguish each point form its neighbours." It should be noted that local observability is a stronger condition than global observability. The former indicates that only the local state space is required to distinguish between states, while the latter may require the system "to travel a considerable distance or for a long time to distinguish between points of Ξ ". The notion of "weak" observability is a weakening of the local observability condition that requires that a given state need only be distinguished from local states rather than from the entire manifold Ξ . Note that local weak observability is a necessary condition for local observability.

A necessary condition for local weak observability is stated in [5] as follows: *if* Σ *is locally weakly observable, then the observability rank condition is satisfied generically*. In other words, if the system fails to satisfy the proposed rank condition, then it will not be locally weakly observable and therefore nor will it be locally observable. The system Σ satisfies the observability rank condition if any of the observability matrices are of rank A (recall that $\mathbf{X} \in R^A$), where the observability matrices are given by

$$O_b = \left[dL^0_{\mathbf{F}} h_b(\mathbf{X}) \quad dL^1_{\mathbf{F}} h_b(\mathbf{X}) \quad \dots \quad dL^{A-1}_{\mathbf{F}} h_b(\mathbf{X}) \right]^T \qquad for \ \ 1 \le b \le B \tag{2}$$

It is also possible to use any combination of A Lie derivatives $L^D_{\mathbf{F}} h_b(\mathbf{X})$ forming a square matrix of dimension A, as an observability matrix for the system. The elements of these matrices (2), $L^D_{\mathbf{F}} h_b(\mathbf{X})$, are the D^{th} repeated Lie derivatives of the b^{th} component of $dh(\mathbf{X})$ with respect to $\mathbf{F}(\mathbf{X},\mathbf{u})$. Specifically, the Lie derivative of a scalar $h(\mathbf{X})$ with respect to a vector $\mathbf{F}(\mathbf{X},\mathbf{u})$ is a vector field defined by

$$L^1_{\mathbf{F}} h(\mathbf{X}) = \frac{\partial h(\mathbf{X})}{\partial \mathbf{X}} \mathbf{F}(\mathbf{X},\mathbf{u}) \tag{3}$$

Similarly the Lie derivative of $dh(\mathbf{X})$ with respect to $\mathbf{F}(\mathbf{X},\mathbf{u})$ is defined by

$$\begin{aligned} L^1_{\mathbf{F}} dh(\mathbf{X}) &= d\left(L^1_{\mathbf{F}} h(\mathbf{X})\right) \\ &= \frac{\partial h(\mathbf{X})}{\partial \mathbf{X}} \frac{\partial \mathbf{F}(\mathbf{X},\mathbf{u})}{\partial \mathbf{X}} + \left[\frac{\partial}{\partial \mathbf{X}} \left(\frac{\partial h(\mathbf{X})}{\partial \mathbf{X}} \right)^T \mathbf{F}(\mathbf{X},\mathbf{u}) \right]^T \end{aligned} \tag{4}$$

The superscript indicates repeated Lie derivatives, which are defined recursively as follows:

$$\begin{aligned} L^0_{\mathbf{F}} dh(\mathbf{X}) &= \frac{\partial h(\mathbf{X})}{\partial \mathbf{X}} \\ L^D_{\mathbf{F}} dh(\mathbf{X}) &= L^{D-1}_{\mathbf{F}} dh(\mathbf{X}) \frac{\partial \mathbf{F}(\mathbf{X},\mathbf{u})}{\partial \mathbf{X}} \\ &\quad + \left[\frac{\partial}{\partial \mathbf{X}} \left(L^{D-1}_{\mathbf{F}} dh(\mathbf{X}) \right)^T \mathbf{F}(\mathbf{X},\mathbf{u}) \right]^T \end{aligned} \tag{5}$$

To establish the observability matrices (2), for a specific non-linear system the exact system model must be specified. In the next section we derive these specific system, process and observation models corresponding to the typical and popular 2D *world-centric* SLAM problem.

IV. SYSTEM MODEL

For two-dimensional (2D) *world-centric* SLAM the system state $\mathbf{X}$ can be represented as,

$$\mathbf{X} = \begin{bmatrix} x_r & M \end{bmatrix}^T \tag{6}$$

where x_r is the robot's global pose $\begin{bmatrix} x & y & \psi \end{bmatrix}^T$ and $M = \begin{bmatrix} m_1, \ldots, m_k, \ldots, m_K \end{bmatrix}^T$ is the vector of observed point features referenced to a global Cartesian reference frame. $m_k = [x_k \quad y_k]^T$ is the k^{th} feature and K is the total number of feature observed at the instant.

A. Process Model

The process model describes the time evolution of the system state vector $\mathbf{X}$. The mobile robot is assumed to be a front wheel steerable car-like vehicle. Its plan view with the relevant kinematics parameters and coordinated frames is as shown in Fig. 1. Assuming that the features in the environment are static the state evolution of the vehicle pose and feature vector is,

$$\begin{aligned} \dot{X} &= \frac{d}{dt}\begin{bmatrix} x & y & \psi & x_1 & y_1 & \ldots & x_K & y_K \end{bmatrix}^T \\ &= \begin{bmatrix} \cos\psi & \sin\psi & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}^T v + \\ &\quad \begin{bmatrix} -a\sin\psi & a\cos\psi & 1 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}^T v\delta \end{aligned} \tag{7}$$

where, a is the distance between the rear axle and exteroceptive sensor (LADAR), $\delta = \frac{\tan(\alpha)}{L}$ is the steering input (α is the steering angle, and L is the wheel base) and v is the vehicle velocity. It is apparent that the process model is highly non-linear and coupled.

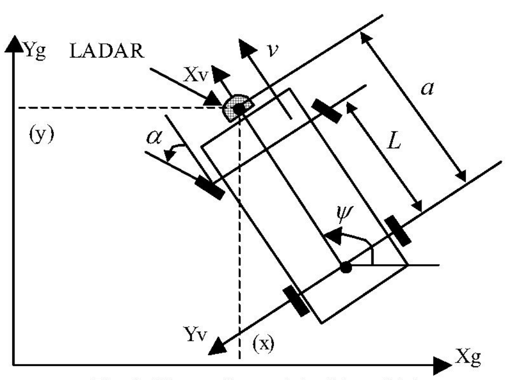

Fig. 1. Kinematics model of the vehicle

To apply the non-linear observability analysis using Lie derivatives, the process model must be in the input affine non-linear function form,

$$\dot{X} = \mathbf{f}_0(X) + \sum_{e=1}^{E} \mathbf{f}_e(X) u_e \tag{8}$$

where E is the dimension of the input vector $\mathbf{u} \in R^E$. This input affine non-linear function (8) also known as normal observation form [10] is essential for the construction of observability matrix (2) using Lie derivatives [5]. However, (7) can be manipulated into the required form (8) through a time-scale transformation technique. We choose a new time-scale, ρ which is the distance along the vehicle path. Note that that $\frac{d\rho}{dt}$ is the speed of the vehicle v. Multiplying (7), by v^{-1} we can obtain the kinematics model in the required input affine process model form,

$$\begin{aligned} &\frac{d}{d\rho}\begin{bmatrix} x & y & \psi & x_1 & y_1 & \ldots & x_K & y_K \end{bmatrix}^T \\ &= \begin{bmatrix} \cos\psi & \sin\psi & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}^T + \\ &\quad \begin{bmatrix} -a\sin\psi & a\cos\psi & 1 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}^T \delta \end{aligned} \tag{9}$$

Note that (9) represents a non-stationary model subjected to a single steer input, $\delta(=u_1)$.

Besides the process model, the system also consists of observation model that derives the output for the measurements obtained from sensors. The next section describes the common observation model used for SLAM system.

B. Observation Model

In typical SLAM problem, the sensor takes range and bearing observation of the point features in the environment. Given the current robot pose x_r and the observed feature position m_k, the observation of the range r_k and bearing θ_k can be modeled as

$$z_k = \begin{bmatrix} r_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} h_1(X) \\ h_2(X) \end{bmatrix} = \begin{bmatrix} \sqrt{(x_k - x)^2 + (y_k - y)^2} \\ \tan^{-1}\left(\frac{(y_k - y)}{(x_k - x)}\right) - \psi + \frac{\pi}{2} \end{bmatrix} \tag{10}$$

where $z_k = [r_k \quad \theta_k]^T$ is the expected sensor measurement of the k^{th} feature with range r and bearing θ respects to the sensor coordinates frame. It is useful to note here that if a known feature $[x_k \quad y_k]^T$ is observed its observation model is similar to (10).

V. Observability Analysis of SLAM System

Now consider the observation model (10) and the process model (9) input δ and one unknown feature, i.e. $k = 1$, the repeated Lie derivatives (the elements of the observability matrix) can be obtained as follows,

$$\begin{aligned} L_{\mathbf{F}}^0 dh_1(\mathbf{X}) &= \frac{\partial h_1(\mathbf{X})}{\partial \mathbf{X}} = \begin{bmatrix} L_{01} & L_{02} & L_{03} & -L_{02} & -L_{03} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{(x_k - x)}{\Delta} & -\frac{(y_k - y)}{\Delta} & 0 & \frac{(x_k - x)}{\Delta} & \frac{(y_k - y)}{\Delta} \end{bmatrix} \end{aligned} \tag{11}$$

$$\begin{aligned} L_{\mathbf{F}}^D dh_1(\mathbf{X}) &= L_{\mathbf{F}}^{D-1} dh_1(\mathbf{X}) \frac{\partial \mathbf{F}(\mathbf{X},\mathbf{u})}{\partial \mathbf{X}} \\ &\quad + \left[\frac{\partial}{\partial \mathbf{X}} \left(L_{\mathbf{F}}^{D-1} dh_1(\mathbf{X}) \right)^T \mathbf{F}(\mathbf{X},\mathbf{u}) \right]^T \\ &= \begin{bmatrix} L_{D1} & L_{D2} & L_{D3} & -L_{D1} & -L_{D2} \end{bmatrix}, \quad D = 1,2,3,4 \end{aligned} \tag{12}$$

$$L_{\mathbf{F}}^{0}dh_2(\mathbf{X})=\frac{\partial h_2(\mathbf{X})}{\partial \mathbf{X}}=\begin{bmatrix}\Lambda_{01} & \Lambda_{02} & \Lambda_{03} & -\Lambda_{01} & -\Lambda_{02}\end{bmatrix}$$
$$=\begin{bmatrix}\frac{(y_k-y)}{\Delta^2} & -\frac{(x_k-x)}{\Delta^2} & -1 & -\frac{(y_k-y)}{\Delta^2} & \frac{(x_k-x)}{\Delta^2}\end{bmatrix} \quad (13)$$

$$L_{\mathbf{F}}^{D}dh_2(\mathbf{X})=L_{\mathbf{F}}^{D-1}dh_2(\mathbf{X})\frac{\partial \mathbf{F}(\mathbf{X},\mathbf{u})}{\partial \mathbf{X}}$$
$$+\left[\frac{\partial}{\partial \mathbf{X}}\left(L_{\mathbf{F}}^{D-1}dh_2(\mathbf{X})\right)^T\mathbf{F}(\mathbf{X},\mathbf{u})\right]^T \quad (14)$$
$$=\begin{bmatrix}\Lambda_{D1} & \Lambda_{D2} & \Lambda_{D3} & -\Lambda_{D1} & -\Lambda_{D2}\end{bmatrix} \quad ,D=1,2,3,4$$

where

$$\Delta=\sqrt{(x_k-x)^2+(y_k-y)^2} \quad (15)$$

By combining the above elements an observability matrix O can be constructed as follows,

$$O=\begin{bmatrix}L_{\mathbf{F}}^{0}dh_1(\mathbf{X})\\ L_{\mathbf{F}}^{0}dh_2(\mathbf{X})\\ L_{\mathbf{F}}^{1}dh_1(\mathbf{X})\\ L_{\mathbf{F}}^{1}dh_2(\mathbf{X})\\ L_{\mathbf{F}}^{2}dh_1(\mathbf{X})\end{bmatrix}=\begin{bmatrix}L_{01} & L_{02} & L_{03} & -L_{01} & -L_{02}\\ \Lambda_{01} & \Lambda_{02} & \Lambda_{03} & -\Lambda_{01} & -\Lambda_{02}\\ L_{11} & L_{12} & L_{13} & -L_{11} & -L_{12}\\ \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & -\Lambda_{11} & -\Lambda_{12}\\ L_{21} & L_{22} & L_{23} & -L_{21} & -L_{22}\end{bmatrix} \quad (16)$$

where the first order Lie derivatives can be expressed as,

$$\begin{aligned}L_{11}&=\Lambda_{01}\Theta \quad ,\Theta=-\frac{(x_k-x)}{\Delta}(\sin\psi+a\delta\cos\psi)\\ L_{12}&=\Lambda_{02}\Theta\\ L_{13}&=\Lambda_{03}\Theta \quad +\frac{(y_k-y)}{\Delta}(\cos\psi-a\delta\sin\psi)\end{aligned} \quad (17)$$

$$\begin{aligned}\Lambda_{11}&=\Lambda_{01}\mathrm{T}_1-L_{01}\mathrm{T}_2\\ \Lambda_{12}&=\Lambda_{02}\mathrm{T}_1-L_{02}\mathrm{T}_2\\ \Lambda_{13}&=\Lambda_{03}\mathrm{T}_1-L_{03}\mathrm{T}_2\\ \mathrm{T}_1&=-\frac{(y_k-y)}{\Delta^2}(\sin\psi+a\delta\cos\psi)\\ &\quad-\frac{(x_k-x)}{\Delta^2}(\cos\psi-a\delta\sin\psi)\\ \mathrm{T}_2&=-\frac{(y_k-y)}{\Delta^3}(\cos\psi-a\delta\sin\psi)\\ &\quad+\frac{(x_k-x)}{\Delta^3}(\sin\psi+a\delta\cos\psi)\end{aligned} \quad (18)$$

Note that (17) and (18) can be represented in terms of their zero order Lie derivatives and the rest of the higher order Lie derivatives is also similarly expressed in terms of the zero order Lie derivatives. It can be shown that for any combination of the Lie derivatives used in the formation of the observability matrix as described in section III, none of these are full rank (i.e. rank of 5). That is *world-centric* SLAM when observing one unknown feature from an unknown vehicle pose is not locally weakly observable. The result can be easily generalized to observing as many unknown features as possible atypical of *world-centric* SLAM. The same conclusion has been arrived at by Andrade-Cetto *et. al.* [7] using linearized models of the *world-centric* SLAM problem. However, as will be shown next, the linearized analysis as used in [7] is inappropriate.

We now investigate the implications on the observability if known features are incorporated in the localization and mapping framework of *world-centric* SLAM. Firstly we consider the scenario of observing a single known feature along with an unknown feature. The modified observation model incorporating the *a priori* known feature point with the unknown feature point is:

$$z_{k,\varepsilon}=\begin{bmatrix}r_k\\ b_k\\ r_\varepsilon\\ b_\varepsilon\end{bmatrix}=\begin{bmatrix}h_1(X)\\ h_2(X)\\ h_3(x_r)\\ h_4(x_r)\end{bmatrix}=\begin{bmatrix}\sqrt{(x_k-x)^2+(y_k-y)^2}\\ \tan^{-1}\left(\frac{y_k-y}{x_k-x}\right)-\psi+\frac{\pi}{2}\\ \sqrt{(x_\varepsilon-x)^2+(y_\varepsilon-y)^2}\\ \tan^{-1}\left(\frac{y_\varepsilon-y}{x_\varepsilon-x}\right)-\psi+\frac{\pi}{2}\end{bmatrix} \quad (19)$$

where $[r_\varepsilon \quad b_\varepsilon]^T$ is the corresponding range and bearing of the known feature point $\varepsilon=[x_\varepsilon \quad y_\varepsilon]^T$. The repeated Lie derivatives of the unknown feature observation are given by (11), (12), (13) and (14), and for the known feature point is as follows:

$$L_{\mathbf{F}}^{0}dh_3(\mathbf{X})=\frac{\partial h_3(\mathbf{X})}{\partial \mathbf{X}}=\begin{bmatrix}G_{01} & G_{02} & G_{03} & 0 & 0\end{bmatrix}$$
$$=\begin{bmatrix}-\frac{(x_\varepsilon-x)}{\Delta} & -\frac{(y_\varepsilon-y)}{\Delta} & 0 & 0 & 0\end{bmatrix} \quad (20)$$

$$L_{\mathbf{F}}^{D}dh_3(\mathbf{X})=L_{\mathbf{F}}^{D-1}dh_3(\mathbf{X})\frac{\partial \mathbf{F}(\mathbf{X},\mathbf{u})}{\partial \mathbf{X}}$$
$$+\left[\frac{\partial}{\partial \mathbf{X}}\left(L_{\mathbf{F}}^{D-1}dh_3(\mathbf{X})\right)^T\mathbf{F}(\mathbf{X},\mathbf{u})\right]^T \quad (21)$$
$$=\begin{bmatrix}G_{D1} & G_{D2} & G_{D3} & 0 & 0\end{bmatrix} \quad ,D=1,2,3,4$$

$$L_{\mathbf{F}}^{0}dh_4(\mathbf{X})=\frac{\partial h_4(\mathbf{X})}{\partial \mathbf{X}}=\begin{bmatrix}\Gamma_{01} & \Gamma_{02} & \Gamma_{03} & 0 & 0\end{bmatrix}$$
$$=\begin{bmatrix}\frac{(y_\varepsilon-y)}{\Delta^2} & -\frac{(x_\varepsilon-x)}{\Delta^2} & -1 & 0 & 0\end{bmatrix} \quad (22)$$

$$L_{\mathbf{F}}^{D}dh_4(\mathbf{X})=L_{\mathbf{F}}^{D-1}dh_4(\mathbf{X})\frac{\partial \mathbf{F}(\mathbf{X},\mathbf{u})}{\partial \mathbf{X}}$$
$$+\left[\frac{\partial}{\partial \mathbf{X}}\left(L_{\mathbf{F}}^{D-1}dh_4(\mathbf{X})\right)^T\mathbf{F}(\mathbf{X},\mathbf{u})\right]^T \quad (23)$$
$$=\begin{bmatrix}\Gamma_{D1} & \Gamma_{D2} & \Gamma_{D3} & 0 & 0\end{bmatrix} \quad ,D=1,2,3,4$$

One of the possible observability matrices that can be formed using a particular combination of the Lie derivatives is:

$$O=\begin{bmatrix} L_{\mathbf{F}}^{0}dh_1(\mathbf{X}) \\ L_{\mathbf{F}}^{0}dh_2(\mathbf{X}) \\ L_{\mathbf{F}}^{0}dh_3(\mathbf{X}) \\ L_{\mathbf{F}}^{0}dh_4(\mathbf{X}) \\ L_{\mathbf{F}}^{1}dh_3(\mathbf{X}) \end{bmatrix}=\begin{bmatrix} L_{01} & L_{02} & L_{03} & -L_{01} & -L_{02} \\ \Lambda_{01} & \Lambda_{02} & \Lambda_{03} & -\Lambda_{01} & -\Lambda_{02} \\ G_{01} & G_{02} & G_{03} & 0 & 0 \\ \Gamma_{01} & \Gamma_{02} & \Gamma_{03} & 0 & 0 \\ G_{11} & G_{12} & G_{13} & 0 & 0 \end{bmatrix} \tag{24}$$

$[G_{11} \quad G_{12} \quad G_{13} \quad 0 \quad 0]$ are terms involving the higher order Lie derivatives and can be obtained using (21). As described for the case of unknown features the higher order Lie derivatives of the known feature observation can also be represented in terms of their zero order Lie derivatives. In all however, the rank of the observability matrix has increased to 4, although it is still rank deficient. Using a similar analysis one can show that the same result holds with one known feature and an arbitrary number of unknown feature observations. In other words *world-centric* SLAM with one known feature is unobservable. However, this result contradicts with the results reported by Andrade-Cetto *et. al.* [7].

Now we extend the analysis to two known feature points. Proceeding in a manner similar to one known feature point, the observation function vector $\mathbf{h}(X)$ of (19) for two known feature points and one unknown feature point is:

$$\begin{aligned} z_{k,\varepsilon,\eta} &= \begin{bmatrix} r_k & b_k & r_\varepsilon & b_\varepsilon & r_\eta & b_\eta \end{bmatrix}^T \\ &= \begin{bmatrix} h_1(X) & h_2(X) & h_3(x_r) & \dots & h_6(x_r) \end{bmatrix}^T \end{aligned} \tag{25}$$

where $h_1(X_r)$ and $h_2(X_r)$ are the observation functions for the unknown feature. $h_3(x_r)$ and $h_4(x_r)$, and $h_4(x_r)$ and $h_5(x_r)$, are the observation functions for the known features $\varepsilon=[x_\varepsilon \quad y_\varepsilon]^T$ and $\eta=[x_\eta \quad y_\eta]^T$ respectively. Using Lie derivatives one can derive an observability O:

$$O=\begin{bmatrix} L_{\mathbf{F}}^{0}dh_1(\mathbf{X}) \\ L_{\mathbf{F}}^{0}dh_2(\mathbf{X}) \\ L_{\mathbf{F}}^{0}dh_3(\mathbf{X}) \\ L_{\mathbf{F}}^{0}dh_4(\mathbf{X}) \\ L_{\mathbf{F}}^{1}dh_5(\mathbf{X}) \end{bmatrix}=\begin{bmatrix} L_{01} & L_{02} & L_{03} & -L_{01} & -L_{02} \\ \Lambda_{01} & \Lambda_{02} & \Lambda_{03} & -\Lambda_{01} & -\Lambda_{02} \\ G_{01} & G_{02} & G_{03} & 0 & 0 \\ \Gamma_{01} & \Gamma_{02} & \Gamma_{03} & 0 & 0 \\ \Upsilon_{01} & \Upsilon_{02} & \Upsilon_{03} & 0 & 0 \end{bmatrix} \tag{26}$$

where $[L_{01} \quad L_{02} \quad L_{03} \quad -L_{01} \quad -L_{02}]$ $[\Lambda_{01} \quad \Lambda_{02} \quad \Lambda_{03} \quad -\Lambda_{01} \quad -\Lambda_{02}]$ are the zero order Lie derivatives for the unknown feature. $[G_{01} \quad G_{02} \quad G_{03} \quad 0 \quad 0]$ $[\Gamma_{01} \quad \Gamma_{02} \quad \Gamma_{03} \quad 0 \quad 0]$ $[\Upsilon_{01} \quad \Upsilon_{02} \quad \Upsilon_{03} \quad 0 \quad 0]$ are the zero order Lie derivatives for the two known features, $\varepsilon=[x_\varepsilon \quad y_\varepsilon]^T$ and $\eta=[x_\eta \quad y_\eta]^T$. Now it can be verified that this observability matrix is of full rank. (i.e. 5).

In general it can be shown that for any two given known feature points and an arbitrary number of unknown features world-centric SLAM is observable. These theoretical results derived are verified through simulations in the next section.

VI. Simulation Results

Simulations are carried out to verify the rank results obtained from the non-linear observability analysis. The robot path and the feature positions of the simulations conducted are shown in Fig. 2. Using Extended Kalman Filter (EKF) [11] as the estimation technique, the simulations are conducted with an initial error in the robot pose. The parameters are shown in Table 1.

TABLE I
PARAMETER VALUES USED IN THE SIMULATION

Parameters		Values
Initial offset	x	$-2m$
	y	$-2m$
	ψ	2°
Steering angle	$0m$ to $100m$	0°
Velocity	$0m$ to $100m$	$3m/s$
Unknown feature position	k	$(130,20)$
Known feature positions	ε	$(120,30)$
	η	$(130,10)$
Control noises	Velocity sigma	$2m/s$
	Steer sigma	2°
Measurement noises	Range sigma	$0.5m$
	Bearing sigma	1°

The simulation results obtained for absolute position errors depicted in Fig. 3 show that in the case of SLAM without known features, the error does not converge to a minimum. Further, Fig. 4 shows the heading error exhibits a constant bias, with nonzero mean error. This simulation thus demonstrates that the SLAM system without known features is unobservable, and as a consequence the initial error in system is not corrected and hence the true pose of robot (nor landmarks) is not obtained, signifying that given the inputs and measurements, the system is unable to distinguish the truth state from the state possesses an offset or error.

In the case of SLAM system with one *a priori* known feature point, the results although for the absolute position error is decreasing to a minimum (Fig. 3), the heading error as depicted in Fig. 4, still exhibits a constant bias, with nonzero mean error. This simple simulation validates the theoretical result established that one *a priori* known feature point is inadequate to correct any initial orientation error. Thus it can be concluded that the system is unobservable with only one *a priori* known feature point.

Another simulation is carried out in a similar manner however with two known feature points as shown in Fig. 3. Fig. 4 shows that the absolute position error converges to a minimum and also for the heading error as shown in Fig. 6. The simulation verifies the theoretical result that for the SLAM system to be locally weakly observable there need be two *a priori* feature points.

These investigations are sufficient to prove that to achieve an observable system in a planar SLAM problem, there exists minimum absolute spatial information in terms of features;

two *a priori* known feature points are required for the SLAM system to be observable.

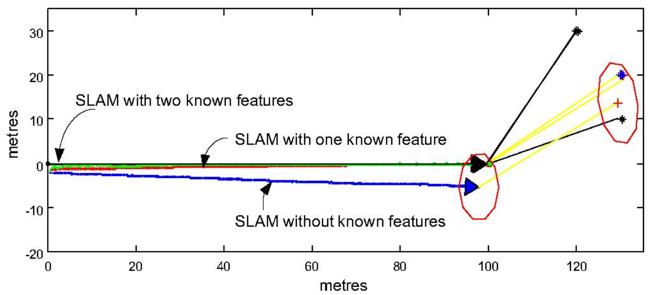

Fig. 2 SLAM with *a priori* known features simulation of truth and estimates robot path.

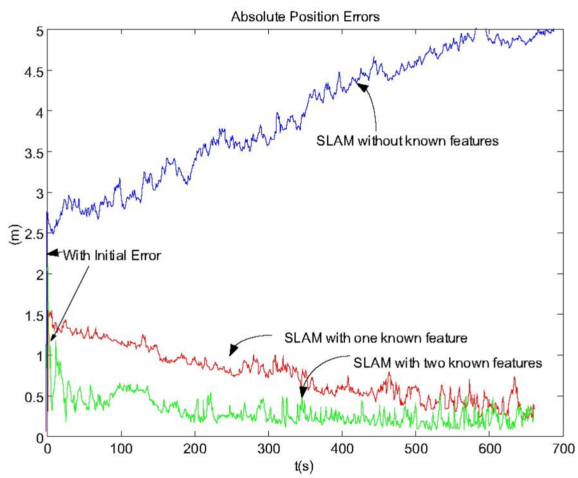

Fig. 3 Absolute position errors on SLAM with a *priori* known features

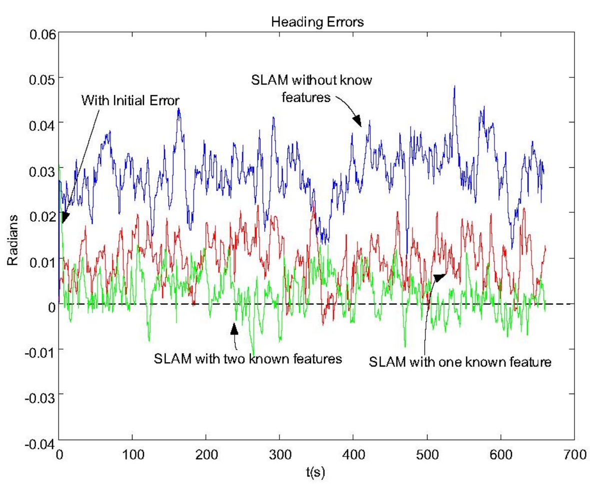

Fig. 4 Heading Errors SLAM with *a priori* known features

VII. CONCLUSIONS

Simultaneous robot localization and absolute mapping per se is not a well-posed problem without using absolute information to deduce the robot pose and the map in the global reference frame. For state estimates in SLAM to converge, it is necessary to reformulate the observation models used. One method is to modify the model to incorporate *a priori* known features positions, which is contrary to the general understanding and meaning implied of SLAM. However, an earlier study has shown that standard SLAM [7] is not observable and suggests the use of *a priori* information to estimate the robot state. The observability of the planar non-linear SLAM system in [7] is analyzed through linearizing the system model and applying the standard observability analysis tool. Due to the non-linearity in planar SLAM system, the obtained results cannot be used to understand the local observability of the system, as the inputs are the important factors that affect and change the system observability.

To accommodate the input effects to the non-linear system, the observability matrix is formulated by using the exact non-linear system equation using Lie derivatives. By satisfying the rank condition of the observability matrix, the system is concluded to be locally weakly observable. With this proper observability analysis, it is established that the typical planar SLAM system is observable when two *a priori* known feature points observation is available, contrary to the analysis done in [7] where one *a priori* known feature point is concluded enough to set the system to be observable.

With these observability studies, a more theoretically sound or justified approach can be devised and also provides a better understanding of the theoretical limitations in the typical SLAM system.

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