Proceedings of the 2009 IEEE International Conference on Mechatronics.
Málaga, Spain, April 2009.

On the Linear and Nonlinear Observability Analysis of the SLAM Problem

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Abstract **- Research in Simultaneous Localization and Mapping (SLAM) has been progressing for almost two decades. Although several researchers attempted recently to investigate its observability (mostly without proofs for the general cases) the established facts have often been left unnoticed or ignored by the research community. In this paper rigorous proofs have been provided as an enlightenment for the observability properties of the general two dimensional SLAM problem incorporating a car like kinematic model in the context of piece-wise constant systems theory and non-linear Lie derivative theory. Observable and Unobservable states of the general *n* landmark SLAM problem have been established with proofs. A comparison of linear and non-linear techniques to evaluate the observability of SLAM is provided using simulations.**

Keywords-SLAM, observability

I. INTRODUCTION

Simultaneous localization and map building (SLAM) [1] and [2] investigates whether a vehicle can achieve localization in an unknown environment while building a map of features, incrementally acquired from its sensors. Successful SLAM solution can complement GPS during satellite occlusion, multi path effects and periodic signal blockage by places having restricted views of the sky. A complete knowledge of the observability, convergence and estimability of SLAM as a robot localization technique is of paramount importance before deploying SLAM in real world industry applications such as mining, surveying and cargo handling.

The work of [3] first highlighted the issue of partial observability of SLAM. [3] shows the partial observability of a simple monobot SLAM and attempts to extend it to complex vehicle models. However, the observability analysis of complex vehicle models is not substantiated with any proofs as such. In this work we seperate the linear combinations of initial SLAM states into observable and unobservable states thus explicitly showing the observability properties of individual SLAM states without any transformation.

Theory of nonlinear observability has been applied for the analysis of observability of SLAM in [4]. The work shows the conditions for the local observability of 2D SLAM for a single landmark SLAM problem and conjectures the general observability properties of the high dimensional SLAM problem. It has stated in this work that in contrast to the work of [3] it is necessary that two known landmarks be observed for the full observability of SLAM. However, in this work we establish that although observing of two known landmarks is necessary for the local observability of SLAM it is not necessary for the global observability along a trajectory in a general control theoretic context. The evidence for the observability properties of SLAM are supported by proper Monte Carlo simulations of EKF SLAM incorporating filter consistency.

This paper is organized as follows. Section II describes the piece-wise constant systems theory and its application to the SLAM problem. The Observability properties of *n* landmark SLAM problem are analysed with proofs. Section III provides proofs of the conditions that must be satisfied for local observability of the *n* landmark SLAM problem. Section IV compares and contrasts the global observability over a trajectory of states and the local observability phenomena. Monte-Carlo simulations are provided for the filter consistency of observable SLAM problems. We conclude that the global observability along a trajectory is quite adequate for good localization results using SLAM.

II. PIECE-WISE CONSTANT SYSTEMS THEORY

A comprehensive treatment of the observability analysis of piece-wise constant systems are given in [5], [8] and [9]. In this theory, estimation problems, which can be assumed piece-wise constant are analysed using general linear system techniques. It has been shown that if certain conditions are satisfied by a piecewise constant system, the observability of the system over several segments can be analysed using a simplified observability matrix known as the stripped observability matrix (STM). If the indirect system (eg. [8]) of SLAM is utilized in the observability analysis the general *n* landmark SLAM problem can be modelled as a piece-wise constant system with linear state space models over segments of time.

A. Problem Formulation

In the indirect form of representation, error states are modelled instead of original states in state space. Let a simple car like mobile robot is moving in an environment while observing range and bearing of *n* landmarks in its environment, estimating landmark location vectors and the robot position vector. Let the continuous error states of the robots's longitudinal and lateral coordinates and the heading be δx, δy and $\delta\theta$. The error states of landmark position coordinates are denoted by δx_1, δy_1, δx_2, δy_2,..., δx_n and δy_n with the usual notation. The indirect SLAM in i^{th} piece-wise constant segment can then be represented by the following linear system (additive noise terms are omitted for clarity).

$$\delta\dot{\mathbf{X}}_n(t) = \mathbf{F}_{n,i}\delta\mathbf{X}_n(t) \quad (1)$$

$$\delta\mathbf{Z} = \mathbf{H}_{n,i}\delta\mathbf{X}_n(t) \quad (2)$$

$$\delta\mathbf{X}_n(t) = \left[\delta x \ \delta y \ \delta\theta \ \delta \mathrm{x}_1 \ \delta y_1 ....\delta \mathrm{x}_\mathrm{n} \ \delta y_\mathrm{n}\right]^T \quad (3)$$

$$\mathbf{F}_{n,i} = \begin{bmatrix} \mathbf{0}_{1x2} & -v\sin(\theta_i) & \mathbf{0}_{1x2n} \\ \mathbf{0}_{1x2} & v\cos(\theta_i) & \mathbf{0}_{1x2n} \\ \mathbf{0}_{(2n+1)x1} & \mathbf{0}_{(2n+1)x1} & \mathbf{0}_{(2n+1)x1} \end{bmatrix} \quad (4)$$

$$\mathbf{H}_{n,i} = \begin{bmatrix} \mathbf{HV}_{1,i} & \mathbf{HL}_{1,i} & .. & .. & .. & \mathbf{0}_{2x2} \\ \mathbf{HV}_{2,i} & \mathbf{0}_{2x2} & \mathbf{HL}_{2,i} & \mathbf{0}_{2x2} & .. & \mathbf{0}_{2x2} \\ .. & .. & .. & .. & .. & .. \\ \mathbf{HV}_{2,i} & \mathbf{0}_{2x2} & .. & .. & .. & \mathbf{HL}_{n,i} \end{bmatrix} \quad (5)$$

$$\mathbf{HV}_{j,i} = \begin{bmatrix} \mathbf{HV}^1_{j,i} & \mathbf{HV}^2_{j,i} & \begin{matrix} 0 \\ -1 \end{matrix} \end{bmatrix} \quad (6)$$

$$\mathbf{HL}_{j,i} = \begin{bmatrix} -\mathbf{HV}^1_{j,i} & -\mathbf{HV}^2_{j,i} \end{bmatrix} \quad (7)$$

where v, $t, x_{v,i}$, $y_{v,i}$, $\theta_{v,i}$, $x_{j,i}$, and $y_{j,i}$ denote the speed of the robot, time, x, y, and heading of the robot, x coordinate of landmark j and y coordinate of landmark j respectively all at segment i. $\mathbf{HV}^1_{j,i}$ and $\mathbf{HV}^2_{j,i}$ are the Jacobians of the range and bearing measurement vector of j^{th} landmark at i^{th} segment. The kinematic model of the robot described in [2] is assumed:

$$\begin{bmatrix} \dot{x} & \dot{y} & \dot{\theta} \end{bmatrix}^T = \begin{bmatrix} v\cos(\theta) & v\sin(\theta) & \omega \end{bmatrix}^T \quad (8)$$

$$\omega = \left(v\tan\gamma/L\right) \quad (9)$$

where ω, γ and L denote the turning rate of the car like robot, steering angle from the heading and the vehicle wheel base line length respectively. Thus, according to the piecewise constant system assumption, the observability matrix of the i^{th} segment of the n landmark SLAM problem is

$$\mathbf{O}_{n,i} = \begin{bmatrix} (\mathbf{H}_{n,i})^T & (\mathbf{H}_{n,i}\mathbf{F}_{n,i})^T & ..... & (\mathbf{H}_{n,i}(\mathbf{F}_{n,i})^{n-1})^T \end{bmatrix}^T \quad (10)$$

B. Observability in a Single Segment

The following results can be shown with proofs.

Conjecture 1*: *2D n landmark SLAM observability matrix when observing range and bearing of n landmarks in any piecewise constant segment is rank deficient by two.

Proof: Consider i^{th} piecewise constant segment. For n=1, $\mathbf{F}_{1,i}$, $\mathbf{H}_{1,i}$ and $\mathbf{O}_{1,i}$ can be deduced from (4), (5) and (10). Thus using a commercial mathematical package it can be shown that rank of $\mathbf{O}_{1,i}$ is 3. Thus $\mathbf{O}_{1,i}$ is rank deficient by 2. Assume that the result is true for p landmarks. i.e. the rank of $\mathbf{O}_{p,i}$ is $2p$+1. From (4) and (5),

$$\mathbf{H}_{p,i}\mathbf{F}_{p,i} = \begin{bmatrix} \mathbf{0}_{1x2} & c_{1,1} & \mathbf{0}_{1x2p} \\ \mathbf{0}_{1x2} & c_{2,1} & \mathbf{0}_{1x2p} \\ ..... & ... & ..... \\ \mathbf{0}_{1x2} & c_{2,p} & \mathbf{0}_{1x2p} \end{bmatrix} \quad (11)$$

and for all $j = 1, 2, ..., p$, $c_{1,j}$ and $c_{2,j}$ are given by

$$c_{1,j} = -\left(\partial z_{1,j}/\partial x\right)v\sin(\theta) + \left(\partial z_{1,j}/\partial y\right)v\cos(\theta) \quad (12)$$

$$c_{2,j} = -\left(\partial z_{2,j}/\partial x\right)v\sin(\theta) + \left(\partial z_{2,j}/\partial y\right)v\cos(\theta) \quad (13)$$

where $z_{1,j}$ and $z_{2,j}$ denote the observation of the range and bearing of j^{th} landmark respectively.

For all positive integers m>1 $\mathbf{H}_{p,i}(\mathbf{F}_{p,i})^m = \mathbf{0}_{2px(2p+3)}$. Hence by elementary row operations $\mathbf{O}_{p,i}$ can be reduced to the following form omitting zero rows.

$$\mathbf{O}_{p,i} = \begin{bmatrix} & \mathbf{H}_{p,i} & \\ \mathbf{0}_{1x2} & 1 & \mathbf{0}_{1x2p} \end{bmatrix} \quad (14)$$

Hence from the assumption above it is thus deduced that $\mathbf{O}_{p,i}$ has $2p$+1 independent rows. When there are p+1 landmarks in the map,

$$\mathbf{H}_{p+1,i} = \left[\begin{array}{c|c} \mathbf{H}_{p,i} & \mathbf{0}_{2px2} \\ \hline \partial\mathbf{z}_{p+1}/\partial(x,y,\theta) \ \ \mathbf{0}_{2px2} & \partial\mathbf{z}_{p+1}/\partial(x_{p+1},y_{p+1}) \end{array}\right] \quad (15)$$

where $\mathbf{z}_{p+1}$ is the range and bearing observation vector of the $p+1^{th}$ landmark. From (11)-(13) it follows that for all positive integers m>1, $\mathbf{H}_{p+1,i}(\mathbf{F}_{p+1,i})^m = \mathbf{0}_{2(p+1)x(2p+5)}$ and $\mathbf{H}_{p+1,i}\mathbf{F}_{p+1,i}$ is similar to (11) with only 3[rd] column having non-zero elements. Hence $\mathbf{O}_{p+1,i}$ can be reduced to the following form by elementary row operations.

$$\mathbf{O}_{p+1,i} = \left[\begin{array}{c|c} \mathbf{H}_{p,i} & \mathbf{0}_{2px2} \\ \hline \mathbf{0}_{1x2} \ \ 1 \ \ \mathbf{0}_{1x2p} & \mathbf{0}_{1x2} \\ \partial\mathbf{z}_{p+1}/\partial(x,y,\theta) \ \ \mathbf{0}_{2x2p} & \partial\mathbf{z}_{p+1}/\partial(x_{p+1},y_{p+1}) \end{array}\right] \quad (16)$$

where $\mathbf{z}_{p+1}$ is the range and bearing observation vector of the p+1[th] landmark. From (14) it can be observed that $\mathbf{O}_{p+1,i}$ has $2p$+1 independent rows and the last two independent non-zero rows are also independent of the first $2p$+1 rows. Hence the rank of $\mathbf{O}_{p+1,i}$ is $2p$+3 (2(p+1)+1) i.e. short by 2 to full rank condition. Since the result is true for n=1 and given the result is true for n=p proves that the result is true for n=p+1, the result is true for every positive integer by the Principle of mathematical induction. Hence the 2D SLAM observability matrix in any piecewise constant segment is rank deficient by two.

C. Observability in Multiple Segments

According to the piecewise constant systems theory, observability of a piecewise constant system can sometimes be improved by including more segments in the analysis. For continuous time systems if $Null(\mathbf{O}_{n,i}) \subset Null(\mathbf{F}_{n,i})$ where the null space of matrix A is denoted by $Null$(A) the observability analysis of multiple segments can be performed by a simplified observability matrix known as the Stripped Observability Matrix [5]. For k piecewise constant segments STM is given by $\mathbf{STM}_k$ as follows.

$$\mathbf{STM}_k = \begin{bmatrix} (\mathbf{O}_{n,1})^T & (\mathbf{O}_{n,2})^T & ..... & (\mathbf{O}_{n,k})^T \end{bmatrix}^T \quad (17)$$

Conjecture 2*: *2D n landmark SLAM problem in error form has the property $Null(\mathbf{O}_{n,i}) \subset Null(\mathbf{F}_{n,i})$.

Proof: Consider i^{th} piecewise constant segment. Since the rank of the observability matrix $\mathbf{O}_{n,i}$ is always rank deficient by two (by ***Conjecture*** 1) the rank of the null space of $\mathbf{O}_{n,i}$ is 2. By inspection of the column space of $\mathbf{O}_{n,i}$ it can be observed that columns 1, 4, 6, 8,.... are linearly dependent and columns 2, 5, 7, 9 are linearly dependent. hence we can deduce that two column vectors constituting the basis of $Null(\mathbf{O}_{n,i})$ are

$\mathbf{n}_1 = [1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ ....]$ and $\mathbf{n}_2 = [0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ ....]$

Hence, $\mathbf{F}_{n,i}(\lambda_1\mathbf{n}_1 + \lambda_2\mathbf{n}_2) = \mathbf{0}_{(2n+1)x1}$ for any real number λ_1 and λ_2. Thus any vector that is in $Null(\mathbf{O}_{n,i})$ is in $Null(\mathbf{F}_{n,i})$ as well. Hence $Null(\mathbf{O}_{n,i}) \subset Null(\mathbf{F}_{n,i})$ for any n. Hence, according to the piecewise constant systems theory we can use $\mathbf{STM}_k$ to analyse the observability of n landmark SLAM problem over multiple segments.

Conjecture 3*: *Addition of multiple segments into the observability matrix of n landmark 2D SLAM does not improve the observability of the SLAM problem.

Proof: From **C*onjecture*** 1 above rank of $\mathbf{O}_{n,i}$ is $2n+1$. Since $\mathbf{STM}_k$ for k number of piecewise constant segments always has $2n+3$ columns we have;

$$2n+1 \le rank(\mathbf{STM}_k) \le 2n+3 \qquad (18)$$

However, by inspection it can be observed that the 1st column of the $\mathbf{STM}_k$ is linearly dependent on columns 4, 6, 8,.... and the 2nd column of the $\mathbf{STM}_k$ is linearly dependent on columns 5, 7, 9,...... Thus,

$$rank(\mathbf{STM}_k) \le 2n+1$$

Hence $2n+1 \le rank(\mathbf{STM}_k) \le 2n+1$. Thus,

$$rank(\mathbf{STM}_k) = 2n+1 \qquad (19)$$

Hence the observability rank condition remains the same regardless of the number of piecewise constant segments added to the observability analysis.

D. Identifying Observable and Unobservable States

Under these circumstances, it is of great significance if it is possible to identify the states that are observable and those that are not observable. The process of distinguishing among observable and unobservable states allows us to design good observers and gives a better insight to the estimation problem at hand.

If $\mathbf{O}_{n,i}$ is reduced by elementary row and column operations to $\mathbf{O}'_{n,i}$ so that $\mathbf{O}'_{n,i} = \left[\begin{array}{c|c} \mathbf{I_o} & \mathbf{O_o} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right]$ where $\mathbf{I_o}$ has the dimension determined by $rank(\mathbf{O}_{1,i})$ and $\mathbf{O_o}$ is the resulting matrix of elementary row and column operations it is possible to find matrices $\mathbf{T}$ and $\mathbf{M}$ such that $\mathbf{O}'_{1,i} = \mathbf{T}\mathbf{O}_{n,i}\mathbf{M}^{-1}$. Let $\mathbf{U}_o = [\mathbf{I}_o \quad \mathbf{O}_o]$ and $\mathbf{U}_u = [\mathbf{O}_o \quad \mathbf{I}_{D-R}]$ where D is the dimension of the SLAM state vector and R is the rank of the observability matrix $\mathbf{O}_{n,i}$ then

$$\mathbf{X}_o = \mathbf{U}_o\mathbf{M}^{-1}\mathbf{X}_n \qquad (20)$$

$$\mathbf{X}_u = \mathbf{U}_u\mathbf{M}^{-1}\mathbf{X}_n \qquad (21)$$

where $\mathbf{X}_o$ and $\mathbf{X}_u$ denote the observable and unobservable states of the SLAM state vector. However, it is important to note that the choice of observable and unobservable states is not unique. Consider the folowing single landmark SLAM problem in segment i.

$$\mathbf{O}_{1,i} = \begin{bmatrix} & \mathbf{H}_{1,i} & \\ \mathbf{0}_{1x2} & 1 & \mathbf{0}_{1x2} \\ & \mathbf{0}_{7x5} & \end{bmatrix} \qquad (22)$$

For the single landmark SLAM problem it is possible to choose $\mathbf{T}$ and $\mathbf{M}$ such that

$$\mathbf{M}^{-1} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (23)$$

$$\mathbf{X}_o = [-\delta x + \delta x_1 \quad -\delta y + \delta y_1 \quad \delta\theta]^T \qquad (24)$$

$$\mathbf{X}_u = [\delta x \quad \delta y]^T \qquad (25)$$

In the same way by examining the matrix transformations it can be shown that the above result generalizes to n landmark SLAM problem as follows.

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{I}_{2n} \\ \mathbf{M}_{21} & \mathbf{0}_{3x2n} \end{bmatrix} \qquad (26)$$

$$\mathbf{M}_{11}^T = \begin{bmatrix} -1 & 0 & -1 & 0 & -1 & .. & 0 \\ 0 & -1 & 0 & -1 & 0 & .. & -1 \\ 0 & 0 & 0 & 0 & 0 & .. & 0 \end{bmatrix}_{3\times 2n} \qquad (27)$$

$$\mathbf{M}_{21} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad (28)$$

Observable states ($\mathbf{X}_o$) are $-\delta x + \delta x_i$, $-\delta y + \delta y_i$ for i=1,2,..,n and $\delta\theta$. $\mathbf{X}_u$ is given by (25). Hence this result shows that only the landmark states relative to the vehicle position and the vehicle orientation are the observable variables. Whereas absolute coordinates of the vehicle are not observable regardless of the number of unknown landmarks being estimated and observed.

Conjecture 4*: *Single vehicle SLAM is observable when observing range and bearing of one known landmark in addition to all the landmarks being estimated.

Proof: Consider i^{th} piecewise constant segment. When observing a known landmark (having the Jacobian $h\mathbf{V}_{m,i}$ with respect to the vehicle states). $\mathbf{H}_{n,i}$ is given by (29). For n=p, $\mathbf{F}_{p,i}$ and $\mathbf{O}_{p,i}$ can be deduced from (4) and (10). $\mathbf{O}_{p,i}$ can be reduced to the same form as (14) in this case as well. However, in this case it is possible to see that there are two additional independent rows in $\mathbf{H}_{p,i}$. Hence the rank of $\mathbf{O}_{p,i}$ is $2p+3$. i.e. full rank. Hence the single vehicle SLAM is observable in any segment when observing one known landmark and all the unknown landmarks that are being estimated.

$$\mathbf{H}_{n,i}=\begin{bmatrix}\mathbf{HV}_{1,i} & \mathbf{HL}_{1,i} & .. & .. & .. & \mathbf{0}_{2x2}\\ \mathbf{HV}_{2,i} & \mathbf{0}_{2x2} & \mathbf{HL}_{2,i} & \mathbf{0}_{2x2} & .. & \mathbf{0}_{2x2}\\ .. & .. & .. & .. & .. & ..\\ \mathbf{HV}_{2,i} & \mathbf{0}_{2x2} & .. & .. & .. & \mathbf{HL}_{n,i}\\ h\mathbf{V}_{m,i} & \mathbf{0}_{2x2} & .. & .. & .. & \mathbf{0}_{2x2}\end{bmatrix} \quad (29)$$

III NONLINEAR OBSERVABILITY ANALYSIS

Piecewise constant systems theory does not take into account the effects of nonlinearities of the process and observation models. In contrast to linear models, the observability of nonlinear models are affected by the inputs as well. However, the piecewise constant systems theory does not take the effects of inputs to the observability into consideration. It has been argued in [4] that these factors can be taken into account if the theory of nonlinear observability analysis is used to analyse the SLAM problem. Although, the observability of SLAM for a single landmark is considered in the work of [4], they have not generalized the nonlinear observability of SLAM into any number of landmarks. Furthermore, it has not compared the nonlinear observability analysis with the linear observability analysis in a rigorous manner. Here, we extend the nonlinear observability result of SLAM to any number of landmarks comparing the linear and nonlinear observability analysis using Monte-Carlo simulations and employing filter consistency measures.

A. Theory of Nonlinear Observability

Before analysing the SLAM problem with nonlinear observability analysis techniques it is important to know the basis of the theory. Nonlinear observability analysis theory was first published in the work of [6]. Let Σ denote the system under consideration with $\mathbf{u}$ as inputs and $\mathbf{z}$ as measurements. The nonlinear function $\mathbf{f}(.)$ provides the evolution of the state $\mathbf{x}$ with time t.

$$\Sigma \quad \begin{cases}\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},\ \mathbf{u})\\ \mathbf{z}=\mathbf{h}(\mathbf{x})\end{cases} \quad (30)$$

[7] shows the following theorem for the observability of Σ.

Theorem 1:If $\mathbf{f}(\mathbf{x},\ \mathbf{u})=\mathbf{g}^0(\mathbf{x})+\sum\mathbf{g}^i(\mathbf{x})u_i$ ***where*** $\mathbf{x}$ ***is a vector of n state variables occupying an open subset*** Ξ ***of*** $\mathbb{R}^n$ $\mathbf{g}^0()$***,....,*** $\mathbf{g}^i()$ ***are n dimensional vector analytical functions in*** Ξ***, the output*** $\mathbf{h}(.)$ ***is an analytic function of*** $\mathbb{R}^p$ ***in*** Ξ ***and*** $\mathbf{u}$ ***is an analytic function of time having distinct scalar controls*** u_i***, then*** Σ ***is locally observable if the matrix*** $\mathbf{O}_\Sigma$ ***given below has full rank (i.e rank n).***

$$\mathbf{O}_\Sigma(\mathbf{d},\mathbf{f},\mathbf{h},n)=\left[(\mathbf{d}L_{\mathbf{f}}^0\mathbf{h})^T \quad (\mathbf{d}L_{\mathbf{f}}^1\mathbf{h})^T \quad ... \quad (\mathbf{d}L_{\mathbf{f}}^{n-1}\mathbf{h})^T\right]^T \quad (31)$$

where $\mathbf{h}(.)=\left[h_1(.) \quad h_2(.) \quad .... \quad h_p(.)\right]^T$. $L_f^m g_i$ is the m^{th} order Lie derivative of $g_i()$ with respect to the function $\mathbf{f}(.)$ and operator $\mathbf{d}$ denote the gradient operator with respect to $\mathbf{x}$. The proof for this theorem is given in [7].

B. Analysis of the SLAM Problem

The n landmark SLAM problem omitting noise terms with usual notation is as follows.

$$\dot{\mathbf{X}}_n(t)=\mathbf{f}(\mathbf{X}_n(t),\ \mathbf{u}) \quad (32)$$

$$\mathbf{Z}=\mathbf{h}(\mathbf{X}_n(t)) \quad (33)$$

$$\mathbf{X}_n(t)=\left[x(t)\ y(t)\ \theta(t)\ \mathrm{x}_1(\mathrm{t})\, y_1(t).........\mathrm{x}_\mathrm{n}(\mathrm{t})\, y_\mathrm{n}(t)\right]^T \quad (34)$$

$$\mathbf{f}(.)=\left[v\cos(\theta) \quad v\sin(\theta) \quad \omega \quad 0 \quad ... \quad 0\right]^T \quad (35)$$

$$\mathbf{h}(.)=[(\mathbf{h}_1)^T \quad (\mathbf{h}_2)^T \quad ... \quad (\mathbf{h}_n)^T]^T \quad (36)$$

$$\mathbf{h}_i=\begin{bmatrix}\sqrt{(x_i-x)^2+(y_i-y)^2}\\ \tan^{-1}\{(y_i-y)/(x_i-x)\}-\theta\end{bmatrix} \quad (37)$$

Using a new time scale ρ which is the distance along the vehicle path it is shown in [4] that the n landmark SLAM problem (32)-(37) is in control affine form. Thus, theorem 1 can be applied to find the nonlinear observability properties of the problem.

C. Nonlinear Observability Rank Condition

Conjecture 5: Observability matrix denoting the nonlinear observability of n landmark SLAM problem is rank deficient by 3.

Proof: Let $\mathbf{O}_n$ denote the observability matrix (as shown in (31)) of the n landmark SLAM problem. When n=1 it is easy to show that rank of $\mathbf{O}_n$ is 2, which is rank deficient by 3. Assume that the result is true for n=p where p is a positive integer and p>1. i.e $\mathbf{O}_p$ has a rank of 2p. Let $\mathbf{f}(n)$ and $\mathbf{h}(n)$ denote the $\mathbf{f}(.)$ of (35) when estimating n landmarks in the state and $\mathbf{h}(.)$ when observing n landmarks respectively.

$$\mathbf{O}_p=\mathbf{O}_p\left(\mathbf{d}_p,\mathbf{f}(p),\mathbf{h}(p),2p+2\right) \quad (38)$$

where $\mathbf{d}_p$ represents the gradient operator with respect to $\mathbf{X}_p$. Now let $\mathbf{O}_{p+1}$ denote the observability matrix (as given in (31)) of the SLAM problem when estimating p+1 landmark locations.

$$\mathbf{O}_{p+1}=\mathbf{O}_{p+1}\left(\mathbf{d}_{p+1},\mathbf{f}(p+1),\mathbf{h}(p+1),2p+4\right) \quad (39)$$

By definition

$$L_{\mathbf{f}(p+1)}^0\mathbf{h}(p+1)=\left[\mathbf{h}^T(p) \quad \mathbf{h}_{p+1}^T\right]^T \quad (40)$$

$$\begin{aligned}\mathbf{d}_{p+1}L_{\mathbf{f}(p+1)}^0\mathbf{h}(p+1)&=\left(\partial/\partial\mathbf{X}_{p+1}\right)L_{\mathbf{f}(p+1)}^0\mathbf{h}(p+1)\\ &=\begin{bmatrix}\mathbf{d}_pL_{\mathbf{f}(p)}^0\mathbf{h}(p) \quad \mathbf{0} \quad \mathbf{0}\\ \mathbf{d}_{p+1}L_{\mathbf{f}(p+1)}^0\mathbf{h}_{p+1}\end{bmatrix}\end{aligned} \quad (41)$$

Therefore, by recursion it can be shown that;

$$\mathbf{d}_{p+1}L_{\mathbf{f}(p+1)}^m\mathbf{h}(p+1)=\begin{bmatrix}\mathbf{d}_pL_{\mathbf{f}(p)}^m\mathbf{h}(p) \quad \mathbf{0} \quad \mathbf{0}\\ \mathbf{d}_{p+1}L_{\mathbf{f}(p+1)}^m\mathbf{h}_{p+1}\end{bmatrix} \quad (42)$$

for any positive integer m. Consider $\mathbf{h}_m$ where m is a positive integer.

$$\mathbf{h}_m=\mathbf{h}_m(x,\ y,\ \theta,\ x_m,\ y_m) \quad (43)$$

By the definition and from recursion it can be shown that;

$$\mathbf{d}_{p+1}L_{\mathbf{f}(p+1)}^r\mathbf{h}_m=\begin{bmatrix}h_{m,r}^1 & h_{m,r}^2 & h_{m,r}^3 & \mathbf{0} & h_{m,r}^4 & h_{m,r}^5 & \mathbf{0}\\ h_{m,r}^6 & h_{m,r}^7 & h_{m,r}^8 & \mathbf{0} & h_{m,r}^9 & h_{m,r}^{10} & \mathbf{0}\end{bmatrix} \quad (44)$$

where $h_{m,r}^1$, $h_{m,r}^2$, ... , $h_{m,r}^{10}$ are functions of x, y, θ, x_m and y_m and r is a positive integer. The positions of the

$h^{i}_{m,r}$ for all i are the same in (44) for all r. Using (39) – (42) $\mathbf{O}_{p+1}$ can be expressed by the following.

$$\mathbf{O}_{p+1}=\begin{bmatrix}\mathbf{O}_{p} & \mathbf{0} & \mathbf{0}\\ \mathbf{d}_{p}L^{2p+3}_{\mathbf{f}(p)}\mathbf{h}(p) & \mathbf{0} & \mathbf{0}\\ \mathbf{d}_{p}L^{2p+4}_{\mathbf{f}(p)}\mathbf{h}(p) & \mathbf{0} & \mathbf{0}\\ & \mathbf{d}_{p+1}L^{0}_{\mathbf{f}(p+1)}\mathbf{h}_{p+1} & \\ & \cdots\cdots\cdots & \\ & \mathbf{d}_{p+1}L^{2p+4}_{\mathbf{f}(p+1)}\mathbf{h}_{p+1} & \end{bmatrix} \tag{45}$$

$\mathbf{d}_{p}L^{2p+3}_{\mathbf{f}(p)}\mathbf{h}(p)$ and $\mathbf{d}_{p}L^{2p+3}_{\mathbf{f}(p)}\mathbf{h}(p)$ can be expanded in $\mathbf{d}_{p}L^{r}_{\mathbf{f}(p)}\mathbf{h}_{m}$, m=1...p and r=2p+3 and 2p+4 . For any given r and m in the above range, since n>1 $\mathbf{d}_{p}L^{r}_{\mathbf{f}(p)}\mathbf{h}_{m}$ terms can have only 5 non zero columns. It immediately follows from the Theorem 1 that $\mathbf{d}_{p}L^{r}_{\mathbf{f}(p)}\mathbf{h}_{m}$ for all m=1...p and r=2p+3 and 2p+4 terms will not contribute any independent rows in $\mathbf{O}_{p}$. This is because the examination of the rank of matrix $\mathbf{O}_{p}$ (with n-1 Lie derivatives) is adequate to determine the observability rank condition according to Theorem 1.

Since it is assumed that the observability matrix $\mathbf{O}_{p}$ is rank defficient by 3, we can find a set of linear transformations on columns corresponding to the vehicle position and those on landmark positions 1,2,...,p that make the columns corresponding to the vehicle pose zero. Since, the terms $\mathbf{d}_{p}L^{r}_{\mathbf{f}(p)}\mathbf{h}_{p}$, r=0,1,2,....,2p+4 contain nonzero elements at columns 1,2,3,2+2p,3+2p, the above assumption establishes that we can find column operations C_{v} on columns 1,2 and 3 and C_{2+2p} and C_{3+2p} on columns 2+2p and 3+2p respectively so that $\mathbf{d}_{p}L^{r}_{\mathbf{f}(p)}\mathbf{h}_{p}$, r=0,1,2,....,2p+4 terms (rows) contain zero elements at columns 1,2 and 3 in $\mathbf{O}_{p}$.

Furthermore, since the Lie derivative is a linear operator and by the equations (44) and (45) we see that $\mathbf{d}_{p+1}L^{r}_{\mathbf{f}(p+1)}\mathbf{h}_{p+1}$, r=0,1,2,....,2p+4 terms in $\mathbf{O}_{p+1}$ take the same form and structure as $\mathbf{d}_{p}L^{r}_{\mathbf{f}(p)}\mathbf{h}_{p}$, r=0,1,2,....,2p+4 except change in nonzero column positions and the fact that instead of x_{p}, y_{p} we have x_{p+1} and y_{p+1}. Thus, we can find column operations C_{v} on columns 1,2 and 3 and C_{4+2p} and C_{5+2p} on columns 4+2p and 5+2p respectively so that $\mathbf{d}_{p+1}L^{r}_{\mathbf{f}(p+1)}\mathbf{h}_{p+1}$, r=0,1,2,....,2p+4 terms (rows) contain zero elements at columns 1,2 and 3.

Now after the column operations are done on $\mathbf{O}_{p+1}$ all the $\mathbf{d}_{p+1}L^{r}_{\mathbf{f}(p+1)}\mathbf{h}_{p+1}$, r=0,1,2,....,2p+4 rows have only non zero columns at columns 4+2p and 5+2p. Hence, these rows can be reduced to only two independent rows.

Hence, it can be concluded that $\mathbf{d}_{p+1}L^{r}_{\mathbf{f}(p+1)}\mathbf{h}_{p+1}$ terms (for all r) add only two independent rows to $\mathbf{O}_{p+1}$.Thus, the rank of $\mathbf{O}_{p+1}$. is 2(p+1). Hence, by the principle of mathematical induction the rank deficiency remains the same for any positive integer p. Hence the observability matrix denoting the nonlinear observability of n landmark SLAM problem is rank deficient by 3.

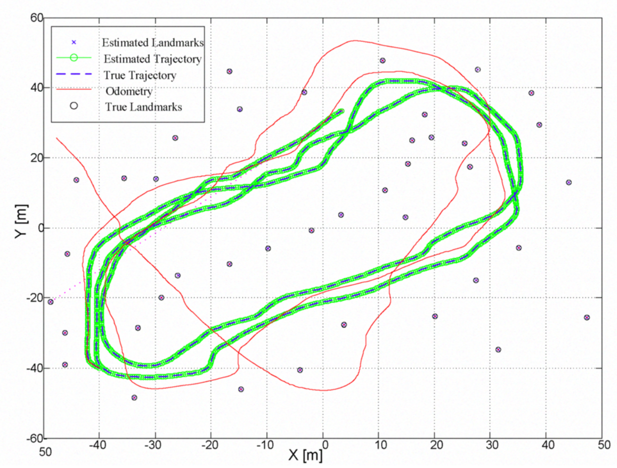

Fig. 1: Robot trajectory and estimated landmarks when observing two known landmarks

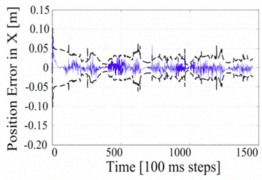

Fig. 2

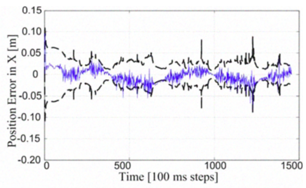

Fig. 3

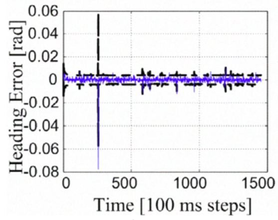

Fig. 4

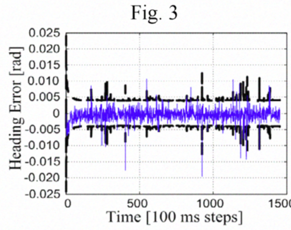

Fig. 5

Fig. 2 and Fig.4 are shown for SLAM with two known landmarks estimation. Fig. 3 and Fig. 5 are for SLAM with one known landmark estimation. Thin lines show the error and thick dotted lines show the 95% confidence regions of error bounds all for 100 Monte-Carlo runs.

Conjecture 6*: *The Observability matrix denoting the nonlinear observability of n landmark SLAM problem becomes full rank when observing range and bearing of two known landmarks.

Proof: The rank of $\mathbf{O}_{1}$ becomes full rank when observing two known landmarks [4]. Now assume that $\mathbf{O}_{p}$ is full rank. This assumption establishes that no linear transformations on columns of $\mathbf{O}_{p}$ exist to make any column a null vector. Now consider $\mathbf{O}_{p+1}$. Since the $\mathbf{d}_{p}L^{2p+3}_{\mathbf{f}(p)}\mathbf{h}(p)$ and $\mathbf{d}_{p}L^{2p+3}_{\mathbf{f}(p)}\mathbf{h}(p)$ can be made zero (shown above) the linearly independent 2p+3 rows (contributed by $\mathbf{O}_{p}$) can reduce all the non zero columns (columns 1,2 and 3) of $\mathbf{d}_{p+1}L^{r}_{\mathbf{f}(p+1)}\mathbf{h}_{p+1}$, r=0,1,2,....,2p+4 rows. Thus, $\mathbf{d}_{p+1}L^{r}_{\mathbf{f}(p+1)}\mathbf{h}_{p+1}$, r=0,1,2,....,2p+4 terms add only two non zero columns to $\mathbf{O}_{p+1}$. Hence the rank of $\mathbf{O}_{p+1}$ is 2p+3+2 (full rank). Thus, by the principle of mathematical induction, the result is true for all positive integers p.

Hence the rank of $\mathbf{O}_{p}$ always becomes full rank, when two distinct known landmarks are observed. Hence the SLAM problem is locally observable when two landmarks together with all the unknown landmarks are observed.

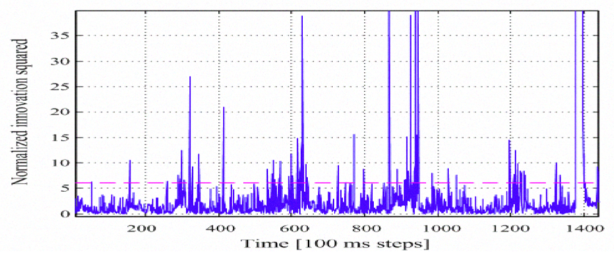

Fig. 6. Normalized X, Y position innovation squared for SLAM when two known landmarks are always observed. The 95% confidence bound is shown by thick dotted line and the normalized position innovation squared is shown by the thin lines.

IV. SIMULATIONS

Simulations are done to compare and contrast linear observability result and the nonlinear observability result assuming a simple car like robot traversing in a simulated environment (Fig. 1). An Extended Kalman Filter (EKF) SLAM approach detailed in [2] has been used to simulate a SLAM algorithm. In this simulation setup, a car like mobile robot navigating in an environment of 100x100m^2 area according to a specified set of speed and steering angle inputs while observing point features in the vicinity using a range bearing sensor is assumed.

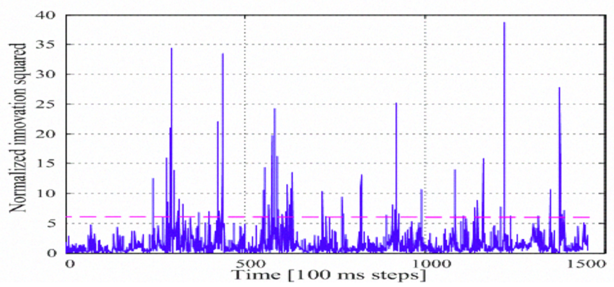

Fig. 7. Normalized X, Y position innovation squared for SLAM when one known landmark is always observed. The 95% confidence bound is shown by thick dotted line and the normalized position innovation squared is shown by the thin lines.

For a comparison between linear and nonlinear observability theories in the simulations; for case (a) it is assumed that one known landmark and all the landmarks included in the state are observed always (for linear observability analysis) and for case (b) it is assumed that two known landmarks and all the landmarks included in the state are observed always (for nonlinear observability analysis). Vehicle position and heading error plots and their 95% confidence intervals averaged over 100 Monte Carlo runs are also plotted for case (a) and (b) respectively in Fig. 2, Fig. 4, (case (a)) and Fig. 3, Fig.5 (Case (b)). Figures 2,3,4 and 5, 6 and 7 show both filters when observing either 1 known landmark and when observing two known landmarks works consistently. The normalized X and Y position innovation squared is plotted in Fig. 6 and 7 for observing two known landmarks and one known landmark (the results are averaged on 100 Monte-Carlo runs). These results show that Normalized innovation squared in both cases remain bounded by the 95% confidence limits thus showing that two filters perform consistently. Hence, it can be concluded based on the Monte-Carlo simulations of SLAM that the SLAM problem is observable when observing at least one known landmark in addition to all the other landmarks those are being estimated.

V. CONCLUSIONS

The observable and unobservable states are clearly identified from the original states of the SLAM state vector. This is an important result which enables one to establish the properties of SLAM in a rigorous way and also allows to design observers accurately. The proofs of the state observability of general *n* landmark SLAM problem using linear and nonlinear theories give a useful insight to the observability of the SLAM problem. The Monte-Carlo simulations conclude the fact that the linear observability theory gives adequate explanation of the observability as the evidence suggests that it ensures filter consistency. The results suggest that in order to have the local observability property of SLAM, it is necessary to observe at least two known landmarks and all the landmarks those (positions) are estimated at the same time. We can conclude based on the facts established that for the global observability of the SLAM problem observing one known landmark and all the landmarks that are being estimated is adequate.

ACKNOWLEDGMENT

This work is supported by the Rio Tinto Centre for Mine Automation.

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