CS120: Intro. to Algorithms and their Limitations Hesterberg & Vadhan

Problem Set 0

Harvard SEAS - Fall 2022 *Due: Wed Sep. 7, 2022 (11:59PM)*

The purpose of this problem set is to reactivate your skills in proofs and programming from CS20 and CS32/CS50. For those of you who haven't taken one or both those courses, the problem set can also help you assess whether you have acquired sufficient skills to enter CS120 in other ways and can fill in any missing gaps through self-study. Even for students with all of the recommended background, this problem set may still require a significant amount of thought and effort, so do not be discouraged if that is the case and do take advantage of the staff support in section and office hours.

For those of you who are wondering whether you should wait and take CS20 before taking CS120, we encourage you to also complete the CS20 Placement Self-Assessment. Some problems there that are of particular relevance to CS120 and are complementary to what is covered below are Problems 2 (counting), 4 (comparing growth rates), 9 (quantificational logic), and 12 (graph theory).

Written answers must be submitted in pdf format on Gradescope. Although LaTeX is not required, it is strongly encouraged. You may handwrite solutions so long as they are fully legible. The `ps0` directory, which contains your code for problems 1a and 1c, must be submitted separately to an autograder on Gradescope. Be sure to pull the starter code from the cs120 GitHub repository.

1. (Binary Trees) In the cs120 GitHub repository, we have given you a Python implementation of a binary tree data structure, as well as a collection of test trees built using this data structure. We specify a binary tree by giving a pointer to its *root*, which is a special *vertex* (a.k.a. *node*), and giving every vertex pointers to its *children* vertices and its *parent* vertex as well as an identifying *key*:

```
class BinaryTree:
    def __init__(self, root):
        self.root: BTvertex = root

class BTvertex:
    def __init__(self, key):
        self.parent: BTvertex = None
        self.left: BTvertex = None
        self.right: BTvertex = None
        self.key: int = key
        self.size: int = None
```

   In CS50, the concept of a Python `class` was not covered. Here, with `BinaryTree` and `BTvertex`, we are using them in the same way as a `struct` in C. An object `v` of the `BTvertex` class contains five attributes, which we list with the type of the object we expect to be named by each attribute (using the Python type annotation syntax). These attributes can be accessed as `v.parent`, `v.left`, `v.right`, `v.key`, and `v.size`. For example, `v.left.key` is the key associated with `v`'s left child. An object of the `BinaryTree` class contains only one attribute, which is the `BTvertex` object that is the root of our binary tree. You can create a `BinaryTree` object as follows:

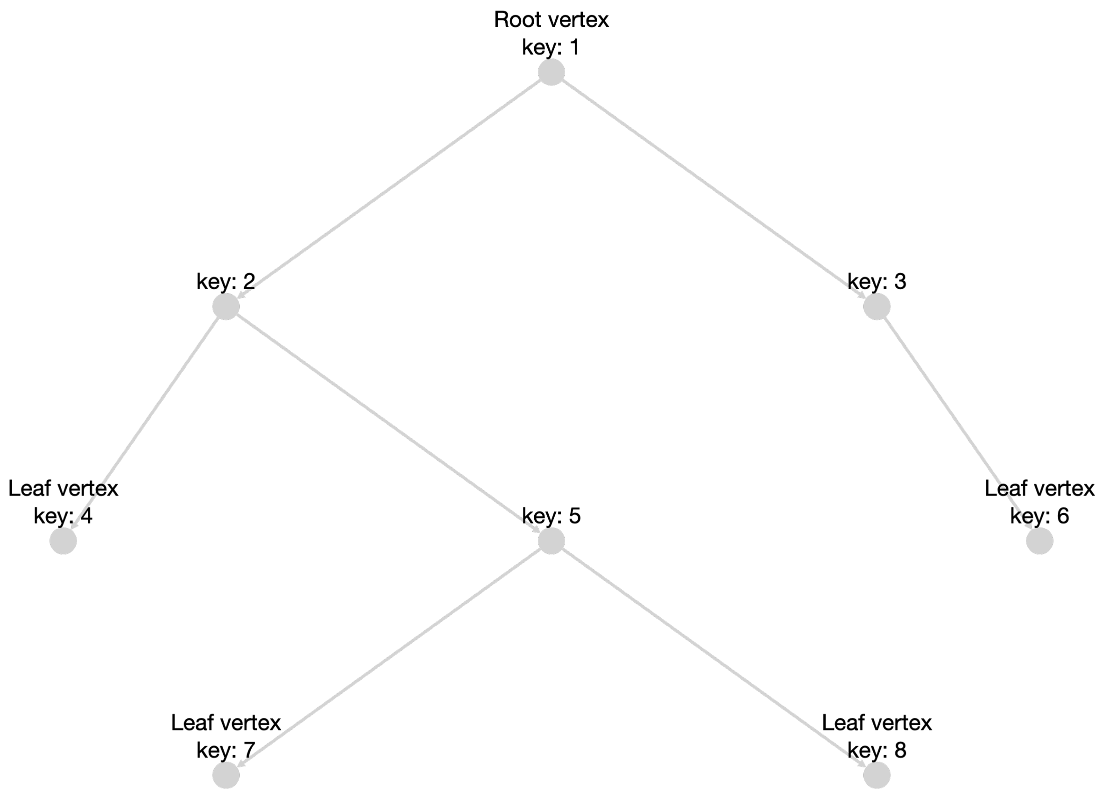
```
root = BTvertex(120)
tree = BinaryTree(root)
tree.root.left = BTvertex(121)
tree.root.right = BTvertex(124)
```

You can then print attributes of the newly created `BinaryTree` object:

```
print(tree.root.key)
>> 120
print(tree.root.left.key)
>> 121
```

Classes are more general than structs because they can also have private attributes and methods that operate on the attributes, allowing for object-oriented programming. However, you won't need that generality in this problem set.

Here is an instance `T` of `BinaryTree`:



A `BinaryTree T` contains only a pointer to its root vertex, `T.root`, which is required to satisfy `T.root.parent==None`. In the above example, the root is the vertex with key 1 (i.e. `T.root.key==1`). A binary tree vertex `v` can have zero, one, or two children, determined by which of `v.left` and `v.right` are equal to `None`. In the above example, the vertex `v` with key 3 has `v.left==None` but `v.right` is the vertex with key 6. A *leaf* is a vertex with zero children, i.e. `v.left==v.right==None`.

A vertex `w` is *descendent* of a vertex `v` if there is a sequence of vertices $\mathtt{v}_0, \mathtt{v}_1, \ldots, \mathtt{v}_k$, $k \in \mathbb{N}$ such that $\mathtt{v}_0 = \mathtt{v}$, $\mathtt{v}_k = \mathtt{w}$, and $\mathtt{v}_i \in \{\mathtt{v}_{i-1}.\mathtt{left}, \mathtt{v}_{i-1}.\mathtt{right}\}$ for $i = 1, \ldots, k$.[1] In the above example, the vertex with key 5 is a descendent of the root (with a path of length 2), but is not a descendent of the vertex with key 3. The sequence $\mathtt{v}_0, \mathtt{v}_1, \ldots, \mathtt{v}_k$ is called a *path* from `v` to `w` and k is the *distance* from `v` to `w`. Taking $k = 0$, we see that `v` is a descendent of itself.

The *vertex set* of a binary tree `T` consists of all of the descendents of `T.root`. The *size* of `T` is its number of vertices. The *height* of `T` is the largest distance from the root to a leaf. The above example has size 8 and height 3.

Given any vertex `v` in a tree, the *subtree* rooted at `v` consists of all of `v`'s descendents. Note that we can remove a subtree and turn it into a new tree `S` by setting `S.root=v` and `v.parent=None`.

For now, the `key` attribute serves to distinguish vertices from each other in our tests and help illustrate what the algorithms are doing. The `BTvertex` class also has a `size` attribute, which is initialized to `None` in all of the test instances; it will be filled in by the program you write in Part 1a.

An instance `T BinaryTree` is *valid* if it satisfies the following constraints:

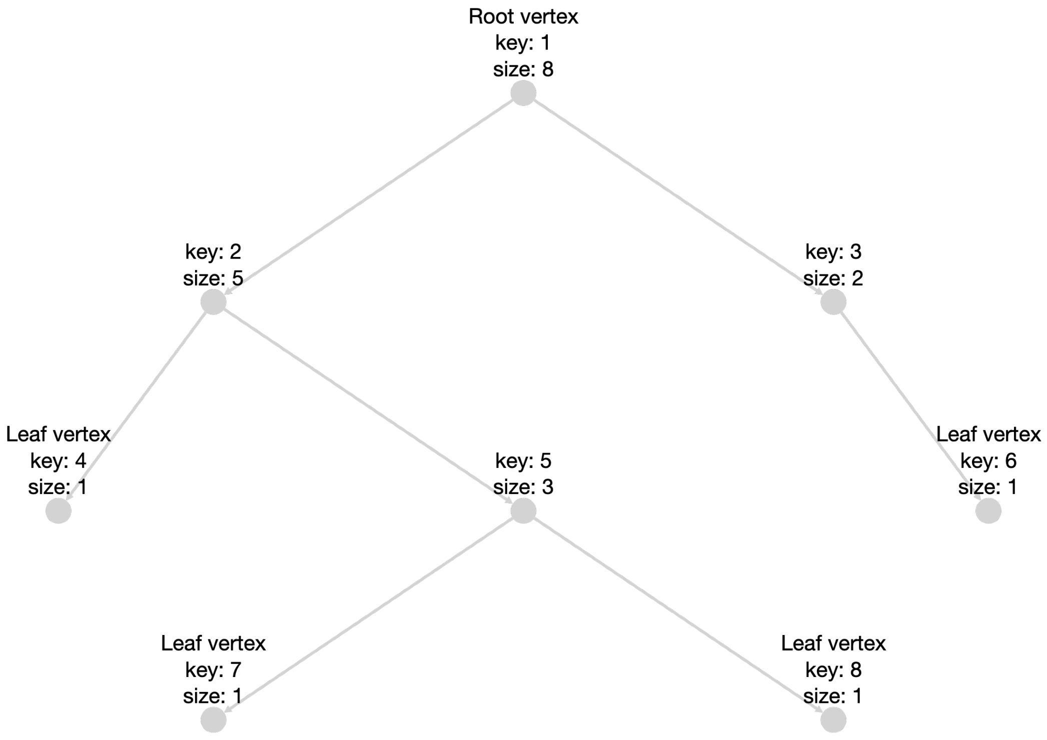
- `T.root.parent==None`
- `T` has finitely many vertices.
- No two vertices `v`, `w` of `T` share a child, i.e. $\{\mathtt{v.left}, \mathtt{v.right}\} \cap \{\mathtt{w.left}, \mathtt{w.right}\} = \emptyset$.

All of the test instances we provide are valid, and furthermore have the property that all of the vertices have distinct keys (which is something we often want, but not always).

(a) (recursive programming) Write a recursive program `calculate_sizes` that given a vertex `v` of a binary tree `T`, calculates the sizes of all of the subtrees rooted at descendents of `v`. After running your program on `T.root`, every vertex `v` in `T` should have `v.size` set to the size of the subtree rooted at `v`. (Recall that the size attributes are initialized to `None`.) We call the resulting tree a *size-augmented* tree.

For example, if `T` is the tree shown above, then calling `calculate_sizes(T.root)` should modify `T` to be the following size-augmented tree:

[1] $\mathbb{N}$ denotes the natural numbers $\{0, 1, 2, 3, \ldots\}$. Since we are computer scientists, we start counting at 0.

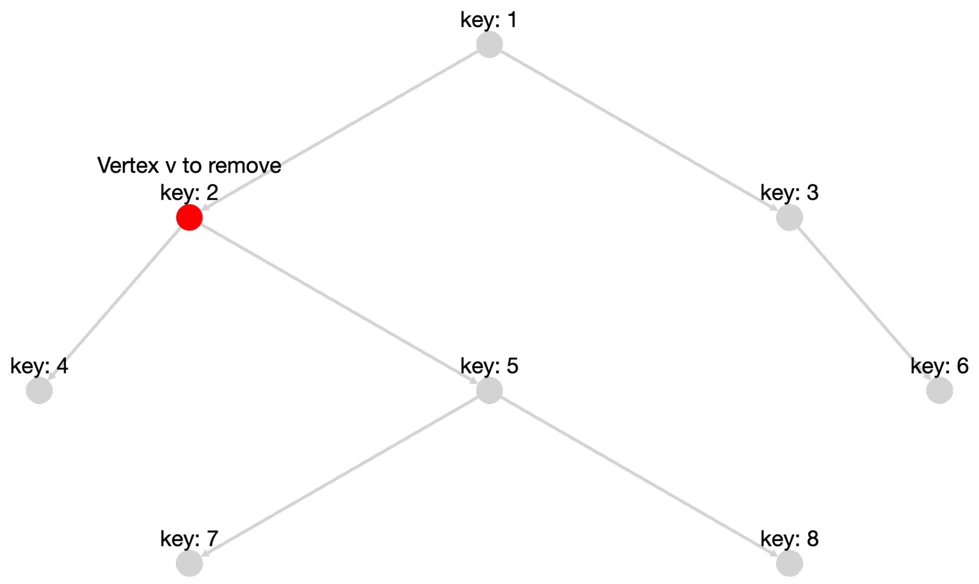


Your program should run in time $O(n)$ when given the root of a tree with n vertices. In a sentence or two, informally justify why your program has such a runtime.

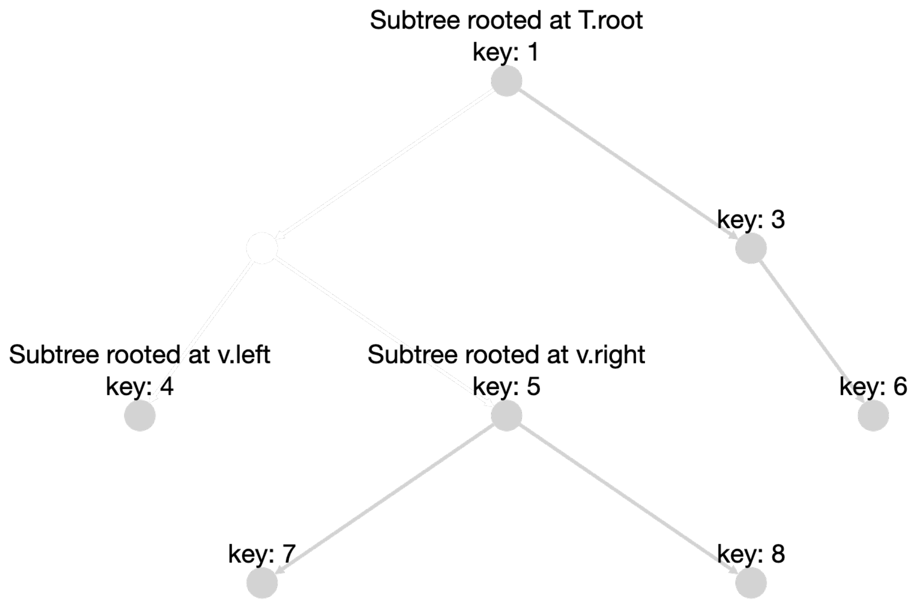
Solution. In the program, each node is touched at most three times - once when approaching from the parent, and once from each child. Also, every node must be touched at most once so that we can perform the basic operation of adding its child nodes' sizes. Therefore we perform a basic operation at least once and at most three times for every node; the runtime is $O(n)$. □

(b) (proofs by contradiction) Removing a vertex `v` from a tree `T` yields up to three disjoint trees: the subtree rooted at `v.left` (unless `v.left==None`), the subtree rooted at `v.right` (unless `v.right==None`), and a tree rooted at `T.root` consisting of all non-descendants of `v` (unless `T.root==v`). For example:

Before:



After:



Prove that in every tree `T` of size n, there exists a vertex `v` such that removing `v` from `T` results in disjoint trees that all have size at most $n/2$.

You may prove this however you like, but a recommended approach is to define a "potential function" ϕ on the vertices of the tree, by setting $\phi(\mathtt{v})$ to equal the size of the largest tree created by removing `v`. Let $\mathtt{v}^*$ be a vertex that minimizes the value of ϕ, i.e. $\mathtt{v}^*$ is a vertex such that $\phi(\mathtt{v}^*) \leq \phi(\mathtt{v})$ for all other vertices `v`. Then we want to prove that $\phi(\mathtt{v}^*) \leq n/2$. Prove this by contradiction. (Hint: try to show that either the parent or one of the children will have smaller potential. If you're feeling stuck, try drawing some pictures!)

Proof. Define $\phi, \texttt{v}, \texttt{v}^*$ as above. Assume for the sake of contradiction that $\phi(\texttt{v}^*) > n/2$. For any binary tree split this way, we create a parent subtree and two child subtrees. Let's say they have sizes T, L, R, where T is the parent and $T+L+R+1 = n$. Without loss of generality, we split this into two cases - $T \geq R, L$ and $L \geq R, T$ (as, R, L can be substituted without issue).

Suppose $T \geq L, R$. Then we examine the parent node of $\texttt{v}^*$, which we call $\texttt{v}'$. The sizes of our three new subtrees when we remove $\texttt{v}'$ instead of $\texttt{v}^*$ are $T-A, L+R+1, A-1$, where A represents some branch of T that becomes most of its own child node. If we can show that all three new sizes are less than T, we have shown that our choice of $\texttt{v}^*$ was not optimal, which is a contradiction. Obviously, $T - A < T$. Next, note that $T+L+R+1 = n \implies L+R+1 < n/2, T$ (from the fact that $T > n/2$). Lastly, $A < T \implies A - 1 < T$. Therefore $\texttt{v}^*$ is not optimal, contradiction.

Now suppose $L \geq T, R$. Then we examine the left and right children of L and call the one with a larger size $\texttt{v}'$. Now removing node $\texttt{v}'$ yields subtrees of size $T+R+1, A, L-A-1$. Clearly $A, L-A-1 < L$, so now we check $T+R+1$. If $L > n/2$, then we require $T+R+1 < n/2$; therefore, $T+R+1 < L$. Therefore, this is another contradiction.

Our cases being exhausted, we conclude that $\phi(\texttt{v}^*) \leq n/2$. □

(c) (from proofs to algorithms) Turn your proof from Part 1b into a Python program that given a root vertex `r` of a *size-augmented* tree `T` with n vertices finds a vertex `v` with $\phi(\texttt{v}) \leq n/2$. Your program should run in time $O(h)$ on all size-augmented trees of height h; again informally justify why your program has such a runtime. (Hint: try to repeatedly reduce the potential function by moving to children. Why don't we need to try moving to parents as in the previous proof?)

Solution. We traverse each height-row at most once (because we only move downwards in our algorithm). However, we do not know how far down the tree we must go; it could be up to $h/2$ rows. Therefore we conclude that our program runs on $O(h)$ time. □

2. (matchings and induction) Later in the course, we will study matching algorithms that are used in practice to match kidney donors to patients. The challenge in general is that some donors are incompatible with some patients (i.e. the patient's body is likely to reject the donated kidney). Suppose we are very lucky and have n donors and n patients where each donor d is incompatible with exactly one patient, denoted $\mathit{incomp}(d)$, and each patient p is incompatible with exactly one donor $\mathit{incomp}(p)$. (Incompatibility is symmetric, so $\mathit{incomp}(d) = p$ iff $\mathit{incomp}(p) = d$.) Let $f(n)$ be the number of ways, under these circumstances, of matching donors to patients so that each donor donates exactly one kidney to a compatible patient and each patient receives exactly one kidney from a compatible donor.

   (a) Show that $f(1) = 0$, $f(2) = 1$, and for all $n \geq 3$, we have

$$f(n) = (n-1) \cdot (f(n-1) + f(n-2)).$$

(Hint: let d be one of the donors, and consider all possible patients p with whom d could be matched. Then consider cases according to whether *incomp*(p) is matched with *incomp*(d) or not.)

Proof. This problem is isomorphic to that of counting derangements. Accordingly, I use similar reasoning. Suppose donor d matches with patient p. There are $n-1$ options for p (because 1 of the patients is incompatible with d). Now we examine two cases: either *incomp*(d) matches with *incomp*(p) or *incomp*(d) does not match with *incomp*(p). In the first case, we need to match $n-2$ patients with $n-2$ donors, where each patient is incompatible with one donor. This is just $f(n-2)$.

Next, we examine the second case - in this instance, there is one patient every remaining donor cannot match with - for every non-*incomp*(p) donor, that is their corresponding incompatible patient, and for the *incomp*(p) donor, it is *incomp*(d). Therefore this is $f(n-1)$. We can see that for any choice of p, there are $f(n-1)+f(n-2)$ ways to proceed; therefore, given that we have $n-1$ ways to choose p, we find

$$f(n) = (n-1)\cdot(f(n-1)+f(n-2)),$$

which is what we desired. □

(b) Prove by strong induction that for all $n \geq 2$, we have

$$\frac{n!}{3} \leq f(n) \leq \frac{n!}{2}.$$

Proof. First, at $n = 2$, we have

$$\frac{2!}{3}1 \leq \frac{2!}{2},$$

which is true. For $n = 3$, we calculate

$$f(3) = 2\cdot(1+0) = 2 \implies \frac{3!}{3} \leq 2 \leq \frac{3!}{2},$$

which is also true. Because our recursion only uses $n-1$ and $n-2$ terms, we can proceed to the inductive steps.
We begin with the first inequality. Assume $\frac{n!}{3} \leq f(n)$. Then we see that

$$f(n+1) = n\cdot(f(n)+f(n-1)) = nf(n)+nf(n-1) \geq n\cdot\frac{n!}{3} + n\cdot\frac{(n-1)!}{3} = \frac{(n+1)!}{3},$$

as desired. With the second inequality, we assume $\frac{n!}{2} \geq f(n)$. Then

$$f(n+1) = nf(n) + f(n-1) \leq n\cdot\frac{n!}{2} + n\cdot\frac{(n-1)!}{2} = \frac{(n+1)!}{2},$$

which is the other part we need. So we are done. □