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PHYS 506 - General Physics III Lab

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Activity Six Extra Credit Problem

We have two statistical distributions, defined as Gaussians. Let the two distributions be:

$$N_1(x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(x-\mu_1)^2}{2\sigma_1}}$$

$$N_2(x) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{(x-\mu_2)^2}{2\sigma_2}}$$

Given these two normalized distributions, we are trying to find the probability of finding x.

This is the normalized product of the two:

$$N_3 \propto N_1 N_2 = \frac{C}{2\pi\sigma_1\sigma_2} e^{-\left[\frac{(x-\mu_1)^2}{2\sigma_1^2} + \frac{(x-\mu_2)^2}{2\sigma_2^2}\right]}$$

where C is some arbitrary normalization constant.

We want N_3 to eventually be in the form of another standard distribution, such that

$$N_3 = \frac{C}{\sigma_3} e^{-\frac{(x-\mu_3)^2}{2\sigma_3^2}}$$

To do this, we can manipulate the term in the exponent, $\left[\frac{(x-\mu_1)^2}{2\sigma_1^2} + \frac{(x-\mu_2)^2}{2\sigma_2^2}\right]$.

$$\begin{aligned}
\frac{(x-\mu_1)^2}{2\sigma_1^2} + \frac{(x-\mu_2)^2}{2\sigma_2^2} &= \frac{\sigma_2^2(x^2 - 2\mu_1 x + \mu_1^2) + \sigma_1^2(x^2 - 2\mu_2 x + \mu_2^2)}{2\sigma_1^2\sigma_2^2} \\
&= \frac{(\sigma_1^2 + \sigma_2^2)x^2 - 2(\mu_1\sigma_2^2 + \mu_2\sigma_1^2)x + (\mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2)}{2\sigma_1^2\sigma_2^2} \\
&= \left(x^2 - 2\frac{\mu_1\sigma_2^2 + \mu_2\sigma_1^2}{\sigma_1^2 + \sigma_2^2}x + \frac{\sigma_2^2\mu_1^2 + \sigma_1^2\mu_2^2}{\sigma_1^2 + \sigma_2^2}\right) = 0
\end{aligned}$$

Then that becomes

$$
\begin{aligned}
x^2 - 2\frac{\mu_1\sigma_2^2 + \mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}x &= -\frac{\sigma_2^2\mu_1^2 + \sigma_1^2\mu_2^2}{\sigma_1^2+\sigma_2^2} \\
x^2 - 2\frac{\mu_1\sigma_2^2 + \mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}x + \left(\frac{\mu_1\sigma_2^2 - \mu_2\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)^2 &= \frac{\mu_1\sigma_2^2 + \mu_2\sigma_1^2 - \sigma_1^2\mu_1^2 - \sigma_1^2\mu_2^2}{\sigma_1^2+\sigma_2^2} \\
\left(x - \frac{\mu_1\sigma_2^2 + \mu_2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2 &= \mu_1 + \mu_2 - \mu_1^2 - \mu_2^2
\end{aligned}
$$

Now we can substitute this back into our expression for N_3:

$$
\begin{aligned}
N_3(x) &= \frac{1}{2\pi\sigma_1\sigma_2}e^{-\left[\left(x-\frac{\mu_1\sigma_2^2+\mu_2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2-\mu_1-\mu_2+\mu_1^2+\mu_2^2\right]/2\sigma_1^2\sigma_2^2} \\
&= \frac{1}{\sigma_1\sigma_2}e^{-\frac{\left(x-\frac{\mu_1\sigma_2^2+\mu_2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2}{2\sigma_1^2\sigma_2^2}} \times \frac{1}{2\pi}e^{\frac{\mu_1+\mu_2-\mu_1^2-\mu_2^2}{2\sigma_1^2\sigma_2^2}} \\
&= \frac{1}{\sigma_1\sigma_2}e^{-\left(x-\frac{\mu_1\sigma_2^2+\mu_2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)^2/2\sigma_1^2\sigma_2^2} \times C
\end{aligned}
$$

which is a new distribution with $\sigma_1\sigma_2 = \sigma_3$, $\mu_3 = \frac{\mu_1\sigma_2^2+\mu_2\sigma_2^2}{\sigma_1^2+\sigma_2^2}$, and C defined such that the distribution is normalized.