Experiment Activity 2: Measuring the Electron Charge-to-Mass Ratio

Luke Abanilla, Eben Quenneville, Augustus Vigorito

Data & Measurements

Goal: e/m_e

Produce free electrons by heating filament

Apply current I to coil to produce $B = \left(\frac{4}{5}\right)^{3/2} \frac{\mu_0 N I}{R}$

Lorentz force: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) = m\vec{a}$

assume $\vec{E} = 0$: $\vec{F} = q\vec{v} \times \vec{B}$

Creates uniform circular motion, emits light due to energy; $a = v^2/r$

Newton's third $m_e a = \frac{m_e v^2}{r} = qvB \Rightarrow \frac{q}{m_e} = \frac{v}{rB}$ → v: calculate by measuring r; use voltage: $q_e V = \frac{1}{2} m_e v^2 \Rightarrow v = \sqrt{\frac{2 q_e V}{m_e}}$, maybe rewrite in q_e/m_e. B: measure

Instrument: use ruler inside to get r

$$\frac{q}{m_e} = \frac{2V}{\left(\frac{4}{5}\right)^3 \left(\frac{\mu_0 N I}{R}\right)^2 r^2}$$

Problem 1. Data Collection

Helmholtz Coil Measurements

Internal Coil Diameter D_{int}	External Coil Diameter D_{ext}
28.5 ± 0.05 c	30.5 ±0.05 cm
28.5	30.8
28.5	30.9
28.5	30.9
28.5	30.7

Type B = 0.05 cm

$D_{best} = 29.63$ cm $= 0.2963$ m

Average all of your coil diameter measurements together and determine the total coil diameter uncertainty (from the Type-A and Type-B uncertainties in your measurements above). Use this average coil diameter, D, and total coil diameter uncertainty, δD, to compute the coil radius, $R = D/2$, and the coil radius uncertainty, $\delta R = \delta D/2$. Give your result for $R \pm \delta R$ below. You will need this value in calculations to follow.

Helmholtz Coil Radius: $R \pm \delta R = 14.8 \pm 0.1$; $N = 130$/coil

$\delta D_{tot} = 2\sqrt{\delta D_A^2 + \delta D_B^2} = 0.1$

$\delta D_A = \sqrt{\left(\frac{1}{n-1} \sum_{i=1}^{n} (x_i - x_{best})^2\right)} = 0.075$ cm

$\delta D_B = 0.05$ cm

Note: You will need to do Problems 3 and 5 to fill out the `Magnetic Field` and `Charge-to-Mass-Ratio` columns of the `Experiment Data` tables below.

Suggestion: It will probably be most efficient to first measure and record the coil currents and beam loop diameters (with uncertainties), then type up these measurements into a `*.txt` or `*.csv` file, then load those values into a Python program and write code to compute the magnetic field and charge-to-mass ratio values (with uncertainties).

Experiment Data (1)

δ 5%

Accelerating Voltage:
$V \pm \delta V = (200 \pm 2)\ V$

Coil Current $I \pm \delta I$	Beam Loop Diameter $d \pm \delta d$ cm	Magnetic Field $B \pm \delta B$	Charge-to-Mass Ratio $e/m_e \pm \delta(e/m_e)$ ($\times 10^{11}$ $\frac{C}{kg}$)
2.30	5.0 ± 0.3	1.81 ± 0.09 mT	1.9 ± 0.1
2.10	5.5 ± 0.3	1.66 ± 0.08 mT	1.9 ± 0.1
1.93	6.0 ± 0.3	1.52 ± 0.08 mT	1.9 ± 0.1
1.80	6.5 ± 0.3	1.42 ± 0.07 mT	1.88 ± 0.09
1.66	7.0 ± 0.3	1.31 ± 0.07 mT	1.9 ± 0.1
1.55	7.5 ± 0.3	1.22 ± 0.06 mT	1.9 ± 0.1
1.47	8.0 ± 0.3	1.16 ± 0.06 mT	1.86 ± 0.09
1.38	8.5 ± 0.3	1.09 ± 0.05 mT	1.87 ± 0.09
1.30	9.0 ± 0.3	1.03 ± 0.05 mT	1.88 ± 0.09
1.24	9.5 ± 0.3	0.978 ± 0.05 mT	1.85 ± 0.09
1.18	10.0 ± 0.3	0.931 ± 0.05 mT	1.85 ± 0.09

Experiment Data (2)

δ 5%

Accelerating Voltage:
$V \pm \delta V = (300 \pm 2)\ V$

Coil Current $I \pm \delta I$	Beam Loop Diameter $d \pm \delta d$ cm	Magnetic Field $B \pm \delta B$ (mT)	Charge-to-Mass Ratio $\lvert e/m_e \pm \delta(e/m_e) \rvert \times 10^{11}\ \frac{C}{kg}$
1.46	10.0 ± 0.3	1.15 ± 0.06 mT	1.81 ± 0.09
1.53	9.5 ± 0.3	1.21 ± 0.06	1.82 ± 0.09
1.61	9.0 ± 0.3	1.27 ± 0.06	1.84 ± 0.09
1.70	8.5 ± 0.3	1.34 ± 0.07	1.85 ± 0.09
1.80	8.0 ± 0.3	1.42 ± 0.07	1.86 ± 0.09
1.92	7.5 ± 0.3	1.57 ± 0.08	1.86 ± 0.09
2.05	7.0 ± 0.3	1.62 ± 0.08	1.87 ± 0.09
2.21	6.5 ± 0.3	1.74 ± 0.09	1.87 ± 0.09
2.38	6.0 ± 0.3	1.88 ± 0.09	1.89 ± 0.09
2.60	5.5 ± 0.3	2.1 ± 0.1	1.89 ± 0.09
2.84	5.0 ± 0.3	2.2 ± 0.1	1.9 ± 0.1

Accelerating Voltage:

$V \pm \delta V =$ (400 ± 2) V

Coil Current $I \pm \delta I$	Beam Loop Diameter $d \pm \delta d$ cm	Magnetic Field $B \pm \delta B$ (mT)	Charge-to-Mass Ratio $\lvert e/m_e \pm \delta(e/m_e) \rvert \times 10^{11}$ $\frac{C}{kg}$
1.69	10.0 ± 0.3	1.33 ± 0.07	1.80 ± 0.09
1.76	9.5 ± 0.3	1.39 ± 0.07	1.84 ± 0.09
1.86	9.0 ± 0.3	1.47 ± 0.07	1.83 ± 0.09
1.97	8.5 ± 0.3	1.55 ± 0.08	1.83 ± 0.09
2.08	8.0 ± 0.3	1.64 ± 0.08	1.86 ± 0.09
2.21	7.5 ± 0.3	1.74 ± 0.09	1.87 ± 0.09
2.37	7.0 ± 0.3	1.87 ± 0.09	1.87 ± 0.09
2.55	6.5 ± 0.3	2.0 ± 0.1	1.87 ± 0.09
2.76	6.0 ± 0.3	2.2 ± 0.1	1.87 ± 0.09
2.99	5.5 ± 0.3	2.4 ± 0.1	1.9 ± 0.1

Experiment Data (4)

Accelerating Voltage:
$V \pm \delta V =$ (500 ± 2) V

Coil Current $I \pm \delta I$	Beam Loop Diameter $d \pm \delta d$ cm	Magnetic Field $B \pm \delta B$ mT	Charge-to-Mass Ratio $\lvert e/m_e \pm \delta(e/m_e) \rvert \times 10^{11}$ $\frac{C}{kg}$
1.72	11.0 ± 0.3	1.36 ± 0.07	1.79 ± 0.09
1.80	10.5 ± 0.3	1.42 ± 0.07	1.80 ± 0.09
1.88	10.0 ± 0.3	1.48 ± 0.07	1.82 ± 0.09
1.97	9.5 ± 0.3	1.55 ± 0.08	1.85 ± 0.09
2.07	9.0 ± 0.3	1.63 ± 0.08	1.85 ± 0.09
2.20	8.5 ± 0.3	1.74 ± 0.09	1.84 ± 0.09
2.33	8.0 ± 0.3	1.84 ± 0.09	1.85 ± 0.09
2.47	7.5 ± 0.3	1.9 ± 0.1	1.87 ± 0.09
2.65	7.0 ± 0.3	2.1 ± 0.1	1.87 ± 0.09
2.85	6.5 ± 0.3	2.3 ± 0.1	1.87 ± 0.09

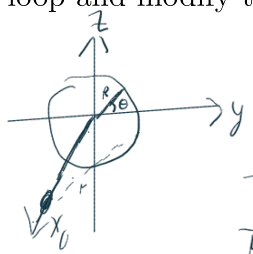
Problem 2. Derivation of Magnetic Field Strength

Derive the expression for the magnetic field due to the Helmholtz coils:

(a.) Use the Biot-Savart Law to derive the general expression for the magnetic field at a point $P = (x,\ 0,\ 0)$ along the axis of symmetry of a circular current-carrying coil with N turns which lies in the y-z-plane and whose center is at the origin $\mathcal{O} = (0,\ 0,\ 0)$. Your result should be

$$\vec{B} = \frac{\mu_0 N I R^2}{2(x^2 + R^2)^{3/2}}\ \hat{x}$$

where $\mu_0 = 1.256\,637\,062\,12(19) \times 10^{-6}\ H/m \approx 4\pi \times 10^{-7}\ H/m$ is the magnetic constant (a.k.a., vacuum permeability), I is the current flowing through the wire, R is the radius of the coil, x is the distance from the center of the coil along its axis of symmetry to the point at which the field is evaluated, and $\hat{x}$ is the unit vector in the x-direction. Hint: Start by deriving the magnetic field expression due to a single current-carrying loop and modify this expression to get the expression for a solenoid of N turns.



Let $\vec{R}$ be the vector in the y-z plane from the origin to $d\vec{l}$, given by $\vec{R} = \langle 0, R\cos\theta, R\sin\theta \rangle$ in the depicted geometry.

Let $\vec{x} = \langle x_0, 0, 0 \rangle$ be the vectot from the origin to the point x_0.

The radial vector from x_0 to a point on the ring is given by

$\vec{r} = \vec{x}_0 - \vec{R} = \langle x_0, -R\cos\theta, -R\sin\theta \rangle$, $|\vec{r}|^3 = (x_0^2 + R^2)^{3/2}$

$d\vec{l} = \frac{d}{d\theta}\vec{R} = \langle 0, -R\sin\theta, R\cos\theta \rangle$.

Now to evaluate the Biot-Savart equation, we first evaluate

$$d\vec{l} \times \vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & -R\sin\theta & R\cos\theta \\ x_0 & -R\cos\theta & -R\sin\theta \end{vmatrix} = R^2\hat{x} + x_0 R\cos\theta\,\hat{y} + x_0 R\sin\theta\,\hat{z}.$$

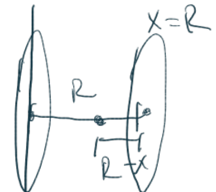
Note: $\hat{y}$ and $\hat{z}$ components cancel by symmetry.

N → number of coils, assumed infinitely thin

$$\vec{B} = \frac{\mu_0 I}{4\pi|\vec{r}|^3}\int_0^{2\pi N} d\vec{l} \times \vec{r} = \frac{\mu_0 I}{4\pi|\vec{r}|^3}(2\pi N R^2)\hat{x} \boxed{= \frac{\mu_0 I N R^2}{2(x_0^2 + R^2)^{3/2}}\hat{x}}$$

(b.) Now consider a pair of identical coils connected in series. Think of the coils in a coordinate system such that $x = 0$ and $x = R$ represent the centers of the two coils. For the case where the current through both coils is flowing in the same direction, the general expression for the total magnetic field along the axis of symmetry can be derived using the Biot-Savart law and the superposition principle. Show that this magnetic field is given by,

$$B = \frac{\mu_0 N I R^2}{2}\left(\frac{1}{(R^2 + x^2)^{3/2}} + \frac{1}{(R^2 + (R - x)^2)^{3/2}}\right).$$



$\vec{B} = \vec{B} + \vec{B}$

$B_{net} = B_1 + B_2$ $\Big|_{x=0}$

$$= \frac{\mu_0 N I R^2}{2(x^2+R^2)^{3/2}} + \frac{\mu_0 N I R^2}{2((R-x)^2+R^2)^{3/2}}$$

$$\boxed{= \frac{\mu_0 N I R^2}{2}\left(\frac{1}{(x^2+R^2)^{3/2}} + \frac{1}{((R-x)^2+R^2)^{3/2}}\right)}$$

(c.) Assume that the current through each coil is flowing in the same direction. Starting from the result of part (b.), show that the magnetic field at $x = R/2$ is given by

$$B = \left(\frac{4}{5}\right)^{3/2} \frac{\mu_0 N I}{R}.$$

$$\vec{B}_{net} = \frac{\mu_0 N I R^2}{2}\left(\frac{1}{(x^2+R^2)^{3/2}} + \frac{1}{((R-x)^2+R^2)^{3/2}}\right)$$

$$= \frac{\mu_0 N I R^2}{2}\left(\frac{1}{\left(\left(\frac{R}{2}\right)^2+R^2\right)^{3/2}} + \frac{1}{\left(\left(R-\frac{R}{2}\right)^2+R^2\right)^{3/2}}\right)$$

$$= \frac{\mu_0 N I R^2}{2}\left(\frac{1}{\left(\frac{5R^2}{4}\right)^{3/2}} + \frac{1}{\left(\frac{5R^2}{4}\right)^{3/2}}\right)$$

$$= \frac{\mu_0 N I R^2}{2}\left(\left(\frac{4}{5R^2}\right)^{3/2} + \left(\frac{4}{5R^2}\right)^{3/2}\right)$$

$$= \frac{\mu_0 N I R^2}{R^3}\left(\frac{4}{5}\right)^{3/2}$$

$$\boxed{= \frac{\mu_0 N I}{R}\left(\frac{4}{5}\right)^{3/2}}$$

Problem 3. Finding δB

Assuming measured values (with uncertainties) of coil current ($I \pm \delta I$) and coil radius ($R \pm \delta R$), use the uncertainty propagation rules to derive an expression for the uncertainty δB in the magnetic field given by

$$B = \left(\frac{4}{5}\right)^{3/2} \frac{\mu_0 N I}{R}.$$

$$\delta B = \sqrt{\left(\frac{\partial B}{\partial I}\delta I\right)^2 + \left(\frac{\partial B}{\partial R}\delta R\right)^2}$$

$$\frac{\partial B}{\partial I} = \left(\frac{4}{5}\right)^{\frac{3}{2}} \frac{\mu_0 N}{R} \quad ; \quad \frac{\partial B}{\partial R} = \left(\frac{4}{5}\right)^{\frac{3}{2}} \left(-\frac{\mu_0 N I}{R^2}\right)$$

$$\Rightarrow \delta B = \sqrt{\left(\frac{4}{5}\right)^3 \left(\frac{\mu_0 N}{R}\right)^2 (\delta I)^2 + \left(\frac{4}{5}\right)^3 \left(-\frac{\mu_0 N I}{R^2}\right)^2 (\delta R)^2}$$

Problem 4. Deriving the Mathematical Expression for e/m

In the `Principle of Operation` section of the Charge-to-Mass Ratio Lab Manual, the derivation of the relation for e/m_e is outlined. Work through this derivation outline and fill in the mathematical details (using Newton's laws, Conservation of Energy, doing all the algebra, etc.) to give a full, step-by-step derivation of the relation

$$\frac{e}{m_e} = \frac{2V}{r^2 B^2}$$

.

Potential energy converted to kinetic: $qV = \frac{1}{2} m_e v^2 \Rightarrow v = \sqrt{\frac{2qV}{m_e}}$

Electron beam $\perp$ B field generated by coils, no E field

$\longrightarrow$ Lorentz force $= qvB$

Circular motion of electrons $\to$ $qvB = \frac{m_e v^2}{r} \Rightarrow qB = \frac{m_e v}{r}$

plugging in $v = \sqrt{\frac{2qV}{m_e}}$: $qB = \frac{m_e}{r}\sqrt{\frac{2qV}{m_e}} \Rightarrow q^{\cancel{2}} B^2 = \frac{m_e^{\cancel{2}}}{r^2} \frac{2qV}{\cancel{m_e}} \Rightarrow qB^2 = \frac{2 m_e V}{r^2}$

$\Rightarrow \frac{q}{m_e} = \frac{2V}{r^2 B^2}$

Problem 5. Finding $\delta\,(\mathbf{e/m})$

Assuming measured values (with uncertainties) of accelerating voltage ($V \pm \delta V$), beam loop radius ($r \pm \delta r$), and magnetic field ($B \pm \delta B$) derive an expression for the uncertainty in e/m_e using the relation

$$\frac{e}{m_e} = \frac{2V}{r^2B^2} \qquad \text{let } Q = \frac{e}{m_e} \quad \text{(for my sanity)}$$

with the uncertainty propagation rules.

$$\delta Q = \sqrt{\left(\frac{\partial Q}{\partial V}\delta V\right)^2 + \left(\frac{\partial Q}{\partial r}\delta r\right)^2 + \left(\frac{\partial Q}{\partial B}\delta B\right)^2}$$

$$\frac{\partial Q}{\partial V} = \frac{2}{r^2B^2}, \quad \frac{\partial Q}{\partial r} = \frac{-4V}{r^3B^2}, \quad \frac{\partial Q}{\partial B} = \frac{-4V}{r^2B^3}$$

$$\Rightarrow \delta Q = \sqrt{\left(\frac{2}{r^2B^2}\delta V\right)^2 + \left(\frac{-4V}{r^3B^2}\delta r\right)^2 + \left(\frac{-4V}{r^2B^3}\right)^2}$$

$$= \sqrt{\frac{4}{r^4B^4}\delta V^2 + \frac{16V^2}{r^6B^4}\delta r^2 + \frac{16V^2}{r^4B^6}\delta B^2}$$

$$= \frac{2}{r^2B^2}\sqrt{\delta V^2 + \frac{4}{r^2}\delta r^2 + \frac{4}{B^2}\delta B^2}$$

Problem 6. Estimating e/m From The Data

Appropriately use, and combine as necessary, the data collected in Problem 1, along with the expressions for $B \pm \delta B$ and $e/m_e \pm \delta(e/m_e)$ that you derived in Problems 2, 3, 4, & 5, to determine an estimated value, with uncertainty, for $e/m_e \pm \delta(e/m_e)$. Show and explain your work.

To calculate the best measurement for $\frac{e}{m_e}$, we use the following:

$$\frac{e}{m_e}\text{best} = \frac{\sum_{j=1}^{m} w_j \left(\frac{e}{m_e}\right)_j}{\sum_{j=1}^{m} w_j},$$

which is a sum over all the independent measurements.

and $w_j = \frac{1}{\delta(e/m_e)^2}$.

Doing this with Python yields $\frac{e}{m_e}_{best} = -1.86 \cdot 10^{11}\ \frac{C}{Kg}$

For uncertainty, we use only the weights:

$$\delta(e/m_e)_{total} = \frac{1}{\sqrt{\sum_{j=1}^{m} w_j}} = 0.01 \cdot 10^{11}\ \frac{C}{Kg}$$

Our final measurement would be

$$\frac{e}{m_e} = (-1.86 \pm 0.01) \cdot 10^{11}\ \frac{C}{Kg}$$

Problem 7. Determining Measurement Accuracy

According to the National Institute of Standards and Technology (NIST), the accepted value of the electron charge-to-mass ratio is

$$e/m_e = (-1.758\ 820\ 010\ 76 \pm 0.000\ 000\ 000\ 53) \times 10^{11}\ C/kg$$

(See: `https://physics.nist.gov/cgi-bin/cuu/Value?esme`).

(a.) Does the value of e/m_e which you found in Problem 6 agree with the accepted value reported by NIST? What is the percent difference between your value for e/m_e and the accepted value? Show your work!

$$\left|\frac{\frac{e}{m_e}\text{best} - \frac{e}{m_e}\text{actual}}{\frac{e}{m_e}\text{actual}}\right| \cdot 100 = \left|\frac{-1.86\cdot 10^{11} + 1.75882001076\cdot 10^{11}}{1.75882001076\cdot 10^{11}}\right| \cdot 100 = 5.75\ \%\ \text{error}$$

To calculate agreement, we use $|x_2 - x_1| \overset{?}{<} |\delta x_1 + \delta x_2|$. In this case, $|(-1.86 + 1.75882001076)\cdot 10^{11}| \not< |(0.01 + 0.00000000053)\cdot 10^{11}|$, so there is disagreement in our measurement and the actual NIST value.

(b.) What does this tell you about your measurements?

/ Our measurements aren't too far off, as indicated by the percent error. However, we had significant uncertainty in in our calculations that imply experimental disagreement.

Design Considerations

Problem 8. Improving the Experiment

Use the work you did in the previous problems to determine the dominant source(s) of uncertainty in your measurements. Answer the following:

(a.) Rank your measurement uncertainties from largest to smallest. What was the dominant source of uncertainty?

$\delta R = 0.1$ cm $\quad \delta$ Beam D $= 0.3$ cm $\quad \delta V \approx 20V$ (5% of $\overbrace{\frac{500+200}{2}}^{\text{average}}$ V)

$\delta \frac{e}{m_e} = 1 \cdot 10^9 \frac{C}{kg} \quad \delta I \approx 0.1A$ (5% of 2A)

So: $\delta \frac{e}{m_e} > \delta V > \delta$ Beam D $> \delta I > \delta R$. The biggest uncertainty in a directly measured quantity was in voltage.

(b.) Given this information, what could you change about the experimental apparatus to improve your measurements?

We could improve the voltage source to have lower type B uncertainty, thus improving our measurements.

(c.) How would you need to modify the apparatus if you replaced the Helmholtz coils which are currently on the device with new coils which have a diameter of 1 ft and 300 $turns$?

1 ft $\approx$ 30.48 cm, which is very close to our existing measurement for the external diameter of the Helmholtz coils. The bigger effect would be increasing the number of turns. This would produce a stronger magnetic field by a factor of over 2. The result would be smaller radius for the beam. To account for these smaller radii, it would be wise to increase the precision of the ruler inside the apparatus.