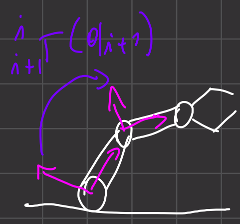
19/02/24

In this course we will mainly focus on manipulators



We saw that we could assign variables to joints

$$\underline{q}(t) = \begin{bmatrix} q_1(t) \\ \vdots \\ q_n(t) \end{bmatrix}$$

in order to control the robot.

There are two important entities:

$$\begin{cases} {}^{i}_{i+1}R\,(q_{i+1}) \\ r_{i+1/i}\,(q_{i+1}) \end{cases}$$

this is what we called forward geometry

In order to control the robot we needed to use the inverse geometric model.

There is always a solution to the forward geometric model.
In the inverse problem we are in trouble if the robot is redundant.
We can have infinite many solutions.

We also have to keep into consideration what happens when time passes.

Typically we have an end effector attached to the end-effector of the robot.

q variables change in time so I might be interested by the trajectory generated by the end effector.

So we have two steps:

- What happens if for a given configuration at a given time instant I assign a set of joint velocities:

$$\dot{q}(t) = \begin{bmatrix} \dot{q}_1(t) \\ \vdots \\ \dot{q}_n(t) \end{bmatrix}$$

If I can compute the velocities then I can compute the angular velocity of the ee wrt any frame of interest.
We can also compute also the linear velocity

Forward kinematic problem

$$\begin{bmatrix} {}^{\phi}\omega_{e/\phi}(t) \\ {}^{\phi}v_{e/\phi}(t) \end{bmatrix} = {}^{\phi}J_{e/\phi}(q)\,\dot{q}(t)$$

input

output

linear transformation

N.B These are all function of time

Computing this solves the forward kinematic problem

Inverse Kinematic Problem

Here control comes clearly into play.

Now we have to revert the problem.

In this case I want to achieve a particular angular velocity at the EE.

$$\dot{X}^* = \begin{bmatrix} \omega^*(t) \\ v^*(t) \end{bmatrix} = \overbrace{J(q)\,\underbrace{\dot{q}(t)}_{\text{output}}}$$

Input

this is the forward model of the robot. Not software

The output is not explicitly expressed.

This expression falls in this type of formulae: $Ax = y$

Now we have to follow rules depending on the shape of J

1) Square → Here you get the solution (excluding sing)
2) Flat (more columns than 6) → Infinite solutions
3) Tall (less columns than 6) → You might not find the solution
   ↳ This is the typical case of mobile robots

In the case of square problems we can compute:

> If square matrix = $\dot{q}(t) = J^{-1}x(t)$

> a flat matrix = We have more solution (the difference between rows and cols number)

The simplest way of solving this is using the least square solution.

$$\dot{q}(t) = \underbrace{J^T(J\,J^T)^{-1}}\,\dot{x}^*$$ → Left pseudoinverse (This is software)

There are variations because the motors of some joints may not be as powerful as required.

This is a closed form formula.

The pseudoinverse has some properties: $J^\# J = \mathbb{1}$

So for square and flat we can find only one solution.

> In tall configuration you have a best effort solution

$$\dot{q} = J^T (J J^T)^{-1} \dot{x}^*$$

In every case we still have problems with singularities.

In this case you get something close to the movement that you desire.

You go from J to $J^\#$ by using SVD

$$A = U \Sigma V^T$$

$$\begin{bmatrix} \sigma_1 & & \emptyset \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$$

$$\sigma_1 > \dots > \sigma_n > 0$$

CHI:

$$\chi = \frac{\sigma_1}{\sigma_n}$$

CONDITION NUMBER OF THE MATRIX

If χ is big then the system will be noisy and the result will be big. This effect gets bigger when you are close to singularities.

In practice you need to adjust the singular values in order to bound the result, and not have jerky motions.

The solution to this problem is to compute $J^\#$ like this:

$$J^\# = J^T (J J^T + \lambda \mathbb{1})^{-1}$$

λ must be added when this is getting critical. λ Generates a wrong solution but keeps everything bounded.

This is called regularization.

The pseudoinverse of A (or J) is basically computed as:

$$A^\# = U \Sigma^\# V^T$$

Orthonormal matrices $\rightarrow \begin{cases} U U^T = \mathbb{1}_n \\ V^T V = \mathbb{1}_m \end{cases}$

$$\Sigma^{\#} = \begin{bmatrix} \frac{1}{\sigma_1} & & \emptyset \\ & \ddots & \\ & & \frac{1}{\sigma_n} \end{bmatrix}$$

When the critical singular values got critical you can play with lambdas.

Every inverse should be done with the pseudoinverse

Inverse geometry problem

I want to find two values:

$\ast$ means desired.

$^{\emptyset}_{e}R^{*}(t)$ $\qquad$ $^{\emptyset}_{e}R(q)$

$r^{*}_{e/\emptyset}$ $\qquad$ $^{\emptyset}r_{e/\emptyset}(q)$

(displacement is a vector)

$$\underbrace{e_L(t)}_{\text{linear error}} = \underbrace{r_{e/\emptyset}}_{\text{actual position}} - \underbrace{r^{*}_{e/\emptyset}}_{\text{desired position}}$$

misalignment between the two frames

$$\rho = v\,\theta$$

ρ: angular error.

My control objective is to $\left\{ \begin{array}{l} \rho \rightarrow \emptyset \\ e_L \rightarrow \emptyset \end{array} \right\}$ Here we want to reduce the distance

N.B. DISPLACEMENT (vector) $\neq$ DISTANCE (scalar)

$$\frac{1}{2}|e_L|^2 = \frac{1}{2} e_L \cdot e_L$$

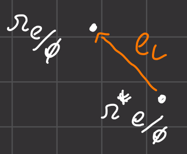
↳ the derivative of this we want negative.

$$\frac{1}{2}\frac{d}{dt}(e_L \cdot e_L) = \frac{1}{2}\, e_L \cdot \dot{e}_L + \frac{1}{2}\,\dot{e}_L \cdot e_L = e_L \cdot \dot{e}_L = e_L \cdot \left[v_{e/\emptyset} - v^{*} \right]$$

This is where I define my control structure.

this is the control signal!

Note that e_L is given



(We want to choose $v_{e/\emptyset}$ so that the scalar product is negative)

This process is called control synthesis

We can express:

$$\underbrace{e_L\cdot[v_{e_\phi}-v^*]}_{\boxed{-\gamma e_L}} = \underbrace{-\gamma(e_L\cdot e_L)<0}$$

If I choose this for the term inside the matrix this will be guaranteed

We want to guarantee that:

$$v_{e/\phi} - v^* = -\gamma e_L$$

output "input" Control parameter input

$$\boxed{v_{e/\phi} = -\gamma e_L + v^*}$$

$[\dot{x}^*]$ feedback control term. feed forward term providing velocity overflow to keep track of the goal.

Angular part

$$\frac{1}{2}|\rho\cdot\rho|^{\frac{1}{2}} = \frac{1}{2}\theta^2$$

this because v is unit vector

$$\frac{d}{dt}\frac{1}{2}(\rho\cdot\rho) = \theta v\cdot\overbrace{(\omega_{e_\phi}-\omega^*)}^{-\gamma_\theta v\theta}$$

I want this to be opposite of v because I want to turn the other way round.

$$\boxed{{}^\phi\omega_{e/\phi} = -\gamma_A v\theta + \omega^*}$$

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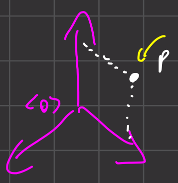


really small robot (sphere)

$$r_{P/\phi} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$v_{P/\phi} = \begin{bmatrix} {}^\phi i_1 & {}^\phi j_1 & {}^\phi k_1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

This is how we wrote this in MCM $\rightarrow$ $\begin{bmatrix} {}^\phi h_1 & {}^\phi h_2 & {}^\phi h_3 \end{bmatrix}$

I want to move P toward P*

The course will be divided in two steps:

1) Review of fundamentals of mechanics

2) Computational Mechanics. (Newton Euler recursive equations)

3) Dynamic models of holonomic robots.

At the end the result will be:

$$\underbrace{A(q)}_{\text{inertia}}\ddot{q} + \underbrace{B(q,\dot{q})}_{\text{coriolis and centripetal}}\dot{q} + \underbrace{C(q)}_{\text{gravity}} = \underbrace{M}_{\text{torques}} + \underbrace{D}_{\text{disturbances}}$$

REMARK

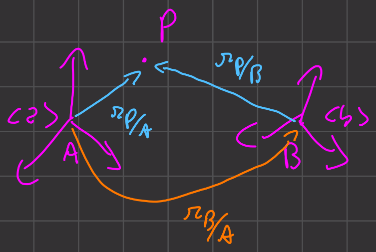
$$\dot{x}^* = J^{\#}\dot{q}$$

4) Control Algorithms. / Fundamentals of Robot dynamics control.

5) Less standard control algorithms. (Dynamic Motion Primitives)

$$\frac{d_a}{dt} u(t) = \frac{d_b}{dt} u(t) + \omega_{b/a} \times u$$

revision of the mother of all the formulas.



$$\frac{d_a}{dt}\left(r_{P/A}\right) = \frac{d_a}{dt}\left(r_{P/B} + r_{B/A}\right) = \underbrace{\frac{d_a}{dt} r_{B/A}}_{v_{B/A}} + \underbrace{\frac{d_b}{dt} r_{P/B}}_{v_{P/B}} + \omega_{B/A} \times r_{P/B}$$

$$\boxed{v_{P/A} = v_{B/A} + v_{P/B} + \omega_{B/A} \times r_{P/B}}$$ 2

$$\frac{d_a}{dt}\left(v_{P/A}\right) = \frac{d_a}{dt}\left[v_{B/A} + v_{P/B} + \omega_{B/A} \times r_{P/B}\right] =$$

$a_{P/A}$

linear operator

$$= \frac{d_a}{dt} v_{B/A} + \frac{d_b}{dt} v_{P/B} + \omega_{B/A} \times v_{P/B} + \left(\frac{d_a}{dt}\omega_{B/A}\right) \times r_{P/B} + \omega_{B/A} \times \left(\frac{d_a}{dt} r_{P/B}\right) =$$

$$= a_{B/A} + a_{P/B} + \omega_{B/A} \times v_{P/B} + \left(\frac{d_a}{dt}\omega_{B/A}\right) \times r_{P/B} + \omega_{B/A} \times \left(v_{P/B} + \omega_{B/A} \times r_{P/B}\right) =$$

$$\boxed{= a_{P/A} = a_{B/A} + a_{P/B} + \underbrace{2 \cdot \left(\omega_{B/A} \times v_{P/B}\right)}_{\text{coriolis}} + \left(\frac{d_a}{dt}\omega_{b/a}\right) \times r_{P/B} + \underbrace{\omega_{B/A} \times \left(\omega_{B/A} \times r_{P/B}\right)}_{\text{centripetal accel.}}}$$

(2)

Proof that $\omega_{b/a} = \omega_{a/b}$

$$\frac{d_a}{dt}\omega_{b/a} = \frac{d_b}{dt}\omega_{b/a} + \cancel{\omega_{B/A} \times \omega_{b/a}}$$

So we can always write $\dot{\omega}_{b/a}$

Point Mass

Ideal mechanical entity characterized by scalar quantity which is the mass. ≥ 0

It's a point in euclidean sense.

Isolated point

A point which is not interacting with any other point in the universe.

Force

Mechanical entity responsible the interaction between points mass

Inertial reference frames

Reference frame which is not rotating wrt. the stars.

Newton Laws

1) An isolated point mass has constant velocity wrt. inertial reference frames.

2) Given a point mass $m \Rightarrow \underbrace{ma = F}_{\text{Inertial quantities}}$

3) P, Q

$$\begin{cases} F_{Q \to P} = -F_{P \to Q} \\ F_{Q \to P} \times F_{P \to Q} = \emptyset \end{cases}$$

$$m \ddot{r}_{P/\emptyset} = F(r_{P/\emptyset}, \dot{r}_{P/\emptyset}, *(t))$$

↪ can be seen as a kind of control action.

↳ something in function of time.

1) Forward Dynamic Problem

Computing $r_{P/\emptyset}$ and $\dot{r}_{P/\emptyset}$ given $F(\cdot,\cdot,\cdot)$

$$r_{P/\emptyset}, \dot{r}_{P/\emptyset} \Big|_{t=t_0}$$

2) Inverse Dynamic Problem ("Control Problem")

I want the point to move with a given trajectory and I want to know the forces to move the point with that trajectory.

Given $r^*_{P/\emptyset}$, $\dot{r}^*_{P/\emptyset}$, $\ddot{r}^*_{P/\emptyset}$ compute F.

Compute the forward dynamic model:

Input:
- $\underline{F}$
- m

Parameter:
- h (sample time)

$$\ddot{\underline{r}}_{P/\emptyset} = \frac{1}{m} \cdot F$$ ← This is valid for a given time istant.

initialization:

$r_{P/\emptyset}(1) = *$

$\dot{r}_{P/\emptyset}(1) = *$

for $i = 1:N$:

$\ddot{x}_{P/\emptyset}(i+1) = \frac{1}{m} \underline{F}(i+1)$

$r_{P/\emptyset}(i+1) = \dot{r}_{P/\emptyset}(i) + h\, \ddot{r}_P(i+1)$

$r_{P/\emptyset}(i+1) = r_{P/\emptyset}(i) + h\, \dot{r}_{P/\emptyset}(i+1)$

end

$$\dot{x} = f(x)$$

$$\frac{x(t+h) - x(t)}{h} = f(x(t))$$

Explicit euler formula.

1) F = -m * g.δ * [zeros(1, Nn); 1:1:N+1] ???

2) Function F = force ($r_{P/\emptyset}$)

EQUILIBRIUM (POINT)

$r^E_{P/\emptyset}$ = constant (w.r.t. some inertial reference frame) $\forall t$

$\dot{r}^E_{P/\emptyset} \equiv \emptyset$ The resultant forces are equal to $\emptyset$.

$\ddot{r}_{P/\emptyset} \equiv \emptyset$

Energy (scalar)

It's related to the capability of some system to generate work.

$$E = \underbrace{T}_{\text{kinetic}} + \underbrace{V}_{\text{Potential}}$$

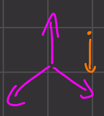
kinetic: $\frac{1}{2}\left(\dot{r}_{P/\emptyset} \cdot \dot{r}_{P/O}\right) m$

Potential: $F_c = \dfrac{\partial V}{\partial r}$

conservative (they do not dissipate energy)

Examples of conservative forces

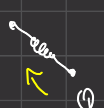
1) Gravity



$F_g = -mg\,\hat{k}$

$V_g = mg\left(r_{P/\emptyset} \cdot \hat{k}\right)$

2) Elastic Force

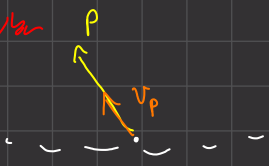


$F_{Q \to P} = -K_E \underbrace{(Q - P)}_{r_{Q/P}}$ (K_E: Constant)

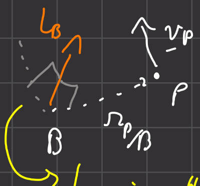
$V_E = \frac{1}{2} K_E \left(r_{Q/P} \cdot r_{Q/P}\right)$

LINEAR MOMENTUM and angular

Linear : $\underline{p} = m \cdot \underline{v}_P$



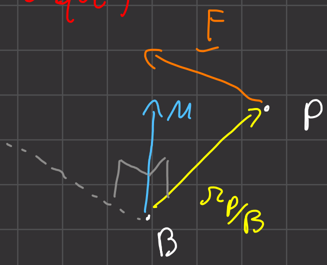
Angular : $\underline{L}_B = m\, r_{P/B} \times v_P$



L_B is orthogonal to $r_{P/B}$ and v_P

Moment of a force (Torque)

$\underline{M} = r_{P/B} \times F$



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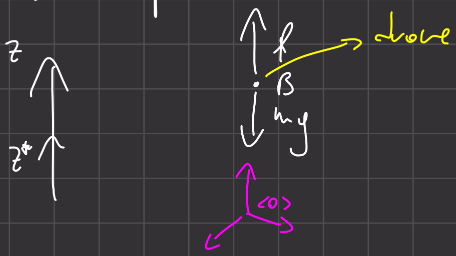
Another example



I build a model

$$^{\phi}m\, \ddot{r}_{B/\phi} = {}^{\phi}F$$ result of external forces

$$^{\phi}r_{B/\phi} = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \qquad {}^{\phi}F = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

So I collapse my model in:

$$m\ddot{z} = F_z$$

Forces in this example:

- gravity : mg
- the propellers : f

Can I control the model?

$$ {}^{\phi}f = \begin{bmatrix} 0 \\ 0 \\ f_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} $$

EXTERNAL FORCES: forces acting on the body

I want to drive my mass with f in my desired position (z^*)

I look $m\ddot{z} = F_z$ and to change z I have to change the $\ddot{z}$

How to change $\ddot{z}$? Changing F_z

F_z is composed by 2 terms f and mg $\begin{cases} m \text{ fixed} \\ g \text{ fixed} \end{cases}$

So I should change f

Law: $f = \underbrace{K_f}_{\text{constant}} \underbrace{\omega}_{\text{Turning speed of the propellers}}$

Propellers tend to push up the point, how to control ω?

$$ u(t) \to \boxed{\text{motor + propeller}} \to \omega $$

$u(t)$: input, what I control, it's reasonable that $\omega(t) = K_\omega u(t)$

I rewrite the equation:

$$ \ddot{z} = \frac{1}{m}\left[-mg + f\right] = \frac{1}{m}\left[-mg + (K_f + K_\omega)u\right] = $$

$$ = \frac{1}{m}\left[-mg + K u\right] $$

So we are in this condition:

$$ u(t) \to \boxed{S} \to z(t) \qquad (d \text{ entering } S \text{ from above}) $$

The gravity for example is a constant disturbance

My control -> my control law: specification to implement, your software, typically dyfl

let's assume your system is in the desired position:

$$z(t) = z^* \quad ⊛$$

Then we can define an error quantity:

$$e(t) = z(t) - z^* = \underbrace{\emptyset}_{\text{This is what I want}}$$

So this means:

$$\dot{e}(t) = \dot{z}(t) - \dot{z}^* = \emptyset$$

$$\ddot{e}(t) = \ddot{z}(t) - \ddot{z}^* = \emptyset$$

⊛ This is valid under this assumption:

$$\ddot{z}(t) = \emptyset$$

So z^* is constant

We can claim:

$$\ddot{z} = \emptyset = \frac{1}{m}\left[-mg + Ku\right]$$

$$\frac{1}{m} K u = g \implies \boxed{u_0(t) = \frac{1}{K} mg} \rightarrow \text{this is control}$$

To know K -> dynamometer
m is another parameter, known with a scale.
g is known
u_0 is the control

What's wrong with this?

To maintain the height of my system my computer must generate u_0

Example:

$m = 0{,}1\,[kg]$

$K = 10^{-3}\,\frac{60}{2\pi}\left[\frac{N}{rad/s}\right]$

$g = 9{,}8\,[m/s^2]$

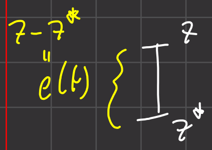
But generally the point is not in that z^*

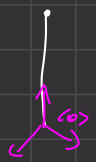
U_0 is still control variable.

By computing in this way this is an open loop signal. It's not good to cancel out disturbances.

That's due to the fact that $z(0) \neq z^*$ so $u(t) = \underbrace{u_0(t)}_{!} + \dots$

Not sufficient: this only cancels out gravity so the point is fixed.
But I want to control the height!





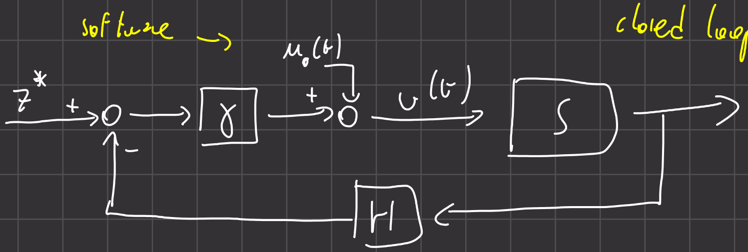
To slow down:

$$\omega_0(t) = K\omega\left[u_0(t) - \gamma e(t)\right] = K\omega\, u(t)$$

If $e(t) > 0$ we are slowing down.

So what we do is this:

closed loop (feedback)

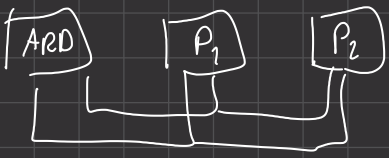


It's still not working because we are only analyzing the physics

To measure what is the output of S you add the block $\boxed{H}$.

They are sensors. How to implement it?

BUS: I^2C



?

SPI $\boxed{ARD} \rightleftarrows \boxed{TOF}$?

CK

For your software

```
for ( ; ; )

    read_i2c( )
    e = z - z*
    u = u_0 - γe      // this could be computed out of the loop before control
    write_i2c( );
    <sync>
```

$e(t) = z - z^*$

$\dot{e}(t) = \dot{z} - \dot{z}^*$

$\ddot{e}(t) = \ddot{z} - \ddot{z}^* = \frac{1}{m}\left[-mg + K u\right] - \ddot{z}^*$

this is the error dynamics

How the control error is going with my action control signal!

I substitute what is u

$= \frac{1}{m}\left[\gamma e(t)\right] = -\frac{\gamma}{m} e(t)$

So: $\ddot{e}(t) + \frac{\gamma}{m} e(t) = \underbrace{= \phi}_{\text{eq point}}$ 2° order linear equation

Is this eq. point asymptotically stable?

I need to compute the poles associated with the system:

$\lambda^2 + \frac{\gamma}{m} = \phi$

strictly negative real parts of poles is required to be asym. stable eq. point.

solutions:

$\lambda_{1,2} = \pm j\sqrt{\frac{\gamma}{m}}$ this is not the case

It means if I apply this control law there is no way to reach undefinetly the goal.

We need to dissipate energy: add another term to check about motion at the target. I have the relative velocity:

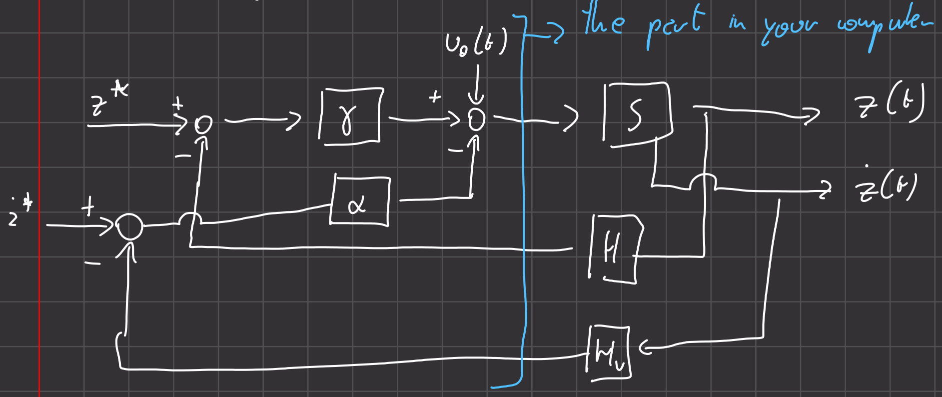
So: $u(t) = u_0(t) - \gamma e(t) - \alpha \dot{e}(t)$ → damping term

they are not real forces, the physics finishes with $u_0(t)$

Now we are considering control forces, they are virtual, but they have physical effects.
The sense of $-\alpha e(t)$ is to decrease $u(t)$ when you are arriving at the goal

But we have to estimate $\dot{e}(t)$: the relative velocity.

You end up with this algorithm:



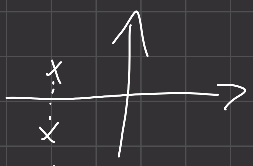
We add a control of velocity

So I find: $\ddot{e}(t) + \frac{\alpha}{m}\dot{e}(t) + \frac{\gamma}{m} e(t) = 0$

↳ closed loop error dynamics

$$\lambda^2 + \frac{\alpha}{m}\lambda + \frac{\gamma}{m} = 0$$

α and $\gamma > 0$ } permanent of signs } cartesio theorem }



strictly negative poles.

$\Rightarrow$ $e(t)$ is an asym. stable equilibrium point. ✓

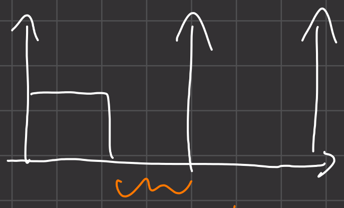
You must find a trade off between α and γ, because if one is greater than the other you can have oscillation or slowed down behaviour.
You should make experiments

$$\begin{cases} \gamma \gg \alpha & \text{oscillations} \\ \alpha \gg \gamma & \text{slow motion} \end{cases}$$

$$u \longrightarrow \boxed{\text{motor + propeller}} \xrightarrow{\omega}$$

↳ Inside here there is closed loop control

$u = \omega^*$
↳ guaranteeing the control movement.



ω measured

Sensor-less feedback control

$\omega_{measured}$ is a voltage and it is used in the back.

For us u is a digital number.

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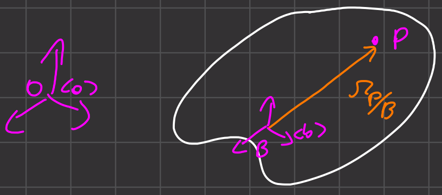
Study motion caused by external events. We need models to develop software.

model → physical property of model

model → control → software → simulation → reality

A robot is a connection of bodies

↳ In this course only rigid bodies



We consider a point B and its reference frame c equal for all points because of rigid body.

Rigid Body constraint

$$|P-B| = \text{constant} \ \forall t$$

Using our convention $|r_{P/B}| = \text{constant} \ \forall t$

We have some considerations derived from mother of formulas.

$$v_{P/\phi} = v_{B/\phi} + \omega_{b/\phi} \times r_{P/B}$$

(because $v_{P/B} = \phi$ for rigid body constraint)

The velocity of the body is the sum of velocities of B w.r.t. ϕ and cross product between angular velocity of frame <b> w.r.t. <o> and the displacement between P and B.

Also for the acceleration

$$a_{P/\emptyset} = a_{B/O} + \dot{\omega}_{b/\emptyset} \times r_{P/B} + \omega_{b/\emptyset} \times (\omega_{b/\emptyset} \times r_{P/B})$$

these expressions hold for any point of the body and for any point which is fixed w.r.t. the body.

Center of Mass

The body is made of stuff connected to each other. If you cut a part of the body -> it has a mass.

We can assume to have $\rho = \rho(p)$ TODO

(density?)

$$dm = \rho \, dV$$

ρ is a positive function

If I sum all the small parts I have the mass:

$$m = \int_V \rho(p) \, dV$$ TODO

$$(C - O) = \frac{\int_V (P-O) \rho \, dV}{\int_V \rho \, dV} = \frac{\int_V (P-O) \rho \, dV}{m}$$

Formula for center of mass

The numerator is a sum of infinite vectors, the sum is weighted by the density.

In our notation:

$$r_{C/\emptyset} = \frac{\int_V r_{P/\emptyset} \rho \, dV}{m}$$

point $\emptyset$ is an arbitrary point ?

TODO

Properties:

1) $$r_{C/\emptyset} = \frac{\int_V (\underbrace{r_{P/B} + r_{B/\emptyset}}_{r_{P/\emptyset} = r_{P/B} + r_{B/\emptyset}}) \rho \, dV}{m}$$

$$= \frac{\int_V r_{P/B}\, \rho\, dV}{m} + \frac{\int_V r_{B/\emptyset}\, \rho\, dV}{m}$$

not depending on ρ so out

TODO

$$= \frac{\int_V r_{P/B}\, \rho\, dV}{m} + \frac{r_{B/\emptyset} \cancel{\int_V \rho\, dV}}{\cancel{m}}$$

$$r_{C/\emptyset} = r_{C/B} + r_{B/\emptyset}$$

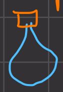
We can express the center of mass wrt. any inertial reference frame <o> by a translation ($r_{B/\emptyset}$)

2) The body is the union of 2 distinct bodies.

Each one will have its own center of mass

$$B = B_1 \cup B_2$$

Example of a bottle



$$B_1 \cap B_2 = \emptyset$$

So
$$r_{C/\emptyset} = \frac{\int_V r_{P/\emptyset}\, \rho\, dV}{m}$$
$$= \frac{\int_{V_1} r_{P/\emptyset}\, \rho\, dV + \int_{V_2} r_{P/\emptyset}\, \rho\, dV}{m}$$
$$= \frac{m_1\, r_{C_1/\emptyset}}{m} + \frac{m_2\, r_{C_2/\emptyset}}{m}$$

Total center of mass is a linear combination of 2 center of masses

$$\underset{(m_1+m_2)}{m}\, r_{C/\emptyset} = m_1\, r_{C_1/\emptyset} + m_2\, r_{C_2/\emptyset}$$

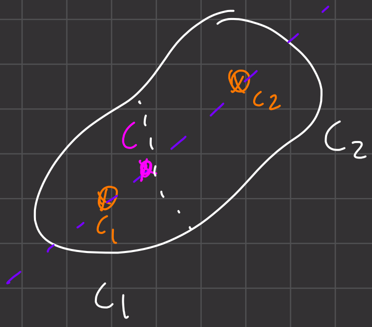
$$(m_1 r_{C/\emptyset} - m_1 r_{C_1/\emptyset}) + (m_2 r_{C/\emptyset} - m_2 r_{C_2/\emptyset}) = \emptyset$$

$$m_1\, r_{C/C_1} + m_2\, r_{C/C_2} = \emptyset$$

m_1 and m_2 are positive scalars and this means r_{C/C_1} and r_{C/C_2} must be parallel.

- So the center of mass lies on the line joining both both center of masses and it's closer to the biggest mass

On the picture:



So we summarize:

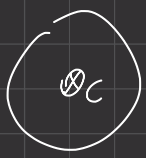
$$r_{C/\emptyset} = \frac{m_1 r_{C_1/\emptyset} + m_2 r_{C_2/\emptyset}}{m}$$

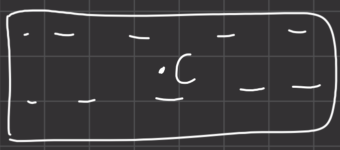
and $r_{C/C_1} \parallel r_{C/C_2}$

We can generalize this formula to n bodies:

$$r_{C/\emptyset} = \frac{1}{m} \sum_{i=1}^{n} m_i \, r_{C_i/\emptyset}$$

REMARK





- C is not a physical point in the body.
- C is the point where we consider all the mass to be concentrated.

For point mass:

$$p = m \, v_{P/\emptyset}$$

For linear movement:

$$L_B = m \, r_{P/B} \times v_{P/\emptyset}$$

I want to define these quantities for the whole body:

$$\begin{cases} dp = v_{P/\emptyset} \, dm \\ dL_B = (r_{P/B} \times v_{P/\emptyset}) \, dm \end{cases}$$

$$L_B = \int_V dL_B = \int (r_{P/B} \times v_{P/\emptyset}) \rho \, dV$$

$$= \int_V r_{P/B} \times [v_{B/\emptyset} + (\omega_{b/\emptyset} \times r_{P/B})] \rho \, dV$$

$$= \int_V (r_{P/B} \times v_{B/\emptyset}) \rho \, dV + \int_V r_{P/B} \times (\omega_{b/\emptyset} \times r_{P/B}) \rho \, dV$$

$$= m (r_{C/B} \times v_{B/\emptyset}) + I_B(\omega_{b/\emptyset})$$

It's a vector depending on the position of the center of mass w.r.t. B, which is an arbitrary point, and the velocity of point w.r.t. $\emptyset$

B is arbitrary so I can choose it as C so $r_{C/C} = \emptyset$

Inertia operator

$$I_B(\omega_{b/\emptyset}) \triangleq \int_V r_{P/B} \times (\omega_{b/\emptyset} \times r_{P/B}) \rho \, dV$$

It's a vector, precisely an operator
↳ transforms input into output.

In this case: sum of infinite vectors derived from that operation.

The input is $\omega_{b/\emptyset}$, so I can see it as:

$$\omega_{b/\emptyset} \rightarrow \boxed{} \rightarrow I_B(\cdot)$$

So I can write:

$$I_B(u) = \int_V r_{P/B} \times (u \times r_{P/B}) \rho \, dV$$

↳ geometric vector

properties

1) The definition has arbitrary choice of B

$$I_B(u) = I_C(u) + m \, r_{C/B} \times (u \times r_{C/B})$$

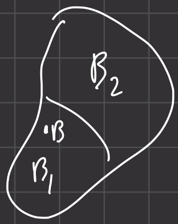
Center of mass

You condense the formula of inertia operator and the second term is considered as if the whole mass is concentrated in a single point, which is the center of mass.

↳ the proof is in the notes

$$r_{P/B} = r_{P/C} + r_{C/B}$$

2)



$$B = B_1 + B_2$$

$$B_1 \cap B_2 = \emptyset$$

We choose point B and we can write:

$$I_B(u) = I_B^{(1)}(u) + I_B^{(2)}(u)$$

We notice the inertia operation is additive

3) $u \cdot I_B(u) \geq 0 \quad \forall u \neq 0$
$\forall B$ TODO

?

This holds for phisical objects

a line is not
a paper sheet depends

there might be some vectors for which the scalar mod is very small, so similar to zero.

DEMONSTRATION

$$u \cdot I_B(u) = u \cdot \int_V r_{P/B} \times (u \times r_{P/B}) \rho \, dV$$

$$= \int_V u \left[r_{P/B} \times (u \times r_{P/B}) \right] \rho \, dV =$$

NOTE

$$a \cdot (b \times c) = c\,(a \times b) = b\,(c \times a)$$

$$= \int_V (u \times r_{P/B}) \cdot (u \times r_{P/B}) \rho \, dV$$

we have scalar prod of the same 2 vectors

$$= \int_V |u \times r_{P/B}|^2 \rho \, dV \geq \emptyset$$ squared mod so non negative quantity

It's $= \emptyset$ when u is aligned with $(P-B)$ but notice.

- u is arbitrary
- we are considering not pathological objects. For those reasons it holds > 0

How do I compute these things?

Algebraic form of "I_B"

$I_B(\mu)$ is a geometric vector

I know it's an integral, integral = sum of bits.

$$^{\phi}I_B(\mu) = \int_V {}^{\phi}\left[r_{P/B} \times (\mu \times r_{P/B})\, \rho\, dV \right]$$

Project the operation on a frame for sake of simpl. calc.

$$= \int_V [{}^{\phi} r_{P/B} \times] \left\{ -[r_{P/B} \times]\, {}^{\phi}\mu \right\} \rho\, dV =$$

NOTE

$$^{\phi}a = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad {}^{\phi}[a \times] = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

- ${}^{\phi}\mu$ does not depend on P so out of the integral
- the − goes at the beginning and

$$-[r_{P/B} \times] = [r_{P/B} \times]^T$$

$$= \underbrace{\left[\int_V {}^{\phi}[r_{P/B} \times]^T\, {}^{\phi}[r_{P/B} \times]\, \rho\, dV \right]}_{\text{this is a 3x3 frame expressed in frame } \phi} {}^{\phi}\mu$$

$$^{\phi}I_B \in \mathbb{R}^{3\times 3}$$

So:

$$^{\phi}I_B = \int_V {}^{\phi}[r_{P/B} \times]^T\, {}^{\phi}[r_{P/B} \times]\, \rho\, dV \in \mathbb{R}^{3\times 3}$$

and $^{\phi}I_B = {}^{\phi}I_B^T$

This matrix changes depending on the position of the body.

It's time varying in general

But I can choose a frame cbz fixed with body:

$$^{b}I_b = \int_V [{}^{b} r_{P/B} \times]^T [r_{P/B} \times]\, \rho\, dV$$

if I consider <b> fixed with body the matrix does not change
because $r_{P/B}$ is constant.

So ${}^{b}I_B$ is constant.

How to link ${}^{\phi}I_B$ and ${}^{b}I_B$? Rotation Matrix

$${}^{\phi}I_B = {}^{\phi}_{b}R \, {}^{b}I_B \, {}^{b}_{\phi}R$$

You will never compute the integral but the important thing is

Property 3 implies:

$$\boxed{\begin{array}{ll} \mu \cdot I_B(\mu) \geq \phi & \iff {}^{b}\mu^T \, {}^{b}I_B \, {}^{b}\mu \\ \forall \mu & \forall {}^{b}\mu \end{array}}$$

${}^{b}I_b$ is strictly positive defined matrix. It means:

${}^{b}I_B = {}^{b}I_B^T \rightarrow$ symmetric

${}^{b}I_B$ has 3 real positive eigenvalues

${}^{b}I_B$ can be diagonalized: $\exists V \in \mathbb{R}^{3\times3}$ s.t. $V^T \, {}^{b}I_B V = \mathrm{diag}(I_1, I_2, I_3)$

$V V^T = \mathbb{1}$ (exmp. $V = R$) Eigenvalues

The eigenvalues: they describe principle directions and how the mass is distributed along the axis.

The smaller ε the more the mass is not distributed.

TODO???

Principle axis of inertia.

Note: $\forall Q = Q^T \geq 0$

We have the property
→ symmetry
→ strictly real positive eigenvalues $v_1 \dots v_m$
→ $v_1, \dots v_m$ are orthonormal vectors.

$$V = [v_1, \dots v_m] \in \mathbb{R}^{n \times n}$$

this V diagonalizes Q, that means:

$$V^T Q V = \mathrm{diag}\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

Definition of eigenvalue

the number which solves the equation

$$\underbrace{A v}_{\text{eigenvector}} = \underbrace{\lambda v}_{\text{eigenvalue}}$$

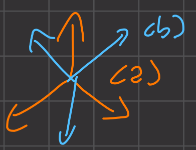
So ${}^{b}I_B \in \mathbb{R}^{3\times 3}$, ${}^{b}I_B = I_B^T$ $\Rightarrow$ $\left.\begin{array}{l} \lambda_1 = I_1 \geq 0 \\ \lambda_2 = I_2 \geq 0 \\ \lambda_3 = I_3 \geq 0 \end{array}\right\}$ They are called principle inertia moments and are a property of the body.

$$V = [v_1, v_2, v_3] \in \mathbb{R}^{3\times 3}$$

$$V V^T = \mathbb{1}_{3\times 3}$$

there exists a set of axis $\langle a \rangle$ for which I have an orthogonal matrix.

graphically:

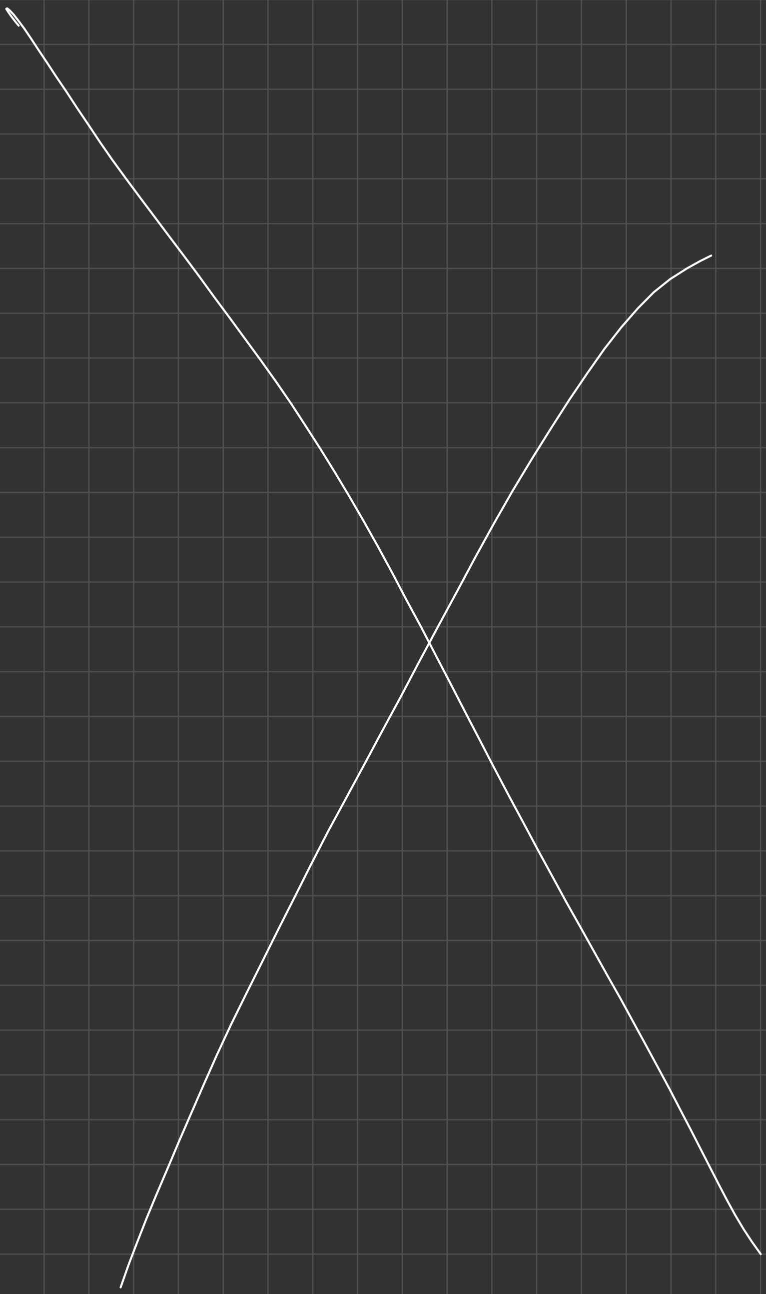


These are the principle inertia direction.

Note:

$$I_B(u) = I_C(u) + m\, r_{c/B} \times (u \times r_{c/B})$$

$$\boxed{{}^{b}I_B = {}^{b}I_C + m\left[{}^{b}r_{c/B}\times\right]^T\left[{}^{b}r_{c/B}\times\right]}$$ stessa dim della precedente.



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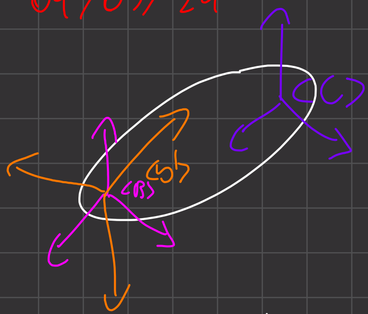


If the inertial reference frame is fixed wrt the frame of the body then the inertial operation is constant

$I_B(\cdot) \to {}^{b}I_B$ -> 3X3 matrix and strictly positive, and symmetric

$$\begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{bmatrix}$$

we may think that these come out the calc. and they define the inertial.

>0

Positive definite.

this has 3 eigenvalues and all strictly positive.

There exists a transformation orthogonal matrix that transforms the inertial into diagonal. the terms on the diagonal are the eigenvalues.

Every strictly positive matrix can be diagonalized ???

there exist as many eigenvectors as many eigenvalues.

$$^{b}I_B \, v_i = \lambda_i \, v_i \qquad i = 1/3$$

$$\left[{}^{b}I_B v_1 \,\vdots\, {}^{b}I_B v_2 \,\vdots\, {}^{b}I_B v_3 \right] = \left[v_1 \lambda_1 \,\vdots\, v_2 \lambda_2 \,\vdots\, v_3 \lambda_3 \right]$$

$$^{b}I_B \underbrace{\left[{}^{b}v_1 \,\vdots\, {}^{b}v_2 \,\vdots\, {}^{b}v_3 \right]}_{V} = \underbrace{\left[{}^{b}v_1 \,\vdots\, {}^{b}v_2 \,\vdots\, {}^{b}v_3 \right]}_{V} \underbrace{\begin{bmatrix} \lambda_1 & \emptyset & \emptyset \\ \emptyset & \lambda_2 & \emptyset \\ \emptyset & \emptyset & \lambda_3 \end{bmatrix}}_{\Delta}$$

N.B

$$v_i^T \cdot v_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \triangleq \delta_{i,j}$$

$$^{b}I_B V = V \Delta$$

orthonormal basis (matrix)

$(V^T V = \mathbb{1})$

$$\left(\begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix} \left[v_1 \; v_2 \; v_3 \right] = \mathbb{1} \right)$$

I multiply for V^T

$$V^T I_B V = \Delta$$

there is a linear transformation to transform I_B into diagonal form.

$\boxed{\text{V is an orthonormal matrix and can be seen in our case as a change of coordinates (change of space)}}$

We changed ⟨B⟩ to ⟨b'⟩ but is still fixed in frame B.

$$V^T \, {}^{b}I_B \, V = {}^{b'}I_B (\Delta)$$

To compute V I use SVD.

$V = {}^{b}_{b'}R$ -> You want this to be right hand so $\det(V) = +1$

If you compute V and $\det \neq 1$ just swap two eigenvectors

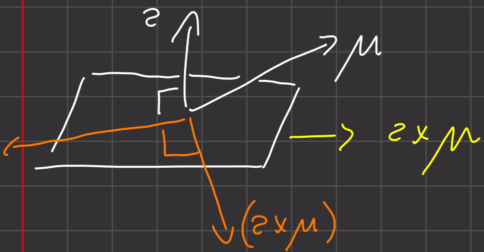
We can now give physical meaning to eigenvalues

$$\Lambda = \begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix} = \begin{bmatrix} {}^{b'}I_{11} & & \emptyset\\ & {}^{b'}I_{22} & \\ \emptyset & & {}^{b'}I_{33}\end{bmatrix}$$

↳ 3 moments of inertia.

$$\boxed{{}^{b}I_{B} = \underbrace{{}^{b}I_{C}}_{\text{strictly positive definite}} + \underbrace{m\left[{}^{b}r_{C/B}\times\right]^T\left[{}^{b}r_{C/B}\times\right]}_{\text{semipositive definite}}}$$

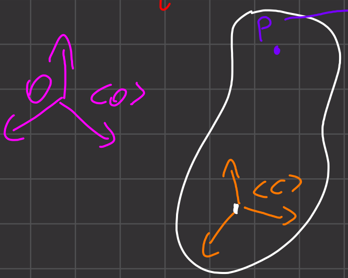
→ rank 2 in general because cross product operator



$$[z\times]^T\underbrace{[z\times]u}_{b}$$

↳ b belongs to the plane and we rotating −(+) 90° on that plane.

Newton equations



P will move following newton second law:

$$\frac{d_0}{dt} v_{P/O} \cdot \underbrace{dm}_{\rho\, dV} = f_P^{Ext} + f_P^{INT}$$

ρ → density, dV → volume

f_P^{Ext}: ↳ force acting on P due to external forces.

f_P^{INT}: ↳ resultant of all the forces which might be generated by all the other points of the body acting on P

To explain this think of two balls connected by a spring and then put the balls in a piece of plastic

The internal force would be the spring and the [illegible]

let's apply this law to all points of the body. So let's do an integral:

$$\int_V \underbrace{\frac{d^o}{dt} v_{p/\phi}}_{v_{c/\phi} + \omega_{b/\phi} \times r_{p/c}} \rho\, dV = \underbrace{\int_V f_p^{ext}\, dV}_{F^{EXT}} + \underbrace{\cancel{\int_V f_p^{int}\, dV}}$$

→ the integral is always equal to $\emptyset$. It's not valid for not rigid bodies.

All this must be compensated otherwise the object would explode

F^{EXT} → resultant of all external forces.

$$\frac{d}{dt}\left(v_{c/\phi} + \omega_{b/\phi} \times r_{p/c}\right) = a_{c/\phi} + \dot{\omega}_{b/\phi} \times r_{p/c} + \omega_{b/\phi} \times \left[\cancel{\frac{d^o}{dt} r_{p/c}}\ \omega_{b/\phi} \times r_{p/c}\right]$$

$$\int_V a_{c/\phi}\, \rho\, dV + \int_V \dot{\omega}_{b/\phi} \times r_{p/c}\, \rho\, dV + \int_V \omega_{b/\phi} \times \left(\omega_{b/\phi} \times r_{p/c}\right) \rho\, dV$$

$$m\, a_{c/\phi} + \dot{\omega}_{b/\phi} \times \underbrace{\int_V r_{p/c}\, \rho\, dV}_{m\, r_{c/c} = \emptyset} + \omega_{b/\phi} \times \Big(\omega_{b/\phi} \times \underbrace{\int_V r_{p/c}\, \rho\, dV}_{m\, r_{c/c} = \emptyset}\Big)$$

$$\Downarrow$$

$$m\, a_{c/\phi} = F^{ext}$$

Formulation of the three newton laws for translational forces.
Forces on the body act as if they act on the center of mass.

$$m\, {}^{\phi}a_{c/\phi} = {}^{\phi}\underline{F}^{EXT}$$

$$m \frac{d}{dt} \underbrace{{}^{\phi}v_{c/\phi}}_{\frac{d\, {}^{\phi}r_{c/\phi}}{dt}} = {}^{\phi}F_{EXT} \quad \Rightarrow \quad X = \begin{bmatrix} {}^{\phi}r_{c/\phi} \\ {}^{\phi}v_{c/\phi} \end{bmatrix}$$

This extracts $v_{c/\phi}$

$$\frac{d}{dt}x(t) = \dot{x}(t) = \overbrace{\begin{bmatrix} \phi_3 & \mathbb{1}_3 \\ \phi_3 & \phi_3 \end{bmatrix}}^{\text{This extracts } v_{c/\phi}} x(t) + \begin{bmatrix} \phi_3 \\ \frac{1}{m}\mathbb{1}_3 \end{bmatrix} u(t)$$

note: $\dot{x}(t) = Ax + Bu$

Remark #1

$$\text{If} \begin{cases} F^{EXT} = \phi & \forall t \\ {}^{\phi}v_{c/\phi}(t_0) = \phi \\ {}^{\phi}r_{c/\phi}(t_0) = {}^{\phi}r_{\phi} \end{cases}$$

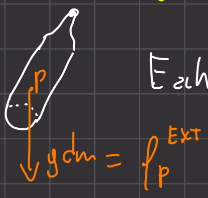
What is the solution of the previous formulas.

$$\Longrightarrow \begin{cases} v_{c/\phi}(t) = \phi \\ r_{c/\phi}(t) = r_{\phi} \end{cases} \forall t$$

→ Equilibrium condition.

Example:

Bottle of water

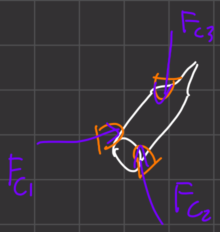


Each point is feeling gravity: $(g\,dm)$

$$\int_V f_p^{EXT}\,dV = \int_V g\,dm = \int_V g\rho\,dV = g\int_V \rho\,dV = gm$$

Gravity acts as a pure force on the center of mass.

I know that on the bottle is acting one external force due to gravity and then I decide to grasp it



let's assume that I can generate these forces

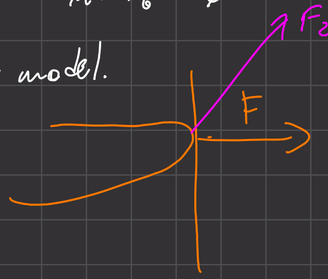
$$F_G = f_{c_1} + f_{c_2} + f_{c_3}$$

↓ grasping force

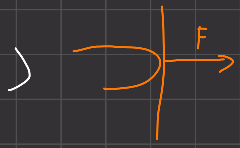
To keep the bottle still I must guarantee that $F_M + F_G = \emptyset$

For these kind of forces you need a contact model.

Point contact: w/ Friction →



Point contact: w/o friction →

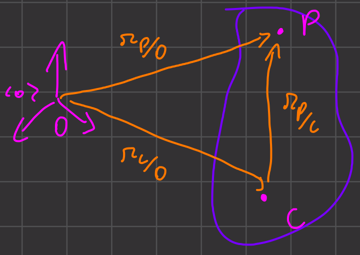


Soft finger contact

there is also some torque that the finger applies to the object not allowing rotation

Euler Equations 06/03/24

they describe the motion of the robot.



$$\frac{d_0}{dt} v_{P/\emptyset}\, dm = F_P^{EXT} + F_P^{INT} \quad \text{and} \quad m_C = r_{P/C} \times f$$

So the moment is:

$$m_C = r_{P/C} \times \frac{d_0}{dt} v_{P/\emptyset}\, dm$$

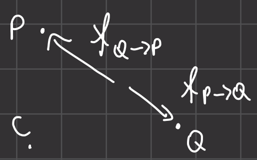
$$= \underline{r}_{P/C} \times f_P^{EXT} + r_{P/C} \times f_P^{INT}$$

$$= m_{P/C}^{EXT} + m_{P/C}^{INT}$$

To have the moment for the whole body while keeping generality, we integrate:

$$\int_V \left(r_{P/C} \times \frac{d_0}{dt} v_{P/\emptyset} \right) \rho\, dV = \int_V m_{P/C}^{EXT}\, dV + \int_V m_{P/C}^{INT}\, dV$$

NOTE:



$$(P-C) \times f_{Q\to P} + (Q-C) \times f_{P\to Q} \underset{\substack{\downarrow \\ \text{newton 3 law}}}{=} \left[(P-C) - (Q-C) \right] \times f_{Q\to P} \underset{\substack{\downarrow \\ \text{newton 3 law}}}{=} (P-Q) \times f_{Q\to P} = \emptyset$$

Valid for every $f_{Q\to P}$, so for every point of a rigid body

$$\frac{d_0}{dt}\left(r_{P/C} \times v_{P/O}\right) = \frac{d_0}{dt} r_{P/C} \times v_{P/O} + r_{P/C} \times \frac{d_0}{dt} v_{P/O}$$

$$\left(r_{P/C} \times \frac{d_0}{dt} v_{P/\emptyset}\right) = \frac{d_0}{dt}\left(r_{P/C} \times v_{P/O}\right) - \underbrace{\frac{d_0}{dt}\left[r_{P/\emptyset} - r_{C/\emptyset}\right] \times v_{P/\emptyset}}_{\left(v_{P/\emptyset} - v_{C/O}\right) \times v_{P/\emptyset}}$$

$$= \frac{d_0}{dt}\left(r_{P/C} \times v_{P/\emptyset}\right) + v_{C/\emptyset} \times \underbrace{v_{P/\emptyset}}_{\cancel{v_{P/C}} + \omega_{b/\emptyset} \times r_{P/C}}$$

$\emptyset \to$ Not moving

$$= \frac{d_0}{dt}\left[r_{P/C} \times \left(v_{C/\emptyset} + \omega_{b/\emptyset} \times r_{P/C}\right)\right] + v_{C/\emptyset} \times \left(\omega_{b/\emptyset} \times r_{P/C}\right)$$

$$\int_V \left(r_{P/C} \times \frac{d_0}{dt} v_{P/\emptyset}\right) \rho \, dV = \int_V \frac{d_0}{dt} [*] \rho \, dV + \int_V v_{C/O} \times \left(\omega_{b/\emptyset} \times r_{P/C}\right) \rho \, dV$$

$$= \frac{d_0}{dt} \int_V r_{P/C} \times \left[v_{C/O} + \omega_{b/\emptyset} \times r_{P/C}\right] \rho \, dV + \cancel{v_{C/\emptyset} \times \left[\omega_{b/\emptyset} \times \underbrace{\int_V r_{P/C} \rho \, dV}\right]}$$

$m \, r_{C/C}$ from def of cent of mass and so always $= \emptyset$

$$= \frac{d_0}{dt} \int_V r_{P/C} \times \underbrace{v_{C/\emptyset}}_{} \rho \, dV + \frac{d_0}{dt} \int_V r_{P/C} \times \left[\omega_{b/\emptyset} \times r_{P/C}\right] \rho \, dV$$

Not depending on ρ

$$= \frac{d_0}{dt}\left[\cancel{\left(\int_V r_{P/C} \rho \, dV\right)}_{\emptyset} \times v_{C/O}\right] + \frac{d_0}{dt} I_C\left(\omega_{b/\emptyset}\right)$$

$$\frac{d_0}{dt} I_C\left(\omega_{b/\emptyset}\right) = M^{ext}$$

$\hookrightarrow$ resultant of external Moments.

$$\frac{d_0}{dt} I_C\left(\omega_{b/\emptyset}\right) = \frac{d_b}{dt} I_C\left(\omega_{b/\emptyset}\right) + \omega_{b/\emptyset} \times I_C\left(\omega_{b/O}\right)$$

$$\frac{d_b}{dt} I_C\left(\omega_{b/\emptyset}\right) = \frac{d_b}{dt} \int_V r_{P/C} \times \left(\omega_{b/\emptyset} \times r_{P/C}\right) \rho \, dV$$

$$= \int_V \frac{d_b}{dt}\left[\underbrace{r_{P/C}} \times \left(\omega_{b/\emptyset} \times \underbrace{r_{P/C}}\right) \rho \, dV\right.$$

always $\emptyset$ in b

$$= \int_V r_{P/C} \times \left[\left(\frac{{}^b d}{dt} \omega_{b/\phi}\right) \times r_{P/C}\right] \rho \, dV = I_C(\dot{\omega}_{b/\phi})$$

but $\boxed{\frac{{}^b d}{dt} \omega_{b/\phi} = \frac{{}^o d}{dt} \omega_{b/\phi}}$ so $\hat{=} \dot{\omega}_{b/\phi}$

Euler equation of motion

$$I_C(\dot{\omega}_{b/\phi}) + \omega_{b/\phi} \times I_C(\omega_b) = M^{EXT}$$

We need to numerically compute these quantities so let's see in algebraic form

Assume to project this equation in Frame $\{\phi\}$

You get $\quad {}^{\phi}I_C \; \dot{\omega}_{b/\phi} + [{}^{\phi}\omega_{b/\phi} \times] + {}^{\phi}I_C \, {}^{\phi}\omega_{b/\phi} = {}^{\phi}M^{EXT}$

unknown

3x3 time varying matrix (depends on body attitude)

First order diff equation.

We need to couple this equation with the step down:

$${}^{\phi}_{b}\dot{R} = [{}^{\phi}\omega_{b/\phi} \times]\, {}^{\phi}_{b}R$$

can be computed offline.

$${}^{b}I_C \; {}^{b}\dot{\omega}_{b/\phi} + [{}^{b}\omega_{b/\phi} \times]\, {}^{b}I_C \, {}^{b}\omega_{b/\phi} = {}^{b}M^{EXT}$$

Same as the one before, but computationally much easier.

comes out from cad

VERY NICE

$${}^{o}_{b}\dot{R} = {}^{o}_{b}R \, [{}^{b}\omega_{b/\phi} \times]$$

NOTE

$$\boxed{\begin{aligned} {}^{a}A\, {}^{a}u &= {}^{a}[A(u)] \\ {}^{b}A\, {}^{b}u &= {}^{b}[A(u)] \end{aligned}}$$

$${}^{a}_{b}R \, {}^{b}A \, {}^{b}u = \underbrace{{}^{a}_{b}R \, {}^{b}A \, {}^{b}_{a}R}_{{}^{a}A} \, {}^{a}u$$

${}^{b}u$

$$\begin{cases} {}^b\dot{\omega}_b = {}^b I_c^{-1} \left[\left[{}^b\omega_{b/0} \times \right] {}^b I_c \, {}^b\omega_{b/\phi} + {}^0 M^{EXT} \right] \\ {}^{\phi}_{b}\dot{R} = {}^0_b R \left[{}^b\omega_{b/\phi} \times \right] \end{cases}$$

↳ not a function of linear or angular acell

You have to check that the vectors are orthonormal

$$\Rightarrow \quad {}^0\hat{i}_B = col_1\left({}^0_b R \left[{}^b\omega_{b/\phi} \times \right] \right), \quad {}^{\phi}\hat{j}_B = col_2(\;\;), \quad {}^{\phi}\hat{k}_B = col_3(\;\;)$$

(You could also use quaternions)

these are non linear differential equations

Conceptually there are xx first order diff equation because you need 3 parameters for orientation.
Angular velocity and orientation can be done with euler angles.

Summarize Newton-Euler for the rigid body

They are two sets of equations describing translation of the body looking at the center of mass.

$$\left.\begin{aligned} & m \frac{d^0}{dt} v_{c/\phi} = F^{ext} \\ & I_c(\dot{\omega}_{b/\phi}) + \omega_{b/\phi} \times I_c(\omega_{b/\phi}) = M^{EXT} \end{aligned}\right\} \text{Forward dynamics}$$

REMARK

All the external forces acting on the bottle are mg $\boxed{F^{EXT} = mg}$

Now let's see for ang. vel.

integrate for every point

$$m_{P/C}^{EXT} \rightarrow M^{EXT} = \int_V r_{P/C} \times g \, dm = \int_V (r_{P/C} \times g)\, \rho \, dV$$

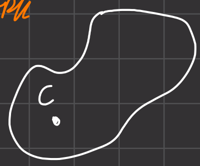
$$m_{P/C}^{EXT} = r_{P/C} \times f_P^{EXT}$$

$$= \cancel{\Big[\underbrace{\int_V r_{P/C} \, \rho dV}_{=0} \Big]} \times g = \phi$$

The gravity is not generating moment.

The inertia matrix can be chosen in smart way:

REMARK



Assume <b> such that $${}^b I_c = \begin{bmatrix} I_1 & & \phi \\ & I_2 & \\ \phi & & I_3 \end{bmatrix}$$

$I_i > \phi$ for rigid bodies. $i \in (1..3)$

We have 3 conditions:

① General structure $\overset{\nearrow \text{principal axis}}{I_1} > \underset{\substack{\downarrow \\ \text{rotations on the intermediate} \\ \text{axis are unstable}}}{I_2} > I_3$

Rotation around the largest inertia moment are stable rotations, and locally asimptotically stable.

② Spherical structure when $I_1 = I_2 = I_3$ Any frame is a principle inertia

Note that geometrically could be not a sphere if made of different density material.

③ Gyroscopic structure $I_1 = I_2 \neq I_3$

$${}^bI_C = \begin{bmatrix} I_1 & & \phi \\ & I_2 & \\ \phi & & I_3 \end{bmatrix} \qquad {}^b\omega_{b/\phi} = \left.\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}\right\} \text{scalar functions.} \qquad M^{EXT} = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}$$

$${}^bI_C \, {}^b\dot{\omega}_{b/\phi} + \left[{}^b\omega_{b/\phi}\times\right] {}^bI_C \, \omega_{b/\phi} = {}^bM^{EXT}$$

$$\begin{bmatrix} I_1 & & \phi \\ & I_2 & \\ \phi & & I_3 \end{bmatrix} \begin{bmatrix} \dot\omega_x \\ \dot\omega_y \\ \dot\omega_z \end{bmatrix} + \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & \phi & -\omega_x \\ -\omega_y & \omega_x & \phi \end{bmatrix} \underbrace{\begin{bmatrix} I_1 & & \phi \\ & I_2 & \\ \phi & & I_3 \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}}_{\begin{bmatrix} I_1\omega_x \\ I_2\omega_y \\ I_3\omega_z \end{bmatrix}} = {}^bM^{EXT}$$

$$I_1\dot\omega_x + \left[-\omega_z I_2\omega_y + \omega_y I_3\omega_z\right] = M_x^{EXT}$$

$$I_2\dot\omega_y + \left[\omega_z I_1\omega_x - \omega_x I_3\omega_z\right] = M_y^{EXT}$$

$$I_3\dot\omega_z + \left[-\omega_y I_1\omega_x + \omega_x I_2\omega_y\right] = M_z^{EXT}$$

$$\begin{cases} I_1\dot\omega_x + (I_3 - I_2)\,\omega_z\omega_y = M_x \\ I_2\dot\omega_y + (I_3 - I_1)\,\omega_x\omega_z = M_y \\ I_3\dot\omega_z + (I_3 - I_1)\,\omega_x\omega_y = M_z \end{cases}$$

The signs depend on the type.