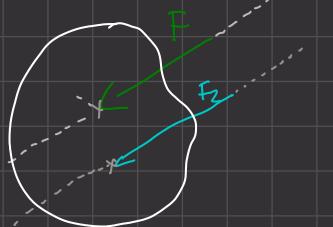
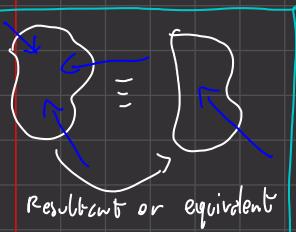


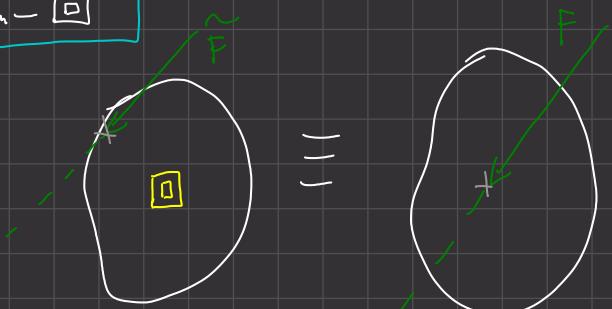
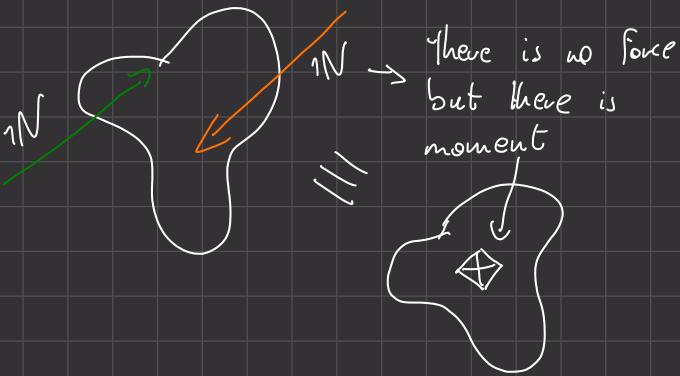
26/04/23



There is no difference in the mechanical effect if we apply  $P$  along the line:

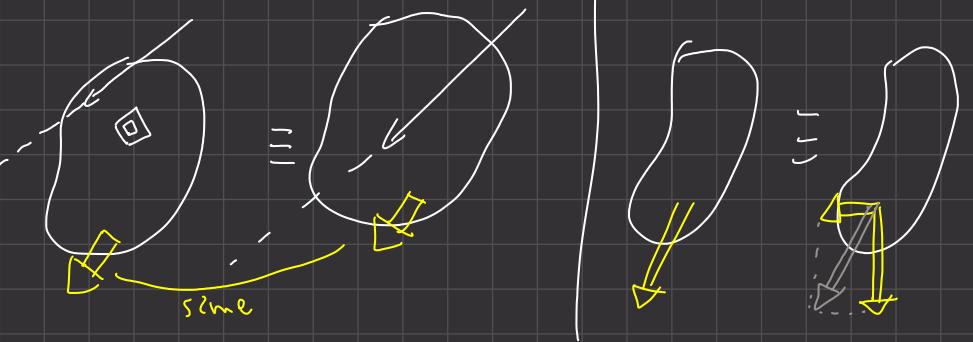


FORCE	MOMENT
↑ - side -	↑
○ - top -	⊗
⊗ - bottom -	□

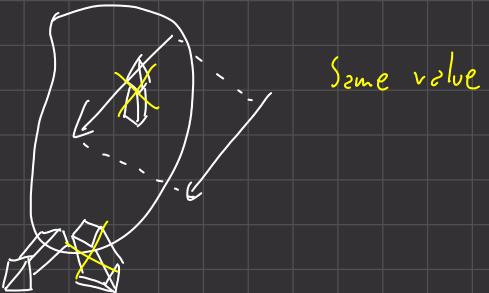


The moment does not allow the object to turn so the two systems are equal.

Let's see another example:

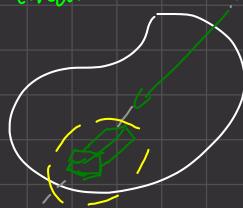


3D



It does not matter where you apply the torque. ? BAH

We can Represent any system with ONE force and ONE moment with the same direction



← this is the canonical representation of a class of equivalence  
or fundamental stat of statics

TOPO dict

The canonical representation is represented by  $\gamma$

$$\gamma = \begin{pmatrix} [N] \\ \vec{F} \\ \vec{m} \end{pmatrix} \equiv (\vec{F} | \vec{m}) \rightarrow N.O.$$

↑ force      ↑ moment

this is invariant with the reference frame.

This is also called WRENCH

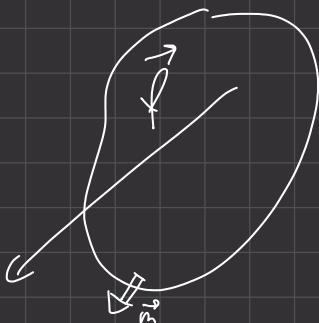
You cannot change that moment with a displacement of the force. The moment must be there to be in the class of equivalence

We also have:

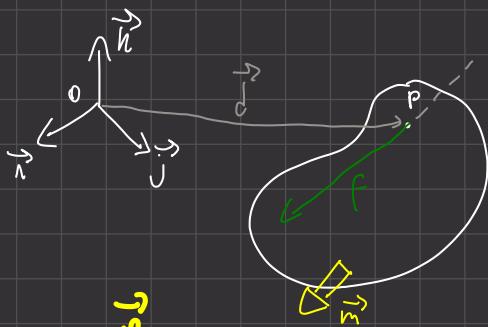
Special wrenches

// pure force  $\varphi = (\vec{F} | \vec{m}) / \vec{m} \perp \vec{F}$  (When the moment is orthogonal to the force)

// pure moment  $\mu = (\vec{o} | \vec{m})$



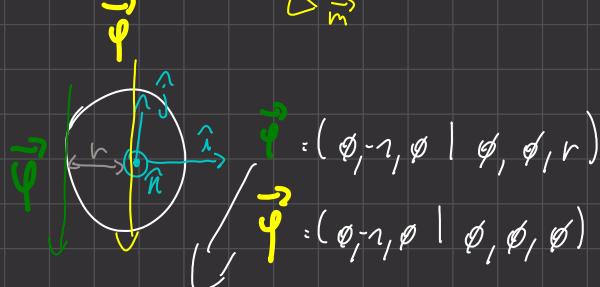
$$\begin{aligned} \vec{f} &= (f_x, f_y, f_z) \\ \vec{m} &= (m_x, m_y, m_z) \end{aligned} \quad \left. \begin{array}{l} \text{This does} \\ \text{not modify} \\ \text{where the} \\ \text{force is applied} \end{array} \right\}$$



$$\vec{d} = \vec{OP}$$

$$\vec{f} = (f_x, f_y, f_z)$$

$$\vec{m} = (m_x, m_y, m_z) + \vec{f} \times \vec{OP}$$

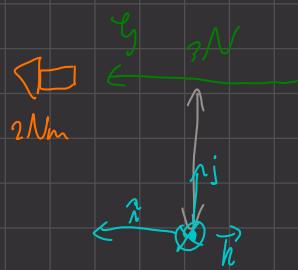


Modulus of  $\gamma N$

Example



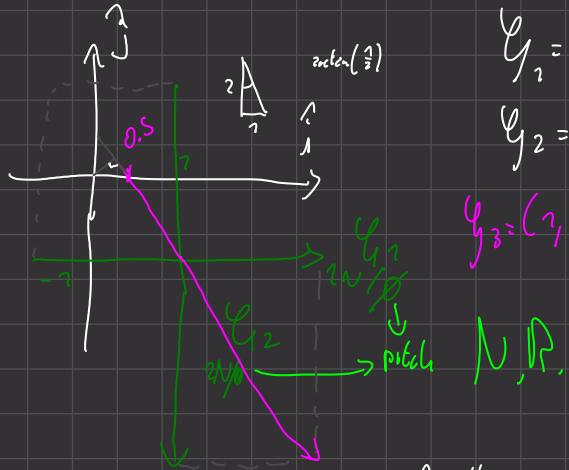
$$\Psi = (\phi, \psi, \theta | 2dc\beta, 2ds\beta, \phi)$$



$$\begin{aligned} \Psi &= (3, \phi, \psi | (2, \phi, \phi) + (\phi, \phi, -3d)) = \\ &= (3, \phi, \psi | 2, \phi, -3d) \end{aligned}$$

Every kind of movement on a rigid body can be replaced with 2 wrenches  
N.B.

28/08/23



$$\begin{aligned} \Psi_1 &= (2, \phi, \theta | 0, 0, 1) \\ \Psi_2 &= (0, -2, \theta | 0, 0, -2) \end{aligned} \quad \text{those are pure forces and can be summed}$$

$$\Psi_3 = (1, -2, \theta | 0, 0, \underbrace{\frac{d\delta s}{2}}_{-1})$$

N.B.

Another way of calculating  $\Psi_3$  is to sum

$$\Psi_1 + \Psi_2 \text{ and it will get the same result.}$$

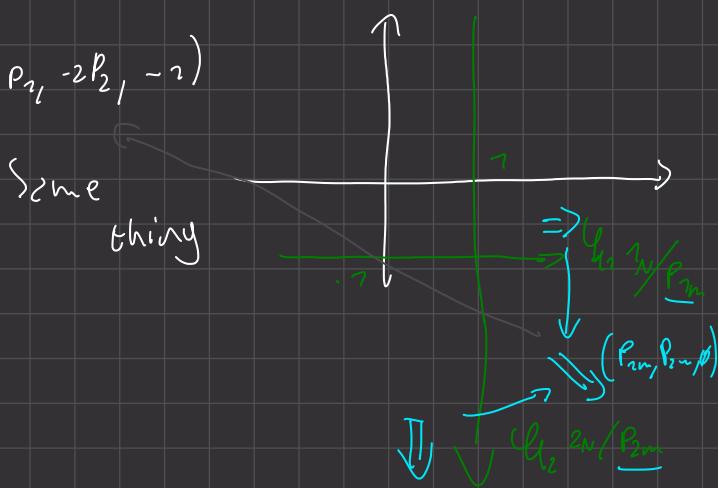
For a plane everything is good but in space lines could be skew. Summing up the components will also work in the space

Pitch is the amount of meters that a screw advances with a full turn.

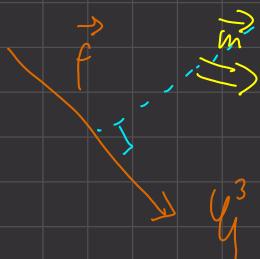
Let's now consider:  $\Psi_1 (2, \phi, \theta | P_1, \phi, 1)$

$$\Psi_2 (0, -2, \theta | \phi, -2P_2, -2)$$

Then  $\Psi_3 = (-1, -2, \emptyset | p_{-1}, -2p_2, -1)$



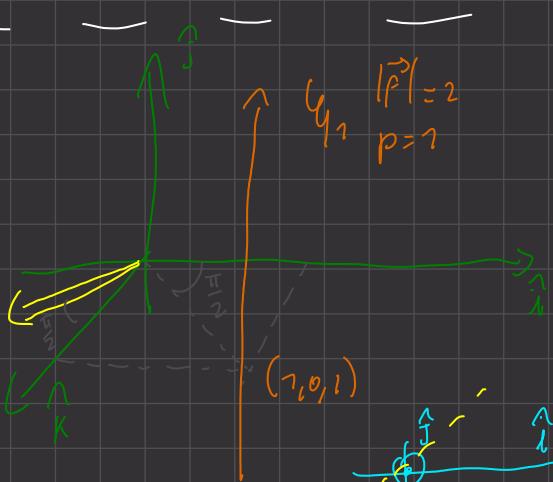
Decomposing the moment



The component along the direction of the force is:  $\frac{\vec{f} \cdot \vec{m}}{|\vec{f}|}$

So to get the moment that's orthogonal to the direction of the force you will do:

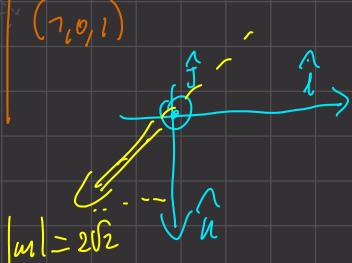
$$\Psi_3 = (\vec{f} \cdot \vec{m}) \rightarrow \vec{f} \rho + \vec{d} \times \vec{f}$$



$$\Psi_1 = (0, 2, 0 | -2, \emptyset, \sqrt{2})$$

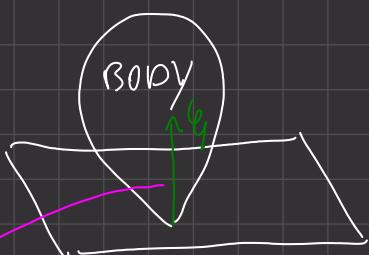
$$2 \cdot \frac{\sqrt{2} \cdot \sqrt{2}}{2}$$

but we also have pitch:



$$\Psi_1 = (0, 2, 0 | (-2, 0, 2) + \vec{f}_p) \\ (0, 2, 0)$$

$$\boxed{\Psi_1 = (0, 2, 0 | -2, 2, 2)}$$



The body is constrained to stay in contact with the plane. The plane is applying a force orthogonal to the direction of the plane. We cannot tell the intensity of the force.

Every interaction now should be described with a wrench. We can immediately say that we have 0 pitch.

$$\mathcal{W} : \left\{ \begin{array}{l} p = 0 \\ \end{array} \right.$$

Now we have an  $\mathbb{R}^3$  subspace in  $\mathbb{R}^6$  where there are many intensities so:

$$\mathcal{W} = (\phi, \theta, \varphi | \phi, \theta, \varphi)$$

Base of the vector space

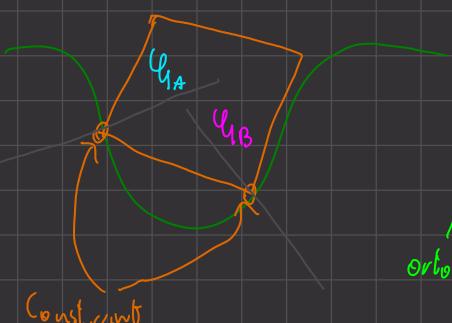
$\mathcal{W}$ RENCH SPACE = WRENCH SYSTEM

If the moment is orthogonal to the force then we have no spinning. This only tells us where the reference frame is.



$$\mathcal{W} = (\phi, \theta, \varphi | b_1, c_1, \varphi)$$

Classic example

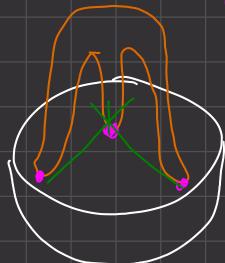


$$\mathcal{W}_C = \lambda_1 \mathcal{W}_A + \lambda_2 \mathcal{W}_B$$

Vector space of constant

Allowed motion is the one which is orthogonal to the vector space.

Contact points to the bowl.



$$\mathcal{W}_A = (a, b, c | 0, 0, 0)$$

$$\mathcal{W}_B = (d, e, f | 0, 0, 0)$$

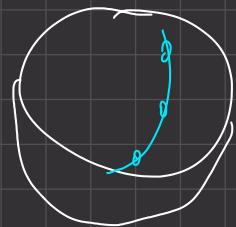
$$\mathcal{W}_C = (g, h, i | 0, 0, 0)$$

If we do a combination of these three we get:

$$\mathcal{W} = \lambda_A \mathcal{W}_A + \lambda_B \mathcal{W}_B + \lambda_C \mathcal{W}_C = (r, s, t | \phi, \theta, \varphi)$$

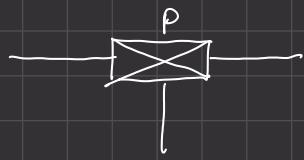
there is no possibility for this system (which is a ball joint) to do any translation because we have forces in any direction to block the movement.

If the three point of contact are on the same plane then they are linear dependent, so we don't have anymore a ball joint, because we have rocking.



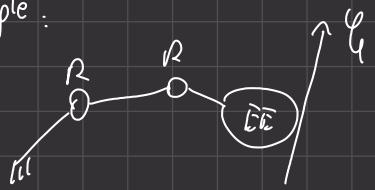
Constraint are of pitch screws

03/20/23



When you apply a wrench you consider an external wrench to the body

For example :



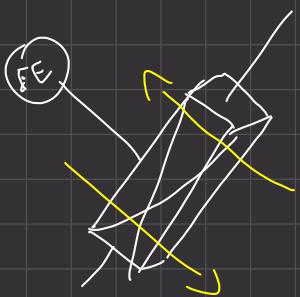
In this case I am applying the wrench between the ee and the ground. Everytime the wrench is applied between two bodies.



If we are now considering only static then now these wrenches are only applying constraints. We need to set on body through our constraints.

We want to understand how the prismatic joint constraint is. If we will try to apply an orthogonal force to the direction of translation then it will be rejected by another from the body.

Constraints of the prismatic joint



If you think of applying a force for every then you will have pure moment. So if the line is infinitely far away you get an infinitely strong moment. For this reason we get infinite pitch.

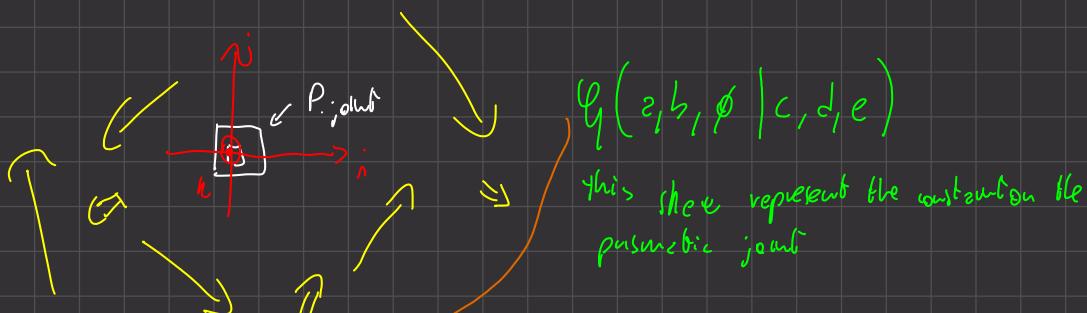
If you fix the moment that you want and start applying the force further and further then the force will go to 0 when the distance will tend to infinity.

When we go really far the position of the force doesn't really change much.

The moment is always on the line between the force and P.

The point is that we should focus only on the moment and not on the force.

You can apply moments with any intensity but you cannot move the sheaf of the prismatic joint.



It doesn't matter where you put the reference frame, the effect is the same.

N.R. In case of constraints we must consider  $\varphi$  as the repelling force or moment, not the applying force.

→ the space in which this 5-dimensional wrench exists is:

$\rightarrow \mathbb{P}$  So we get:  $\varphi \subset \mathbb{P}$  which is a 5-system.  
Geometric P

If we want to find the base:

$$\mathbb{P} = \text{span}(\varphi_x, \varphi_y, M_x, M_y, M_z)$$

If we want to express all the vectors of the base of the constraint:

$$\varphi_x = (1, 0, 0 | 0, 0, 0)$$

$$\varphi_y = (0, 1, 0 | 0, 0, 0)$$

$$\varphi_z = (0, 0, 1 | 0, 0, 0)$$

$$M_x = (0, 0, 0 | 0, 0, 1)$$

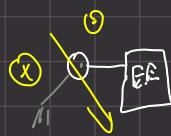
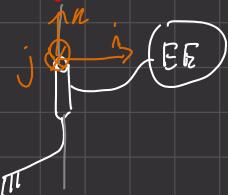
$$M_y = (0, 0, 0 | 0, 1, 0)$$

$$M_z = (0, 0, 0 | 1, 0, 0)$$

Constraints of the prismatic joint contain all the forces orthogonal to the direction of the shuttle of the prismatic and all the moments.

It doesn't really matter the position of the reference frame but needs to be on the direction of the joint.

Revolut joint



Applying here forces the door will not open.

The line in the revolut joint is called the invariant of the joint.  
Also the direction of the prismatic is called invariant.

All the lines that intersect the invariant of the revolut do not move the hinge.

If we want to apply moments that are orthogonal to the direction of the hinge will not work.

Only moments that are parallel to the direction of the hinge or forces that do not intersect the hinge and they will move the revolut.

The best reference frame to take is along the line of the revolut and with one axes on the line.

For the wrenches we have need to make a distinction:  
the ones who are parallel to the direction of the joint:

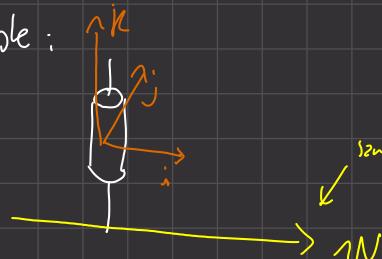
$$\Psi_1 = (\phi, \psi, \gamma | b, c, \theta) \quad \leftarrow \text{This is a subset of the next one}$$

And the ones that intersect the hinge of the door:

$$\Psi_2 = (d, e, f | l, m, \phi)$$

Note that the space is 5-dimensional, but it's not because we have 5 degrees of freedom but because they are independent from each other

For example:

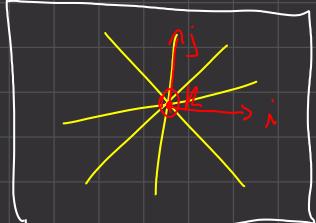


$$\Psi_1 (1, 0, 0 | 0, -\pi/2, 0)$$

1N

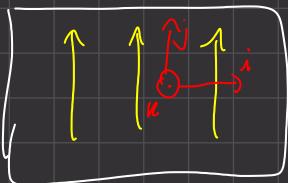
## Examples of systems dimension

Have all the forces directions go through the center.



$$\mathcal{Q}_1 = (\alpha, b, \phi | \emptyset, \emptyset, \emptyset)$$

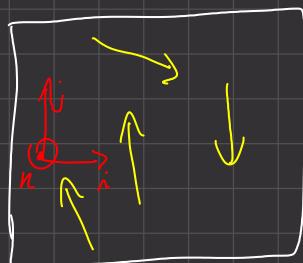
This is a 2-system.



$$\mathcal{Q}_1 = (\phi, \alpha, \beta | \emptyset, \emptyset, b)$$

2-system.

N.B. look at the reference

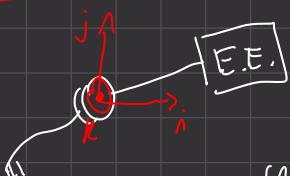


$$\mathcal{Q}_1 = (\alpha, b, \phi | \emptyset, \emptyset, c)$$

3-system

## Spherical joint

AN INFINITE PITCH FORCE IS A MOMENT

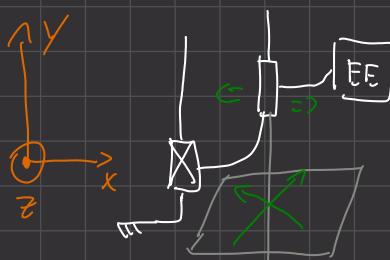


Any pure force going through the center will be rejected. So:

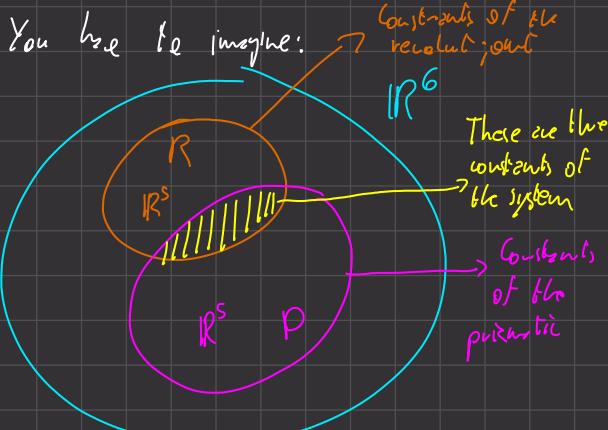
$$\mathcal{Q}_1 = (\alpha, b, c | \emptyset, \emptyset, \emptyset)$$

3-system

More than one joint in series



Try to visualize the constraint forces we can see.

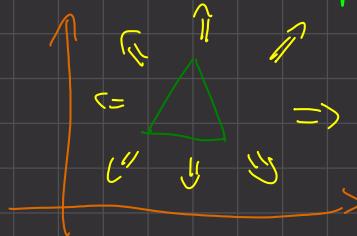


The structural constraint will be:  $\mathcal{Q}_1 = (\alpha, \phi, b | m, \emptyset, n)$

OS/10/23

Planar mechanism is something that stays on a plane while moving. To do this you need constraints and so wrenches.

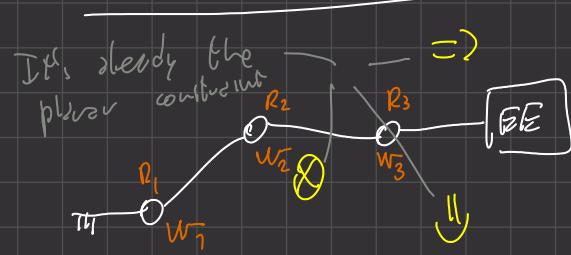
This constraint is called planar constraint.



this moment spans a 2 dimension vector space.

We need to find the number of vectors that keep the object on the plane.

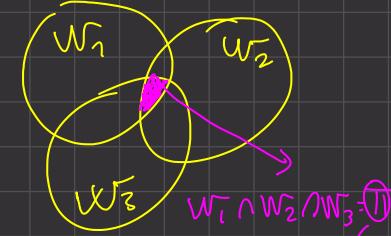
$$\phi \rightarrow \Psi = (\phi, \theta, z | b, c, \rho) \rightarrow \text{planar constraint}$$



$$3 | R$$

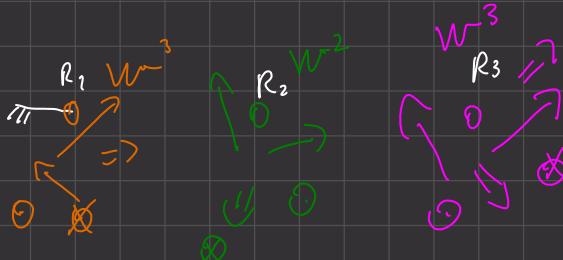
$$W_1 = \begin{cases} \text{span } (q_1, q_2) & (q_3 \text{ not curving } R_1) \\ \text{system} \end{cases}$$

We have 3 invariants



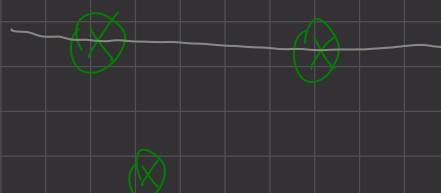
Planar constraint

We are in a planar constraint



The wrenches and the forces are by plane

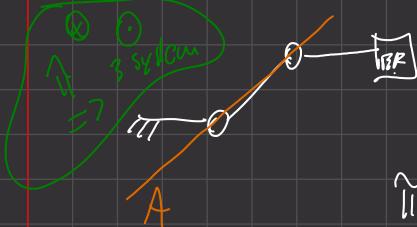
The vertical lines do not belong to the invariance because



Only two can be on the same place.

If we now consider

$$2 | R$$



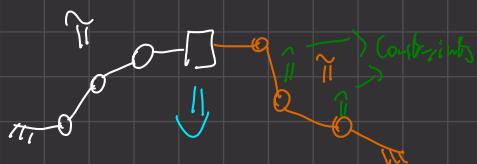
we call here / in the constraints so  
now it's a system

$$\tilde{\Pi} \cap A = \emptyset \rightarrow \tilde{\Pi} + A \text{ is a 2-q system.}$$


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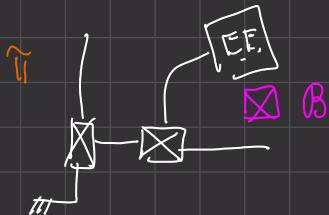
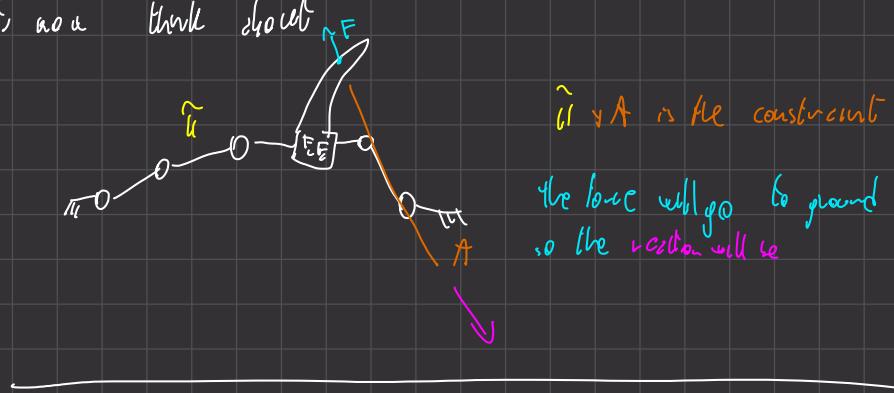
Let's now see this 2D parallel system

These two (we mean  
both what's inside  
the parallel invariant  
lines)



$\tilde{\Pi}$  and  $\tilde{\Pi}$  are applied in parallel so at the end the system is behaving like there was only one  $\tilde{\Pi}$ . So the constraint is the same but the forces are distributed in a different way.

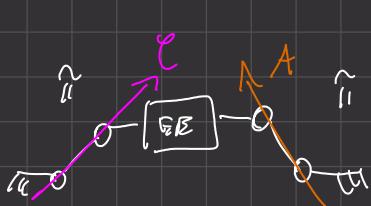
Let's now think about



$\tilde{\Pi} + B$  is a q-system.

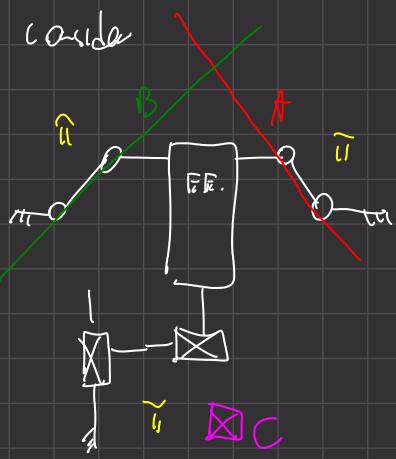
They don't need to be orthogonal but it's easier because the momenta are decoupled

---



$\tilde{\Pi} + A + C$  is a S-system.

If you consider

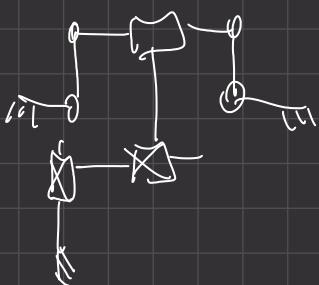


$\text{I}_1 + \text{A} + \text{B} + \text{C}$  is a 6-system.

C cannot be produced by A and B couple so it's another constraint.

If they were parallel then this would be 5-system.

If you consider

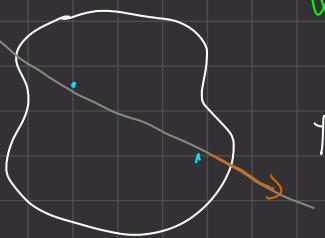


This is a system and the span of the intersection is C  
so

$$\text{I}_1 + \text{C} \cap \text{I}_1 + \text{A} + \text{B} = \text{span}(C)$$

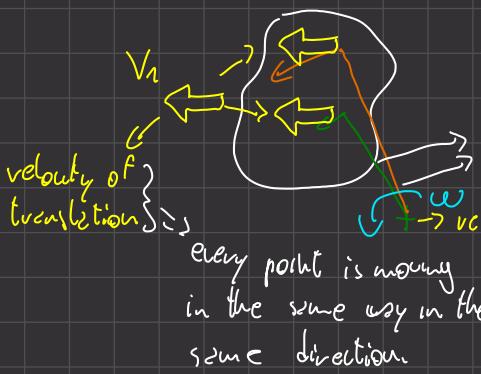
20/10/23

We are talking about instantaneous speeds



There can be only one speed in one direction in the body

Chasles realized that whatever is the distribution of velocities the body is spinning around a line in space and translating along the line.



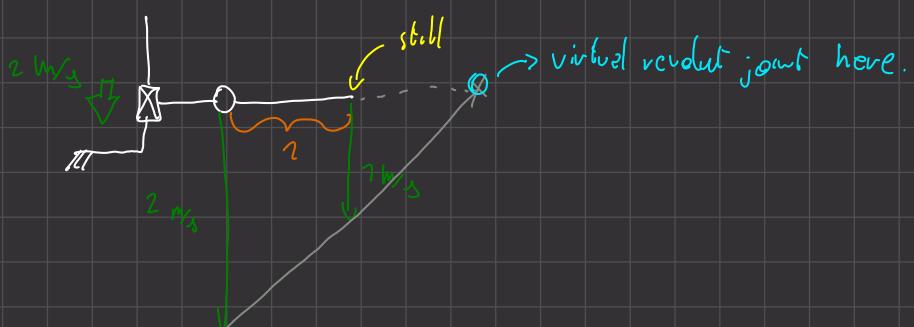
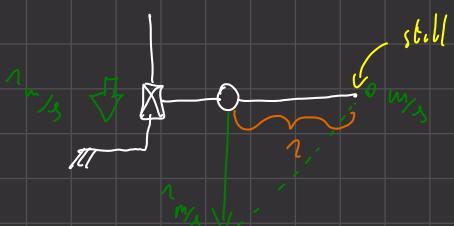
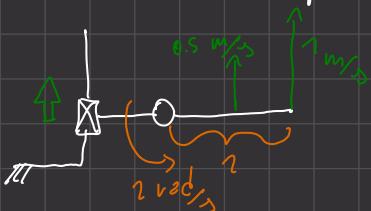
Obviously the further you are from the hinge the more speed you have.  $\omega$ , angular velocity is the same.

$V_1 = (\omega_1, v_1)$  is the combination of the two velocities.

Let's now consider another hinge.

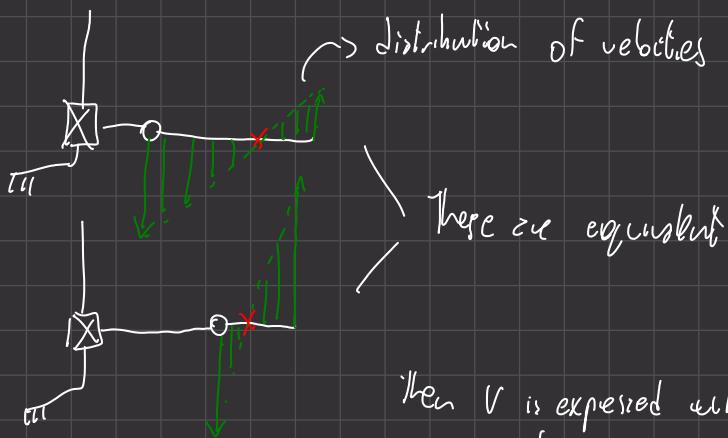
I can have the same motion by rotating around another hinge and changing the translational direction I will get the same movement.

Let's do another example:



If now I move the joint along the "arm" I can get different combinations of  $\omega$  and  $v$  such that the point will have the same movement

$$(\omega_1, v_1) = (\omega_2, v_2) = (\omega_3, v_3) \rightarrow \text{class of equivalence}$$



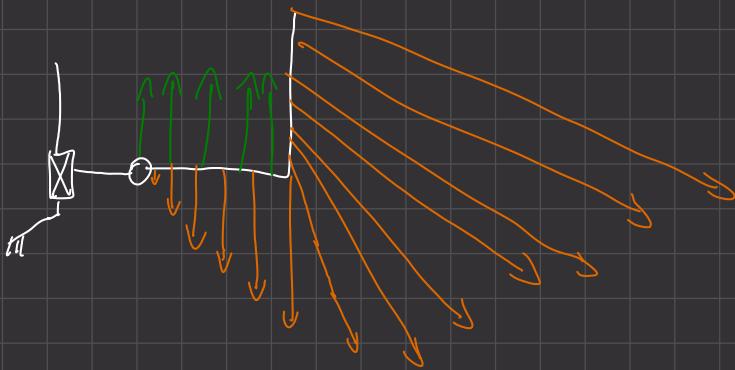
Then  $v$  is expressed with a parallelogram



We are building a class of equivalence  $\rightarrow$  economical representation

↪ twist:  $\xi [v, \omega]$

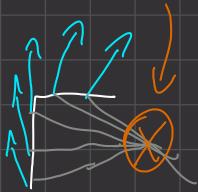
This is a representation of speed in the body and this distribution works for instantaneous movement.



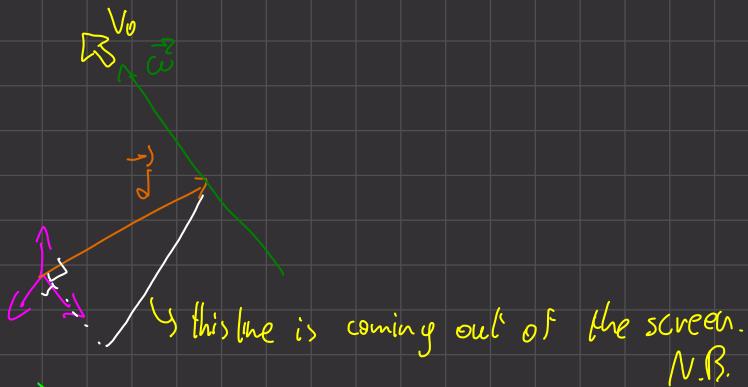
If you add the two then you get the absolute speed.

the point in which the endpoints of the resultant of the speeds meet is called:  
IAR (Instantaneous Axis of Rotation)

For example:



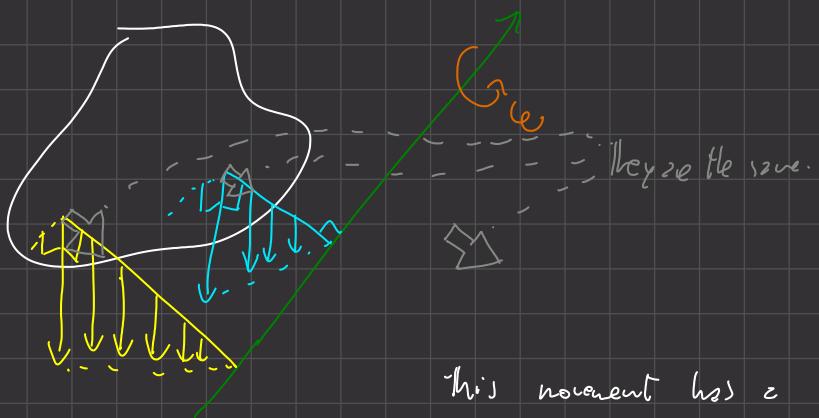
$$\xi = (\vec{w} | \vec{v})$$



$$V = V_0 + \vec{d} \times \vec{\omega}$$

$$V_\theta = \rho \omega \downarrow_{\text{pitch}}$$

Speed is a free vector, the rotations are not. You can add translations with parallelism but not the rotations.



this movement has a pitch.

Now we can pose the question: what is the velocity of a body?

Now we have tools to describe this movement.

$$\vec{v}_e = \left( \vec{l} \mid \vec{m} \right)$$

$$\vec{v}_e = \left( \vec{\omega} \mid \vec{v} \right)$$

line matters just direction matters

Vectors in  $\mathbb{R}^6$ 
 $P = \frac{|m|}{|\vec{l}|}$ 
 $N.B.$

$P = \frac{|v|}{|\vec{\omega}|}$

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$$\vec{v}_e = \vec{v}_{EE} \quad / \quad P=0 \quad / \quad \omega = \gamma \rightarrow$$

If you want to model the joint is convenient to give a twist with  $\omega = \gamma$

Now if we take a motor and attach it to the joint with 300 rpm

$$\text{then } \vec{v}_{em} = \vec{v}_{300}$$

We are using a unit length vector like  $(1, 0, 0)$  or  $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)$

So we are normalizing just the omega:  $\vec{\omega} (\vec{\omega} \mid \vec{v})$   
↳ s.t.  $|\vec{\omega}| = 1$

We use omega for direction and V for intensity.

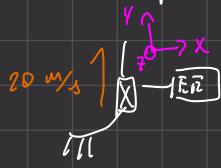
direction position of spinning of line.

$$\vec{v}_e = (\phi, \psi, \gamma \mid \theta, \phi, \psi)$$

motorize the R joint  
motor speed = 200 RPM

speed of the EE =  $200 \text{ deg/min}^6$

Polar metric joint

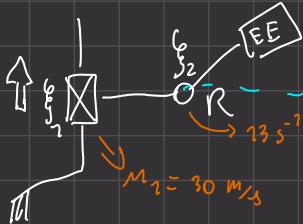


$$\dot{\theta}_j = (\vec{\omega} | \vec{V}) \quad r = \infty$$

$$\dot{\theta}_m = 20 \dot{\theta}_j = 20 (\vec{\omega} | \vec{V})$$

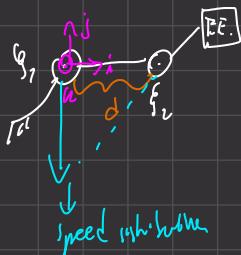
We prefer  $|\vec{V}| = r$

$$\dot{\theta} = 20 (\phi, \theta, \rho | \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)$$



$$\dot{\theta} = 20 \dot{\theta}_1 + 73 \dot{\theta}_2$$

Movement is linear in velocity  
TAR will be on this line



$$\dot{\theta}_{EE} = \omega_1 \dot{\theta}_1 + \omega_2 \dot{\theta}_2$$

$$\dot{\theta}_1 = (\phi, \theta, \gamma | 0, 0, 0)$$

$$\dot{\theta}_2 = (\theta, \phi, \gamma | \dot{\phi}, -\dot{\theta}, \dot{\gamma})$$

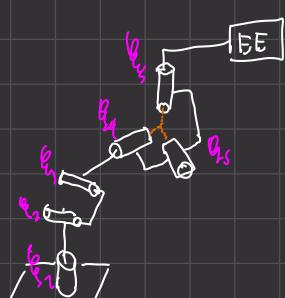
You get this value by trying to have  $r_2$  fixed and moving  $\gamma$  line above the origin of the axis.

Focused  
kinematic  
problem



$$\dot{\theta} = (0, 0, \gamma | z_1 h, 0) \quad d = \sqrt{z_1^2 + h^2}$$

Let's now consider a 6R robot



$$\dot{\theta}_{EE} = 2 \dot{\theta}_1 - 5 \dot{\theta}_2 + 20 \dot{\theta}_3 + 72 \dot{\theta}_4 - 7 \dot{\theta}_5 + 3 \dot{\theta}_6$$

Motor	speed
1	2
2	-5
3	20
4	12
5	-7
6	3

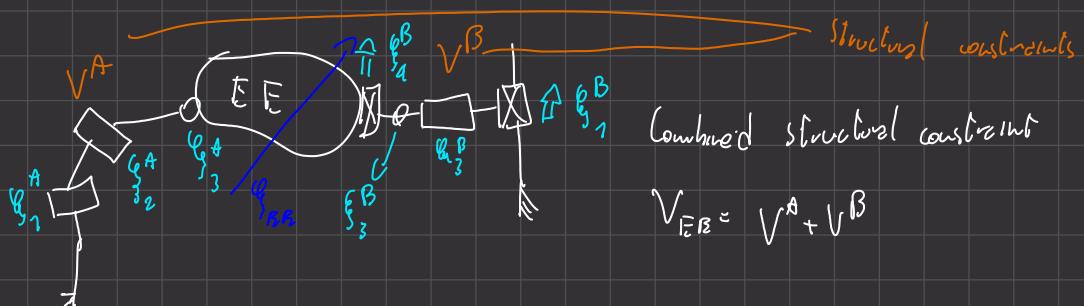
## Geometric Jacobian

$6 \times 6$  matrix

Joint velocities  $\rightarrow$  EEE velocities

$$\begin{bmatrix} \dot{\xi}_1 & \dot{\xi}_2 & \dot{\xi}_3 & \dot{\xi}_4 & \dot{\xi}_5 & \dot{\xi}_6 \end{bmatrix}_{\text{EEE}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \\ \omega_6 \end{bmatrix}$$

depends on configuration parameters only



We can imagine an imaginary kinematic chain (mechanism)



Let's now make an analogy with velocity and discuss what happens.

The mechanism is consistent between the two chains so:

$$\dot{\xi}_{\text{EEE}} = \omega_1^A \dot{\xi}_1^A + \omega_2^A \dot{\xi}_2^A + \omega_3^A \dot{\xi}_3^A = V_1^B \dot{\xi}_1^B + V_2^B \dot{\xi}_2^B + V_3^B \dot{\xi}_3^B + V_{EE}^B$$

$M^A = \text{span}(\dot{\xi}_1^A, \dot{\xi}_2^A, \dot{\xi}_3^A) \rightarrow$  If they are linearly independent then it's a 3D system.

The  $\dot{\xi}_{\text{EEE}}$  create = span of the feasible movements.

Any feasible or allowable  $\dot{\xi}_{\text{EEE}} \in M^A$

Vector space in  $\mathbb{R}^6$ .

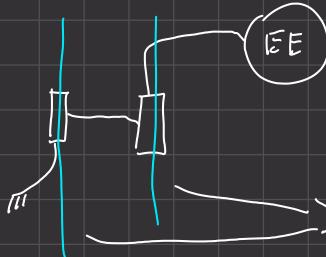
$M^B = \text{span}(\dot{\xi}_1^B, \dot{\xi}_2^B, \dot{\xi}_3^B, \dot{\xi}_4^B) \rightarrow$  if they are linearly independent then we have a 4D system.

So Any feasible  $\dot{\xi}_{\text{EEE}} \in M^B$

Concluding  $\dot{\xi}_{\text{EEE}} \in M^A \cap M^B$

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## Systemic Reasoning



Our way to abstracting the mechanism.

If we extract the systemic view we get:



The twist is the material thing and the pitch is zero like the speed is coming. The pitch is a geometrical property of the line.

The abstraction is that now we start from lines and geometry.

The advantage is that now we can use linear algebra.

If we now use pitch we can use vector. and vector spaces (systems).

If vectors in  $\mathbb{R}^6$  then they are twists and they can create 2 space.

For example the two lines create a vector space.

If we do a linear combination of the two lines then we get another line we get a resulting line:

$$p=0$$

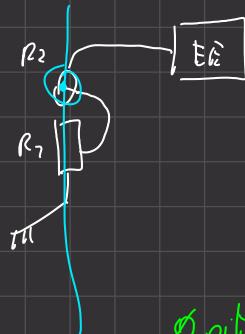
We should try to not rely on numeric projections but learn to reason with vectors.

The one line represent the feasible speeds that the ee. can do.

The blue lines represent the distribution of velocities in the body in some way.

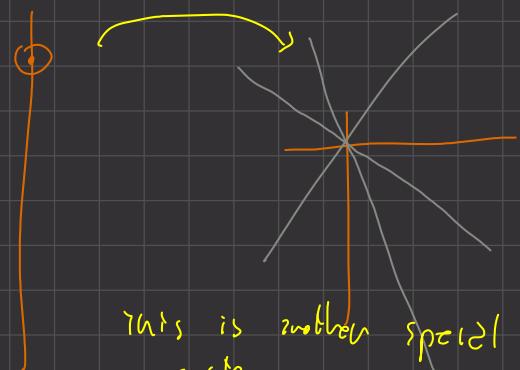
The strange thing is that we can pull the resulting line every where in the space by playing with the intensity of the red lines.

Let's do another example



example

seen from the side



This is another special system

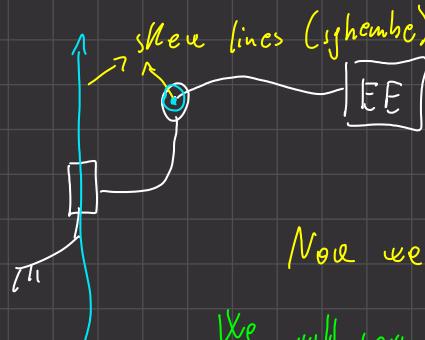
You can generate any line that passes through the intersection of the lines and the angle will be determined by the intensity of those.

Keep in mind that we are always talking about instantaneous movements.

Note: if you change the configuration you change the geometry

For example in the previous example if you rotate  $v_1$  the plane will change.

Let's now consider lines that are not touching:



finite pitch.

Now we have a general system

We will now get a finite pitch N.B.

linca

These are all the trajectories created by the two lines.

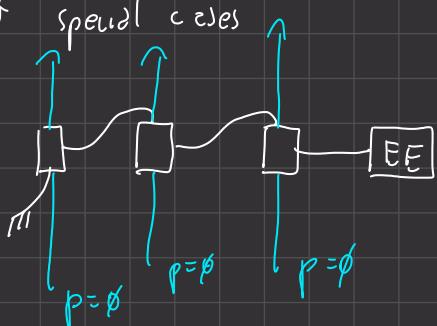
All these lines have a non-zero pitch.

To look deeper in these things look at ruled surfaces.

Using screws we generate ruled surfaces

Now we will step into the three systems.

First special cases



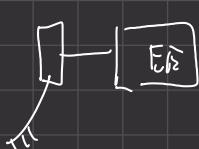
These 3 lines are linearly dependent so this system is a 2-system.

this is a 3 degree of freedom mechanism

BUT

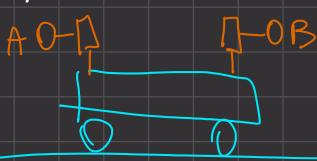
the degrees of freedom of the EE. is 2.

The missing dof is now gone in one of the other lines.



The degree of freedom of e.e. =  
 $\dim(\text{span}(\Phi_R)) = 7$

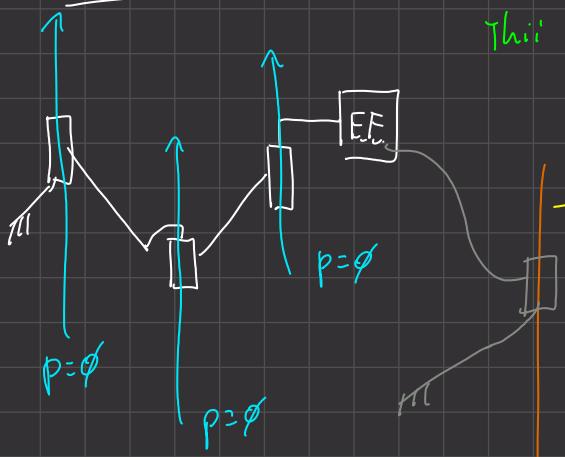
Example of degrees of freedom



Degrees of freedom of the system: 3

Dof of A = 2

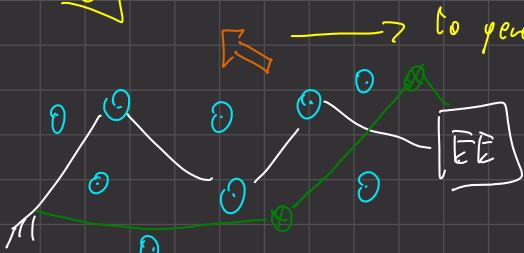
Dof of B = 2



This is now a 3-system.

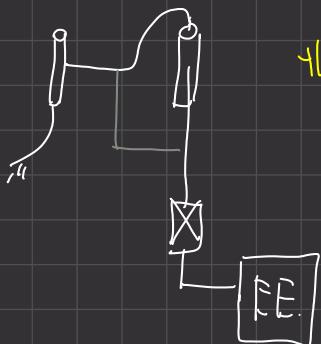
→ We can find a line parallel to the others that can represent the motion.

( ) seen from above

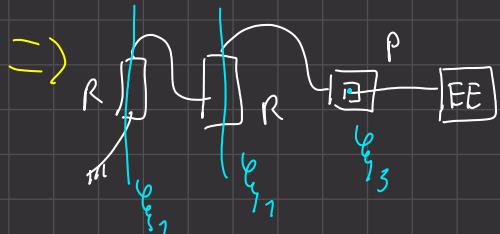


→ To generate this translation you can take any two virtual hinges

In this system you can generate  $\theta$  or infinite pitch motions.

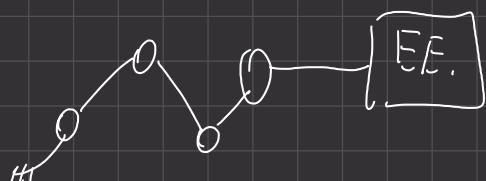
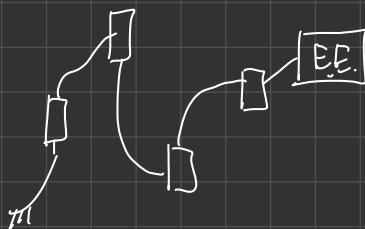


this is still a 2-system.



NB To visualize the lost degree of freedom imagine Fixing the E.E.  
Now  $R_2$  will be still able to move

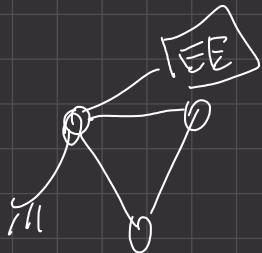
A singularity is when you lose some degree of freedoms.



N.B. For the system the dof is 4

FOR THE EE IS 3

Another example is :



Dof of the EE is still 3

This is the case of  
Wrench the case of

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Constraint  $w$  wrenches up to 6 sys

EE. Allowable inst. motion (velocity) twists up to 6 sys

N.B.

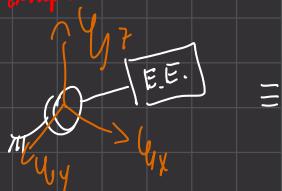
Constraints are reciprocal to velocity  
"orthogonal"

$w_{EE}$   
 $A_{EE}$

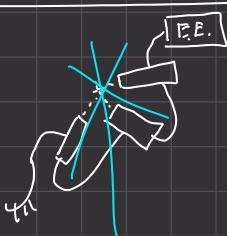
N.B.

$$\dim w_{EE} + \dim A_{EE} = 6$$

Joint example



=



Any pure force passing through the center of the joint belongs to the constraint.

$$W = \text{span}(\varphi_x, \varphi_y, \varphi_z) \quad 3 \text{ sys} \quad \left. \begin{array}{l} \text{These systems are the same lines.} \end{array} \right\}$$

$$A_0 = \text{span}(p_x, p_y, p_z) \quad 3 \text{ sys} \quad \left. \begin{array}{l} \text{These systems are orthogonal} \\ (\text{they are not "working together"}) \end{array} \right\}$$

twist, with pitch  $\rho$  so no everything is brought with rotation

If we due  $\omega$  a general wrench we get:  $\varphi = (z, b, c | 0, 0, 0)$

$$\varphi_x = (1, 0, 0 | 0, 0, 0)$$

$$\varphi_y = (0, 1, 0 | 0, 0, 0)$$

$$\varphi_z = (0, 0, 1 | 0, 0, 0)$$

$$\varphi \in W$$

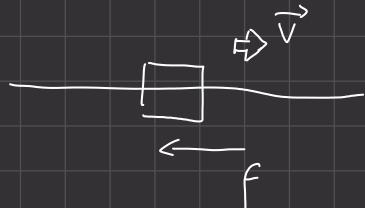
$pGA$

General  $p = ( \overset{\omega}{\underset{m,n}{|}} \underset{0,0,0}{})$

Power =  $\vec{\omega} \cdot \vec{m} + \vec{v} \cdot \vec{f}$



power =  $\omega m \rightarrow$  because in the same direction



$$\text{power} = vf$$

The question is how do we calculate the instantaneous power in the E.E.

If we take any twist and consider where the pitch could be 0



We can calculate the reciprocal product similar to the scalar product

RECIPROCAL PRODUCT (gives us the power of the E.E.)

$$\varphi \circ \xi = (\vec{f} | \vec{m}) \circ (\vec{\omega} | \vec{v}) = \vec{f} \cdot \vec{v} + \vec{\omega} \cdot \vec{m}$$

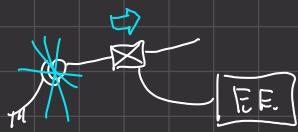
↓  
scalar in  $\mathbb{R}^3$

Let's now try to apply this to the spherical joint:

$$\rho \circ \varphi = \left( l, m, n \mid \begin{matrix} 0, 0, 0 \\ z, b, c \end{matrix} \right) \circ \left( \begin{matrix} \phi, \theta, \psi \\ \varphi, \psi, \phi \end{matrix} \right) = (l, m, n)(\phi, \theta, \psi) + (\phi, \psi, \theta)(z, b, c) = \varphi$$

no power

Now let's attach a prismatic joint to the spherical joint:



$$W = \text{span}(\varphi_1, \varphi_2)$$

$$A_0 = \left. \begin{array}{l} (1, 0, 0 | 0, 0, 0) = \varphi_x \\ (0, 1, 0 | 0, 0, 0) = \varphi_y \\ (0, 0, 1 | 0, 0, 0) = \varphi_z \\ (0, 0, 0 | 1, 0, 0) = \tau_x \end{array} \right\} = \text{span}(\varphi_x, \varphi_y, \varphi_z, \tau_x)$$

$$\text{dim}(A_0) = 4$$

$$\text{dim}(W^\perp) = 2$$

$$W^\perp = \text{span}(\varphi_1, \varphi_2)$$

Dof of system = number of joints

Dof of E.E. = You also have to notice if some Dofs are linear dependent.

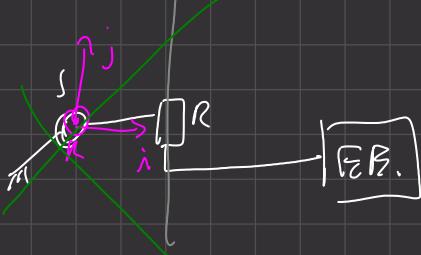
Any force whose direction passes through the center of the spherical joint and is orthogonal to the direction of the prismatic belongs to  $W^\perp$ .

$\varphi = (\phi, \underline{z, b} | \theta, \psi, \phi) \rightarrow 2\text{-system}$   
 $\hookrightarrow$  C.N. nob C.S. to be 2 system.

$$\text{At the end } A_0 = \text{span}((l, m, n | p, \theta, \phi))$$

$$\xi = (l, m, n | p, \theta, \phi)$$

$$\varphi_{EE} \circ \xi_{EE} = (\phi, \underline{z, b} | \theta, \psi, \phi) \circ ((l, m, n | p, \theta, \phi)) = (\phi, z, b) \cdot (\theta, \psi, \phi) + (\phi, \psi, \theta) \cdot (l, m, n) = \varphi$$



$$W = \text{span}(\xi_1, \xi_2)$$

$$\dim(A_0) = 4 \Rightarrow \dim(W) = 2$$

$$S = \left\{ \begin{array}{l} \xi_1 = (1, \phi, \theta | \phi, \theta, \phi) \\ \xi_2 = (\phi, 1, \theta | \theta, \phi, \theta) \\ \xi_3 = (0, 0, 1 | \phi, \theta, \phi) \end{array} \right.$$

$$R = \left\{ \xi_4 = (\phi, 1, \theta | \theta, \phi, \phi) \right\}$$

Not linear dependent

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

You can find a minor of rank(4)

Links in the constraint must go through the center of the spherical and be in the  $i, j$  plane.

$$\xi_{EE} = (z, h, c | \phi, \theta, \psi)$$

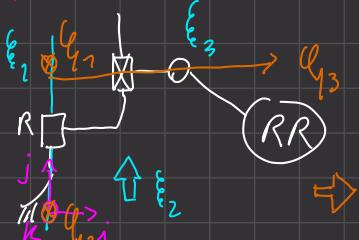
$$\xi_{EE} \circ \xi_{EE} = \phi$$

$$\xi_1 = (1, 0, 0 | 0, 0, 0)$$

$$\xi_2 = (0, 1, 0 | 0, 0, 0)$$

$$\xi_{BB} = (z, h, \phi | 0, 0, \theta)$$

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$$\xi_1 = (0, z, \theta | 0, 0, 0)$$

$$\xi_2 = (0, 0, 0 | 0, 0, 1)$$

$$\xi_3 = (\phi, 0, 0 | z, h, \theta)$$

- open chain mechanism

\* count the joints  $\rightarrow 3$  joints  $\rightarrow \text{total dof} = 3$

\* check if joints are linearly independent or dependent

$$\dim(\text{span}(\xi_1, \xi_2, \xi_3)) = 3 \quad \text{or} \quad \dim(\xi_1, \xi_2, \xi_3) = 3$$

- if  $\text{dof} = 3 \rightarrow W$  is a 3-system

$$\hookrightarrow \xi_1, \xi_2, \xi_3 \text{ s.t. } W = \text{span}(\xi_1, \xi_2, \xi_3)$$

$A, W$  are reciprocal  $\Leftrightarrow$  take any  $\xi \in A$  and  $\xi \in W \rightarrow \xi \circ \xi = \phi$

- Is there any pure force or moment reciprocal to all the joints
- Moments are always orthogonal to prismatic joints, and orthogonal to resultant if intersecting the moment or parallel
- Pure forces are orthogonal to resultants if intersecting <sup>V</sup> the moment and orth. to prismatic if orthogonal to the moment.
- Moment have a direction but no line. So for one direction there is only one moment

Let's now write  $\mathbf{q}_s$ :

$$\left. \begin{array}{l} \mathbf{q}_1 = (\phi, \theta_1, \omega, \dot{\theta}, \dot{\phi}, \dot{\omega}) \\ \mathbf{q}_2 = (\theta, \theta_2, \phi, \dot{\theta}, \dot{\phi}, \dot{\theta}) \\ \mathbf{q}_3 = (1, \phi, \theta, \dot{\phi}, \dot{\theta}, \dot{\omega}) \\ \mathbf{q}_4 = (\theta, \phi, \theta, \dot{\theta}, \dot{\phi}, \dot{\theta}) \end{array} \right\} \text{Base} \quad \left. \begin{array}{l} \mathbf{q}_{EE} = (r, \phi, b, \omega, \dot{\phi}, \dot{\omega}) \\ \mathbf{q}_{EB} = (\theta, \theta_1, r, \dot{\theta}, \dot{\theta}_1, \dot{b}) \end{array} \right\}$$

If we change the reference frame the numbers change but not the concepts.

If you parameterize everything then the reciprocal product becomes  $\phi$ .

$$\mathbf{q}_1 = (0, 1, \theta, \dot{\theta}, \dot{\phi}, \dot{\omega})$$

$$\mathbf{q}_2 = (0, \theta, \theta, \dot{\theta}, \dot{\theta}, \dot{\theta})$$

$$\mathbf{q}_3 = (\phi, \theta, \theta, \dot{\phi}, \dot{\theta}, \dot{\theta})$$

$$\mathbf{q}_{EE}^0 \quad \mathbf{q}_{EB} = (\phi, \theta, \theta, \dot{\phi}, \dot{\theta}, \dot{\theta})$$

$$\vec{\omega} \times \vec{j} + \vec{p} \vec{\omega}$$

$\nwarrow$  pitch?

parametrization

$$\left( \begin{matrix} \omega \\ t \\ \theta \end{matrix} \right) \cdot \left( \begin{matrix} I_{\theta} & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & 0 \end{matrix} \right) = \frac{m}{s} \left[ \begin{matrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{matrix} \right]$$

$$\left( \begin{matrix} \omega \\ t \\ \theta \end{matrix} \right) \cdot \left( \begin{matrix} I_{\theta} \\ \sqrt{\omega^2 + t^2} \\ \sqrt{\omega^2 + t^2} \end{matrix} \right) = \frac{m}{s} \left[ \begin{matrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{matrix} \right]$$

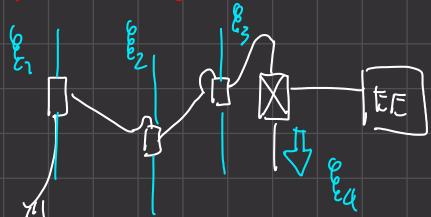
↳ pitch

$$(\omega, t, \theta) - p(\phi, \alpha, r) = \vec{\omega} \times \vec{j}$$

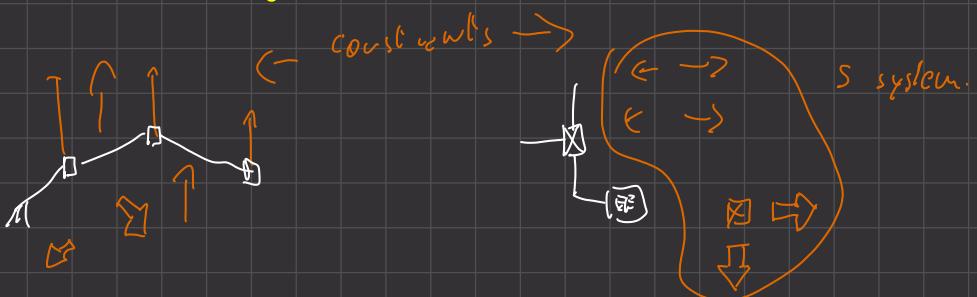
We have projected the components of the moment on the axis and then we substituted this from \_\_\_\_\_ to get the position of the line in space

The order of movements is very important because the end result is different.

### SCARA ROBOT Example | RRR|P

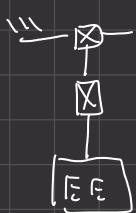
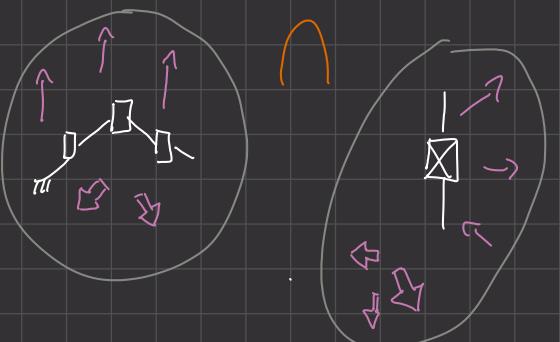
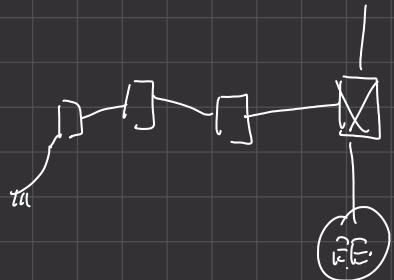


Constraints of



For the constraint you need to do the intersection of the two

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→ This force will move the system

To be in the constraint of the whole chain the constraint has to be in all the joints of the system.

If we interested the two previous systems we get ~~any~~ movement in the horizontal place: the dimension of this system is 2.

The end effector so won't be able to make any rotation around any line parallel to the horizontal place.

When we operate the other joints you move the other dof?

If you combine the speeds of this motors you can get any speed of the end effector.

$A$  is a-system  $\xi_{EE} \in A$

$W: 2\text{-dys}$   $\gamma_{EE} \in W$

$$A = \text{span}(\xi_1, \xi_2, \xi_3, \xi_4)$$

$$W = \text{span}(\mu_x, \mu_y)$$

The speeds of the motors are:  $-\omega_1$ ,  $-\omega_2$ ,  $-\omega_3$ ,  $-V_4$  } scalars

The generic feasible EE twist can be described fully by the 4 scalars.

This is true because the 4 joints are linearly independent.

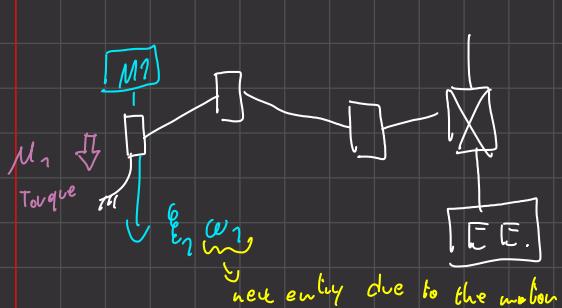
$$\omega_{\text{EE}} \circ \gamma_{\text{EE}} = \emptyset = \omega_1 \epsilon_{1,0} \gamma_{\text{EE}} + \omega_2 \epsilon_{2,0} \gamma_{\text{EE}} + \omega_3 \epsilon_{3,0} \gamma_{\text{EE}} + \omega_3 \epsilon_{q,0} \gamma_{\text{EE}}$$

2 scalar equation

$$\gamma_{\text{EE}} = m_x \mu_x + m_y \mu_y$$

Now we must find a force that is perpendicular to every wrench in the system to find an allowable motion.

---



Every upper class needs as many motions as joints if there are no singularities

$\leftarrow W$

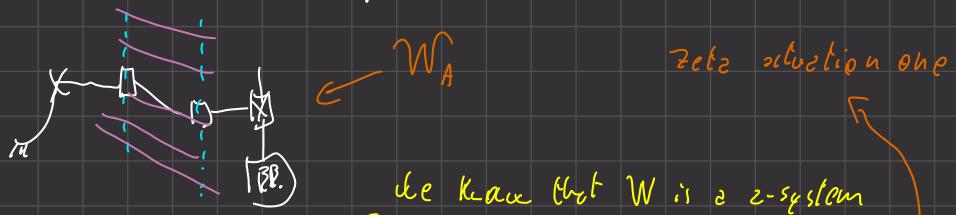
Redundant means that there are more dof in the chain than in the EE.

$[M_1]$  compounds the angle between the first joint and the ground.

torque moves through the

To evaluate how much the robot we will look at the first motor.

Looking at the first motor we get:



Now we work out  $W_{\alpha_1} = 3. \text{system} = W + \text{span}(\epsilon_{\alpha_1})$

$$W \subseteq W_{\alpha_1}$$

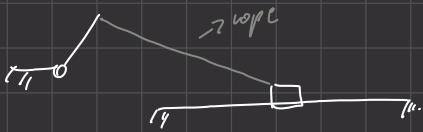
wrench orthogonal to the remaining joints  
but it wasn't already in  $W$ .

The next constraint is any line passing through the invariants of the wrenches and orthogonal to the direction of the prismatic.

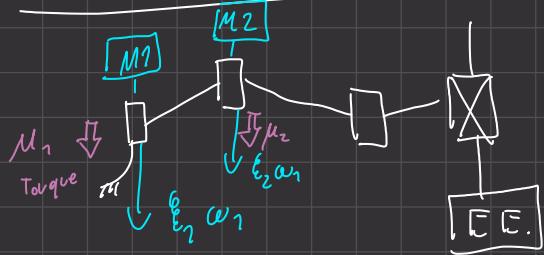
Note the intersection between  $W_{\alpha_1} \cap W$  is any moment produced by a pair of these faces

## Mechanical gain

NOTE



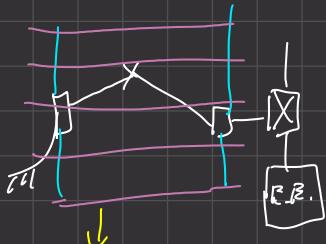
In this configuration if you turn 1 degree the crank, you will need 2 different mechanical movement in the slider instead of another configuration.



If you take the second motor you have to forget the first one. Now we look the second motor and now:

$$W_{a_2} \text{ is } 3\text{-sys} = W + \text{span}(\epsilon_{1a_2})$$

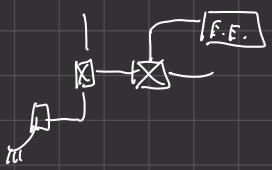
Now the system is



there are the new  $\epsilon$  added to the constraint.

You can also see these lines as the movement that you apply to the end effector when you move the locked motor.

The cylindrical robot  $a_2$  like this:



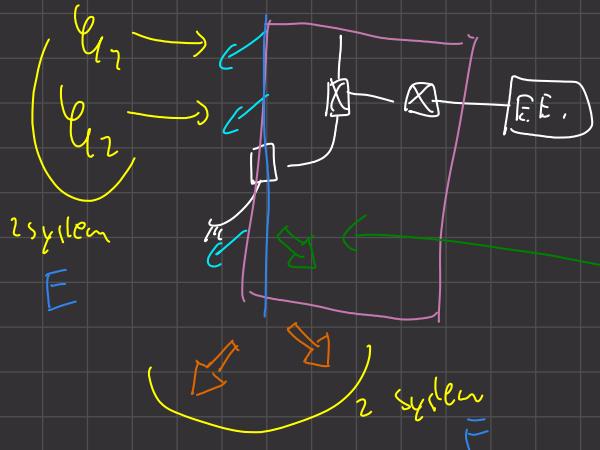
$$A = \text{span}(\epsilon_1, \epsilon_2, \epsilon_3) = \text{span}(\varphi_1, \tilde{\tau}_2, \tilde{\tau}_3)$$

$$\dim(A) = 3$$

$$\dim(W) = 6 - \dim(A) = 6 - 3 = 3 \quad \square$$

$W_{a_1, a_2}$  3-system of wrenches

$$W = \text{span}(\varphi_1, \varphi_2, \varphi_3)$$



the intersection between the two systems must be  $\perp$  because  $W$  is a system.

The interaction is still there

$$E \cap F = \text{span}(\varphi_1) \quad 1\text{-sys}$$

$$E \times F = E + \text{span}(\mu_1) \quad 3\text{-sys}$$

$$W = \text{span}(\varphi_1, \varphi_2, \mu_1) = \text{span}(\varphi_1, \mu_1, \mu_2)$$

If I miss  $\mu_1$  then I miss a piece of the system

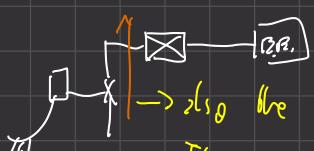


$$V_1 = W + \text{span}(\varphi_2)$$

$\uparrow$   
constraint applied to the e.e. with the joint locked.

$\varphi_{2e_1}$  is a moment pointing upwards or downwards.

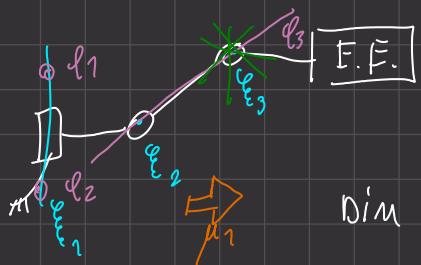
Remember that for the moment it does not matter where it is placed.



$\rightarrow$  also the position of this ac does not matter.

It is required to be the other two joints. (orthogonal to  $\varphi_1$  and same dir as  $\mu_1$ )

full-cycle mobility = trajectory + turning of the e.e.



R1RR1

$$W = \text{span}(\mu_1, \ell_1, \varphi_1, \ell_3)$$

$$\dim W = 6 - m \quad \text{f.e.} = 6 - \text{span}(\mu_1, \ell_1, \varphi_1, \ell_3) = 2$$

Remember any moment orthogonal to the rotators, invariant is part of the constraint. Any force parallel or intersecting the invariant, it's orthogonal.

If you change configuration, the R is persistent.  
to RPP

### Duality of W and A.

---

Now we lock the first motor.

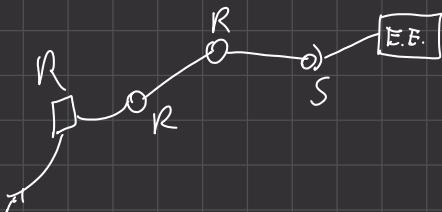
We are applying a moment like:  that is reciprocal to the other two joints so it will go directly to the E.E.

If we lock the second motor then the new moment is not reciprocal to the last.

The added constant is that a new star of forces is added.

The meaning of the star of forces is that the E.E. can rotate around that point. This new star is composed by lines that intersect the one from the base R so reciprocal.

R1R1S

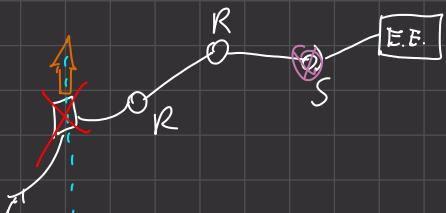


$\Rightarrow$  3 non coplanar with lines intersecting at the center of S



$$W = \emptyset$$

Let's now lock the first motor:

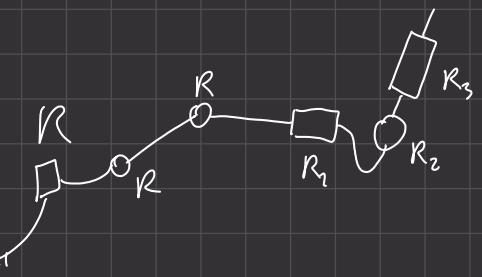
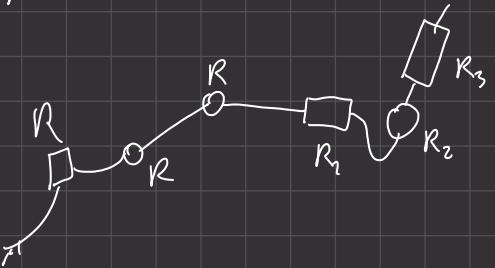
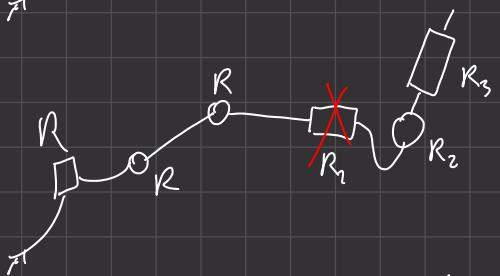
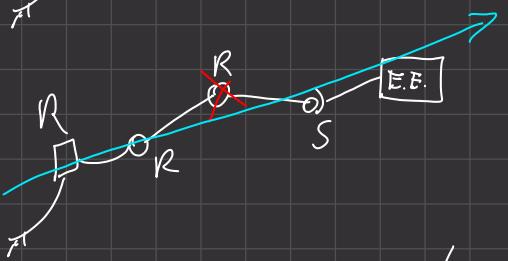
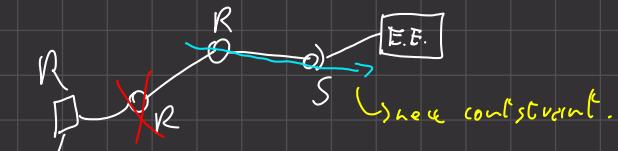


Case 1:  $R_1$  activated

$$V_1 = \text{span}(\underline{\ell_1})$$

This is where the motor will act on the e.e. of the robot.

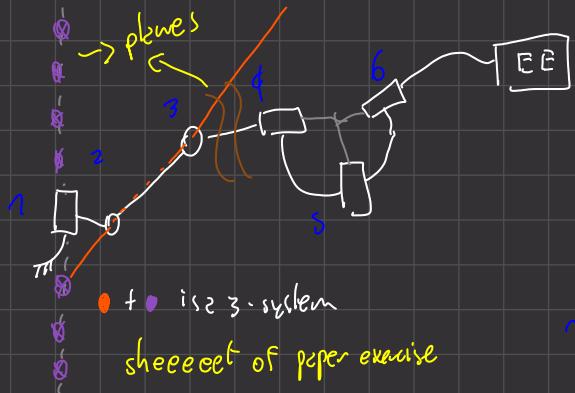
If you lock the second joint:



FINISH  
NO TIME

If you lock one by one the joints in the sequence at the end you get that the e.e. cannot move.

07/11/23



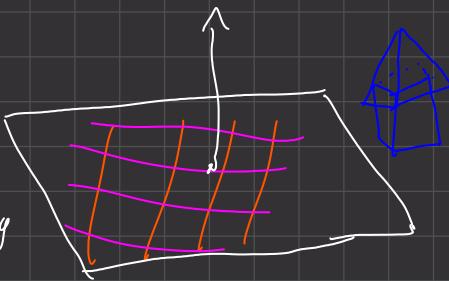
$V$  is 2 d system

$$V = W + \text{span}(L_{12})$$

To calculate the constraint on the spherical joint you take 2 plane and on that plane every line and one moment that is perpendicular to that plane is part of the constraint.

4-system:

is created in the spherical joint by blocking one of the two rotation

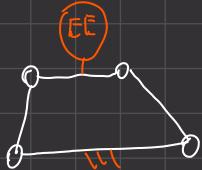


If you find the intersection between the constraint in the first 3 joints and the sphere you find the final constraint.

### Purely parallel robots

It's common in the industry to have closed draws if the ee will have less degrees of freedom.

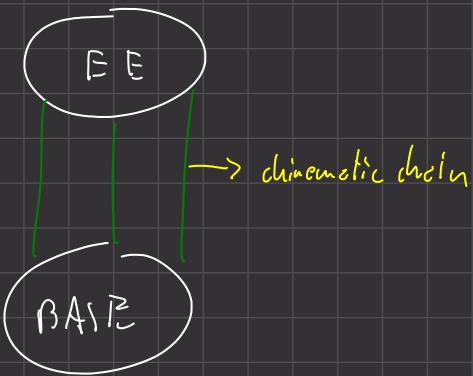
The most common is the 4-bar linkage.



We would wanna know how the ee. will move.

03/12/23

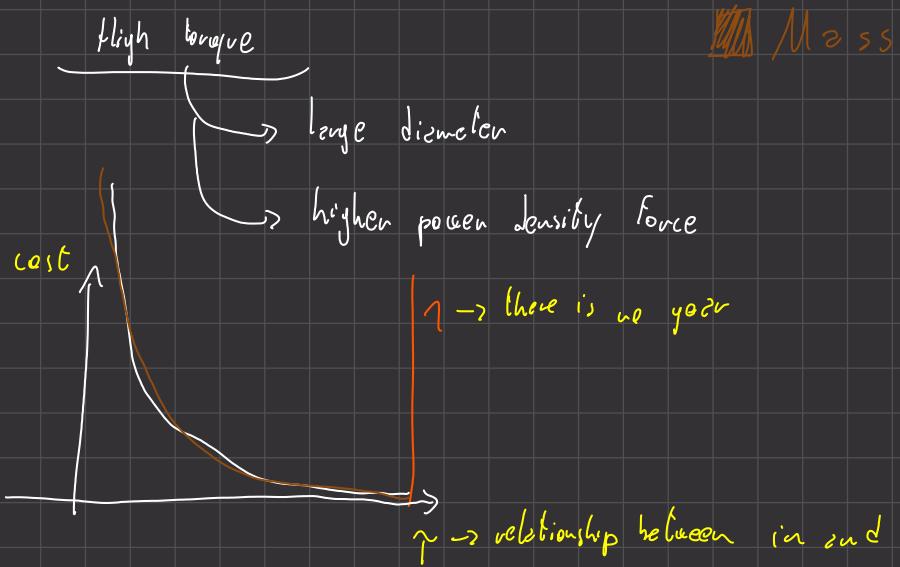
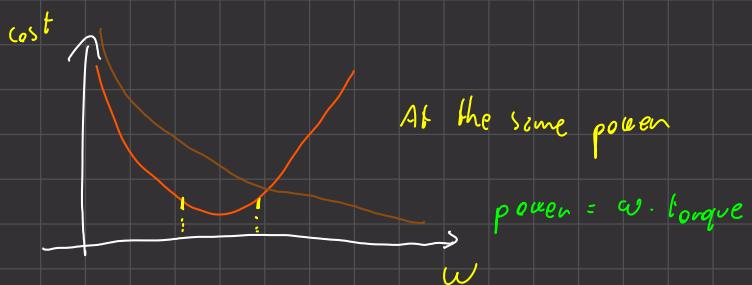
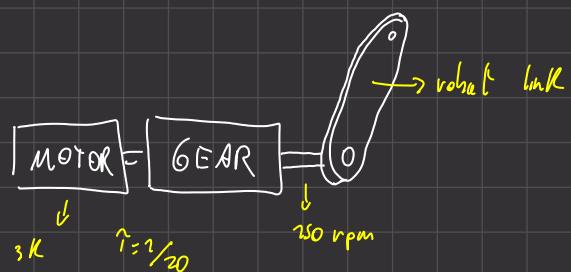
### Purely parallel mechanisms



By having closed chains the movement of the ee. is reduced. Having two chains moving on end effector needs to have some elasticity somewhere because you will never have perfect synchronization.

You can also have this compliance because the motors have some play inside the gearbox.

Slow shaft is the output from the gears of the motor.

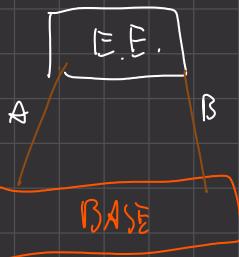


If you go to FAULHABER you will find every graph that you want.

If you go to MOOG or PARKER you see individual motors.

By having multiples heuristic draws the def will get lower.

Workflow for closed loop chains.



If we delete key B then we fall in the already considered case.

We can work out the  $W_A$  and the  $\tilde{w}$ , and always be effected by  $W_A$ .

Also  $A_A$  can be worked out.

The same thing can be done for the  $B$  chain  $\{ \overset{W^B}{A_B} \}$

We know that constraints add up in rigid bodies.

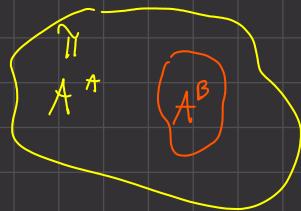
Let's work out some "numbers":

$$W^A = \text{span}(\varphi_1^A, \dots, \varphi_N^A)$$

$$W^B = \text{span}(\varphi_1^B, \dots, \varphi_N^B)$$

$$W = W^A + W^B = \text{span}(\varphi_1^A, \varphi_N^A, \varphi_1^B, \varphi_N^B)$$

If you evaluate the allowable motions of something on the plane constraint and then add to that the constraint that that object must go in 2 line we have:



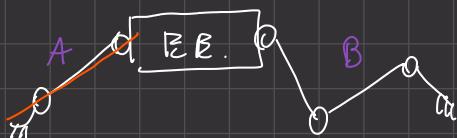
$$\text{So } A = A^A \cap A^B$$

We still have the rule of the eight:

$$\dim(A) + \dim(W) = 6$$

examples

number of legs of this type  
 $\gamma|RR|$        $\gamma|R\bar{R}|$



$$\dim W_B = 3$$

$$\dim W_A = 4$$

$$W_A = \tilde{\Pi} + \text{span}(\varphi^A)$$

$$W_B = \tilde{\Pi}$$

Note that if the joints become coplanar then they become dependent

$$W = \underbrace{\tilde{\Pi} + \tilde{\Pi}}_{\tilde{\Pi}} + \text{span}(\varphi^A)$$

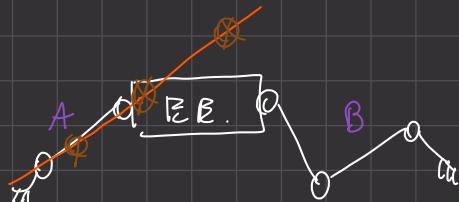
The mechanical meaning is that the load is split between the two links so the strength required by each leg is less.

FORCE = CURRENT

$$\frac{b}{g}_{EE} = \frac{b_A + b_B}{A} = \frac{b_1^A + b_2^B + b_3^B}{B}$$

$\text{Dim}(W) = 4 \Rightarrow \text{Ran}(A) = 6 - 4 = 2 \text{ d.o.f of E.E.}$

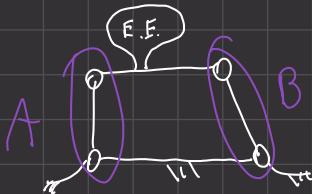
All these moments are reciprocal to the constraint.



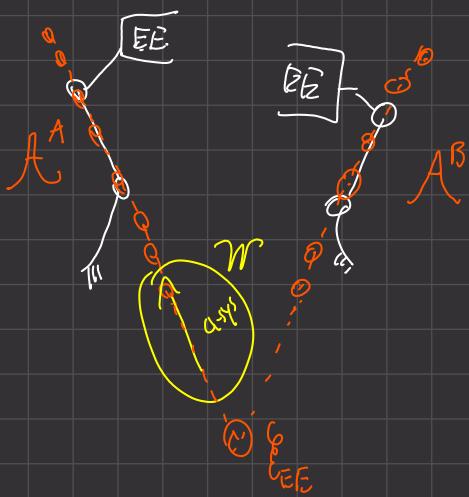
Note if you add a revolute joint along this line then the E.E. will move instantaneously.

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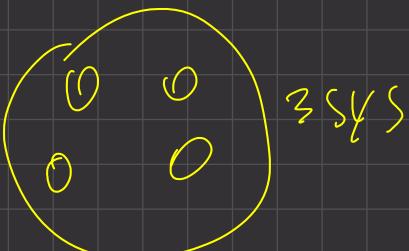
#### 4-bar linkage



As always we split the two ways. first we analyse the A chain:



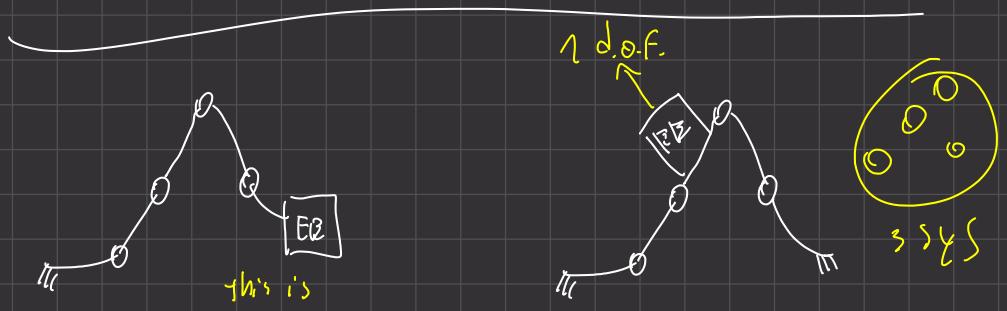
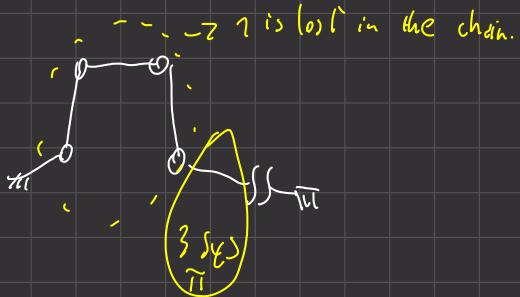
$$A^A \cap f^B = A_{EE}$$



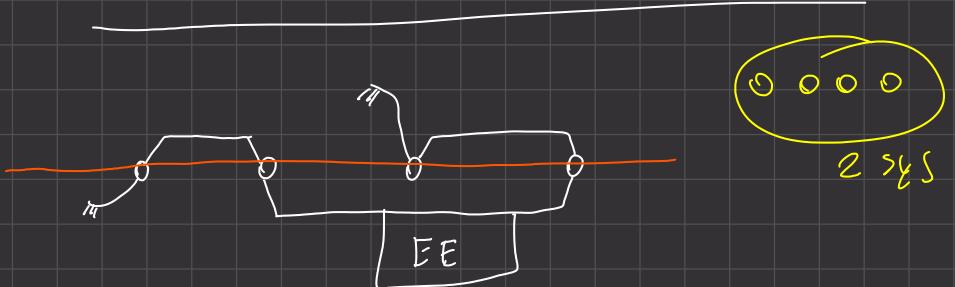
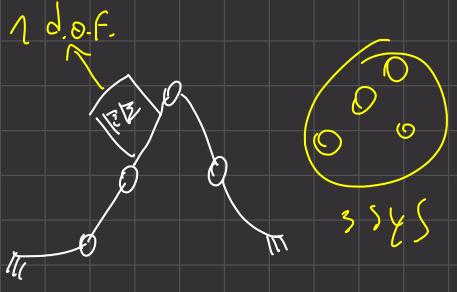
The only way for having A non null is to have the joints dependent to each other.

If you put the lines parallel to each other then you do not have 3 moments, but 2 translation because the twist goes to infinity.

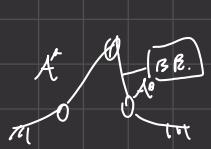
If the intersection is null then the joints are linear independent.



You have 4 degrees of freedom in the chain and one is in the chain

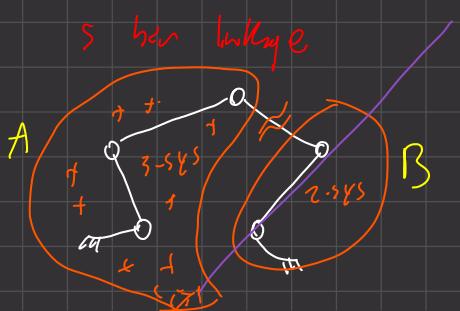


To see the two d.o.f. in this example you have to imagine the two lines in the  $\begin{pmatrix} \textcircled{1} \\ \textcircled{2} \end{pmatrix}$  example as overlapping. So in this case the overlapping is everything in the line so any virtual hinge along the line is good so it will be a 2-system.



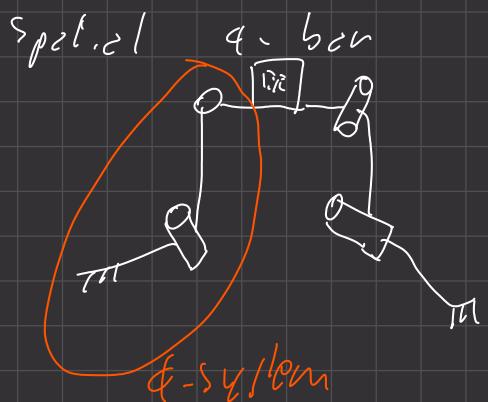
$$\lambda^1 \cap \lambda^2 = \emptyset \rightarrow (\phi_1, \phi_2, \phi_3 | \phi_4, \phi_5, \phi_6)$$

$\phi$  vector or null twist

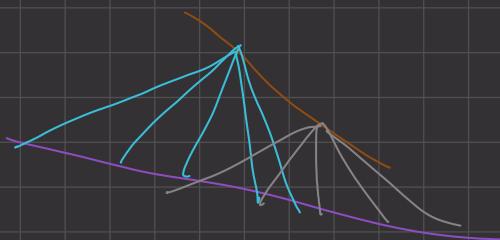


the intersection is any axis of rotation  $\wedge$  with the line so 2-system

If you look the and effector plan = 3-system will become 2 with the 2 distributed in the chain

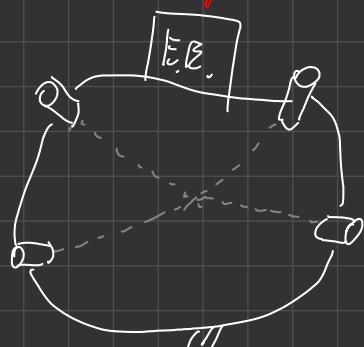


2<sup>1</sup> constraints  $\rightarrow$  To see this you have to imagine two slave lines



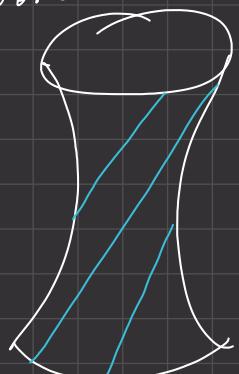
There are many slave constraints so 2-sys / 3m.

### Spherical 4-bar linkage



This will span the source of expense

### Hyperboloid



### Ruled surface

Any line belongs to a 3 system of lines

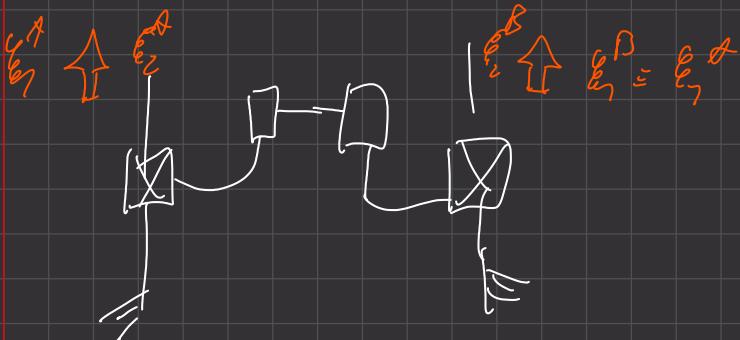
This is used in the Denavit mechanism

Look it up.

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$$\mathcal{E}_{\text{fix}}^A = \text{span}(\mathcal{E}_1^A, \mathcal{E}_2^A) \quad \mathcal{E}_{\text{fix}}^B = \text{span}(\mathcal{E}_1^B, \mathcal{E}_2^B)$$

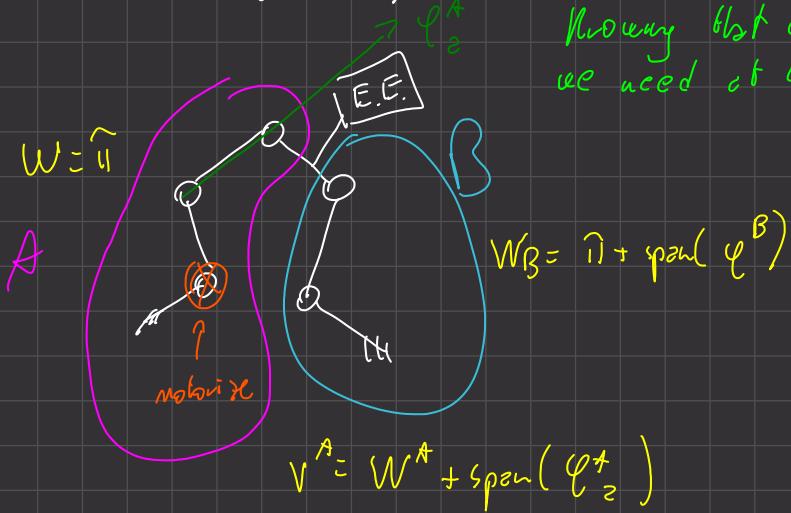
$$\mathcal{E}_{\text{link}} = \mathcal{E}_{\text{fix}}^A \cap \mathcal{E}_{\text{fix}}^B = \text{span}(\mathcal{E}_2)$$



S-B AR linkage

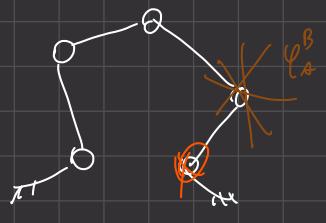
$$2 - \text{dof} \quad \text{dim}(A_{\text{fix}}) = 2$$

Knowing that we have 2 dof we need at least 2 motors.



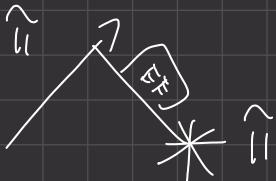
$$\mathcal{E}_2^A = \text{span}(\mu_x, \mu_y, \varphi_z, \mathcal{E}_2^A)$$

There when motor is locked.



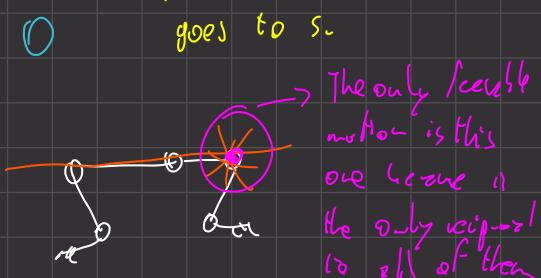
$$W_B = W^B + \text{span}(\varphi_A^B)$$

$$\mathcal{L}_A^B = \text{span}(\mu_x, \mu_y, \ell_z) \varphi^R \varphi^B$$

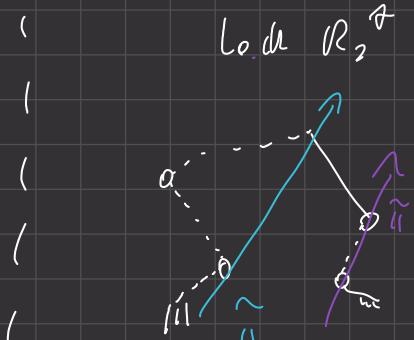
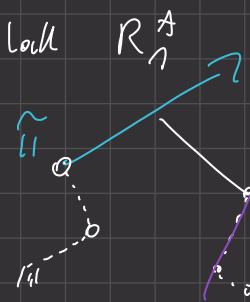
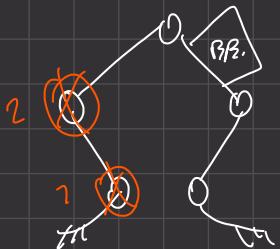


The EE is fully constrained.

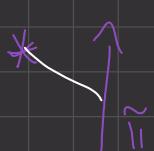
geometrically is this configuration



N.B. Most situations are good.



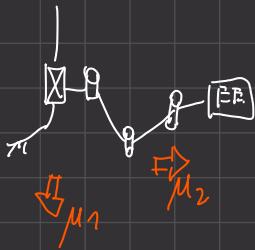
If we were join them



So good situation

Tripreon - 3P (RRR)

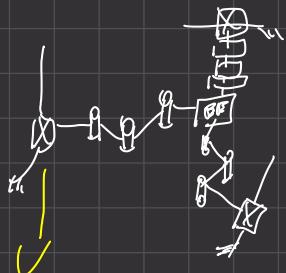
6 joints and 6.D.F.E.E.  $\Rightarrow$  D.O.F.E.E. = 4



$$W_L = \text{span}(\mu_1, \mu_2)$$

$$\dim(W_L) = 6 - 4 = 2$$

Now let's combine the three (eqs).

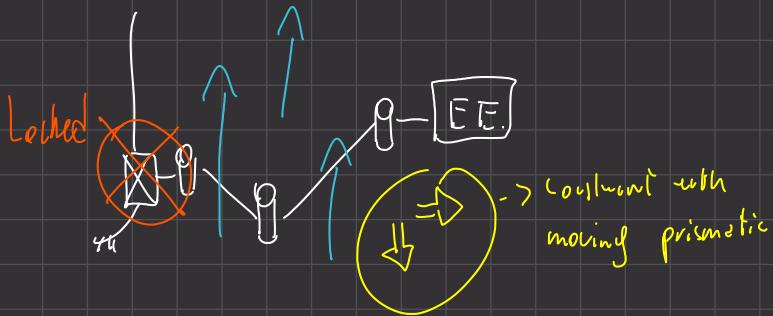


$$W = W^A + W^B + W^C =$$

$$= \text{span}(\mu_1, \mu_2, \mu_3) = 3 - 3 = 0.$$

If you look the prismatic than place constant.

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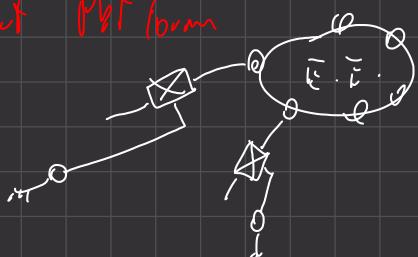
$$V^A = W^A + \text{span}(\mu_p^A)$$

structural constraint

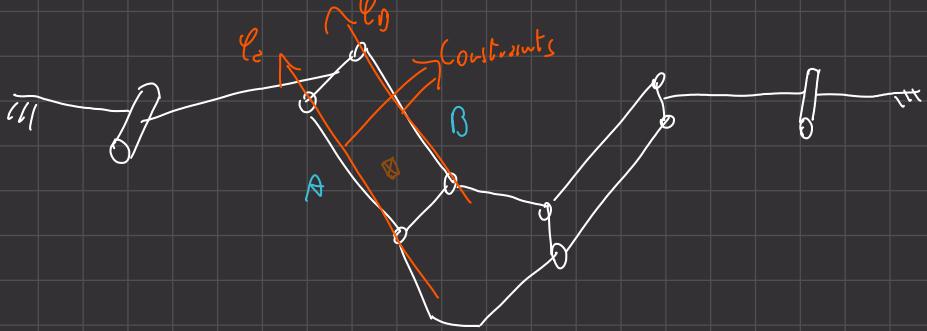
actuated constraint

There are the actuated constraint. the difference between them is the reaction of the "constraint" of the object.

Stewart Platform



In robots you use repeatability and not absolute accuracy

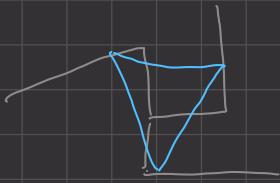


$$W_A = \text{span}(\ell_A) \quad \text{there also contain the moment orthogonal to the plane.}$$

$$W_B = \text{span}(\ell_B)$$

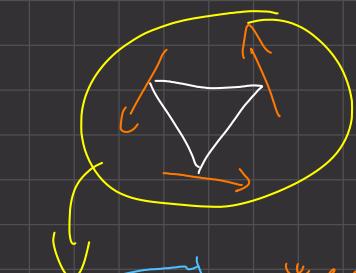
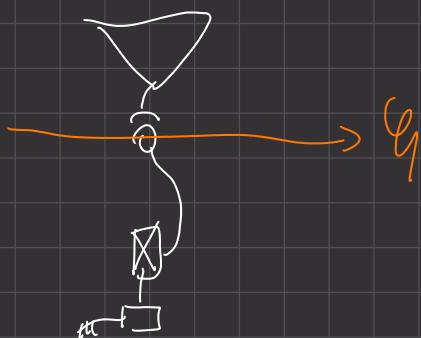
You have three moments in different direction as space so the E-E will never rotate.

30/11/23



$$\ell_0 \in (\mathbf{f}(m) \circ (\omega | v)) \quad \text{geometric, clockwise}$$

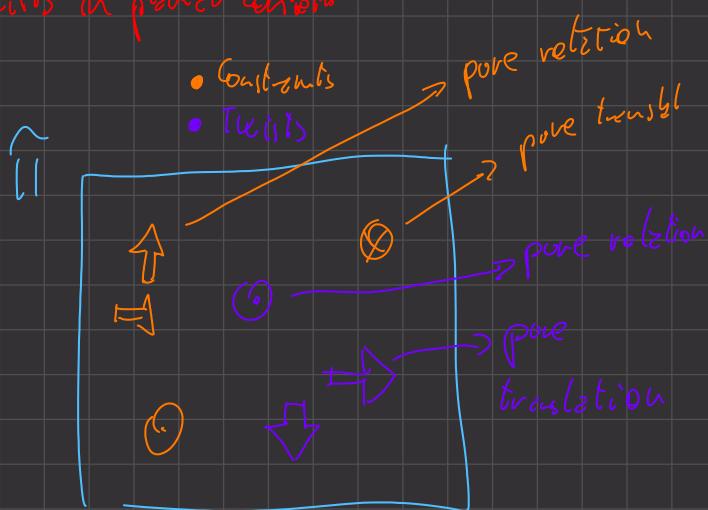
$$(\mathbf{f}(m) \cdot (v | \omega)) \quad \begin{bmatrix} \ell_1 & \ell_2 & \dots \end{bmatrix} \begin{bmatrix} \ell \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \end{bmatrix}$$



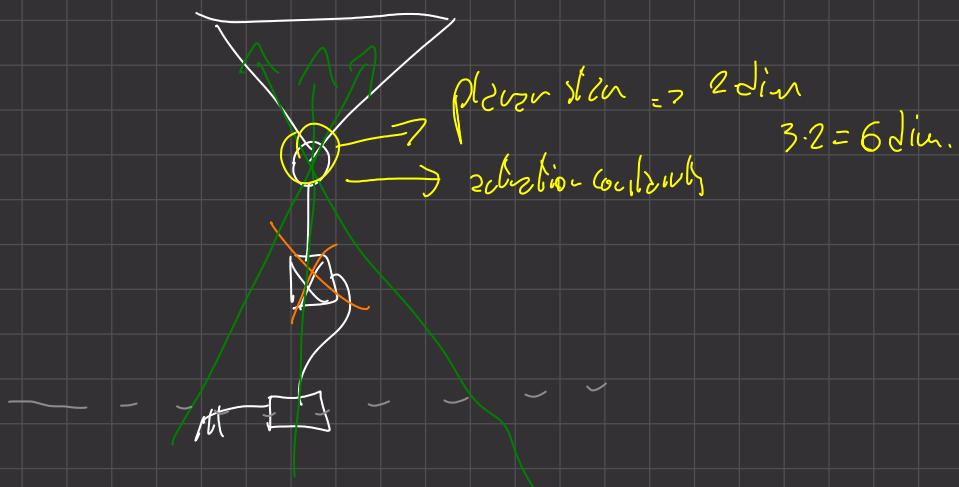
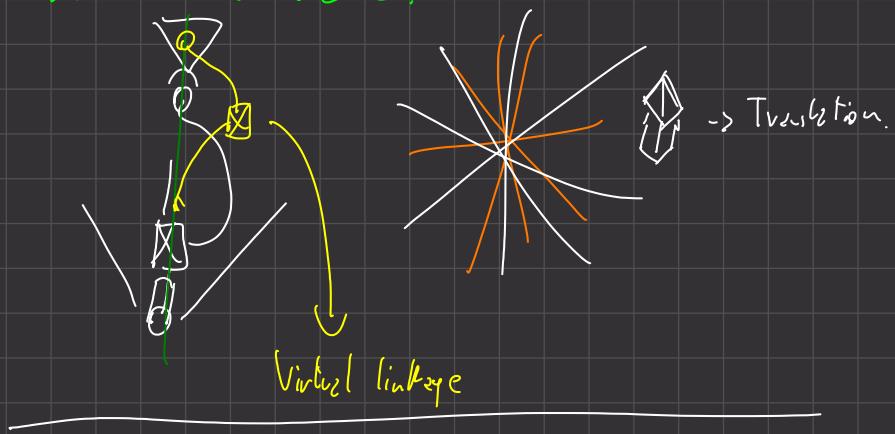
If the constraint is the second or third on the first and third if there goes to infinity

N.B

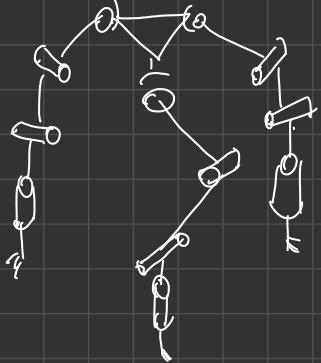
Traits in planar motion



Planar star : dim = 2.



N.B



Two slave lines span  
2 cylinders

N.B.