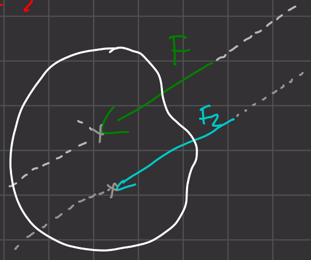
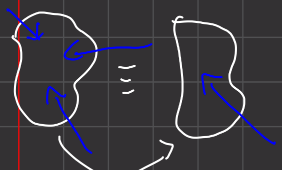
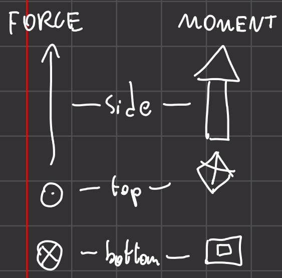
26/04/23

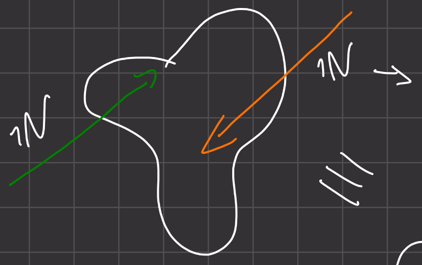


There is no difference in the mechanical effect if we apply F along the line:

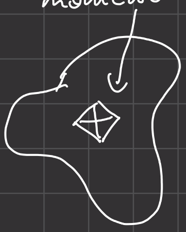


Resultant or equivalent



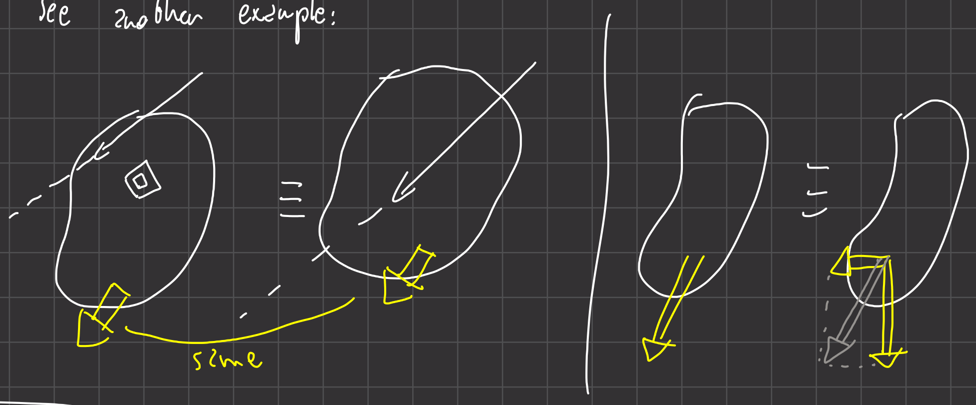


There is no force but there is moment

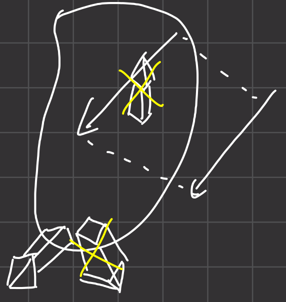


The moment does not allow the object to turn so the two systems are equal.

Let's see another example:



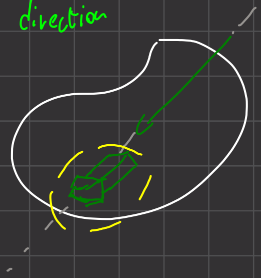
3D



Same value

It does not matter where you apply the torque.?? BAH

We can Represent any system with ONE force and ONE moment with the same direction



← this is the canonical representation of a class of equivalence or fundamental stat of statics

$\boxed{\text{TODO check}}$

The canonical representation is represented by $\mathcal{G}$

$$\mathcal{G} = (\underset{[N]}{\vec{f}}, \underset{[Nm]}{\vec{m}}) \equiv (\vec{f} \mid \vec{m})$$

force — moment

This is also called WRENCH

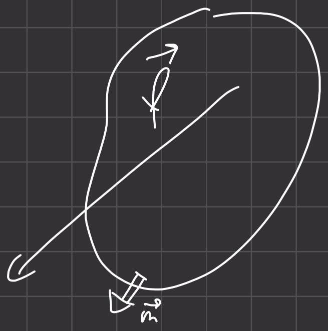
N.B. this is invariant with the reference frame.

You cannot change that moment with a displacement of the force. The moment must be there to be in the class of equivalence

We also have:

Special wrenches

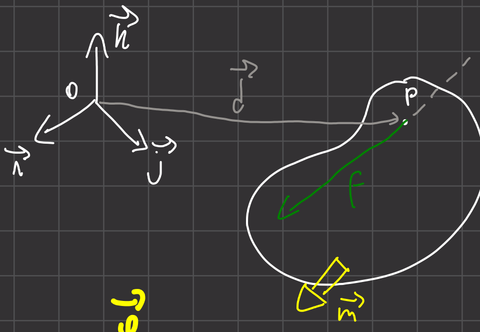
- pure force $\varphi = (\vec{f} \mid \vec{m}) \,/\, \vec{m} \perp \vec{f}$ (When the moment is ortogonal to the force)
- pure moment $\mu = (\vec{0} \mid \vec{m})$



$\vec{f} = (f_x, f_y, f_z)$

$\vec{m} = (m_x, m_y, f_z)$

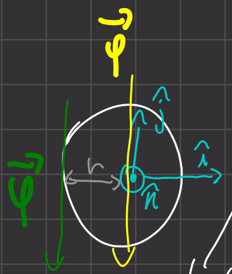
} this does not codify where the force is applied



$\vec{d} = \vec{OP}$

$\vec{f} = (f_x, f_y, f_z)$

$\vec{m} = (m_x, m_y, m_z) + \vec{f} \times \vec{OP}$



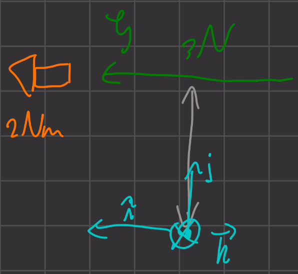
$\vec{\varphi} = (0, -1, 0 \mid 0, 0, r)$

$\vec{\varphi} = (0, -1, 0 \mid 0, 0, 0)$

Modulus of 1N

Examples

$$\varphi = (\emptyset, \emptyset, 2 \mid 2d\,c\beta, 2d\,s\beta, \emptyset)$$



$$\psi = (3, \emptyset, \emptyset \mid (2, \emptyset, \emptyset) + (\emptyset, \emptyset, -3d)) =$$

$$= (3, \emptyset, \emptyset \mid 2, \emptyset, -3d)$$

$\boxed{\text{Every kind of movement on a rigid body can be replaced with a wrench N.B.}}$

28/09/23

$$\psi_1 = (1, \emptyset, \emptyset \mid 0, 0, 1)$$
$$\psi_2 = (\emptyset, -2, \emptyset \mid 0, 0, -2)$$

these are pure forces and can be summed

$$\psi_3 = (1, -2, \emptyset \mid 0, 0, \underbrace{d\sqrt{5}}_{2})$$

pitch N.B.

Another way of calculating ψ_3 is to sum $\psi_1 + \psi_2$ and it will get the same result.

For a plane everything is good but in space lines could be skew. Summing up the components will also work in the space

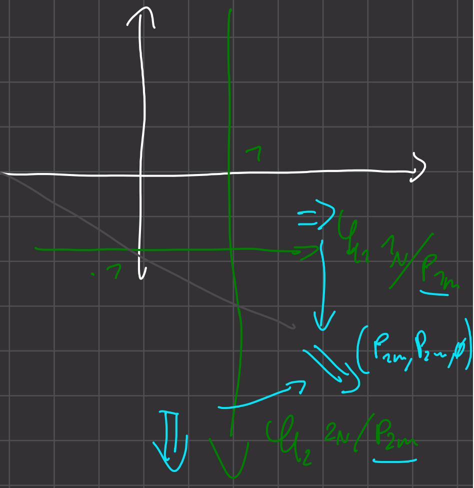
$\boxed{\text{Pitch is the amount of meters that a screw advances with a full turn.}}$

Let's now consider: $\psi_1(1, \emptyset, \emptyset \mid P_1, \emptyset, 1)$

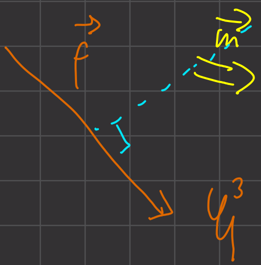
$\psi_2(\emptyset, -2, \emptyset \mid \emptyset, -2P_2, -2)$

Then $\psi_3 = (1,-2,\emptyset \mid p_{2_1}, -2p_2, -1)$

Same thing



Decomposing the moment



The component that's on the direction of the force is:

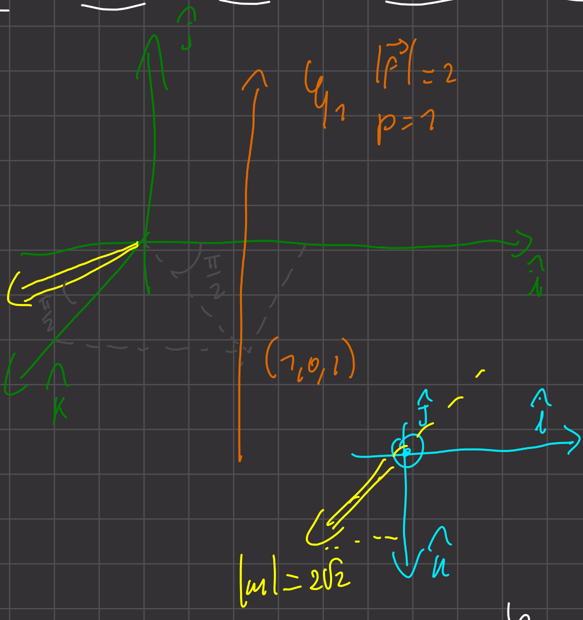
$$\frac{\vec f \cdot \vec m}{|\vec f|}$$

So to get the moment that's orthogonal to the direction of the force you will do:

$$\vec f - \frac{\vec f \cdot \vec m}{|\vec f|}$$

$\psi = (\vec f \mid \vec m)$

↪ $\vec f p + \vec d \times \vec f$



$$\psi_1 = (0,2,0 \mid -2, \emptyset, 2)$$

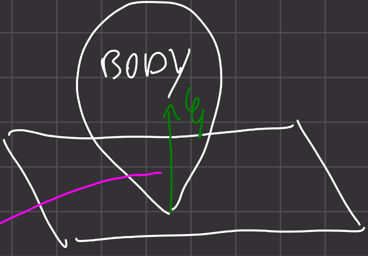
$2 \cdot \frac{\sqrt2 \cdot \sqrt2}{2}$

but we also have pitch: 1

$$\psi_1 = (0,2,0 \mid (-2,0,2) + \vec f p)$$

$(0,2,0) \cdot 1$

$$\boxed{\psi_2 = (0,2,0 \mid -2,2,2)}$$



The body is constrained to stay in contact with the plane.
the plane is applying a force orthogonal to the direction of the plane. We cannot tell the intensity of the force.

Every interaction now should be described with a wrench.
We can immediately say that we have $\emptyset$ pitch.

$$\psi : \left\{ p = \emptyset \right.$$

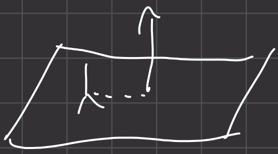
Now we have an $\mathbb{R}^1$ subspace in $\mathbb{R}^6$ where there are many intensities so:

$$\underbrace{\psi = (\emptyset, \emptyset, a \mid \emptyset, \emptyset, \emptyset)}$$

Base of the vector space

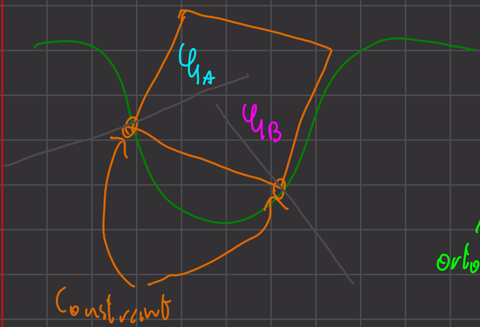
WRENCH SPACE = WRENCH SYSTEM

If the moment is orthogonal to the force then we have no spinning. It's only telling me where the reference frame is.



$$\psi = (\emptyset, \emptyset, a \mid b, c, \emptyset)$$

Chair example



$$\psi_C = \lambda_1 \psi_A + \lambda_2 \psi_B$$

vector space of constraint

Allowed motion is the one wich is orthogonal to the vector space.

Contact points to the bowl.

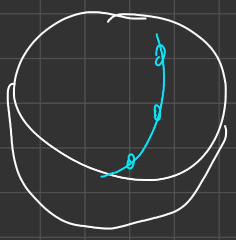
$$\psi_A = (a, b, c \mid 0, 0, 0)$$
$$\psi_B = (d, e, f \mid 0, 0, 0)$$
$$\psi_C = (g, h, i \mid 0, 0, 0)$$

If we do a combination of these three we get:

$$\psi = \lambda_A \psi_A + \lambda_B \psi_B + \lambda_C \psi_C = (o, p, q \mid \emptyset, \emptyset, \emptyset)$$

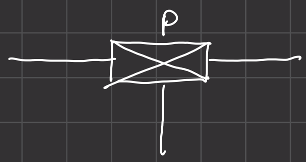
there is no possibility for this system (wich is a ball joint) to do any translation because we have forces in any direction to block the movement.

If the three point of contact are on the same plane then they are linear dependent, so we don't have anymore a ball joint, because we have rocking.



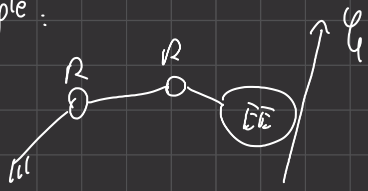
Constraint are ϕ pitch screws

03/20/23



When you apply a wrench you consider an external wrench to the body

For example:



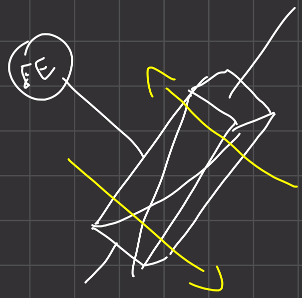
In this case I am applying the wrench between the ee and the ground.
Everytime the wrench is applied between two bodies.

→ this is applied between the base and the ee.

If we are now considering only static then now these wrenches are only applying constraints. We need to act on body throug our constraints.

We want to understand how the prismatic joint constraint us. If we will try to apply an ortogonal force to the direction of translation then it will be rejected by another from the body.

Constraints of the prismatic joint



If you think of applying a force far away then you will have pure moment. So if the line is infinitely far away you get a infinitely strong moment. For this reason

We get infinite pitch.

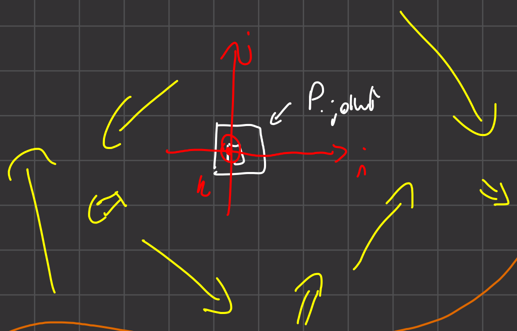
If you fix the moment that you want and start applying the force further and further then the force will go to ϕ when the distance will tend to infinity.

When we go really far the position of the force doesn't really change much.

The moment is always on the line between the force and ϕ.

The point is that we should focus only on the moment and not on the force.

You can apply moments with any intensity but you cannot move the shuttle of the prismatic joint.



$\varphi(a, b, \phi \mid c, d, e)$

this shee represent the constraint on the prismatic joint

It doesn't matter where you put the reference frame, the effect is the same.

N.B. In case of constraints we must consider φ as the repelling force or moment, not the applying force.

→ the space in which this 5 dimensional wrench exists is:

→ $\mathbb{P}$ calligraphic P

So we get: $\varphi \subset \mathbb{P}$ wich is a 5-system.

If we want to find the base:

$$\mathbb{P} = \text{span}(\varphi_x, \varphi_y, \mu_x, \mu_y, \mu_z)$$

If we want to express all the vectors of the base of the constraint:

$\varphi_x = (1,0,0 \mid 0,0,0)$ $\quad$ $\varphi_y = (0,0,0 \mid 0,1,0)$

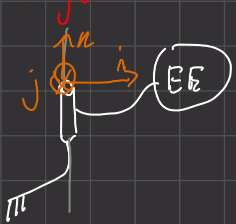
$\varphi_y = (0,1,0 \mid 0,0,0)$ $\quad$ $\varphi_z = (0,0,0 \mid 0,0,1)$

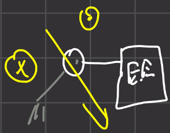
$\varphi_x = (0,0,0 \mid 1,0,0)$

Constraints of the prismatic joint contain all the forces ortogonal to the direction of the shuttle of the prismatic and all the moments.

It doesn't really matter the position of the reference frame but needs to be on the direction of the joint.

Revolut joint





Applying these forces the door will not open.

The line in the revolut joint is called the INVARIANT of the joint.
Also the direction of the prismatic is called INVARIANT.

All the lines that intersect the invariant of the revolut do not move the hinge.

If we want to apply moments that are ortogonal to the direction of the hinge will not work.
Only moments that are parallel to the direction of the hinge or forces that do not intersect the hinge and they will move the revolut.

The best reference frame to take is along the line of the revolut and with one axes on the line.

For the wrenches we now need to make a distinction:
the ones who are parallel to the direction of the joint:

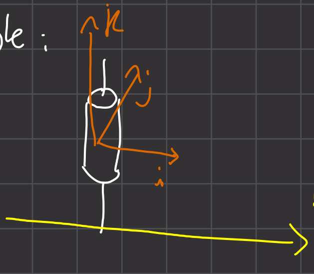
$$\varphi_1 = (\emptyset, \emptyset, a \mid b, c, \emptyset)$$ ← This is a subset of the next one

And the ones that intersect the hinge of the door:

$$\varphi_2 = (d, e, f \mid l, m, \emptyset)$$

Note that the space is 5-dimensional, but it's not because we have 5 to values but because they are indipendent from each other

For example:

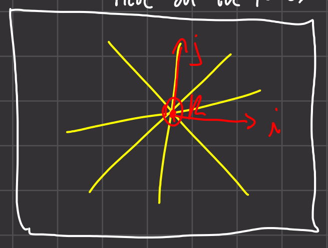


same direction of i

$$\varphi(1,0,0 \mid 0,-1d,0)$$

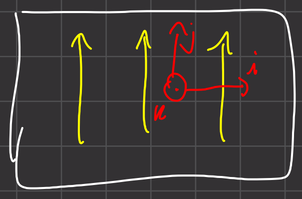
Examples of systems dimension

Here all the forces directions go through the center.



$\mathcal{G}(a,b,\emptyset \mid \emptyset,\emptyset,\emptyset)$

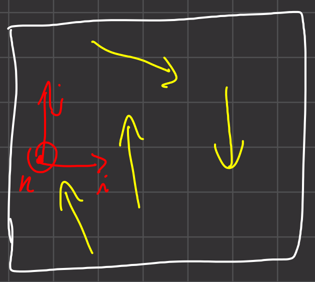
this is a 2-system.



$\mathcal{G}:(\emptyset,a,\emptyset \mid \emptyset,\emptyset,b)$

2-system.

N.B. look at the reference frame

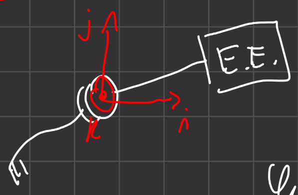


$\mathcal{G}:(a,b,\emptyset \mid \emptyset,\emptyset,c)$

3-system

Spherical joint

$\boxed{\text{AN INFINITE PITCH FORCE IS A MOMENT}}$

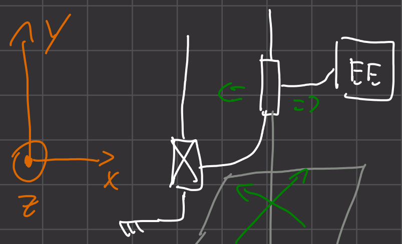


Any pure force going through the center will be rejected. So:

$\mathcal{G}:(a,b,c \mid \emptyset,\emptyset,\emptyset)$

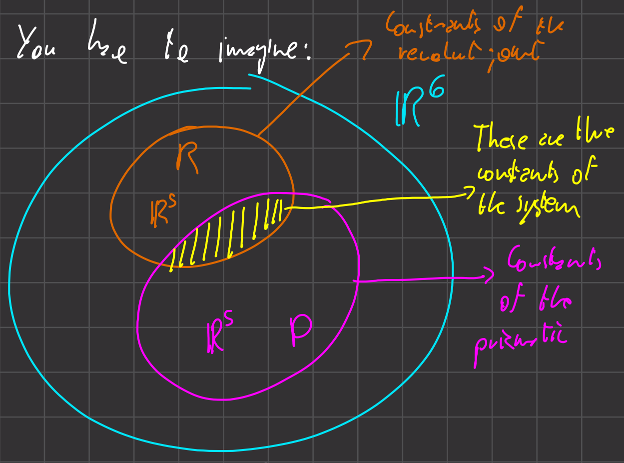
3-system

More than one joint in series



I we try to analyze the constraining forces we can see.

You have to imagine:

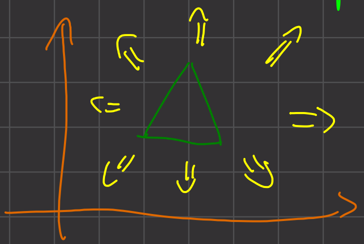


The structural constraint will be: $\mathcal{G}:(a,\emptyset,b \mid m,\emptyset,n)$

05/20/23

Planar mechanism is something that stays on a plane while moving. To do this you need constraints and so wrenches.

This constraint is called planar constraint.

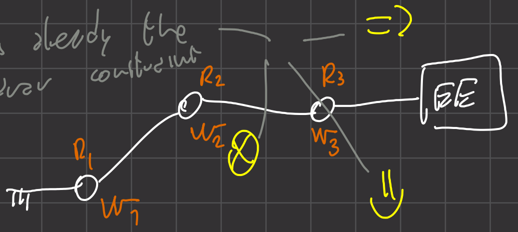


This moment span a 2 dimension vector space.

We need to find the number of vectors that keep the object on the plane.

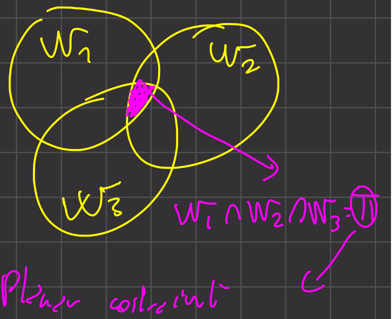
phi $\to \varphi = (\phi, \phi, a \mid b, c, \phi) \to$ planar constraint

It's already the planar constraint



$3|R|$

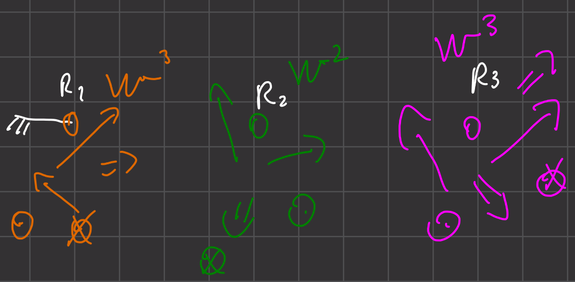
We have 3 intersects



$W_1 = \left(\text{span}(\varphi_1, \varphi_5) \,/\, \varphi_3 \text{ not turning } R_1 \right)$

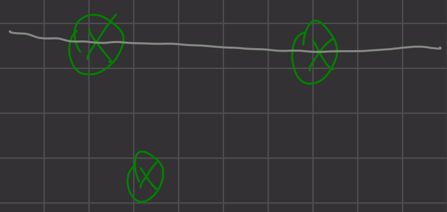
s-system

We are in a planar constraint



The common things are the forces on the plane

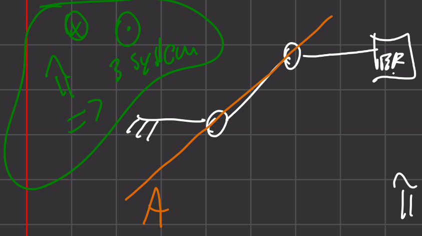
The vertical lines do not belong to the intersection because



Only two can be on the same plane.

If we now consider

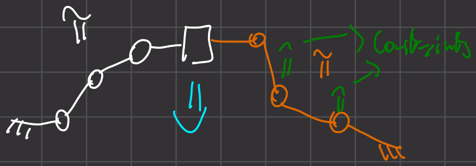
$2|R|$



we add here / in the constraints so now it's a 4 system

$\tilde{\Pi} \cap A = \emptyset \Rightarrow \tilde{\Pi} + A$ is a 4 system.

Let's now see this 2(PPR) system → these two lines mean that what's inside has parallel invariant lines

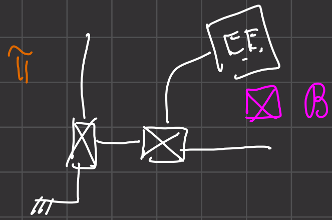


$\hat{\Pi}$ and $\tilde{\Pi}$ are applied in parallel so at the end the system is behaving like there was only one $\tilde{\Pi}$. So the constraint is the same but the forces are distributed in a different way.

Let's now think about

$\tilde{\Pi} + A$ is the constraint

the force will go to ground through A
so the reaction will be

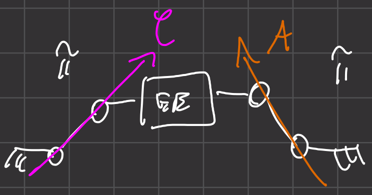


$\tilde{\Pi} + B$ is a 4-system.

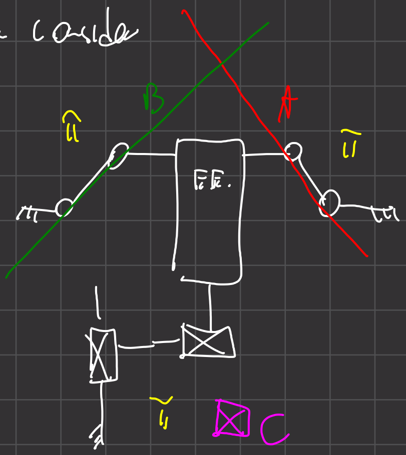
They don't need to be ortogonal but it's easier because the movements are decoupled

---

$\tilde{\Pi} + A + C$ is a 5-system.



Let's now consider

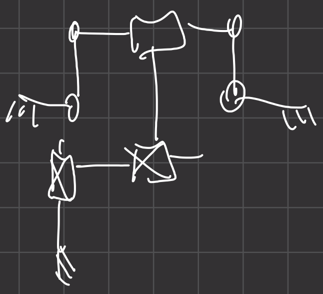


$\hat{\Pi} + A + B + C$ is a 6-system.

C cannot be produced by A and B couple so it's a revolute constraint.

If they were parallel then this would be 5-system.

If you consider

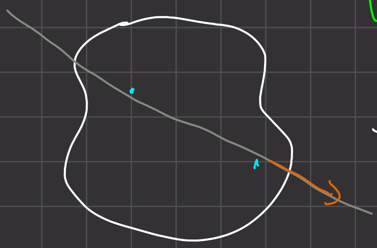


It's a 5 system and the span of the intersection is C

so

$\hat{\Pi} + C \cap \hat{\Pi} + A + B = \text{span}(C)$

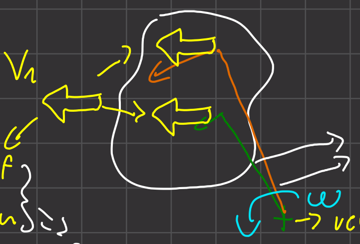
10/10/23



We are talking about instantaneous speeds

There can be only one speed in one direction in the body

Chasles realized that whatever is the distribution of velocities the body is spinning around a line in space and translating along the line.



every point is moving in the same way in the same direction.

ω → revolute joint around which we have the velocity of rotation

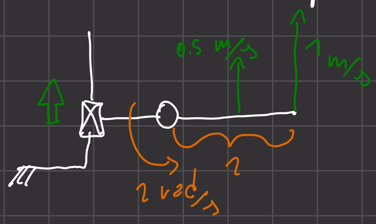
Obviously the farther you are from the hinge the more speed you have. ω, angular velocity is the same.

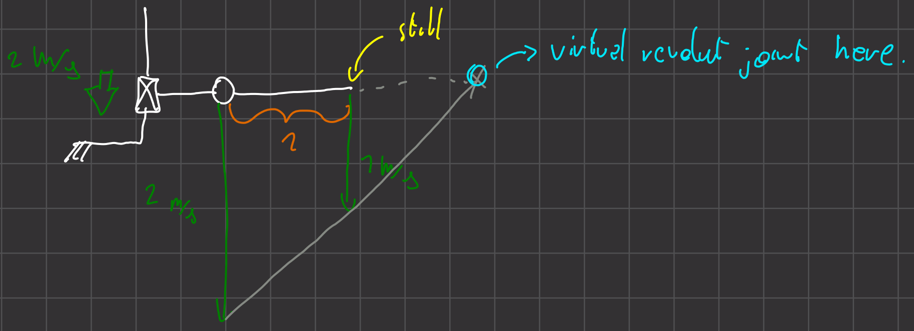
$V_1 = (\omega_1, v_1)$ is the combination of the two velocities.

Let's now consider another hinge.

I can have the same motion by rotating around another hinge and changing the translating direction I will get the same movement.

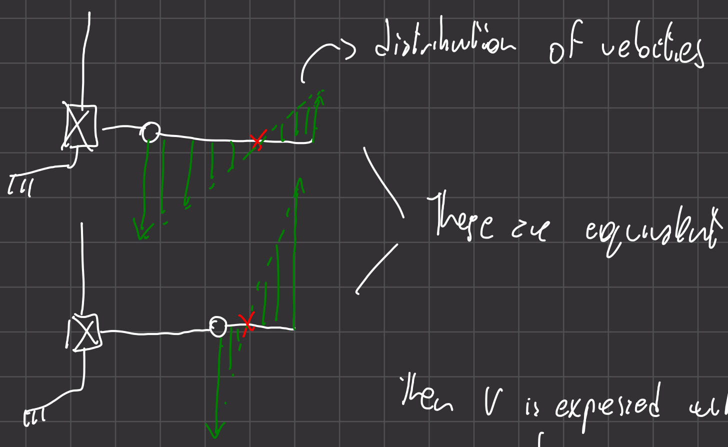
Let's do another example:





If now I move the r joint along the "arm" I can get different combinations of ω and v such that the point will have the same movement

$$(\omega_1, v_1) = (\omega_2, v_2) = (\omega_3, v_3)$$ → class of equivalence

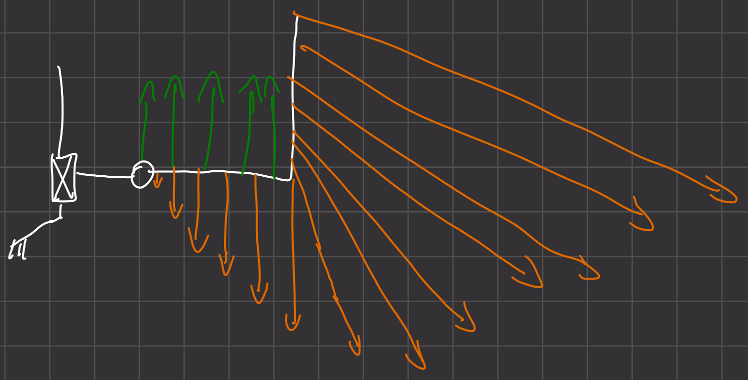


Then v is expressed with a parallelogram

We are building a class of equivalence → canonical representation

↳ twist : $\xi\,[x,]$

This is a representation of speed in the body and this distribution works for instantaneous movement.

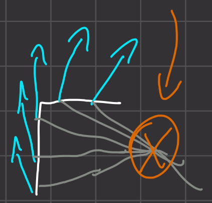


If you add the two then you get the actual speed.

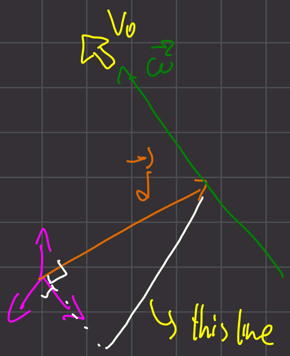
The point in which the ortogonals of the resultant of the speeds meet is called:

IAR (Istantaneous Axis of Rotation)

For example:



$\xi = (\vec{\omega} \mid \vec{v})$

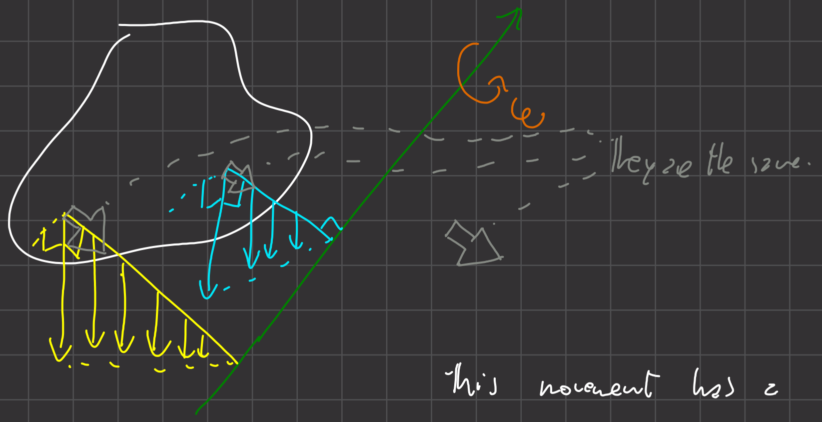


$$V = V_0 + \vec{d} \times \vec{\omega}$$

$$V_0 = p\,|\omega|$$

↘ pitch

Speed is a free vector, the rotations are not. You can add translations with parallelogram but not the rotations.



this movement has a pitch.

Now we can pose the question: What is the velocity of a body?

Now we have tools to describe this movement.

Vectors in $\mathbb{R}^6$

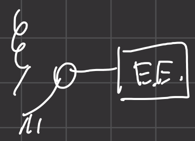
$$\zeta = (\vec{l} \mid \vec{m}) \qquad p = \frac{|m|}{|l|}$$

$$\xi = (\vec{\omega} \mid \vec{v}) \qquad p = \frac{|v|}{|\omega|}$$

line matters | just direction matters

NB

12/10/23



ξ $p=0$ / $\omega = 1 \rightarrow$

If you want to model the joint is convenient to give a twist with $\omega = 1$

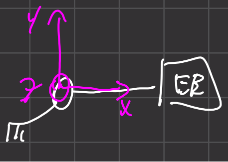
Now if we take a motor and attach it to the joint with 300rpm

then $\xi_m = \xi\, 300$

We are using a unit lenght vector like $(1,0,0)$ or $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$

So we are normalizing just the omega: $\xi(\vec{\omega} \mid \vec{v})$
↳ s.t. $|\vec{\omega}| = 1$

We use omega for direction and V for intensity.

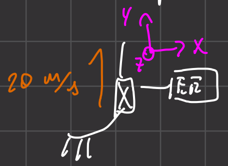


$$\xi = (\overbrace{0,0,1}^{\text{direction}} \mid \overbrace{0,0,0}^{\text{position of point of line.}})$$

motorize the R joint
motor speed = 200 RPM

speed of the EE = $200\,\xi \in \mathbb{R}^6$

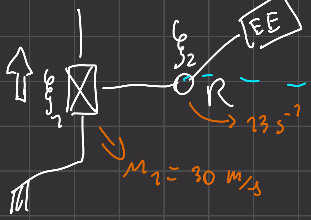
Prismatic joint



$\xi = (\vec{0} \mid \vec{v})$ $\quad p = \infty$ $\qquad \xi_m = 20\,\xi = 20(\vec{0} \mid \vec{v})$

We prefer $|\vec{v}| = 1$

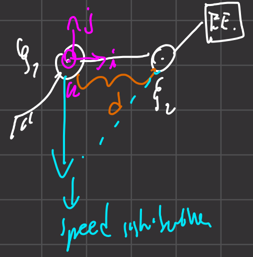
$\xi = 20(0,0,0 \mid 0,1,0)$



Movement is linear in velocity

$$\boxed{\xi = 20\,\xi_1 + 13\,\xi_2}$$

IAR will be on this line

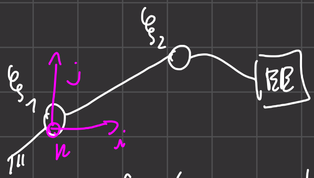


$\xi_{EE} = \omega_1 \xi_1 + \omega_2 \xi_2$ $\qquad$ Forward kinematic Problem

$\xi_1 = (0,0,1 \mid 0,0,0)$

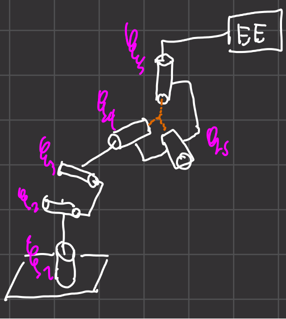
$\xi_2 = (0,0,1 \mid 0,-d,0)$

You get this value imagining to have r_2 fixed and moving a line above the origin of the axis.



$\xi = (0,0,1 \mid a,b,0)$ $\qquad d = \sqrt{a^2+b^2}$

Let's now consider a 6R robot



$\xi_{EE} = 2\xi_1 - 5\xi_2 + 10\xi_3 + 12\xi_4 - 7\xi_5 + 3\xi_6$

Motor	Speed
1	2
2	-5
3	10
4	12
5	-7
6	3

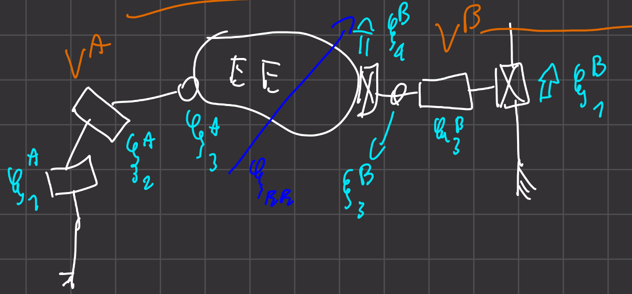
Geometric Jacobian

6 x 6 matrix

Joint velocities <-> E.E. velocities

$$\xi_{EE} = \underbrace{\begin{bmatrix} | & | & | & | & | & | \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 & \xi_5 & \xi_6 \\ | & | & | & | & | & | \end{bmatrix}}_{\text{depends on configuration parameters only}} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \\ \omega_6 \end{bmatrix}$$

Structural constraints



Combined structural constraint

$$V_{EE} = V^A + V^B$$

We can imagine an imaginary kinematic chain (mechanism)

Let's now make an analogy with velocities and discuss what happens.

The mechanism is consistent between the two chains so:

$$\xi_{EE} = \omega_1^A \xi_1^A + \omega_2^A \xi_2^A + \omega_3^A \xi_3^A = V_1^B \xi_1^B + \omega_2^B \xi_2^B + \omega_3^B \xi_3^B + V_4^B \xi_4^B$$

$M_o^A = \text{span}(\xi_1^A, \xi_2^A, \xi_3^A)$ -> If they are linear independent then it's a 3 system.

The ξ_i create a span of the feasible movements.

Any feasible or allowable $\xi_{EE} \in M_o^A$

Vector space in $\mathbb{R}^6$.

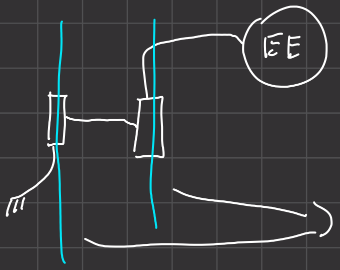
$M_o^B = \text{span}(\xi_1^B, \xi_2^B, \xi_3^B, \xi_4^B)$ -> if they are linear independent then we have a 4-system.

So Any feasible $\xi_{EE} \in M_o^B$

Concluding $\xi_{EE} \in M_o^A \cap M_o^B$

17/10

Systemic Reasoning



Our way to abstracting the mechanism.

If we extract the systemic view we get:

The twist is the material thing and the pitch is zero while the speed is coming. The pitch is a geometrical property of the line.

The abstraction is that we start from lines and geometry.

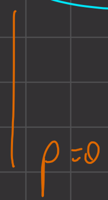
The advantage is that now we can use linear algebra.

If we now use pitch we can use vectors. and vector spaces (systems).

If vectors in $\mathbb{R}^6$ then they are twists and they can create a space.

For example the two lines create a vector space.

If we do a linear combination of the two lines then we get another line we get a resulting line:



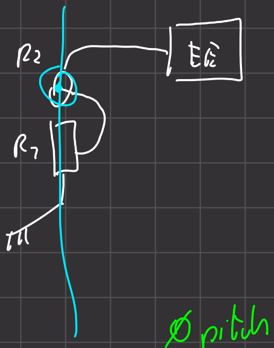
We should try to not rely on numeric projections but learn to reason with vectors.

The wire line represent the feasible speeds that the ee. can do.

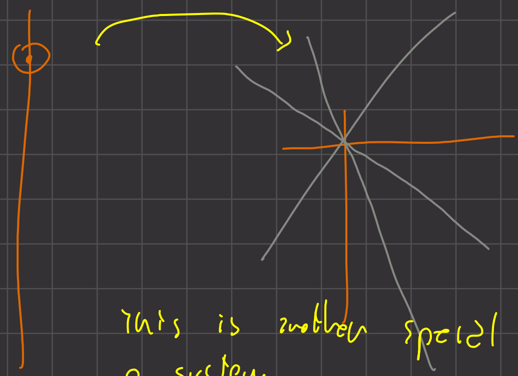
The blue lines represent the distribution of velocities in the body in some way.

The strange thing is that we can put the resulting line everywhere in the space by playing with the intensity of the red lines.

Let's do another example



∅ pitch



This is another special
a system

You can generate any line that passes through the intersection of the lines and the angle will be determined by the intensity of those.

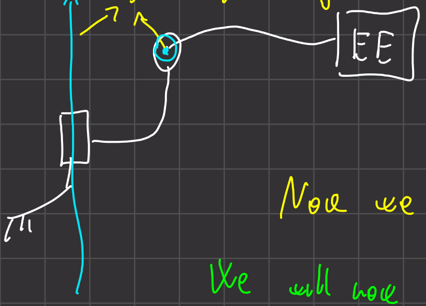
Keep in mind that we are alrdy talking about instantaneous movements.

Note: if you change the configuration you change the geometry

For example in the previous example if you rotate r_1 the plane will change.

Let's now consider lines that are not touching:

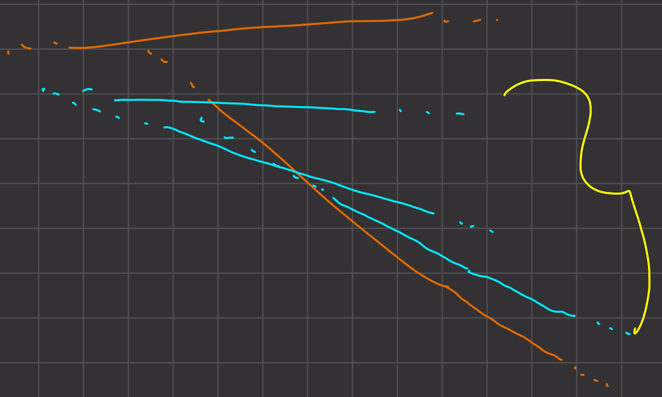
skew lines (sghembe)



Finite pitch.

Now we have a general system

We will now get a finite pitch N.B.



These are all the linear combinations created by the two lines.

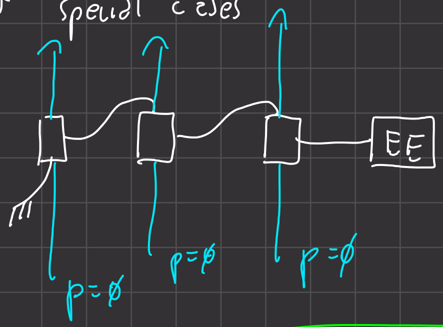
All these lines have a non-zero pitch.

To look deeper in these things look at ruled surfaces.

Using screws we generate ruled surfaces

Now we will step into the three systems.

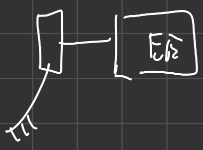
First special cases



These 3 lines are linearly dependant so this system is a 2-system.

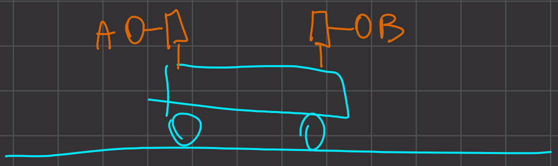
NB this is a 3 degree of freedom mechanism
BUT
the degrees of freedom of the EE. is 2.

The missing dof is now gone in one of the other lines.



The degree of freedom of e.e. =
$= \dim(\mathrm{span}(\$_R)) = 1$

Example of degrees of freedom

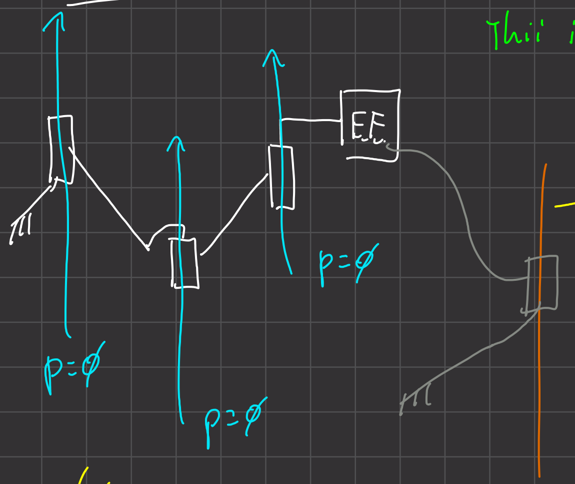


Degree of freedom of the system = 3

Degree of freedom of A = 2

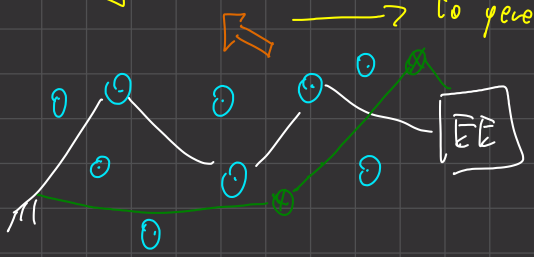
DoF of B = 2

This is now a 3-system.



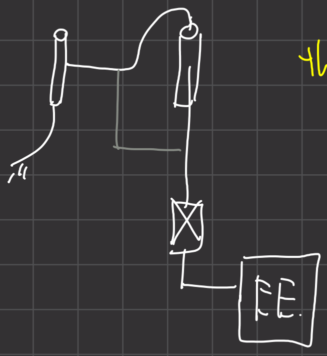
We can find a line parallel to the others that can represent the motion.

seen from above

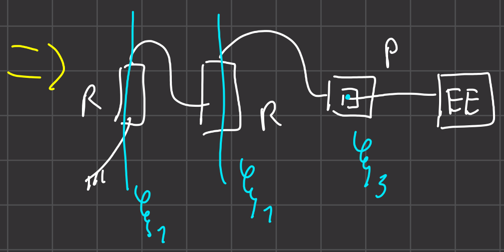


To generate this translation you can take any two virtual hinges

In this system you can generate 0 or infinite pitch motions.

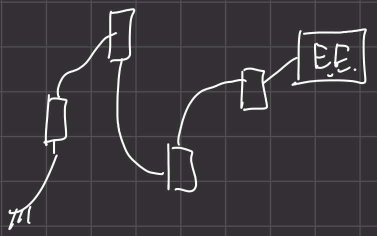


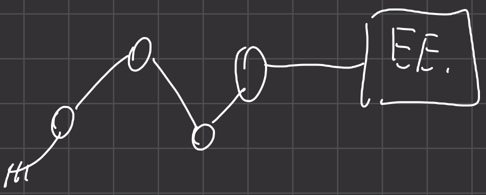
This is still a 2-system.



N.B. To visualize the lost degree of freedom imagine fixing the E.E. Now R_2 will be still able to move

A singularity is when you lose some degree of freedoms.

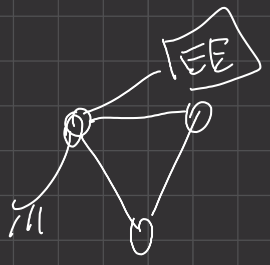




N.B. For the system the dof is 4

FOR THE EE is 3

Another example is :



Dof of the EE is still 3

Twist is the name of ξ

Wrench the name of ξ

29/10/23

Constraint $\mathcal{W}$ → E.E. ← $\mathcal{A}$ Allowable inst. motion (velocity)

wrenches up to 6 sys

twists. up to 6-sys

N.B

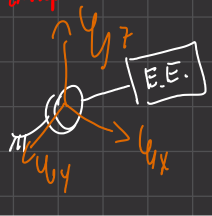
Constraints are reciprocal to velocity
"ortogonal"

$\left.\begin{matrix}\mathcal{W}_{EE} \\ \mathcal{A}_{EE}\end{matrix}\right\}$ generic

N.B.

$$\boxed{\dim \mathcal{W}_{EE} + \dim \mathcal{A}_{EE} = 6}$$

Spherical example



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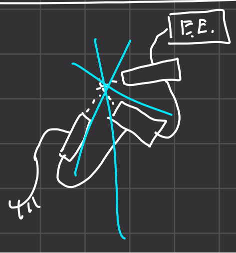


Any pure force passing through the center of the joint belongs to the constraint.

$W = span(\varphi_x, \varphi_y, \varphi_z)$ 3 sys

$A_0 = span(p_x, p_y, p_z)$ 3 sys

- These systems are the same lines.
- These systems are orthogonal (they are not "working together")

twist, with pitch ϕ so no translating is brought with rotation

If we draw a general wrench we get: $\varphi = (a, b, c \mid 0, 0, 0)$ (with f over a, b, c and m over $0, 0, 0$)

$\varphi_x = (1,0,0 \mid 0,0,0)$

$\varphi_y = (0,1,0 \mid 0,0,0)$

$\varphi_z = (0,0,1 \mid 0,0,0)$

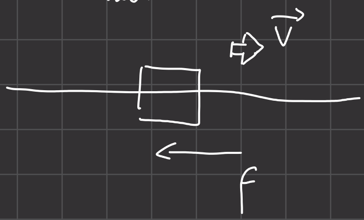
$\varphi \in W$

PGA

Generic $p = (\underbrace{l, m, n}_{\omega} \mid \underbrace{0, 0, 0}_{V})$

Power $= \vec{\omega} \cdot \vec{m} + \vec{v} \cdot \vec{F}$

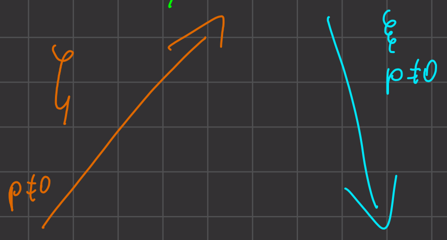
power $= \omega\, m$ → because in the same direction



power $= vf$

The question is how do we calculate the instantaneous power in the E.E.

If we take any twist and wrench where the pitch could be $\neq 0$



We can calculate the reciprocal product similar to the scalar product

---

RECIPROCAL PRODUCT (gives us the power of the EE.)

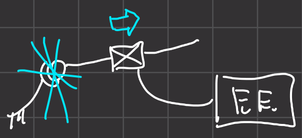
$$\xi \circ \zeta = (\vec{f} \,|\, \vec{m}) \circ (\vec{\omega} \,|\, \vec{v}) = \underbrace{\vec{f}\cdot\vec{v} + \vec{\omega}\cdot\vec{m}}_{\text{scalar in } \mathbb{R}^3}$$

Let's now try to apply this to the spherical joint:

$$\rho \circ \varphi = (l,m,n \,|\, 0,0,0) \circ (a,b,c \,|\, \emptyset,\emptyset,\emptyset) = (l,m,n)(\emptyset,\emptyset,\emptyset) + (\emptyset,\emptyset,\emptyset)(a,b,c) = \emptyset$$

no power

Now let's attach a prismatic joint to the spherical joint:



$$\mathcal{W} = \text{span}(\varphi_1, \varphi_2)$$

$$\mathcal{A} = \left.\begin{matrix} (1,0,0 \,|\, 0,0,0) = \varphi_x \\ (0,1,0 \,|\, 0,0,0) = \varphi_y \\ (0,0,1 \,|\, 0,0,0) = \varphi_z \\ (0,0,0 \,|\, 1,0,0) = \tau_x \end{matrix}\right\} = \text{span}(\varphi_x, \varphi_y, \varphi_z, \tau_x)$$

$$\text{DIM}(\mathcal{A}) = 4 \Rightarrow \text{Dim}(\mathcal{W}) = 2 \Rightarrow \mathcal{W} = \text{span}(\varphi_1, \varphi_2)$$

DoF of system = number of joints
DoF of EE. : you also have to notice if some DoFs are linear dependent.

Any force whose direction passes through the center of the spherical joint and is orthogonal to the direction of the prismatic belongs to $\mathcal{W}$.

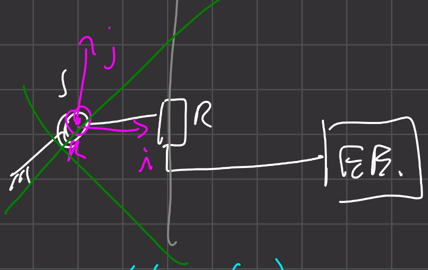
$$\varphi = (\emptyset, \underbrace{a, b}\,|\, \emptyset,\emptyset,\emptyset) \rightarrow \text{2-system}$$

↳ C.N. not C.S. to be 2 system.

At the end $\mathcal{A} = \text{span}(l,m,n \,|\, p,\emptyset,\emptyset)$

$$\xi = (l,m,n \,|\, p,\emptyset,\emptyset)$$

$$\varphi_{EE} \circ \xi_{EE} = (\emptyset,a,b \,|\, \emptyset,\emptyset,\emptyset) \circ (l,m,n \,|\, p,\emptyset,\emptyset) = (\emptyset,a,b)\cdot(\emptyset,\emptyset,\emptyset) + (\emptyset,\emptyset,\emptyset)\cdot(l,m,n) = \emptyset$$



$W = \text{span}(\varphi_1, \varphi_2)$ $\qquad$ $\text{Dim}(A_o) = 4 \Rightarrow \dim(W) = 6-4 = 2$

$$S = \begin{cases} \xi_1 = (1,0,0 \mid 0,0,0) \\ \xi_2 = (0,1,0 \mid 0,0,0) \\ \xi_3 = (0,0,1 \mid 0,0,0) \end{cases}$$

$$R = \{ \xi_4 = (0,1,0 \mid 0,0,d)$$

Not linear dependent

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

You can find a minor of rank(4)

Lines in the constraint must go through the center of the spherical and be in the i, j plane.

$$\xi_{EE} = (a,b,c \mid 0,0,d) \qquad \varphi_{EE} \circ \xi_{EE} = 0$$

$$\varphi_x = (1,0,0 \mid 0,0,0)$$

$$\varphi_y = (0,1,0 \mid 0,0,0)$$

$$\varphi_{EE} : (a,b,0 \mid 0,0,0)$$