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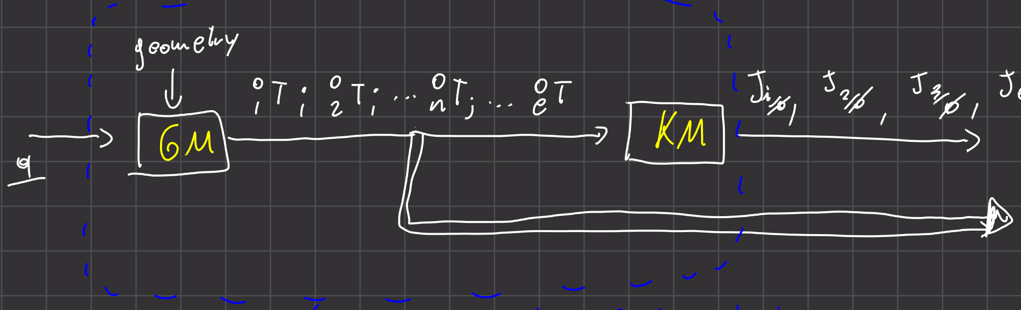
$$J_{i/0}(q) = J_{i/0}\left({}^0_1T(\underline{q});\ {}^0_2T(\underline{q});\ {}^0_3T(\underline{q}) \dots {}^0_mT\,T(\underline{q})\right)$$

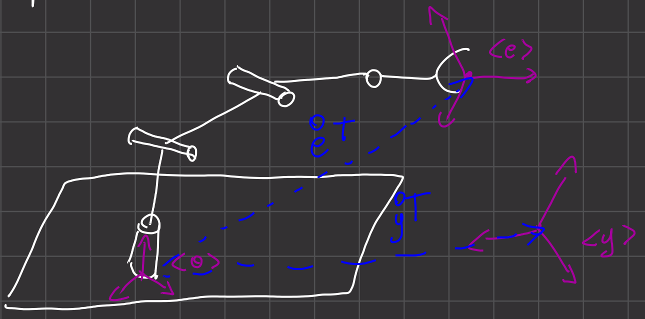
The info that we need to describe a Jacobian is not only the angle of every joint but also the distances and all geometrical distances and all the Transf. matrices contain this information. Inside J there are all these geometric distances. J contain all the parameters of the geometry. This connects to the geometric model.

We can now show a new block that can take the info from the GM.



Geometric/Kinematic model of the robot

We have the means for building two blocks. One to know where the frames start and on the basis of that we can work on the kinematic model. The Jacobians provide the link from the cartesian to joint space?



<g> goal frame

${}^0_gT$ desired transformation matrix

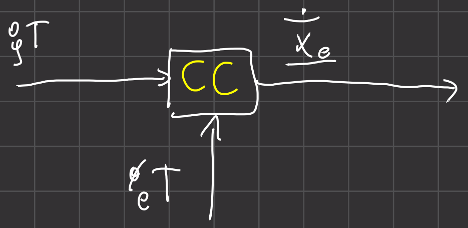
We want to make <e> approach <g>

<e> depends on how the robot moves and is constrained by that. Let's think for a moment that <e> was not constrained. First we want to understand how a frame not constrained should move to reach <g>

If I want to go to position of <g> with <e> how should I move? For this we should have a vector in the direction from <e> to <g>. But keep in mind that this is not the only solution because any vector orthogonal to the distance of <e> would turn but get closer if there is a component towards <g>. In addition to this examples there are infinite vectors that work. We will also see solutions for rotation.

The block outputs the desired velocity for the frame

We now introduce the Cartesian Controller



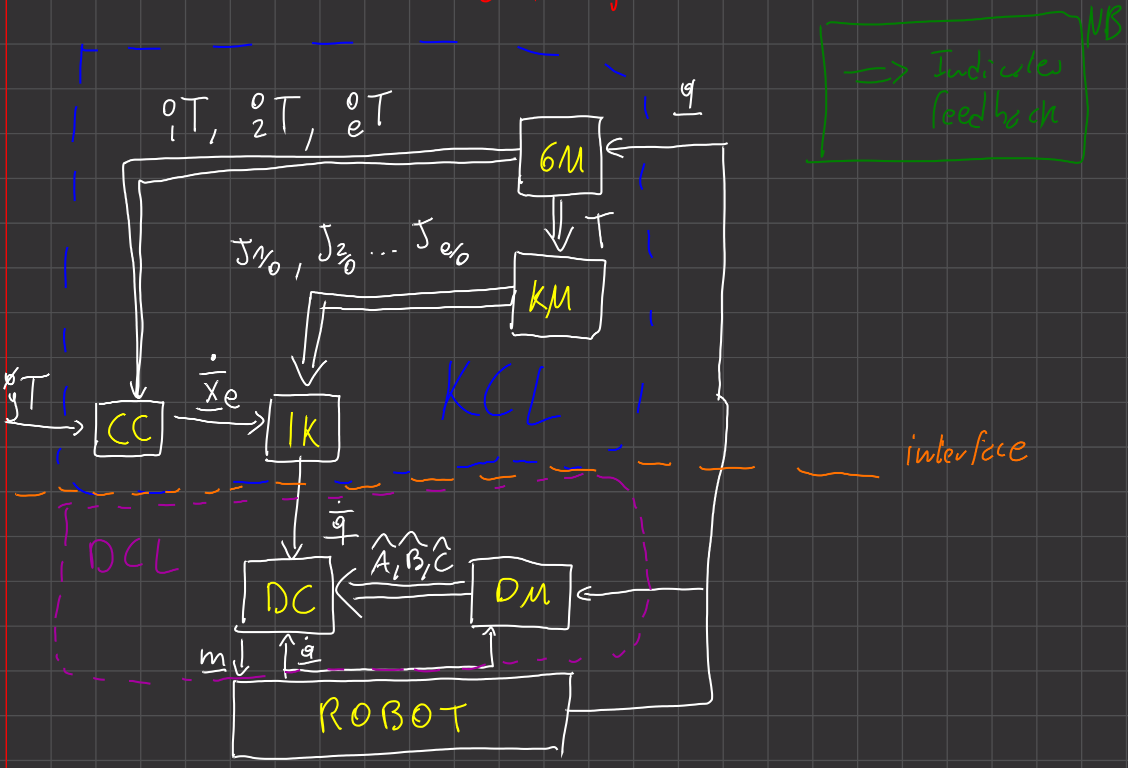
$\dot{x}_e \in \mathbb{R}^6$

This outputs the desired cartesian velocity ($\dot{x}$), but we need the desired joint velocity vector ($\dot{q}$).
We have seen that Jacobian maps from joint to cartesian space.
We will see that the inverse of Jacobian will offer in some way the inverse, even if the Jacobian won't be square. This block will output the velocity that best approximate the desired velocity.

If we have a 4-degree of freedom robot we can go in a subset of the 6-dof space.
In the other hand if we have a more than 6-dof robot sometimes we cannot garantee the velocity required. For example think when your arm is stretched and you want to extend even more. You can't.

We are still missing the inverse kinematic block. That provides the best approximation to this problem. Remember the workspace of the robot.

The overall control scheme is the following:



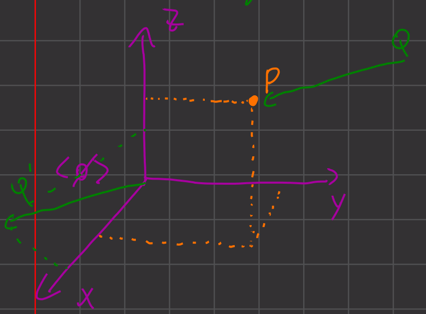
The robot wants torque (m). From the robot we measure q and $\dot{q}$. The first thing we do in the DCL is building a controller that builds the desired velocity. The objective of the DCL is to follow as close as possible the reference velocity. Once you have done this you need to drive the ee to the goal. Now I need to generate $\dot{\bar{q}}$. From geometry we move

position of the frames and with RM their movements with Jacobian.
Feeding this to the Jacobian we see how the system should move in space.
We get the desired velocity that gets mapped to the desired velocity with the inverse kinematic.

TODO

CH. 1 Geometric Foundamentals

Coordinate Systems (Frame)



Oriented tripod of ortogonal vectors.

X, Y, Z are placed in the right hand rule.
- thumb = x
- Index = y
- Middle = z

This is the right handed frame

With the concept of frame we can describe vectors and points. For example

$$^{\phi}P = \begin{bmatrix} x_p \\ y_p \\ z_p \end{bmatrix}$$

The components are the lenght of the vectors that allow me to reach P in the reference of this particular frame.
We also have vectors.
If we imagine Q in φ of the frame it's easy to identify the components.
This is called a projected vector.

$$^{\phi}(P-Q) = {}^{\phi}P - {}^{\phi}Q$$

TODO ?

The point on wich I apply the force is very important, but when I want to know the distance the point on wich is applied is not very important.

Often we can also say:

Projected vector ??

$$^{\phi}(P-Q) = {}^{\phi}P - {}^{\phi}Q \triangleq {}^{\phi}\underline{v}_{p,q}$$

the vector that joins q to p

We can now recall the grassman rule to join point and vectors

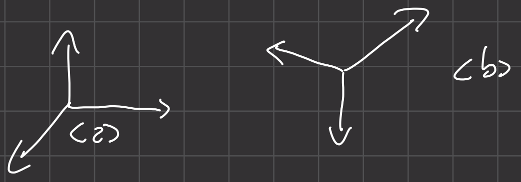
Grassman Rule

$$^{\phi}P = {}^{\phi}Q + {}^{\phi}\underline{v}_{P,Q}$$

In the frame unit vectors are immediately representable:

x	$\phi\, i_\phi$	$[1\ 0\ 0]^T$
y	$\phi\, j_\phi$	$[0\ 1\ 0]^T$
z	$\phi\, k_\phi$	$[0\ 0\ 1]^T$

Let's now consider two frames



Set of information needed to describe how <b> is with respect to <a> is

$$\left\{ {}^a\underline{O}_b,\ \underbrace{{}^a\underline{i}_b,\ {}^a\underline{j}_b,\ {}^a\underline{k}_b}_{\text{unit vectors of b projected on } \langle a\rangle \text{ to code the rotation of frame } \langle b\rangle \text{ in respect to } \langle a\rangle.} \right\}$$

This information is encoded into the rotation or orientation matrix.

Orientation matrices (Rotation Matrices)

Let's consider a vector v and represent it as a function of unit vectors of <b>.

$\underline{v} = \underline{i}_b x_b + \underline{j}_b y_b + \underline{k}_b z_b$ i, j, k form a base.

If now I perform projection of $\underline{v}$ on <a>

$${}^a\underline{v} = {}^a\underline{i}_b x_b + {}^a\underline{j}_b y_b + {}^a\underline{k}_b z_b$$

Now representing as a vector-matrix multiplication we get:

$${}^a\underline{v} = \underbrace{\left[{}^a\underline{i}_b \quad {}^a\underline{j}_b \quad {}^a\underline{k}_b \right]}_{} \begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix}$$

referred frame $\longrightarrow a$
referring frame $\longrightarrow b$

${}^a_bR$ = ROTATION MATRIX

$${}^a\underline{v} = {}^a_bR\ {}^b\underline{v}$$

This is another way of seeing the multiplication.

In Matlab this will be reported like a_R_b.

It's important that in the vector we include the name of the frame on which the vector should be projected on. For example b_distance.

So a moltiplication could be:

$$a\_distance = {}_{a}R_{b} \ \ b\_distance.$$

R is a ortho-normal matrix
columns are orthogonal to each othere
unitary lenght

$$ {}^{a}_{b}R \ {}^{a}_{b}R^{T} = I_{n \times n}$$

Given this the matrix has some properties:
Recalling that:

$$\det(A\,B) = \det(A)\,\det(B)$$
$$\det(A) = \det(A^T)$$

Beeing ortho-normal means:

$$\begin{cases} {}^{a}_{b}R \ {}^{a}_{b}R^{T} = I_{3 \times 3} \\ {}^{a}_{b}R^{T} \ {}^{a}_{b}R = I_{3 \times 3} \end{cases} \Rightarrow {}^{a}_{b}R^{T} = {}^{a}_{b}R^{-1} = {}^{b}_{a}R$$

this because of $\det(R) = 1$

This is important because:

$$\forall M: \ M M^T = I \Rightarrow \det(M) = \pm 1$$

For Rot. Mat the determinant equals 1 → $\det({}^{a}_{b}R) = 1$

this to preserve the direction meaning the right handed frame.

For example

$$\begin{array}{ccc} x & y & z \text{ on } b \\ \end{array}$$
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \det = -1$$

We can see that y and z are the same while x is the opposite, but this is impossible if we consider a right handed frames. So this cannot happen. $\det({}^{b}_{a}R) = -1$ does not respect right-hand rule.

From these properties we can also conclude that the transpose is the inverse.
It also makes sense if you think about it geometrically.

$$SO(n) = \{ R \in \mathbb{R}^{n \times n} : R^T R = I_{n \times n}, \ \underbrace{\det(R) = +1}_{\text{Special for this limitation}} \}$$

Special ortogonal group

This group has some properties : N.B

TODO on what? MULT DIV

- CLOSURE : given any two elements R_1, R_2 of $SO(n)$ then also $R_1 \cdot R_2 \in SO(n)$
- IDENTITY : there exists an element I of $SO(n)$ such that $IR \in SO(n)$ for any R
- INVERSE : for any $R \in SO(n)$ there exists R^{-1} such that $RR^{-1} = I$
- ASSOCIATIVITY : given three elements R_1, R_2, R_3 in $SO(n)$ it follows that $(R_1 R_2) R_3 = R_1 (R_2 R_3)$

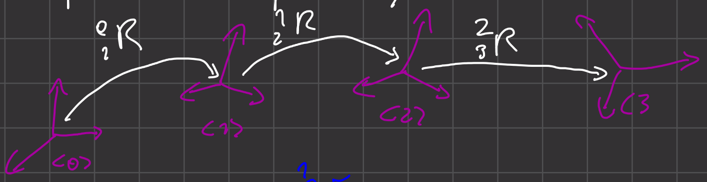
A member of $SO(3)$ is a Rotation Matrix

A consideration about ${}^{a}_{b}R^{T} = {}^{b}_{a}R$ is that this is also equivalent to:

$$ {}^{a}_{b}R = \left[{}^{a}\underline{i}_{b} \quad {}^{a}j_{b} \quad {}^{a}k_{b} \right] = \begin{bmatrix} {}^{b}i_{a}^{T} \\ {}^{b}j_{a}^{T} \\ {}^{b}k_{a}^{T} \end{bmatrix} $$

On the transpose you will se what you see on columns now on the rows.

Our manipulator is composed by a tree of frames:



$$ {}^{0}\underline{v} = \underbrace{{}^{0}_{1}R\,\overbrace{{}^{1}_{2}R\,\underbrace{{}^{2}_{3}R\,{}^{3}v}_{{}^{2}v}}^{{}^{1}v}}_{{}^{0}v} $$

The notation helps to check if everything is done correctly

N.B. This holds for every v so for definition ${}^{0}_{1}R\,{}^{1}_{2}R\,{}^{2}_{3}R = {}^{0}_{3}R$

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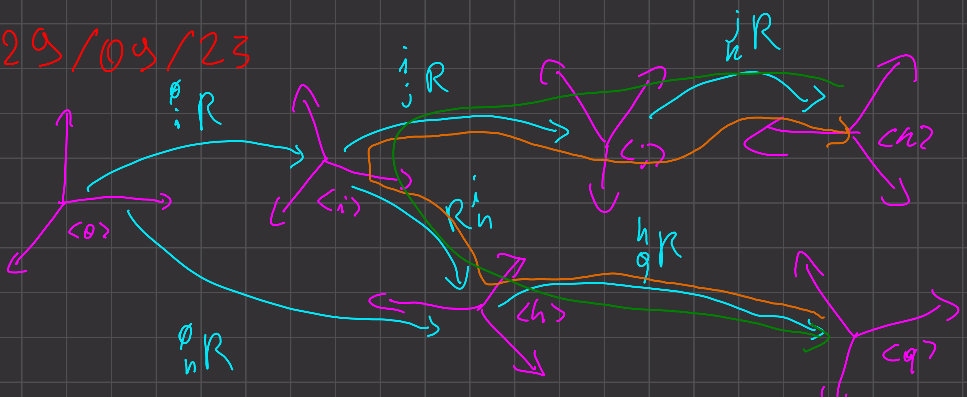


Here we have some reference frames and suppose that we know some rotation matrices.

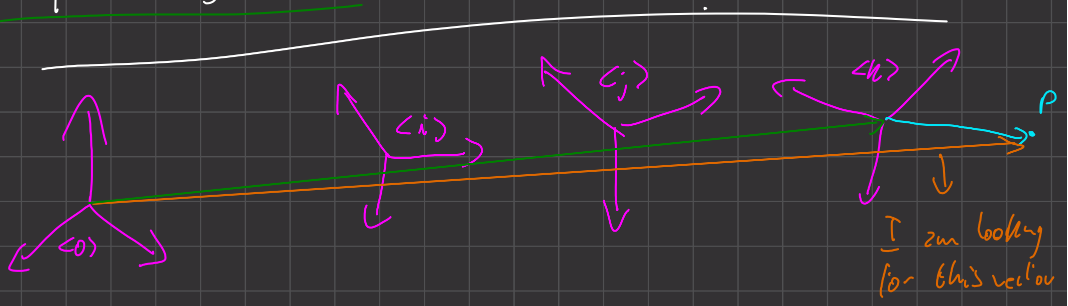
postmoltiplication

How do you compute: $\overrightarrow{{}^q_hR = {}^h_qR^T \, {}^i_hR^T \, {}^i_jR \, {}^j_hR = {}^q_hR \, {}^h_iR \, {}^i_jR \, {}^j_hR}$

We can also do the same thing following this path:

premoltiplication

$$\overleftarrow{{}^q_hR = {}^h_qR^T \, {}^i_hR^T \, {}^i_jR \, {}^j_hR}$$



To get from P to P the position of all the frames is an essential parameter. I could for example use another vector P and now the description of P becomes P + P. This can be done iteratively going back towards <0>.

$${}^kp = {}^k(P - O_h)$$

$${}^0p = {}^0O_h + {}^0(P - O_h)$$

$${}^0p = {}^0O_h + {}^0_hR \, {}^hp$$

Omogeneous Coordinates

$${}^k\underline{\bar{p}} = \begin{bmatrix} {}^h\underline{p} \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

$${}^0\underline{\bar{p}} = {}^0_hT \, {}^h\underline{\bar{p}} \qquad {}^0_hT \triangleq \begin{bmatrix} {}^0_hR & {}^0O_h \\ 0_{3\times1} & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}$$

So if i do

$$\underbrace{\begin{bmatrix} {}^0_hR & {}^0O_h \\ 0_{3\times1} & 1 \end{bmatrix}}_{{}^0_hT} \underbrace{\begin{bmatrix} {}^h\underline{p} \\ 1 \end{bmatrix}}_{{}^h\bar{p}} = \underbrace{\begin{bmatrix} {}^0_hR \, {}^hp + {}^0O_h \\ 1 \end{bmatrix}}_{{}^0\bar{p}}$$

Inverse transformation matrix

The increase in dimension is a gimmick to obtain the previous formula with a matrix by vector multiplication.

We know that:

$${}^a_bT^{-1} = {}^b_aT$$

N.B.

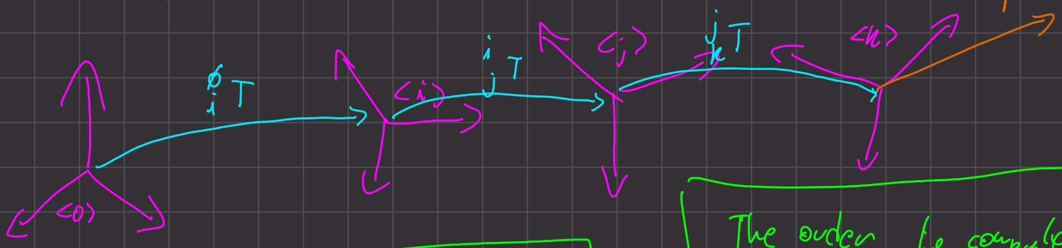
BUT

$${}^a_bT^{-1} \neq {}^a_bT^T$$

$$\boxed{{}^a_bT^{-1} = \left[\begin{array}{c|c} {}^a_bR^T & -{}^a_bR^T\,{}^aO_b \\ \hline 0_{3\times1} & 1 \end{array}\right]}$$

The origin of a on b would be the opposite vector and to put it in the right direction I need a minus in front.

This is very handfull because if we consider the original diagram



$$\boxed{{}^0\bar{p} = \underbrace{{}^0_iT\;{}^i_jT\;{}^j_kT\;{}^j_kT}_{{}^0_kT}\;{}^k\bar{p}}$$

The order to compute the transformation matrix is from <0> to <e>

It's also possible to describe a vector using homogeneous coordinates. It's done by justaposing $\emptyset$ to the vector.

$$p = \begin{bmatrix} {}^np \\ \emptyset \end{bmatrix}_{4\times1}$$

This because the origin in this case does not come into play.

The inertial frame it's not different from any other frame.

Three parameter representation of rotation matrices

^a_bR - Rotation matrices are ortonormal. — vectors are ortogonal to each other; modulus = 1 for every vector.

So I have:

dot product

$$\begin{cases} {}^a i_b^T \, {}^a i_b = 1 \\ {}^a j_b^T \, {}^a j_b = 1 \\ {}^a k_b^T \, {}^a k_b = 1 \end{cases} \qquad \begin{cases} {}^a i_b \, {}^a j_b^T = 0 \\ {}^a j_b^T \, {}^a k_b = 0 \\ {}^a k_b^T \, {}^a i_b^T = 0 \end{cases}$$

^a_aR satisfy all these equations

We have 6 constraints and 9 numbers so we only have to pick 3 that are unknown.

For this reason we can talk about 3 number representation of rotation matrices.

The easiest of these representation is called

Pronounced Oiler

Euler angles

They express that any rotation can be represented by three rotations between 3 different axes.

We can choose between:
- extrinsic rotation
- intrinsic rotation

- I describe the final orientation with rotation around a fixed coordinate system. This unfortunately is not so easy to implement because when we multiply R. matrices we are not doing intrinsic rotation. In this case we first rotate around the original x, then around the original y and finally around the original z.

- This is easier to implement. We first rotate around x, then the rotated y and then the rotated z. When we multiply with Rot. Matrices we are doing intrinsic rotations.

Intrinsic angles can be divided in:

- proper Euler angles
  z-x-z; y-x-y; x-z-x; ....
- Tait-Bryan angles
  x-y-z; z-y-x; ...

These represent the sequence of axes around which you do the rotation

$$\boxed{z - y' - x''}$$

this is the most used one and we will use this one

$${}^{a}_{b}R = \underset{\text{YAW}}{R_z(\psi)}\ \underset{\text{PITCH}}{R_y(\theta)}\ \underset{\text{ROLL}}{R_x(\phi)}$$

the apex stresses that we are talking about intrinsic rotation

It's possible to demonstrate that an extrinsic rotation x-y-z it's the same as an intrinsic rotation z-y-x.

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$${}^{a}_{b}R = R_z(\psi)\ R_y(\theta)\ R_x(\phi) =$$

$$= \begin{bmatrix} c\psi c\theta & -s\psi c\phi + c\psi s\theta s\psi & s\psi s\phi + c\psi c\phi s\theta \\ s\psi c\theta & c\psi c\phi + s\phi s\theta s\psi & -c\psi s\phi + s\theta s\psi c\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}$$

Not needed to know by heart

It's usefull to know the structure to know the inverse.

$$\theta = \underbrace{\text{atan2}}_{\text{software function to compute atan on 4 quadrants instead of two.}}\left(-r_{3,1}, \sqrt{r_{11}^2 + r_{21}^2}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

If $\cos\theta \neq 0$: $\psi = \text{atan2}(r_{21}, r_{11})$
$\phi = \text{atan2}(r_{32}, r_{33})$

If $\cos\theta = 0$ it's an interesting configuration because leads to a singularity.

In the sequence of rotation if we rotate in pitch of $\pm\frac{\pi}{2}$ the x axis becomes the same as the z axis of the original frame. We have infinite combination of first and third axis that end up representing the same output. We get a singularity when the pitch is $\pm\frac{\pi}{2}$.

Suppose now that I want to control roll to a certain angle, when i get $\frac{\pi}{2}$ in pitch yaw and roll become the same. It's an ill defined situation.

The use of Euler angles is fine but we must have regard for pitch being close to $\frac{\pi}{2}$

Vector operations

- Linear combination

$$\underline{v} = c_1 \underline{v}_1 + c_2 \underline{v}_2 + \dots c_n \underline{v}_n$$

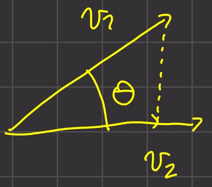
If they are linear independent they can form the base of your n-based dimension frame.

If I take into consideration the projection of a frame I also need to projection the same frame all the components.

$$^{a}\underline{v} = c_1 \, ^{a}\underline{v}_1 + c_2 \, ^{a}\underline{v}_2 + \dots c_n \, ^{a}\underline{v}_n$$

- Scalar product

$$(\underline{v}_1 \cdot \underline{v}_2) = (\underline{v}_2 \cdot \underline{v}_1) = |v_2||v_1| \cos\theta$$



$$(v_1 \cdot v_2) = \underbrace{^{a}\underline{v}_1^T \, ^{a}\underline{v}_2}_{\text{projected on } \langle a \rangle}$$

I can consider $(\underline{v}_1 \cdot) = (\underline{v}_1^T)_{1\times 3}$

linear operation

- Vector product

$$(\underline{v}_1 \wedge \underline{v}_2) = (\underline{v}_1 \times \underline{v}_2) = \begin{cases} \overbrace{|v_1 \times v_2|}^{\text{modulus}} = |v_1||v_2| \sin(\theta) \\ \underline{v}_1 \times \underline{v}_2 \perp (\underline{v}_1, \underline{v}_2) \end{cases}$$

↳ minimum angle

the direction is ortogonal to both v_1 and v_2 according to the right hand rule

- Properties:
  - Anti-commutative: $(\underline{v}_1 \times \underline{v}_2) = -(\underline{v}_2 \times \underline{v}_1)$
  - $^{a}(\underline{v}_1 \times \underline{v}_2) = \underbrace{^{a}[\underline{v}_1 \times]}_{\substack{\text{Operator} \\ 3\times 3}} \, ^{a}v_2 = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} {^{a}v_2}$

    Components of $^{a}\underline{v}_1$

    ↳ This matrix is skew-simmetric

    For this reason
  - $^{a}[\underline{v}_1 \times]^T = -^{a}[\underline{v}_1 \times]$ → axial vector

    sometimes this can also be represented like: $[\underline{v}_1 \times] = S(v_1)$
  - Inverse mapping: vex operator → $\text{vex}(S(\underline{v}_1)) = \underline{v}_1$

$$\forall M : M \in \mathbb{R}^{3\times 3},\ M = -M^T$$
$$M\underline{y} = \text{vex}(M) \times \underline{y}$$

Any skew-simmetric has an axial vector. Vex is needed to get x, y and z

Power of skew-symmetric 3x3 real matrices associated with a unit vector

$h \in \mathbb{R}^{3\times 1}$

$[\underline{h}\times] = \underline{h}\times$

$[\underline{h}\times]^2 = (\underline{h}\,\underline{h}^T - I)$

$[\underline{h}\times]^3 = -[\underline{h}\times]$

$[\underline{h}\times]^4 = -[\underline{h}\times]^2$

$[\underline{h}\times]^5 = [\underline{h}\times]$

$[\underline{h}\times]^6 = [\underline{h}\times]^2$

$\vdots$

After the 4th power they repeat from the first one.

- In general we state that:

$$[\underline{h}\times]^{2i+1} = (-1)^i [\underline{h}\times]$$

Rule for odd numbers (skew symmetric)

$$[h\times]^{2(i+1)} = (-1)^i (h h^T - I)$$

Rule for even numbers (symmetric)

$(I - hh^T)$ projects a vector on the plane ortogonal to h (tono Co..)

- $(I - \underline{h}\,\underline{h}^T)$ is the projection operator defined by h

$(I - \underline{h}\,\underline{h}^T)\underline{h} = \emptyset$ because $(h - h\underbrace{h^T h}_{1}) = \emptyset$

- $\text{trace}([h\times]^{2i+1}) = 0$

$tr([\underline{h}\times]^{2(i+1)}) = (-1)^{(i+1)} \cdot 2$

tr $\to$ sum of all the elements on the main diagonal in a matrix

- $A \in \mathbb{R}^{n\times n}$

$A = A^T$ symmetrical

$A = -A^T$ skew symmetrical

$\forall A \in \mathbb{R}^{n\times n}$ $\quad A = \underbrace{\frac{A + A^T}{2}}_{\text{simmetric}} + \underbrace{\frac{A - A^T}{2}}_{\text{skew-simmetric}}$

no symm or skew-symm

Linear Operators between Vectors

Consider: • frame <b>
• any projected vector ${}^b\underline{v}$
• ${}^b\underline{\omega}$ subject to ${}^b\underline{\omega} = {}^bL\,{}^b\underline{v}$ — linear operator $L \in \mathbb{R}^{3\times3}$

$$ {}^a\underline{\omega} = {}^a_bR\,{}^b\underline{\omega} = {}^a_bR\,{}^bL\,{}^b\underline{v} = \boxed{{}^a_bR\,{}^bL\,{}^a_bR^T}\,{}^a\underline{v} = {}^aL\,{}^a\underline{v} $$

$$ \boxed{{}^aL = {}^a_bR\,{}^bL\,{}^a_bR^T} $$

this is the operation to change the operator from one frame to one other.

For example:

Vector product operator

$$ {}^b(\underline{\omega} \times \underline{v}) = {}^b[\underline{\omega}\times]\,\underline{v} $$

$$ {}^a(\underline{\omega} \times \underline{v}) = {}^a_bR\,{}^b[\underline{\omega}\times]\,{}^a_bR^T\,{}^a\underline{v} = {}^a[\underline{\omega}\times]\,{}^a\underline{v} $$

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Matrix exponential operator

$$ {}^bL = e^{{}^bA} \qquad {}^bA \in \mathbb{R}^{3\times3} $$

$$ {}^bL = \sum_{k=0}^{\infty} \frac{({}^bA)^k}{k!} $$

As any other linear operator we can write:

$$ {}^b\underline{\omega} = {}^bL\,{}^b\underline{v} = e^{{}^bA}\,{}^b\underline{v} $$

Now we want to know ${}^aL$:

$$ {}^aL = {}^a_bR\,{}^bL\,{}^a_bR^T = {}^a_bR\,e^{{}^bA}\,{}^a_bR^T $$

Now if we substitute the definition of the exponential:

$$ = {}^a_bR\left(\sum_{k=0}^{\infty}\frac{({}^bA)^k}{k!}\right){}^a_bR^T = \sum_{k=0}^{\infty}\frac{{}^a_bR\,({}^bA)^k\,{}^a_bR^T}{k!} $$

$$ = \sum_{k=0}^{\infty}\frac{({}^a_bR\,{}^bA\,{}^a_bR^T)^k}{k!} $$

this step is possible because:

$$ \underbrace{({}^a_bR\,{}^bA\,\cancel{{}^a_bR^T})(\cancel{{}^a_bR}\,{}^bA\,\cancel{{}^a_bR^T})\cdots(\cancel{{}^a_bR}\,{}^bA\,\cancel{{}^a_bR^T})(\cancel{{}^a_bR}\,{}^bA\,{}^a_bR^T)}_{k\ \text{times}} $$

Identity Identity Identity

To conclude

$$ {}^{a}L = e^{{}^{a}_{b}R\, {}^{b}A\, {}^{a}_{b}R^{T}} = e^{{}^{a}A} $$

Properties of the matrix exponential

- $M e^{\theta M} = e^{\theta M} M \qquad \theta \in \mathbb{R}$
- $(e^{\theta M})^{T} = e^{\theta M^{T}}$
- $(e^{\theta M})^{-1} = e^{-\theta M}$

Exponential representation of rotation matrixes

Every $e^{\theta[\underline{h}\times]} \in SO(3)$, so this is a rotation matrix

To prove this:

$$ \left(e^{\theta[\underline{h}\times]}\right)^{T} e^{\theta[\underline{h}\times]} = e^{\theta[\underline{h}\times]^{T}} e^{\theta[\underline{h}\times]} = $$

$$ = e^{-\theta[\underline{h}\times]} e^{\theta[\underline{h}\times]} = I_{3\times3} $$

This determinant is either +or -1. If $\theta=0$ the determinant is +1. There is a continuity rule also for the determinant. It cannot jump from +1 to -1. For this continuity rule we can state that the determinant is always +1.

So we have det = 1 and the product for the transpose is $I_{3\times3}$ so it's a Rotation Matrix.

Angle axes parameters.

$\theta \in \mathbb{R}$ angle

$h \in \mathbb{R}^{3\times1}$, $|h| = 1$ axis

together they are the angle-axis representation of the rotation matrix $e^{\theta[\underline{h}\times]}$

n gives you the axis around which to rotate and θ gives you the quantity.

Rotation Vector

$\rho = \theta h \in \mathbb{R}^{3\times1}$ is called the rotation vector associated to the rotation matrix $e^{\theta[\underline{h}\times]}$

Angle axes vector

$$\underline{v} = \begin{bmatrix} \underline{h} \\ \theta \end{bmatrix} \in \mathbb{R}^{4\times 1}$$

This is also called 4D-vector associated to the matrix $e^{\theta[\underline{h}\times]}$

Let's now consider how the exponential is computed:

$$e^{\theta[\underline{h}\times]} = \sum_{k=0}^{\infty} \frac{(bA)^k}{k!} =$$

We split even and odd powers of this infinite summation:

$$= I_{3\times 3} + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \dots\right)[\underline{h}\times] + \left(\frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right)[\underline{h}\times]^2$$

We can do this for the properties of skew-simmetrical matrixes

Rodrigues Formula

$$\boxed{= I_{3\times 3} + \underbrace{\sin\theta}_{\text{the first sum converges to sin}}[\underline{h}\times] + \underbrace{(1-\cos\theta)}_{\text{the second one converges to } (1-\cos\theta)}[\underline{h}\times]^2}$$

N.B. → h is always a unit vector

From the Rodrigues formula we can find the inverse mapping from a rotation matrix to the angle axes mapping.

Equivalent Angle-axis inverse problem

$$R \xrightarrow{?} \theta, \underline{h}$$

On the diagonal you get the square of the components.

$$(\underline{h}\underline{h}^T - I)$$

Let's look at the $tr(R) = tr\left(I + \sin\theta[\underline{h}\times] + (1-\cos\theta)[\underline{h}\times]^2\right) =$

$$= 3 + 0 + (1-\cos\theta)(1-3) = 3 - 2(1-\cos\theta) = 1 + 2\cos\theta$$

So: $\theta = \cos^{-1}\left(\frac{tr(R)-1}{2}\right)$

Like this we can find more than one parameter

For example if:

$\theta = \emptyset$ then h is arbitrary

$\theta = \pi$ then

Let's try it:

$$e^{\pi[h\times]} = I_{3\times3} + 2(\underline{h}\,\underline{h}^T - I) = 2hh^T - I_{3\times3} =$$

$$= \begin{bmatrix} 2h_1^2 - 1 & 2h_1h_2 & 2h_1h_3 \\ 2h_1h_2 & 2h_2^2 - 1 & 2h_2h_3 \\ 2h_1h_3 & 2h_2h_3 & 2h_3^2 - 1 \end{bmatrix} = R$$

From this we can conclude that the modulus of h =

$$|h_i| = +\sqrt{\frac{r_{ii}+1}{2}} \qquad \forall_i = 1,2,3$$

To find the sign of this vector we can select a random component $\bar{i} \in 1,2,3$.

$$h_{\bar{i}} = \pm\sqrt{\frac{r_{\bar{i}\bar{i}}+1}{2}}$$

The generic $h_j = \text{sgn}(h_{\bar{i}})\,\text{sgn}(r_{\bar{i}j})\sqrt{\frac{r_{jj}+1}{2}} \qquad \forall_j \in (1,2,3) \setminus \bar{i}$

If you pick $\bar{i} = 1$
then you try for 2 and 3

What about all the other cases?

Let's consider $R - R^T$

First we expand R and R^T with the Rodriguez formula.
we are left with

$$R - R^T = \sin\theta[h\times] - \sin\theta[h\times]^T$$

$$R - R^T = \cancel{I_{3\times3}} + \sin\theta[h\times] + \cancel{(1-\cos\theta)[h\times]^2}$$
$$- \cancel{I_{3\times3}} - \sin\theta[h\times] - \cancel{(1-\cos\theta)[h\times]^2}$$

symmetrical
=

$$= 2\sin\theta[\underline{h}\times]$$

$$\frac{R - R^T}{2} = \sin\theta[\underline{h}\times] \qquad \text{so} \qquad \sin\theta\,\underline{h} = \text{vex}\left(\frac{R - R^T}{2}\right)$$

This extracts the
axial vector from the
skew symmetrical Matrix

The goal at the end will be to make the rotation matrix converge to I and the norm of the rotation vector to $\emptyset$.

Both yaw-pitch-roll and rotation matrixes show that the mapping from R to these 2/3 parameters is not unique

Euler theorem

It states that any 3d frame can be aligned with any other arbitrary fixed frame by one rotation θ around one axis.

08/10/23

Unit quaternion

These are defined by $q \in \mathbb{R}^{4\times1}$

$$q \triangleq \begin{bmatrix} \mu \\ \varepsilon \end{bmatrix} \quad \text{where} \begin{cases} \mu \in \mathbb{R} \\ \varepsilon \in \mathbb{R}^{3\times1} \end{cases}$$

$$\mu^2 + \|\varepsilon\|^2 = 1$$ ← this is why it's called unit quaternion

The relation between this and the rotation vector are the following:

Trigonometric Properties

First let's recall:

$$\text{I} \quad \sin(\varphi+\beta) = \sin\varphi\cos\beta + \sin\beta\cos\varphi$$

$$\text{II} \quad \cos(\varphi+\beta) = \cos\varphi\cos\beta - \operatorname{sen}\varphi\operatorname{sen}\beta$$

} Postaferesi Formulas

Let's now consider:

$$\varphi = \beta = \frac{\theta}{2}$$

Now we get what appears in the Rodriguez formula:

$$\text{I} \quad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\text{II} \quad \cos\theta = \cos^2\frac{\theta}{2} - \operatorname{sen}^2\frac{\theta}{2}$$

Now if we do

$$\cos\theta = \cos^2\frac{\theta}{2} \underbrace{- \operatorname{sen}^2\frac{\theta}{2} + \cos^2\left(\frac{\theta}{2}\right)}_{=1} - \cos^2\left(\frac{\theta}{2}\right)$$

$$1 + \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)$$

Now if we add and subtract $\operatorname{sen}^2\left(\frac{\theta}{2}\right)$ here we get

$$1 - \cos\theta = 2\operatorname{sen}^2\left(\frac{\theta}{2}\right)$$

Rodriguez V2

$$R = I + \sin\theta [\underline{h}\times] + (1-\cos\theta)[\underline{h}\times]^2 =$$

$$= I + 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)[\underline{h}\times] + 2\sin^2\left(\frac{\theta}{2}\right)[\underline{h}\times]^2$$

Now if we define $\boxed{\begin{cases} \mu \triangleq \cos\frac{\theta}{2} \\ \underline{\varepsilon} \triangleq \sin\frac{\theta}{2}\,\underline{h} \end{cases}}$ we can write: ???

$\sin\theta$ goes inside $[\underline{h}\times]$

$\rightarrow$ This is a not singular representation of the rotation matrix.

$$R = I + 2\mu[\underline{\varepsilon}\times] + 2[\underline{\varepsilon}\times]^2$$

If $\theta = 0$ then $\varepsilon = 0$. There is no undefinition of the axis vector $\underline{h}$ because I am using sin. It's always well defined.

Inverse mapping: from $R \to$ quaternions

To compute the inverse mapping we do:

$$tr(R) = 3 + 0 + 2\,tr([\underline{\varepsilon}\times]^2)$$

$$tr(R) = 3 + 2\,tr(\underline{\varepsilon}\underline{\varepsilon}^T - \underline{\varepsilon}^T\underline{\varepsilon} I)$$

$$tr(R) = 3 + 2(\|\varepsilon\|^2 - 3\|\varepsilon\|^2)$$

$$tr(R) = 3 - 4\|\varepsilon\|^2$$

$$tr(R) = 4\mu^2 - 1$$

$$\mu = \frac{1}{2}\sqrt{tr(R)+1}$$

Let's now work on ε

$$R - R^T = \cancel{I} + 2\mu[\varepsilon\times] + \cancel{2[\varepsilon\times]^2} +$$
$$\cancel{-I} + 2\mu[\varepsilon\times] - \cancel{2[\varepsilon\times]^2}$$

$$R - R^T = 4\mu[\varepsilon\times]$$

$$\frac{R-R^T}{2} = 2\mu[\varepsilon\times]$$

$$\varepsilon = \frac{1}{2\mu}\,vex\left(\frac{R-R^T}{2}\right) \quad \text{if } \mu \neq 0$$

ε eigenvector of the matrix $\frac{R-R^T}{2}$ and then pick the one with eigenvalue = 1
↳ autovalore

if $\mu = 0$

Flipping the signs will give you the other

or values?

Recalling euler angles, there the problem was that if we have the sequence x-y-x if $y=0$ then we have two times the same x

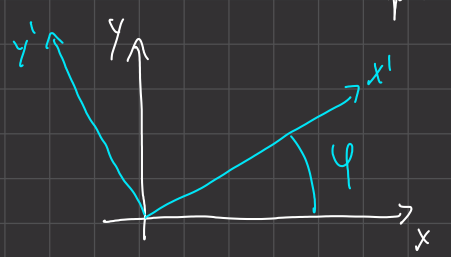
Let's also recall how an elementary rotation is written. So fo example a rotation around the z axis of φ is:

$$R_z(\varphi) = \begin{bmatrix} \cos\varphi & -\text{sen}\,\varphi & 0 \\ \text{sen}\,\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\varphi & 0 & \text{sen}\,\varphi \\ 0 & 1 & 0 \\ -\text{sen}\,\varphi & 0 & \cos\varphi \end{bmatrix}$$

$$R_x[\phi] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\text{sen}\,\phi \\ 0 & \text{sen}\,\phi & \cos\phi \end{bmatrix}$$

To find them for example:



Right hand rule. the rotation vector is out of the page.

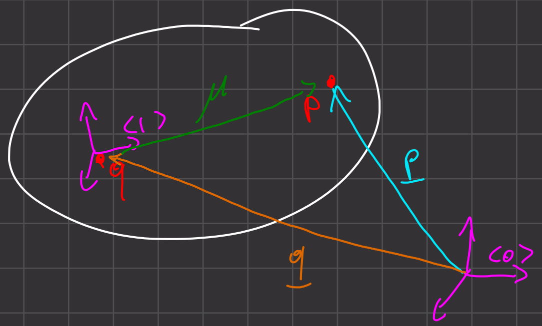
So these are the x' coordinates on the x axis

These the y' on the y

Thes z' on the z.

Angular velocity vector

Let's consider a generic object with specific points where we have some frames



$\rho = q + \mu$

If we now start projecting:

$${}^0p = {}^0q + {}^0\mu = {}^0q + {}^0_1R\,{}^1\mu$$

Now we want to consider:

$${}^0\dot{p} \triangleq \frac{d}{dt}\left({}^0_1R\,{}^1p\right) \neq {}^0_1R\,\frac{d}{dt}\,{}^1p$$

this is the correct one because we are watching from frame <0>
In the second one we are calculating from the frame <1>

This ambiguity will be solved with ${}^0V_{P/0}$ -> this means seen from 0

Using this notation the previous becomes:

$${}^0\dot{p} \triangleq \underbrace{\frac{d}{dt}\left({}^0_1R\,{}^1p\right)}_{= {}^0\underline{V}_{P/0}} \neq \underbrace{{}^0_1R\,\frac{d}{dt}\,{}^1p}_{= {}^0V_{P/1}}$$

Let's continue:

$${}^0\dot{p} = {}^0\dot{q} + \frac{d}{dt}\left({}^0_1R\,{}^1\mu\right)$$

q is the position of frame <1> to <0>
μ position from p to <1>

$$= {}^0\dot{q} + {}^0_1\dot{R}\,{}^1\mu + {}^0_1R\,{}^1\dot{\mu}$$

$$= \underbrace{{}^0\dot{q}}_{} + \underbrace{{}^0_1\dot{R}\,\underbrace{{}^0_1R^T\,{}^0_1R}_{I_{3\times3}}\,{}^1\mu}_{} + \underbrace{{}^0_1R\,{}^1\dot{\mu}}_{}$$

This tells how <1> is changing in respect to <0>

This term takes into account that <1> is changing orientation to <0>

how μ changes in respect to <1> then projected on <0> } this is 0 if in the rigid space of frame <1>

Now R is an element of $SO(3)$ so:

$$R R^T = I_{3x3}$$

So if now I do:

$$\frac{d}{dt}\left({}^0_1R \, {}^0_1R^T \right) = 0 = {}^0_1\dot{R} \, {}^0_1R^T + {}^0_1R \, {}^0_1\dot{R}^T$$

$$\underbrace{{}^0_1\dot{R} \, {}^0_1R^T} = - {}^0_1R \, {}^0_1\dot{R}^T = \underbrace{- \left({}^0_1\dot{R} \, {}^0_1R^T \right)^T}$$

skew-simmetric matrix

We can always associate an axial vector to a skew symmetrical matrix.

$$\left({}^0_1\dot{R} \, {}^0_1R^T \right) = {}^0\underline{\omega}_{1/0} \times \underline{\mu}$$???

Definition of angular velocity vector

$\forall$ $R \in SO(3)$ there exists a unique $\omega \in \mathbb{R}^{3x1}$ such that

$$\dot{R} R^T = [\underline{\omega} \times]$$

Strapdown equation

So if we have:

$${}^0_1\dot{R} \, {}^0_1R^T = [{}^0\underline{\omega}_{1/0} \times]$$

$$\Downarrow$$

$${}^0_1\dot{R} = [{}^0\underline{\omega}_{1/0} \times] \, {}^0_1R$$

$${}^0_1\dot{R} = {}^0_1R \, [{}^1\underline{\omega}_{1/0} \times]$$

inverse = transpose

$$\underline{[{}^0\omega_{1/0}] = {}^0_1R [{}^1\omega_{1/0}] {}^0_1R^T} \cdot {}^0_1R$$

Transformation of the operator

Example:

$$R_z(\alpha) = \begin{bmatrix} \cos(\alpha) & -\operatorname{sen}(\alpha) & 0 \\ \operatorname{sen}(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\dot{\alpha} \neq 0$

$$\dot{R}_z(\alpha) = \begin{bmatrix} -\operatorname{sen}(\alpha) & -\cos(\alpha) & 0 \\ \cos(\alpha) & -\operatorname{sen}(\alpha) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If we directly calculate:

$$\dot{R}_z(\alpha)\, R_z^T(\alpha) = \dot{\alpha} \cdot \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [\underline{\omega} \times]$$

$$\text{So } \omega = \begin{bmatrix} 0 \\ 0 \\ \dot{\alpha} \end{bmatrix}$$

When you have the angular velocity (ω) you cannot directly integrate it but use something like the strapdown matrix.

Except when you are constantly rotating around one axis, meaning that I don't change axis in respect to wich I rotate.

To better say in general you have to apply the strapdown for every infinitesimal rotation and compute ω.

Angular velocity properties and composition rules

$$ {}^1_0\dot{R}\, {}^1_0R^T = [{}^1\underline{\omega}_{0/1} \times] = \text{N.B.}$$

$$ {}^0_1R\, [{}^1_0\dot{R}\, {}^1_0R^T]\, {}^0_1R^T \triangleq [{}^0\omega_{0/1}]$$

$$= {}^0_1R\, {}^1_0\dot{R} (\cancel{{}^1_0R^T\, {}^0_1R^T}) = {}^1_0R^T\, {}^1_0\dot{R} \longrightarrow \quad A^T B = (B^T A)^T$$

$$= ({}^1_0\dot{R}^T\, {}^1_0R)^T = ({}^0_1\dot{R}\, {}^0_1R^T)^T = -({}^0_1\dot{R}\, {}^0_1R^T) =$$

skew simmetric

strapdown

$$= -[{}^0\underline{\omega}_{1/0} \times]$$

N.B.

$$\boxed{{}^0\underline{\omega}_{0/1} = -{}^0\omega_{1/0}}$$

We can conclude

the angular velocity is a relative concept

Let's see the composition:

$$ {}^0_2R = {}^0_1R\, {}^1_2R$$

$$ {}^0_2\dot{R}\, {}^0_2R^T = \left[\frac{d}{dt}({}^0_1R\, {}^1_2R)\right] ({}^0_1R\, {}^1_2R)^T$$

$$= \left({}^0_1\dot{R}\,{}^1_2R + {}^0_1R\,{}^1_2\dot{R}\right){}^1_2R^T\,{}^0_1R^T$$

$$= \underbrace{{}^0_1\dot{R}\,{}^0_1R^T}_{\text{strapdown}} + \underbrace{{}^0_1R\left[\underbrace{{}^1_2\dot{R}\,{}^1_2R^T}_{\text{strapdown}}\right]{}^0_1R^T}_{\text{Rotated to be on } 0}$$

this would be:

$${}^0\omega_{1/0} + {}^0\left[{}^1\omega_{2/1}\right]$$

From this follows:

$$\boxed{\underline{\omega}_{2/0} = \underline{\omega}_{1/0} + \underline{\omega}_{2/1}}$$

Galilean velocity composition

Angular velocity and the rotation vector

$$\underline{\omega} = \dot{\theta}\underline{h} + \text{sen}\,\theta\,\dot{\underline{h}} + (1-\cos\theta)\,\underline{h}\times\dot{\underline{h}}$$

Base of a 3-dimensional space. $\Rightarrow$ h and $\dot{h}$ are ortogonal because h is unitary in length and can only rotate, not change the length.

If $\dot{h} = 0$, means that the axis around wich one frame is rotating respect to the other is not changing, and i can integrate.

Otherwise if $\dot{h} \neq 0$ integrating ω it does not give θ, and we are rotating around many axes.

Mapping between ω and the 4-D axis vector

$$\underline{v} = \begin{bmatrix} \underline{h} \\ \theta \end{bmatrix}$$

$$\dot{\underline{v}} = \begin{bmatrix} -\dfrac{\sin\theta}{2(1-\cos\theta)}[\underline{h}\times]^2 - \dfrac{1}{2}[\underline{h}\times] \\ \underline{h}^T \end{bmatrix} \underline{\omega}$$

This works only if $\cos\theta \neq 1$

You cannot integrate $\dot{R}$ from the strapdown because otherwise you won't be in SO(3). One solution could be to normalize the matrix. If you want to know the new orientation it could be convenient to find the mapping of yaw-pitch-roll angles.

If we know how yaw-pitch, roll derivative behave in relation to ω then we are good.

When you know yaw-pitch-roll you can now do a forward euler integration. What you will get is another so(3) element.

Angular velocity vector and time derivative of minimal parametrization vectors

$$^{a}\omega_{b/a} = {}^{b}L_{b/a}\,\dot{\bar{\varphi}} \qquad \bar{\varphi} = \begin{bmatrix}\psi\\ \theta\\ \phi\end{bmatrix}$$

N. B. The order is yaw-pitch-roll (z-y-x)

$$^{a}\omega_{b/a} = \begin{bmatrix}0\\0\\1\end{bmatrix}\dot{\psi} + R_z(\psi)\begin{bmatrix}0\\1\\0\end{bmatrix}\dot{\theta} + R_z(\psi)R_y(\theta)\begin{bmatrix}1\\0\\0\end{bmatrix}\dot{\phi} =$$

$$= \begin{bmatrix}0 & -s\psi & c\psi c\theta\\ 0 & c\psi & s\psi c\theta\\ 1 & 0 & -s\theta\end{bmatrix}\begin{bmatrix}\dot{\psi}\\ \dot{\theta}\\ \dot{\phi}\end{bmatrix}$$

$$\det(L_{b/a}) = -\cos(\theta)$$

← This mappings becomes undefined whenever $\theta = \frac{\pi}{2}$ or better I cannot find the inverse meaning that from omega I don't know the derivatives of ψ, θ, ϕ

26/10

Unit quaternions and the angular velocity vector

$$\begin{cases}\dot{\mu} = -\frac{1}{2}\,\underline{\varepsilon}^T \omega \\ \dot{\underline{\varepsilon}} = \frac{1}{2}\left(\mu I_{3\times3} - [\underline{\varepsilon}\times]\right)\underline{\omega}\end{cases}$$

Inverse mapping

$$\underline{\omega} = 2(\mu\dot{\underline{\varepsilon}} - \underline{\varepsilon}\,\dot{\mu}) + 2\,\underline{\varepsilon}\times\dot{\underline{\varepsilon}}$$

Keep in mind that doing a forward euler integration you could spoil the relation:

$$\mu^2 + \|\varepsilon\|^2 = 1$$

If you then use this not unit quaternion you would end up with a non rotation matrix so you need to normalize the unit quaternion before calculating the rotation matrix.

Euler angles do not have these problems, but they suffer from singularities.

For example to describe the rotation of an object to another with a unit quaternion is not too easy, while using yaw-pitch-roll it's easier.

23/10 – KINEMATICS

Revise rotation

Vectors can be described as pure geometrics so an abstract thing.

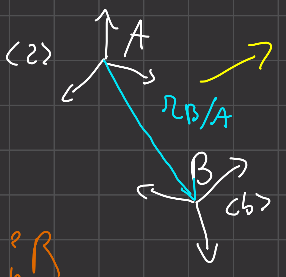
The geometric idea is useful to make models.

Then we have the algebraic representation

We are going to learn how to handle moving vectors in space.

We are gonna observe these moving vectors from a frame called observer.

Each one of the body composing a robot is home of one reference frame.



translation vector between the two reference frame.

A, B, C are points
$\langle a \rangle, \langle b \rangle$ are ref. frame

${}^a_b R$

${}^a_b R$ this represent the axis of frame $\langle b \rangle$ as seen from frame $\langle a \rangle$

The axis of frame $\langle a \rangle$ will be described in two ways:

$$\begin{cases} (\underline{i}_1, \underline{j}_1, \underline{k}_1) \\ (\underline{e}_1^1, \underline{e}_2^1, \underline{e}_3^1) \end{cases}$$

in frame 1

first vector

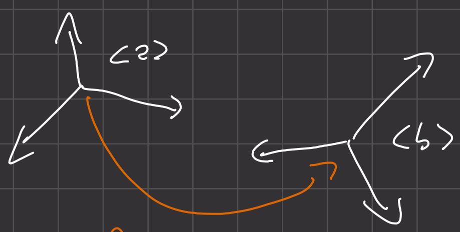
For example in frame B this becomes:

$$\begin{cases} (i_1, j_1, k_1) \\ (e_1^2, e_2^2, e_3^2) \end{cases}$$

So R can be written as:

$${}^a_bR = \left[\left(e_i^1 \cdot e_j^2\right)\right] = \begin{bmatrix} e_1^1\, e_1^2 & e_1^1\, e_2^2 & e_1^1\, e_3^2 \\ e_2^1\, e_1^2 & * & * \\ e_3^1\, e_1^2 & * & * \end{bmatrix}$$

${}^a e_1^2 = {}^a i_2$ ${}^a e_2^2 = {}^a j_2$ ${}^a e_3^2 = {}^a k_2$



${}^a_bR \rightarrow$ this is also a way to indicate how to rotate vectors.

We can also define an homogeneous transformation:

$${}^a_bT = \left[\begin{array}{ccc|c} & {}^a_bR & & {}^a r_{b/a} \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

sometimes this is called H

Let's suppose that we put a vector ${}^b m$.
We might be interested to transform this vector in frame <a>
To do that first we have to write it in homogeneous coordinates
So ${}^b\bar{m} = \begin{bmatrix} {}^b m \\ 1 \end{bmatrix}$

$${}^a_bT\,{}^b\bar{m} \triangleq {}^a\bar{m} = \begin{bmatrix} {}^a_bR\,{}^b m + {}^a r_{b/a} \\ 1 \end{bmatrix} \rightarrow {}^a m$$

Assumptions:

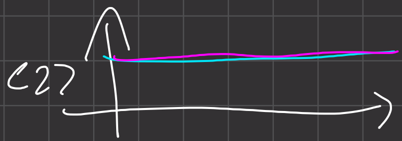
- All measurements are consistent
- Non consider relativistic effects

Let's consider a vector that is changing its lenght in time.

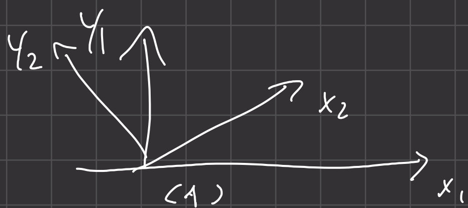
$u(t)$

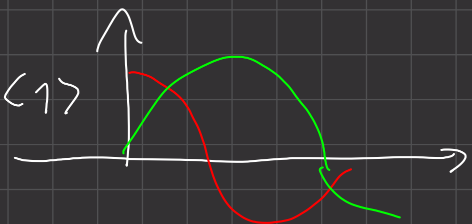
Carosel example

If we consider the frame of the axis the orientation of the vectors is constant so if we map them



Now if I add another observer:





Now if I take the derivates taken from the first frame I get $\frac{d_2}{dt} x_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

If I take the derivate from frame 1:

$$\frac{d_1}{dt} x_2 = \begin{bmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{bmatrix}$$

the derivatives are not absolute

$$\mu(t) \rightarrow \frac{d_a}{dt}\mu(t) = \underbrace{\frac{d}{dt}\sum_{i=1}^{3} \mu_i(t)\, e_i^a}\ \{x_1 i_1 + y_1 j_1 + z_1 k_1$$

in frame a

scalar functions

$$\mu(t) \rightarrow \frac{d_b}{dt}\mu(t)$$

$$= \underbrace{\sum_{i=1}^{3}\left(\frac{d}{dt}\mu_i(t)\, e_i^a\right)} + \underbrace{\sum_{i=1}^{3}\mu_i(t)\cdot\frac{d_a}{dt}e_i^a}$$

these are the same common for all the observer.

equivalent

these are 0 because represent how the frame changes with respect with himself.

$$= \frac{d_b}{dt}\sum_{i=1}^{3} {}^b\mu_i(t)\, e_i^b = \underbrace{\sum_{i=1}^{3}\left(\frac{d_b}{dt}{}^b\mu_i(t)\right)e_i^b} + \underbrace{\sum_{i=1}^{3}{}^b\mu_i(t)\,\frac{d_b}{dt}e_i^b}$$

We can write that:

$${}^b\left(\frac{d_b \mu}{dt}\right) = \begin{pmatrix} \frac{d}{dt}{}^b\mu_1 \\ \frac{d}{dt}{}^b\mu_2 \\ \frac{d}{dt}{}^b\mu_3 \end{pmatrix}$$

We now would want to compute the time derivative in frame a of a quantity expressed in frame b.

$$\frac{d_a}{dt}\mu(t) = \frac{d_a}{dt}\sum_{i=1}^{3}{}^b\mu_i(t)\, e_i^b =$$

vector in b composed by derivatives of scalar functions that are invariant to the reference frame.

$$= \underbrace{\sum_{i=1}^{3}\left(\frac{d_a}{dt}{}^b\mu_i(t)\right)e_i^b}_{\frac{d_b}{dt}\mu(t)} + \sum_{i=1}^{3}{}^b\mu_i(t)\left(\frac{d_a}{dt}e_i^b\right)$$

So now we can write:

$$ {}^b\left(\frac{d_b \mu}{dt}\right) = \begin{pmatrix} \frac{d}{dt}{}^b\mu_1 \\ \frac{d}{dt}{}^b\mu_2 \\ \frac{d}{dt}{}^b\mu_3 \end{pmatrix} = \frac{d_a}{dt}\begin{pmatrix} {}^b\mu_1(t) \\ {}^b\mu_2(t) \\ {}^b\mu_3(t) \end{pmatrix} $$

Now we know that:

$$ \frac{d_a}{dt}\mu(t) = \frac{d_b}{dt}\mu(t) + \sum {}^b\mu_i(t)\left(\frac{d_a}{dt}{}^b e_i\right) $$

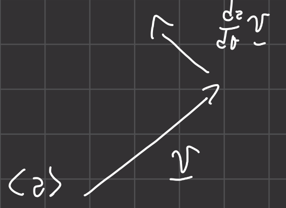
NOTE

Consider a vector $\underset{\text{arbitrary}}{v}$ s.t. it's $|v| = \underline{v}\cdot\underline{v} = 1$

Let's now compute the time derivative:

$$ \frac{d}{dt}(\underline{v}\cdot\underline{v}) = \emptyset \Rightarrow \cancel{2}\,\underline{v}\cdot\frac{d}{dt}v = 0 $$

this tells us that the time derivative of a vector must be orthogonal to the vector. The vector can only rotate, not change its length.



We can think of associating a cross product to $\left(\frac{d}{dt}v\right)$.

So we assume that I can find a vector ω that:

$$ v\cdot\underbrace{\frac{d_a}{dt}\underline{v}}_{\underline{\omega}\times\underline{v}} $$

Assume $\exists\ \omega_1 \neq \omega_2$ so $\omega_1 \times v = \omega_2 \times v$

$$(\omega_1 - \omega_2) \times \underline{v} = 0 \quad \forall v$$ so ω is unique

ω represent the angular velocity of v to respect to b.

Now we have a way to describe:

$$\sum {}^{b}\mu_i(t) \underbrace{\left(\frac{d}{dt}\, e_i^b\right)}_{\underline{\omega} \times e_i^b}$$

$\omega_{b/a}$ → the angular velocity of frame b in respect to a.

$$\frac{d_a}{dt}\,\mu(t) = \frac{d_b}{dt}\,\mu(t) + \sum \omega_{b/a} \times (\mu_i\, e_i^b) =$$

$$= \frac{d}{dt}\,\mu(t) + \omega_{b/a} \times \underbrace{\sum_{i=1}^{3} (\mu_i \cdot e_i^b)}_{\underline{\mu(t)}}$$

We have just derived the law that associates a derivative from a frame to another.

the mother of all laws

$$\boxed{\frac{d_a}{dt}\,\mu(t) = \frac{d_b}{dt}\,\mu(t) + \omega_{b/a} \times \mu(t)}$$

→ value of vector

how the observer rotates.

Algebraic representation

this means scalar multiplying what's inside [ ] by e_1^b, e_2^b, e_3^b

$${}^{a}\left[\frac{d}{dt}\,\mu(t)\right] = {}^{a}_{b}R\ {}^{b}\left[\frac{d_b}{dt}\,\mu(t)\right] + {}^{a}\left[\omega_{b/a} \times \mu(t)\right]$$

this is what they are

$$\frac{d}{dt}\begin{bmatrix} {}^a\mu_1(t) \\ {}^a\mu_2(t) \\ {}^a\mu_3(t) \end{bmatrix} \qquad \frac{d}{dt}\begin{bmatrix} {}^b\mu_1(t) \\ {}^b\mu_2(t) \\ {}^b\mu_3(t) \end{bmatrix} \qquad \underbrace{[\omega_{b/a}\times]}_{3\times 3}\, {}^a\mu(t)$$

skew symmetric

$$\frac{d_a}{dt}\mu = \sum\left(\frac{d}{dt}\,{}^a\mu_i\right) e_i^a + \dots$$

this is always a standard derivative because these quantities are scalar

$$\frac{d_b}{dt}\mu = \sum\left(\frac{d}{dt}\,{}^b\mu_i\right) e_i^b$$

$$\frac{d_a}{dt}\mu = \frac{d_a}{dt}\left(\sum {}^b\mu_i\, e_i^b\right) = \sum\left(\frac{d}{dt}\,{}^b\mu_i\right){}^b e_i + \sum {}^b\mu_i\left(\frac{d_a}{dt} e_i^b\right)$$

Let's now consider an example:

$$\frac{d_a}{dt} e_i^b = \underbrace{\frac{d_b}{dt} e_i^b}_{\text{this is } \emptyset} + \left(\omega_{b/a} \times e_i^b\right) \qquad \forall i = 1,2,3$$

$${}^a\!\left(\frac{d_a}{dt} e_i^b\right) = {}^a_b\dot{R} = \left[{}^a\omega_{b/a}\times\right]{}^a_bR$$

Properties:

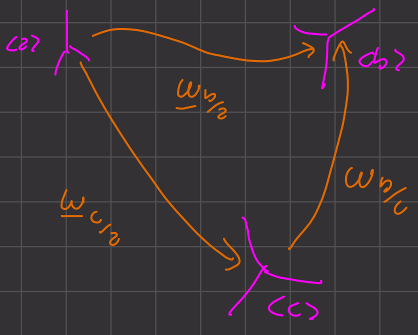
1) ${}^b_aR\,{}^a_bR = \mathbb{1}$ (identity) $\rightarrow \frac{d}{dt}\left({}^b_aR\,{}^a_bR\right) = [\emptyset] \Rightarrow {}^b_a\dot{R}\,{}^a_bR + {}^b_aR\,{}^a_b\dot{R} = [\emptyset]$.

Same trickery as before

$${}^b_a\dot{R}\,\cancel{{}^a_bR} = -{}^b_aR\,{}^a_b\dot{R} = -{}^b_aR\left[{}^a\omega_{b/a}\right]\cancel{{}^a_bR}$$

$${}^b_a\dot{R} = -{}^b_aR\left[{}^a\omega_{b/a}\right]$$

2) Composition of angular velocities



$$\begin{cases} \dfrac{d_a}{dt}\mu = \dfrac{d_b}{dt}\mu + \omega_{b/a} \times \mu \\ - \\ \dfrac{d_c}{dt}\mu = \dfrac{d_b}{dt}\mu + \omega_{b/c} \times \mu \\ + \\ \dfrac{d_a}{dt}\mu = \dfrac{d_c}{dt}\mu + \omega_{c/a} \times \mu \end{cases}$$

$$\cancel{\frac{d_c}{dt}\mu} + \cancel{\frac{d_a}{dt}}\mu - \cancel{\frac{d_a}{dt}} = \cancel{\frac{d_c}{dt}}\mu + \frac{d_b}{dt}\mu - \frac{d_b}{dt}\mu + \left[\left(\omega_{c/a} + \omega_{b/a}\right) - \omega_{b/a}\right] \times \mu$$

$$\left[\left(\omega_{c/a} + \omega_{b/c}\right) - \omega_{b/a}\right] \times \mu = \emptyset$$

We assumed μ arbitrary so this implies that the square brackets is a null vector:

$$\Downarrow$$

$$\omega_{b/a} = \omega_{c/a} + \omega_{b/c}$$

Angular velocities are additive quantities.

Very easy corollary:

$$\underbrace{\omega_{a/a}}_{\emptyset} = \omega_{b/a} + \omega_{a/b} \Rightarrow \omega_{a/b} = -\omega_{b/a}$$

Knowing this we can immediately write the previous equation

$${}^b_a\dot{R}\,\cancel{{}^a_bR} = -{}^b_aR\,{}^a_b\dot{R} = -{}^b_aR\left[{}^a\omega_{b/a}\right]\cancel{{}^a_bR} = {}^b_aR\left[{}^a\omega_{a/b}\right]$$

NOTE

$${}^aA = {}^a_bR\,{}^aA\,{}^b_aR$$