Throughput and Pricing of Ridesharing Systems

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***Abstract*—We introduce a queueing model for passengers waiting to get transport by drivers participating in a ridesharing system. The effect of available drivers' mobility pattern, their willingness to accept rides in a given location, and the incentives offered by the platform, on system throughput is considered. We characterize the largest set of passenger arrival rates which can be served by a fixed number of drivers under any policy dictating the mobility pattern of available drivers. It turns out that any rate in this set can be achieved by offering appropriate constant but region-dependent rewards to drivers for passenger pick up. Moreover, it is shown that dynamic rewards which scale in proportion to the number of passengers waiting for pick up in each region, not only maximize throughput but also significantly decrease pick up delays.**

***Index Terms*—crowdsourcing, ridesharing, pricing, queueing model, stability**

I. Introduction

The past few years have witnessed the growth of ridesharing platforms, such as Uber and Lyft, which offer personal transportation services by matching potential passengers with self-employed drivers who transport the former to their requested destinations. The platform operates a two-sided market: one between the platform and passengers who pay the forner a fee per ride, and another betwen the drivers and the platform who retains a fixed part of the ride fee.

Each participant has its own goals: the passengers aim for low cost transport with small pick up delays, while the drivers and platform each maximize their own revenues. The overall premise [1] is that these can be organized in a virtuous cycle where: lower pick up delays attracts more passengers and demand for rides is increased $\rightarrow$ driver revenues are increased $\rightarrow$ more drivers participate $\rightarrow$ lower delays and lower cost, and so on.

A key to efficient operation, is the ability to dynamically match demand (the passengers) to supply (the drivers) in the presence of short-term stochastic fluctuations. This ability is controlled by: *1. Assignment of passenger rides to drivers*: after a potential passenger submits his ride request, the platform decides which driver will serve[1] him. Travel time for pick up, and pick up location also plays a role in the decision.

2. Pricing of rides. The ride fee may depend on pick up and destination locations, time of day. In periods, where there is a surge of potential passengers in a region with too few drivers to accomodate it, the ride fees are multiplied by some factor which depends on the size of demand excess. This type of dynamic pricing is called *surge pricing* in Uber and *Prime Time* in Lyft. The goal is to discourage more passengers requesting rides and at the same time to attract more drivers, in the location and for the duration of surge.

3. Driver strategy. Drivers optimize their revenues by moving to locations with higher chances of being assigned rides by the platform. The location-dependent riding fee may play a role in this decision.

The growth of ridesharing platforms has been accompanied by a surge of recent activity in models which attempt to predict and/or explain equilibrium behavior in the long and short-term. The authors in [2] model the long-term outcomes in car ownership and market size. [3] explores the long and short- properties of surge pricing in systems with stochastic fluctuations and show that it leads to better utilization and all stakeholders are better off. A more detailed short-term queueing model is incorporated in [4] where it is found that dynamic prices do not offer any advantages over static ride fees (i.e., depending on location but not on time) other that being more robust under uncertainty on demand. [5] explores the role of assignment of passenders to drivers, where maximum utilization policies are proposed which differ from assigning the driver closest to the passenger. The work in [6] studies the effect of demand topology on the equilibria and pricing under restrictive symmetry assumptions. In [7] the queueing of drivers awaiting for potential passengers in each location is analyzed and optimal mobility patterns are calculated.

In this paper, we study the maximum passenger throughput that can be carried by a ridesharing system for a fixed number of participating drivers, and the pricing mechanism to enforce it. So instead of focusing on platform profits as done in [4], [6], we consider a system-wide criterion similar but not identical to the criteria studied in [5], [7]. We introduce a queueing model for passengers, where the effect of topology, location-specific demand, number of drivers, free driver mobility pattern, drivers' willingness to accept a ride, pricing, and drivers' revenue maximization is explicitly modeled without symmetry assumptions. Our analysis is mainly concerned with stability, although numerical analysis suggests that our stable polices produce also good delay results.

Based on the model, we determine the stability region, i.e., the largest set of passenger arrival rates which can result to stable queues under the best possible driver strategy for these rates. In Theorem 4 we give a characterization of this region using a system of linear equations. In Theorem 5 we establish that for any set of arrival rates, the driver mobility pattern

[1] This decision does not depend on destination.

which requires the least number of drivers for stability, can be enforced if the platform gives incentives to drivers in the form of location-specific rewards for passenger pick up. In reality, these rewards correspond to the ride fee from each location. Although, unstable passenger queues will likely not occur in reality, as passengers find alternative transportation means, the reason they do this may very well be due to the load in that location lying outside the stability region.

The best driver strategy, as well as the rewards that achieve it, depend on the demand pattern and topology. We propose a policy, where the platform sets the reward for passenger pick up from a location to be proportional to the number of passengers waiting for a ride in that location. In Theorem 6 we show that this policy yields stable queues for any rate within the stability region. It is interesting to note that this reward mechanism which resembles surge-type of pricing, is essentially a type of MaxWeight algorithm, first proposed by Tassiulas in [8] for scheduling transmissions in multihop wireless networks.

The paper is organized as follows. The basic model, definitions and notation is established in Section II. Section III deals with the case of drivers with static stability patterns. There, Theorems 1-3 characterize the maximum achievable throughput for various classes of policies. In Section IV we deal with the case of dynamic mobility patterns. Theorem 4 gives the basic characterization. Theorem 5 in the same section establishes the existence of rewards which enforce the optimal system behavior to drivers as their best strategies. In Section IV-D we show that MaxWeight type of policies maximize throughput. The proof is based on the fluid limit approach in [7], and most details are delegated to the Appendix. In Section V we illustrate the results of the paper using a toy example. There we also give simulations results which illustrate the the delay improvement exhibited by MaxWeight policy. We conclude with a discussion in Section VI.

II. Model

Consider a system of $D = \{1, \ldots, d\}$ drivers serving passengers in $R = \{1, \ldots, r\}$ regions. Let $X_m(t) \in R$ be the region where the m-th driver is located at time t, and let $X(t) = (X_m(t), m \in D)$. If t is the time the m-th driver transits to region i, he becomes 'busy' or stays 'free' with probability $\Theta_i(t)$, independently of past choices. All drivers use the same probability in each region as they are interchangeable. Nevertheless, we allow $\Theta(t) = (\Theta_i(t), i \in R)$ to depend in an arbitrary manner on the drivers' past locations, the outcomes of randomization, and passenger queues in all regions. At this point we are concerned about describing the outcome of choices – not about how these are taken; this is considered in Section IV-C.

A driver who becomes busy in region i is assigned to a waiting passenger –if there is one– from i going to destination region j with probability q_{ij}. The driver transits to j after an exponential time of rate μ_{ij}, which represents the travel time from i to j, and then the passenger leaves the system. We let $1/\mu_i = \sum_j q_{ij}/\mu_{ij}$ denote the average travel time for drivers offering rides originating at $i \in R$. If there is no passenger in the queue at the time of arrival, a driver who becomes busy still makes the transition as above (i.e., according to q_{ij}, μ_{ij}), as if he was transporting a real passenger. In this case we think of the driver to be transporting a *virtual passenger*. The effect this virtual service has on throughput will be considered, and as it turns out, it is not a mere technical issue.

If the driver chooses to stay free he then has a choice of which region to move next. Let $P_{ij}(t)$ be the transition probability, of a free driver which has arrived at i, to move to j, and define $P(t) = (P_{ij}(t), i, j \in R)$. Since drivers within the same region are interchangeable, we let them move according to the same transition matrix. As with $\Theta(t)$, $P(t)$ may depend on the past in an arbitrary way. The chosen transition will take place after an exponential travel time with rate μ_{ij}.

$P(t)$ is selected from a set $\mathcal{P}$ of available transition matrices, which in what follows, we assume is one of the following two extremes:

Static mobility pattern: free drivers perform a random walk according to some fixed transition probability matrix $p = (p_{ij}, i, j, \in R)$, e.g., derived from historical experience but not depending on more recent states. In this case, the set of available transition matrices $\mathcal{P}$ is the singleton $\{p\}$.

Dynamic mobility pattern: free drivers can pick any region i with randomization allowed. In this case, $\mathcal{P}$ is the set of all $R \times R$ stochastic matrices.

After the driver arrives to the next region, he again randomly decides to become busy or stay free, makes an appropriate transition after some random time and so on.

Passengers arrive to region i according to a Poisson process $A_i(t), t \geq 0$ with rate λ_i, and wait –until they are offered a ride– at the queue of region i, whose length is given by:

$$Q_i(t) = Q_i(0) + A_i(t) - \int_0^t 1_{\{Q_i(s-)>0\}} B_i(ds)\,, i \in R\,, \quad (1)$$

where $B_i(t)$ is the number of drivers which have become busy up to time t, at region i. Note that the queue is not reduced when a driver becomes busy on a virtual customer.

The *policy* processes $P = (P(t) \in \mathcal{P}, t \geq 0), \Theta = (\Theta(t) \in [0,1], t \geq 0)$ which along with external randomization determine the drivers' actions (i.e., busy/free decisions and regions visited while free), depends (i.e., it is adapted) on the histories

$$(Q_i(s), X_m(s), B_i(s), i \in R, m \in D, 0 \leq s < t)\,. \quad (2)$$

Conditional on (P, Θ), the drivers' locations $X_1(\cdot), \ldots, X_d(\cdot)$ are independent. Moreover, all travel times, passenger arrival processes, and randomization decisions are mutually independent. The space of all policies is denoted by Π. The set of *work conserving* policies is $\Pi_w = \{(P, \Theta) \in \Pi : Q_i(t-) \neq 0 \Rightarrow \Theta_i(t) = 1$ for all $i, t\}$. The set of *policies with no virtual service* is $\Pi_0 = \{(P, \Theta) \in \Pi : Q_i(t-) = 0 \Rightarrow \Theta_i(t) = 0$ for all $i, t\}$. Finally, for any $p \in \mathcal{P}, \theta \in [0,1]^R$, the *static policy* (p, θ) is the policy $P(t) = p, \Theta(t) = \theta$, for all $t \geq 0$.

A central object of study is the set of passenger arrival rates $\lambda = (\lambda_i, i \in R)$ for which there exists a policy in Π which keeps passenger queues stable.

Definition 1. *The* stability region Λ *is the set of passenger arrival rates λ for which there exists a policy in Π such that $\lim_{t\to\infty} \sum_i Q_i(t)/t = 0$, with probability one. If only policies in Π_0 ($\Pi_0 \cap \Pi_w$) are considered then the corresponding stability region is denoted by Λ_0 (Λ_{0w}).*

III. STATIC MOBILITY PATTERN

We first consider the case of static mobility given by a transition probability matrix $p = (p_{ij}, i, j \in R)$, followed by free drivers.

A. Driver balance equations

$$\text{Let } \lim_t \frac{E[B_i(t)]}{t} = b_i, \ \lim_t \frac{E[F_i(t)]}{t} = f_i, \ i \in R \tag{3}$$

be the average rate of busy and free drivers respectively, leaving region i. Since the aggregate rate of drivers leaving and entering i balance,

$$b_i + f_i = \sum_j b_j q_{ji} + \sum_j f_j p_{ji}\,, \quad i \in R\,. \tag{4}$$

holds. By Little's law, the average number of busy and free drivers in region i is b_i/μ_i and f_i/ν_i, respectively, where $1/\nu_i = \sum_j p_{ij}/\mu_{ij}$ is the mean time of travel for a free driver in region i. Since the total number of drivers is d,

$$\sum_i \frac{b_i}{\mu_i} + \sum_i \frac{f_i}{\nu_i} = d \quad \text{holds.} \tag{5}$$

The relation between time-average rates of busy and free drivers out of any region and the driver balance equations is formalized in the following lemma.

Lemma 1. *For any policy where the limits in* (3) *exist, these satisfy the driver balance equations* (4)*, and* (5)*.*

Conversely, if the driver balance equations (4) *and* (5) *hold for some $b_i, f_i \geq 0$, a static policy (p, θ) and an initial driver placement exist for which* (3) *hold.*

Proof. We show the converse as the proof for the other direction essentially follows the argument laid above. Let $b_i, f_i, i \in R$ satisfy (4) and (5), and consider any static policy (p, θ) where $\theta_i = b_i/(b_i + f_i)$, for i with $b_i + f_i \neq 0$. Under this policy, $X_1(t)$ is a semi Markov process and

$$\pi_i = \frac{1}{d}\left(\frac{b_i}{\mu_i} + \frac{f_i}{\nu_i}\right), \forall i\,,$$

is a stationary distribution as implied by (4). Under this distribution, the average number of busy (free) drivers in i is b_i/μ_i (f_i/ν_i) and Little's law implies the rate of busy (free) drivers leaving i is b_i (f_i). This also gives the limit in (3) under some appropriate distribution for $X_1(0)$. □

B. Stability

Lemma 1 can be used to determine if stable policies exist for a given arrival vector λ, by checking the existence of nonnegative $f = (f_i, i \in R), b = (b_i, i \in R)$ which satisfy (4) and (5) with $\lambda \leq b$. Indeed, Lemma 1 then implies the existence of a static policy with the time-average busy rate at i exceeding λ_i for all i, and so $\lambda \in \Lambda$.

Theorem 1.

$$\Lambda = \left\{\lambda \in \mathbb{R}_+^R \,\middle|\, (4) \textit{ and } (5) \textit{ hold for some } b = (b_i, i \in R), \right.$$
$$\left. f = (f_i, i \in R), \textit{ with } b \geq \lambda, f_i \geq 0, \textit{ for all } i \in R \right\}. \tag{6}$$

Proof. It remains to show that Λ is a subset of the right hand side of (6). If $\lambda \in \Lambda$ then (4), (5) hold by Lemma 1 for the stabilizing policy, and

$$0 = \lim_t \frac{E[Q_i(t)]}{t} \geq \lambda_i - b_i\,,$$

i.e., $\lambda \leq b$. □

Since the stabilizing policy in Theorem 1 is static, it both allows virtual service and it is not work conserving. As is this might not be desirable, in the sequel we characterize the stability region Λ_{0w} achieved by the policy in $\Pi_0 \cap \Pi_w$. This is given in the next theorem which also implies that work conservation does not impede stability.

Theorem 2.

$$\Lambda_0 = \Lambda_{0w} = \left\{\lambda \in \mathbb{R}_+^R \,\middle|\, (4) \textit{ and } (5) \textit{ hold for } b_i = \lambda_i, i \in R, \right.$$
$$\left. \textit{for some } f = (f_i, i \in R) \geq 0 \right\}. \tag{7}$$

Proof. Let $\lambda \in \Lambda_0$ and consider the stabilizing policy in Π_0, for which Lemma 1 implies (4),(5) hold for $b, f \geq 0$ as in (3). Since no virtual service is allowed, $b = \lambda$, and thus Λ_0 is a subset of the set on the righthand side of the second equality in (7) which we denote by Λ'.

As $\Lambda_{0w} \subset \Lambda_0$, to conclude the proof it remains to show $\Lambda' \subset \Lambda_{0w}$. Let $\lambda \in \Lambda'$ and $f_i \geq 0, i \in R$ satisfy (4), (5) with $b_i = \lambda_i$, for all $i \in R$. Assume the time-average rate of busy (free) drivers in region i resulting from the sole policy in $\Pi_0 \cap \Pi_w$ is λ'_i (f'_i), as given by the corresponding limit in (3). These satisfy,

$$\lambda'_i + f'_i = \sum_j \lambda'_j q_{ji} + \sum_j f'_j p_{ji}, \quad i \in R$$
$$\sum_i \frac{\lambda'_i}{\mu_i} + \sum_i \frac{f'_i}{\nu_i} = d\,. \tag{8}$$

Since no drivers will idle if there are unserved customers, $f'_i = 0$ whenever $\lambda'_i < \lambda_i$. Moreover, the absence of virtual service implies $\lambda'_i \leq \lambda_i$ for all i. Using contradiction we show that $\lambda'_i = \lambda_i$ for all i.

Assume $\lambda'_j < \lambda_j$ for some j and let $\bar{B}$ be the largest subset of R for which $f'_i \leq f_i$. It is not empty, as $\lambda'_i < \lambda_i$ implies

$f_i' = 0 \le f_i$. The latter also implies $\lambda_i' = \lambda_i$ for all $i \in \bar{B}^{\complement}$. Lemma 2 below implies $\bar{B} = R$.

Lemma 2. *Let $B \subset R$ with $f_i' \le f_i$ for all $i \in B$, and $\lambda_i' = \lambda_i$ for all i in $B^{\complement}$. Then there exists i in $B^{\complement}$ with $f_i' \le f_i$.*

Proof. The flow of drivers between B and $B^{\complement}$ is balanced, so

$$\begin{aligned}\sum_{(i,j)\in B^{\complement}\times B}(\lambda_i' q_{ij} + f_i' p_{ij}) &= \sum_{(i,j)\in B^{\complement}\times B}\left(\lambda_j' q_{ji} + f_j' p_{ji}\right)\\ &\le \sum_{(i,j)\in B^{\complement}\times B}(\lambda_j q_{ji} + f_j p_{ji})\\ &= \sum_{(i,j)\in B^{\complement}\times B}(\lambda_i q_{ij} + f_i p_{ij})\\ &= \sum_{(i,j)\in B^{\complement}\times B}(\lambda_i' q_{ij} + f_i p_{ij})\end{aligned}$$

Thus, $\sum_{(i,j)\in B^{\complement}\times B} f_i' p_{ij} \le \sum_{(i,j)\in B^{\complement}\times B} f_i p_{ij}$ and so there must exist $i \in B^{\complement}$ with $f_i' \le f_i$. □

Thus, $f_i' \le f_i$ for all $i \in R$, which contradicts (8) as

$$\sum_i \frac{\lambda_i'}{\mu_i} + \sum_i \frac{f_i'}{\nu_i} < \sum_i \frac{\lambda_i}{\mu_i} + \sum_i \frac{f_i}{\nu_i} = d\,.$$

This concludes the proof of the theorem. □

In general Λ is a strict superset of $\Lambda_0 = \Lambda_{0w}$ as virtual service increases the routing flexibility of free drivers.

Theorem 2 gives information about which policies achieve maximal throughput, however it is not very informative about how the system parameters affect achievable throughput. Next, under the condition that p has a single recurrent class denoted by R_p, we give a sufficient condition for $\lambda \in \Lambda_{0w}$, in which the effect of passenger demand, number of drivers, and driver mobility pattern is more apparent. Let $\pi = (\pi_i, i \in R)$ be the (unique) stationary distribution of a Markov chain with transition rate $v_i p_{ij}$ from i to j, for all i, j.

Theorem 3. $\lambda \in \Lambda_{0w}$ *if and only if*

$$\sum_i \frac{\lambda_i}{\mu_i} + \sum_{i,j}\lambda_i q_{ij} T_{ji} - \min_{k\in R_p}\frac{\sum_{i,j}\lambda_i q_{ij} f_k^{ij}}{\pi_k \nu_k} < d\,, \quad (9)$$

where T_{ji} is the mean time for a free driver to visit j from i if no passengers were around, and $(f_k^{ij}, k \in R)$ is the solution of (4) *with respect to f for $b_i = 1, b_k = 0, k \in R \setminus \{i\}$, $q_{ij} = 1$, i.e., the rate of free drivers if a unit flow of passengers travelled from i to j (and no other passengers existed).*

Condition (9) is more intuitive: the first (leftmost) term is the minimum number of drivers necessary to serve the demand λ, if free drivers could be transported with infinite speed towards an unserved passenger. The second term accounts for the number of free drivers travelling in the system separately for each origin/destination pair i, j. From i there is a flow of busy drivers transported to j with rate $\lambda_i q_{ij}$. Then, the freed drivers wander around until they get busy again at i. The first two terms give a conservative upper bound on the minimum number of drivers needed to stabilize λ, because the free drivers may serve passengers from any origin/destination pair. Thus, the waste in free drivers is reduced by the third term.

For the proof we need the following definition and series of lemmas.

Definition 2. *For given λ, a solution $(f_i^0, i \in R)$ of* (4)*, with respect to $(f_i, i \in R)$ with $b_i = \lambda_i, i \in R$ is* minimal *for λ if* $\min_{i\in R_p} f_i^0 = 0$.

Lemma 3.
1) *For two solutions f^1, f^2 of* (4) *(with respect to f) for given λ, either $f^1 \ge f^2$ or $f^1 \le f^2$.*
2) *Any two solutions as above, differ by a multiple of the stationary distribution π.*
3) *For each b there is a unique f which solves* (4) *and* (5)*.*
4) *There is a unique minimal solution for each λ.*

Proof. 1) Without loss of generality, assume $f_j^1 \le f_j^2$ for some j and let $\bar{B}$ be the nonempty set $\{i \in R | f_i^1 \le f_i^2\}$. Lemma 2 implies $\bar{B} = R$, i.e., $f^1 \le f^2$.

2) If f^1, f^2 are two such solutions then,

$$f_i^1 - f_i^2 = \sum_j (f_j^1 - f_j^2) p_{ji}\,, i \in R\,,$$

and so $f_i^1 = f_i^2 - t\pi_i\nu_i, i \in R$, for some t.

3) (5) and the previous statement imply $f^1 = f^2$.

4) Let f^1, f^2 be two minimal solutions. 1) implies $f_i^1 = f_i^2 = 0$ for some $i \in R \Rightarrow f^1 = f^2$. But $f_i^1 = 0 \Rightarrow f_i^2 = 0$; otherwise $f_j^1 < 0$ for some $j \in R_p$. Hence $f^1 = f^2$. □

Lemma 4. (4), (5) *(for $b = \lambda$) have a solution $f_i, i \in R$, with $f_i > 0, i \in R_p$, if and only if*

$$\sum_i \frac{\lambda_i}{\mu_i} + \sum_i \frac{f_i^0}{\nu_i} < d\,, \quad (10)$$

where $(f_i^0, i \in R)$ is the minimal solution for λ.

Proof. Assume (9) holds first and let $f = (f_i, i \in R)$ satisfy (4), (5) for $b = \lambda$. In particular, (5) implies $\sum_i \frac{f_i}{\nu_i} > \sum_i \frac{f_i^0}{\nu_i}$. Then by Lemma 3 we have $f_i > f_i^0, i \in R_p$, and so $\min_{i\in R_p} f_i > \min_{i\in R_p} f_i^0 = 0$.

Now assume the existence of f which satisfies the conditions for the converse. Since both f, f^0 solve the driver balance equations, and $\min_{i\in R_p} f_i > 0 = \min_{i\in R_p} f_i^0$, by Lemma 3 we must have $t > 0$ and so $f \ge f^0$ with $f_i > f_i^0, i \in R_p$. This and (5) imply (10). □

Lemma 5. *The minimal solution $f^0 = (f_l^0, l \in R)$ for λ is given by*

$$f_l^0 = \sum_{ij}\lambda_i q_{ij} f_l^{ij} - \pi_l \nu_l \min_{k\in R_p}\frac{\sum_{ij}\lambda_i q_{ij} f_k^{ij}}{\pi_k \nu_k}\,, l \in R\,. \quad (11)$$

Proof. By Lemma 3 any solution $f \geq 0$ of the driver balance equations has $f_i = f_i^0 + t\pi_i\nu_i$ for $t \geq 0$. Thus, $\min_{k\in R_p} f_k^0 = 0$ implies

$$f_l^0 = f_l - t\pi_l\nu_l \min_{k\in R_p} \frac{f_k}{\pi_k\nu_k} \,. \tag{12}$$

To get (11) we construct a solution f with an explicit form. For any $i, j \in R$ assume a unit flow of passengers arrives at region i with destination j, and no passengers arrive for any other origin-destination pair. If free drivers move according to p, the rate f_k^{ij} at which they leave region $k \in R$ satisfies (4) for $b_i = 1, b_k = 0, k \in R \setminus \{i\}$ and $q_{ij} = 1$. Now, suppose the number of drivers is just enough such that f^{ij} is minimal for the unit flow. Since the rate of drivers which become free at the destination is equal to the rate of passengers, $f_j^{ij} = 1$. Note also that the mean time a driver stays free is equal to the mean time T_{ij} to reach j starting from i. (Drivers become busy as soon as they reach i because there is no idling.) Thus, the mean number of free drivers is $f_k^{ij} T_{ji} = T_{ji}$, and so

$$\sum_k \frac{f_k^{ij}}{\nu_k} = T_{ji}\,, i,j \in R\,. \tag{13}$$

Now because of linearity, a solution f to the driver balance equations can be obtained by superposing the solutions f^{ij} for each origin-destination pair i, j. Substituting $\sum_{ij} \lambda_i q_{ij} f^{ij}$ for f in (12) and using (13) yields (11). □

Proof of Theorem 3. Lemma 4 gives a necessary and sufficient condition (10). Now plugging (11) of Lemma 5 in (10), yields (9).

To show necessity, note that if $\lambda \in \Lambda_{0w}$ then the limits in (3) satisfy $b_i = \lambda_i$ for all i. Thus, (4), (5) hold for $b = \lambda$. □

IV. Dynamic mobility pattern

A. Driver balance equations

Similarly to the static mobility case, the limits

$$\lim_t \frac{E\left[B_i(t)\right]}{t} = b_i,\ \lim_t \frac{E\left[F_{ij}(t)\right]}{t} = f_{ij}\,, \tag{14}$$

with $F_{ij}(t)$ being the number of free drivers which have moved to j after i up to time t, satisfy

$$b_i + \sum_j f_{ij} = \sum_j b_j q_{ji} + \sum_j f_{ji}\,, i \in R \tag{15}$$

$$\text{and } \sum_i \frac{b_i}{\mu_i} + \sum_{i,j} \frac{f_{ij}}{\mu_{ij}} = d\,. \tag{16}$$

The following lemma parallels Lemma 1.

Lemma 6. *For any policy where the limits in* (14) *exist, they satisfy the driver balance equations* (15)*, and* (16)*.*

Conversely, if the driver balance equations (15) *and* (16) *hold for some* $b = (b_i, i \in R) \geq 0, f = (f_{ij}, i,j \in R) \geq 0$, *there exists a static policy* (p, θ) *and an initial driver placement for which* (14) *hold.*

Proof. The first statement is straightforward and follows a similar argument to Lemma 1. The second (converse) statement follows by Lemma 1 using the static policy (p, θ) with $p_{ij} = f_{ij}/\sum_k f_{ik}$ and $\theta_i = b_i/\left(b_i + \sum_k f_{ik}\right)$. □

B. Stability

In this section we give an algebraic characterization of the stability region.

Theorem 4.

$$\Lambda = \Lambda_0 = \Lambda_{0w} = \Big\{ \lambda \in \mathbb{R}_+^R \Big| \text{ (15) } \textit{and} \text{ (16) } \textit{hold}$$
$$\textit{for } b_i = \lambda_i, i \in R, \textit{ for some } f = (f_{ij}, i,j \in R) \geq 0 \Big\}\,. \tag{17}$$

Proof. Let Λ'' denote the set on the righthand side of the last equality in (17). We first show $\Lambda \subset \Lambda''$. Let $\lambda \in \Lambda$ and consider the stabilizing policy for which the limits $b = (b_i, i \in R), f = (f_{ij}, i,j \in R)$ in (14) satisfy (15),(16). Thus, defining $f'_{ij} = f_{ij} + (b_i - \lambda_i)q_{ij}$ for all i, j, gives

$$\begin{aligned}\lambda_i + \sum_j f'_{ij} &= b_i + \sum_j f_{ij} \\ &= \sum_j b_j q_{ji} + \sum_j f_{ji} = \sum_j \lambda_j q_{ji} + \sum_j f'_{ji}\,,\end{aligned}$$

for all i. Moreover, $\sum_i \frac{\lambda_i}{\mu_i} + \sum_{i,j} \frac{f'_{ij}}{\mu_{ij}} = \sum_i \frac{b_i}{\mu_i} + \sum_{i,j} \frac{f_{ij}}{\mu_{ij}} = d$, with $f'_{ij} \geq f_{ij} \geq 0$, and so $\lambda \in \Lambda''$.

We next show $\Lambda'' \subset \Lambda_{0w}$. For $\lambda \in \Lambda''$ let $f = (f_{ij}, i,j \in R) \geq 0$ be such that (15),(16) hold. The argument in the proof of Theorem 2 implies that the $\Pi_0 \cap \Pi_w$ policy with the static mobility pattern $p_{ij} = f_{ij}/f_i$ where $f_i = \sum_k f_{ik}$, is stable, i.e., $\lambda \in \Lambda_{0w}$.

The trivial observation $\Lambda_{0w} \subseteq \Lambda_0 \subseteq \Lambda$ concludes the proof. □

The first two equalities in (17) imply that throughput cannot increase by serving virtual passengers. This is not the case under static mobility patterns, as $\Lambda_0 \subsetneq \Lambda$ in general.

C. Reward-optimizing drivers

In this section we restrict to policies $(P, \Theta) \in \Pi$ which will be selected by reward-maximizing drivers, if the platform offers rewards for choosing the busy action *and* and a waiting passenger exists. Note that no reward is given for serving a virtual passenger. More specifically, the rate at which reward is accrued at time t to any driver who is busy in region i is $R_i(t) = r_i$ if $Q_i(t-) \neq 0$ and 0 otherwise. In section IV-D we consider the case where $R(t) = Q(t-)$.

If $\lambda \in \Lambda$, do appropriate rewards exist which result to stability? To tackle this question, we consider reward-maximizing drivers who if choose to become busy in region i, they may be denied service with some probability $1 - \phi_i$ (as should happen when no passengers exist in a region) and then decide where to transit next (as free drivers). To facilitate the analysis, we assume the drivers take into account the event of service denial only through its probability. This is accurate as long as drivers decide what to do at region i based on long-term empirical experience –not what happened in recent visits as this would correspond to a Markov decision problem with partial state information, which is a much harder problem–. The imposition of rewards leads the drivers to select busy b_i and free rates f_i out of any region i which maximize the time-average reward rate, i.e., $b = (b_i, i \in R), f = (f_{ij}, i, j, \in R)$ is an optimal solution of

$$\text{DRIVER}(r, \phi) : \max \sum_i \frac{r_i}{\mu_i} \phi_i b_i$$

$$\text{s.t. } \phi_i b_i + \sum_j f_{ij} = \sum_j \phi_j b_j q_{ji} + \sum_j f_{ji} \,, \tag{18}$$

$$\sum_i \frac{\phi_i b_i}{\mu_i} + \sum_{i,j} \frac{f_{ij}}{\mu_{ij}} = d \,, \tag{19}$$

$$\sum_j f_{ij} \geq (1 - \phi_i) b_i \,, \forall i \in R \,, \tag{20}$$

$$\text{over } b_i, f_{ij} \geq 0, i, j, \in R \,.$$

The third constraint comes from the fact that the rate of free drivers is at least $(1 - \phi_i) b_i$ due to the busy drivers which are denied service.

$\phi = (\phi_i, i \in R)$ itself is determined by the requirement that only a proportion ϕ_i of busy drivers at i gets to serve customers, such that the average service rate does not exceed λ_i, i.e.,

$$\phi_i = \min \left(\frac{\lambda_i}{b_i}, 1 \right) \,, \text{ for all } i \in R \,. \tag{21}$$

Theorem 5. *If $\lambda \in \Lambda$ there exist reward rates $r = (r_i, i \in R)$, service probabilities $\phi = (\phi_i, i \in R)$ and busy rates $b = (b_i, i \in R)$ such that i) b is a maximizer of DRIVER(r, ϕ), and ii)* (21) *holds.*

Proof. Let $\rho(\phi_1)$ be the optimal value of

$$\max \quad \rho \tag{22}$$

$$\text{s.t. } \lambda_i \rho \leq b_i \phi_1 \,, \text{ for all } i \in R \,, \text{(18)-(20) hold for } \phi_i = \phi_1 \,,$$

$$\text{over } \rho, b_i, f_{ij} \geq 0, i, j, \in R \,.$$

Notice that there exists a feasible solution with $b = \lambda$, since $\lambda \in \Lambda$, and so $\rho(1) \geq 1$. Since $\rho(0) = 0$ and $\rho(\cdot)$ is continuous, there exists $\phi_* \in (0, 1]$ with $\rho(\phi_*) = 1$.

Now, let b^* be an optimal solution for $\phi_1 = \phi_*$, with $\lambda_i = b_i^* \phi_*$ for all i. This is always possible since for any maximizer b, f, the solution $b^* = (\lambda_i / \phi_*, i \in R), f^* = (f_{ij} + (b_i - b_i^*) q_{ij}, i, j \in R)$ is also a maximizer with the desired property.

For the dual optimal pair $\zeta^* = (\zeta_i^*, i \in R)$ of b^*, f^*, we have that b^*, f^* is a maximizer of the dual function of (22), which is just the optimal value of DRIVER$((\zeta_i^* \mu_i, i \in R), (\phi_*, \ldots, \phi_*))$. Hence, $r = (\zeta_i^* \mu_i, i \in R), b = b^*, \phi = (\phi_*, \ldots, \phi_*)$ have the properties in the statement. ☐

D. Queue-size proportional rewards

In this section we consider a policy arising from the reward rates $R(t) = Q(t-)$. More specifically, we consider policies where the busy rates $b(t) = (b_i(t), i \in R)$ maximize DRIVER$(Q(t-), (1, \ldots, 1))$ at any time t. As the maximizers are nonunique, this definition does not uniquely define a policy. Any such policy which does not employ virtual service, i.e., is a member of Π_0 is referred to as a MaxWeight policy. MaxWeight policies always exist as drivers take zero reward if $Q_i(t-) = 0$ and so do not gain more by serving virtual passengers. In this policy, drivers get higher rewards for serving regions with more customers waiting. Note however that we assume drivers do not take into account the variability of rewards, i.e., do not take into account the effect of their actions (or of the other drivers) on the evolution of queues. This is a valid assumption in a system with very long passenger queues, where the relative change of queue sizes is slow.

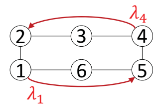


Fig. 1: Simulated topology

In this section we show that MaxWeight policies are stable for any set of passenger arrival vectors within the stability region. The proof is based on the fluid limits technique for establishing stability, where the set of trajectories of a relevant deterministic system are shown to stay bounded. The relevant system here is the following:

$$\dot{q}_i(t) = \lambda_i - b_i(t) \,, \text{ if } q_i(t) > 0 \,, \tag{23}$$

where $b(t) = (b_i(t), r \in R)$ are the rates of busy drivers resulting from the MaxWeight policy with $R(t) = q(t-)$.

Proposition 1. *If $\lambda \in \Lambda$ then $q(0) = 0$ implies $q(t) = 0$ for each $t \geq 0$.*

Proof. As is the case with MaxWeight type of policies (e.g., see [9]), this is established using a quadratic Lyapunov function, where it suffices to show that $\sum_i q_i(t)^2 / \mu_i$ decreases from any nonzero state $q(t)$. Now for any $q(t) \neq 0$, we have

$$\frac{d}{dt} \sum_i \frac{q_i(t)^2}{\mu_i} = 2 \left(\sum_i \frac{q_i(t)}{\mu_i} \lambda_i - \sum_i \frac{q_i(t)}{\mu_i} b_i(t) \right) \leq 0 \,,$$

since $\lambda \in \Lambda$ corresponds to a feasible solution in DRIVER$(q(t-), (1, \ldots, 1))$ and $b(t)$ is optimal, by the definition of MaxWeight. ☐

Theorem 6. *The MaxWeight policy is stable for any $\lambda \in \Lambda$.*

Proof. In Lemma 7 and Proposition 2 it is shown that appropriately scaled versions of the system as defined in the Appendix, converge to trajectories of (23). Proposition 1 implies $q(t) = 0$ for all $t \geq 0$ if $q(0) = 0$. By Theorem 3 in [10], or Theorem 3 in [9], this and uniform integrability of $\{Q_i(t)/t, t > 0\}$ in turn implies $\lim_t \sum_i E[Q_i(t)]/t = 0$. ☐

V. Numerical Exposition

We give a numerical example of the stability region of the system depicted in Fig. 1. Passengers take rides from region 1 to 5, and from 4 to 2 with rate λ_1 and λ_4, respectively. A trip to an adjacent region takes unit time.

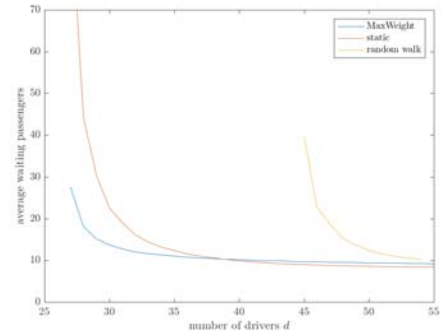
In the case where the free drivers move according to a symmetric random walk (i.e., $p_{ij} = 1/2$ for all $|i - j| \mod 6 = 1$), and the probability of becoming busy is chosen by the $\Pi_0 \cap \Pi_w$ policy, the stability condition (9) in Theorem 3 is

$$10(\lambda_1 + \lambda_4) - 12 \min(\lambda_1, \lambda_4) \leq d\,. \tag{24}$$

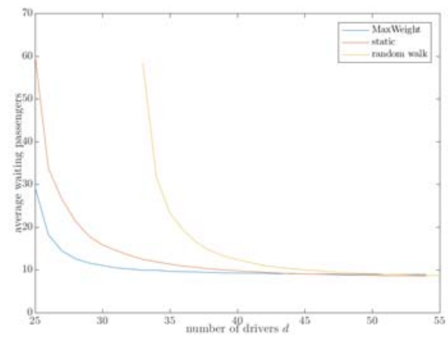
Also, by minimizing[2] the second term in the left-hand side of (10) over all minimal solutions f^0 which satisfy (15), (16) for $b = \lambda$, we get the optimal stability condition for policies in $\Pi_0 \cap \Pi_w$, by combining Theorems 4 and 3:

$$2(\lambda_1 + \lambda_4) + 2 \max(\lambda_1, \lambda_4) \leq d\,. \tag{25}$$

As expected, (25) is less strict than (24) because drivers who complete their rides are routed primarily to the closest region with waiting passengers, so as to minimize idling, e.g., free drivers moving from region 2 to 1 instead of heading towards the more remote region 4. This can significantly decrease the number of drivers necessary for stability. E.g., for $\lambda_1 = 3, \lambda_4 = 5$, the random walk policy requires 44 drivers where only 26 are necessary in (25). More symmetric demands, e.g., $\lambda_1 = \lambda_4 = 4$ reduce the drivers for both policies, and random walk performs better in the previous case. In Fig. 2, the average number of waiting passengers in the system is depicted, after simulating the topology in Fig. 1. As expected from Theorem 6, MaxWeight has the optimal stability region (25), as the static policy. MaxWeight has resulted to lower delay for a low number of drivers d, while the optimal static policy performs marginally better when d is large. It should be noted though, that MaxWeight is a much simpler policy since it does not require any estimation of arrival rates nor any optimization to be performed by the platform.



(a) Unbalanced demand: $\lambda_1 = 3, \lambda_4 = 5$.



(b) Balanced demand: $\lambda_1 = \lambda_4 = 4$.

Fig. 2: Average number of waiting passengers obtained by simulating three policies.

VI. Discussion

In the present paper we studied the stability of passenger queues for a given number of drivers, and interpret surge pricing as a MaxWeight type of dynamic scheduling policy. In our analysis we made the simplifying assumption that drivers are fully rational and able to solve the DRIVER problem. This is not realistic, as drivers tend to be more myopic and/or take other factors in their optimization. On the other hand, it is well known from the analysis of MaxWeight type of algorithms, that its stability is very robust under noisy and delayed information, or inexact computation. Thus, we believe the assumption of exact computations is not critical in terms of stability.

Another simplifying assumption is that the travel costs between regions are not accounted for in reward maximization. A little thought reveals that constant costs per unit of time are covered in the formulation of DRIVER$(\cdot, \cdot)$ as the optimal policy is unchanged by reducing the rewards of each action by an amount proportional to the respective travel time.

In our model, the waiting time from request for a ride until pick up is not explicitly taken into account, as there is no assignment taking place between passengers and drivers in different regions. Nevertheless, as free drivers will take longer time to reach waiting passengers in remote regions, the placement of drivers is also important in our model.

[2] The solution can be obtained by solving a transshipment problem involving free drivers. In our example, this can be done by inspecting each basic feasible solution.

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APPENDIX: FLUID LIMIT ANALYSIS

We consider the dynamic mobility pattern case since it is more interesting; the case of static mobility is ommited as its analysis is simpler and similar arguments are employed. Let $f = (f_{ij}, i, j \in R) \geq 0, b = (b_i, i \in R) \geq 0$ satisfy (15), (16). Lemma 6 implies that every set of f, b as above corresponds to a static policy (p, θ) where the stationary rate of busy drivers leaving region i is $b_i, i \in R$. Since the average reward (given in the objective function in DRIVER$(r, (1, \ldots, 1))$) is linear in b, for any choice of rewards r there will be at least one optimal basic feasible solution, which corresponds to one of the deterministic policies $(p^m, \theta^m), m = 1, \ldots, M$ (with $\theta_i^m \in \{0, 1\}$).

For each region i and m, let $(U_i^m(l), l = 0, 1, \ldots)$ be an i.i.d. sequence of random variables, where $P[U_i^m(l) = j] = \theta_i^m q_{ij} + (1 - \theta_i^m) p_{ij}^m$ for all $i, j \in R$. $U_i^m(l)$ is the region to which the l-th driver entering region i hops next. Let $(N_{ij}^{r,w}(t), t \geq 0), i, j \in R, r \in D, w \in \{0, 1\}$ be unit rate Poisson processes, independent of all other primitives.

Define the following set of processes

$$
\begin{aligned}
Z_{ij}^{m,r}(t) &= \int_0^t \mu_{ij} I_i^r(s-) 1\{V_i^{m,r}(s-) = j\} dT_m(s) \\
B_{ij}^r(t) &= N_{ij}^{r,1}\left(\sum_m \theta_i^m \; Z_{ij}^{m,r}(t)\right), \\
B_{ij}(t) &= \sum_{r \in D} B_{ij}^r(t), B_i(t) = \sum_j B_{ij}(t) \\
F_{ij}^r(t) &= N_{ij}^{r,0}\left(\sum_m (1 - \theta_i^m) Z_{ij}^{m,r}(t)\right) \\
F_{ij}(t) &= \sum_{r \in D} F_{ij}^r(t), \\
I_i^r(t) &= I_i^r(0) + \sum_j B_{ji}^r(t) + \sum_j F_{ji}^r(t) \\
&\quad - \sum_j B_{ij}^r(t) - \sum_j F_{ij}^r(t), \\
V_i^{m,r}(t) &= U_i^{m,r}\left(\sum_j B_{ji}^r(t) + \sum_j F_{ji}^r(t)\right)
\end{aligned}
$$

where $I_i^r(0)$ are Bernoulli random variables with $\sum_i I_i^r(0) = 1$ for each $r \in R$, and independent of the other primitives. Now, $T_m(t)$ is the amount of time in $[0, t]$ where policy (p^m, θ^m) is actuated. By the definition of MaxWeight, $\dot{T}_m(t) > 0$ if (f^m, b^m) maximizes $\sum_i Q_i(t-) b_i$ over f, b satisfying (18),(19), and $\dot{T}_m(t) = 0$ if m is not a maximizer. Thus,

$$
T_m(t) - T_m(s) \leq t - s, \sum_m T_m(t) = t, s, t \geq 0. \tag{26}
$$

For any vector of queue lengths $q = (q_i, i \in R)$ let MW$(q) \in \{1, \ldots, M\}$ denote the set of maximizers.

Now for every $n \geq 1$, define the scaled processes

$$
\begin{aligned}
\bar{A}_i^n(t) = \frac{A_i(nt)}{n}, \bar{Q}_i^n(t) = \frac{Q_i(nt)}{n}, \bar{T}_m^n(t) = \frac{T_m(nt)}{n}, \\
\bar{B}_{ij}^n(t) = \frac{B_{ij}(nt)}{n}, \bar{F}_{ij}^n(t) = \frac{F_{ij}(nt)}{n},
\end{aligned}
$$

and $\bar{B}_i^n(t) = \sum_j \bar{B}_{ij}^n(t)$ for each $t \geq 0$.

We let $D[0, +\infty)$ denote the space of right-continuous functions $f : [0, +\infty) \to \mathbb{R}$ with left limits, equipped with the metric of uniform convergence over compact sets, i.e., $f_n \xrightarrow{\text{u.o.c.}} f$ if and only if for $\sup_{s \in [0,t]} |f_n(s) - f(s)| \to_n 0$ for each $t \geq 0$.

Lemma 7. *There exist a subsequence $(n_k, k \geq 1)$ such that*

$$
\begin{aligned}
\bar{A}_i^n(\cdot) \xrightarrow{u.o.c.} \lambda_i \cdot, \bar{T}_i^n \xrightarrow{u.o.c.} \bar{T}_i, \bar{B}_{ij}^n \xrightarrow{u.o.c.} \bar{B}_{ij}, \\
\bar{F}_{ij}^n \xrightarrow{u.o.c.} \bar{F}_{ij}, \bar{Q}_i^n \xrightarrow{u.o.c.} q_i, i, j \in R,
\end{aligned}
$$

as $n \to \infty$, almost surely, for some Lipschitz continuous limit processes $\bar{T}_i, \bar{B}_{ij}, \bar{F}_{ij}, q_i$ with

$$
\frac{1}{n}\int_0^{nt} I_i^r(s-) dT_m(s) \xrightarrow{u.o.c.} \pi_{im}(t) t, \forall i, m, \tag{27}
$$

$$
\bar{B}_{ij}(t) = \xi_i \sum_m \theta_i^m \pi_{im}(t) q_{ij} t. \tag{28}
$$

$$
\bar{F}_{ij}(t) = \xi_i \sum_m (1 - \theta_i^m) \pi_{im}(t) p_{ij}^m t, \tag{29}
$$

$$
\frac{1}{n}\int_0^t \sum_i \bar{Q}_i^n(s-) d\bar{B}_i^n(s) \xrightarrow{u.o.c.} \int_0^t \sum_i q_i(s) d\bar{B}_i(s), \tag{30}
$$

for some probability measure $(\pi_{im}(t), i \in R, m = 1, \ldots, M)$ on $R \times \{1, \ldots, M\}$, and $\bar{B}_i(t) = \sum_j \bar{B}_{ij}(t)$.

Proof. The arguments are standard and the reader is referred to [7] for more details. The convergence of $\bar{A}_i^n$ is because of the strong law of large numbers (SLLN) and Lemma 4.1 of [7]. Since T_m is Lipschitz continuous, $(\bar{T}_m^n, n \geq 1)$ is an equicontinuous family of functions. By the Arzela-Ascoli theorem [11], there exist a converging subsequence, and let $\bar{T}_m$ denote any such limit. We pick a subsequence such that the convergence is obtained for all $m \in \{1, \ldots, M\}$. Now,

$$
\bar{B}_{ij}^{n,r}(t) = \sum_m \theta_i^m \frac{Z_{ij}^{m,r}(nt)}{n} + \frac{1}{n}\tilde{N}_{ij}^{r,1}\left(\sum_m \theta_i^m Z_{ij}^{m,r}(t)\right), \tag{31}
$$

where $\tilde{N}_{ij}^{r,1}$ is the martingale part of $N_{ij}^{r,1}$. Since $Z_{ij}^{m,r}(t) \leq \mu_{ij} t$, the second term in (31) is a square integrable martingale which converges to zero in probability, so by Doob's inequality the convergence is uniform in $[0,t]$. One can pick a subsequence over which the the second term converges to zero u.o.c., almost surely. (The selection of this subsequence is done before (n_k) is formed, i.e., the latter is a subsequence of the former.) Now, the SLLN for renewal processes implies the proportion of time a busy driver spends in transiting from i to j relative to the time spent while busy in i, converges to q_{ij} times the proportion of the respective sojourn times, i.e.,

$$\frac{\int_0^{nt} I_i^r(s-)1\{V_i^m(s-)=j\}dT_m(s)}{\int_0^{nt} I_i^n(s-)dT_m(s)} \underset{n}{\rightarrow} \frac{q_{ij}/\mu_{ij}}{1/\xi_i}, \text{ a.s..} \quad (32)$$

Notice that the limit is not dependent on m , as $P[U_i^m(l)=j]=q_{ij}$. Furthermore, since the denominator is Lipschitz continuous, there is a subsequence such that (27) holds for $\pi_{im}(t)$ as described in the statement (due to (26)). As the left-hand side in (27) is increasing and $\pi_{im}(t)$ is continuous, Lemma 4.1 in [7] implies the convergence is u.o.c. Putting (32), (27) in (31) and summing over $r \in D$, yields $\bar{B}_{ij}^n \xrightarrow{\text{u.o.c.}} \bar{B}_{ij}$ over a subsequence, where the limit is given by (28). The same analysis applied to $\bar{F}_{ij}^{n,r}$ gives the convergence of $\bar{F}_{ij}^n$ to (29). Also,

$$\frac{1}{n}\int_0^{nt} 1\{Q_i(s-)>0\}dB_{ij}(ds)\,, t\geq 0\,, \quad (33)$$

is Lipschitz continuous, so it converges u.o.c., over some subsequence. This and the convergence of $\bar{A}_i^n$ yield the convergence of $\bar{Q}_i^n$ to a Lipschitz continuous function q_i.

The convergence in (30) is shown by similar arguments as above and Lemmas 4.4, 4.1 in [7]. □

By defining,

$$b_i(t)=\dot{\bar{B}}_i(t)=\xi_i\sum_m \theta_i^m\pi_{im}(t)\,,$$
$$f_{ij}(t)=\dot{\bar{F}}_{ij}(t)=\xi_i\sum_m(1-\theta_i^m)\pi_{im}(t)p_{ij}^m\,, \quad (34)$$

for each $i,j\in R$ it is easy to check the equalities in (18),(19) are satisfied (for $b_i=b_i(t), f_{ij}=f_{ij}(t)$). The limit reward rate is $\sum_i q_i(t)b_i(t)$, by (30). In the next Proposition, we show that $b(t)=(b_i(t), i\in R)$ maximizes the reward rate at almost all t.

Proposition 2. *The fluid limit* $(q_i, i\in R)$ *of* $((\bar{Q}_i^n, i\in R), n\geq 1)$ *along any subsequence satisfies* (23) *where* $b(t)\in Co(b^m, m\in MW(q(t)))$*, for almost all* $t\geq 0$.

Proof. Let t be any time where $q(t)\neq 0$ and the derivatives of $q_i, \bar{T}_m, i\in R, m=1,\ldots,M$ exist.

If $q_i(t)>0$ then there exists $\epsilon>0$ with $q_i(u)>\epsilon$ for all $u\in[t,t+\delta]$. Thus, Lemma 7 implies $Q_i(nt+nu)>n\epsilon/2$ for all $u\in[0,\delta]$, and so the number of passengers departed from region i during $(nt, nt+n\delta]$, is given by $B_i(nt+n\delta)-B_i(nt)$.

Hence,

$$q_i(t+\delta)=q_i(t)-\left[\bar{B}_i(t+\delta)-\bar{B}_i(t)\right] \quad (35)$$

and taking the limit $\delta\to 0$ gives (23).

Since $\dot{\bar{T}}_m(t)>0$ for at least one m, there is a $\delta>0$ where $\mathrm{MW}(q(t+\delta))=\mathrm{MW}(q(t))$. Let $m\notin\mathrm{MW}(q(t))$ and so $m\notin\mathrm{MW}(q(u))$ for all $u\in[t,t+\delta]$. There exists an $\epsilon>0$ and $m^*\in\mathrm{MW}(q(t))$ with $\sum_i b_i^m q_i(u)<\epsilon+\sum_t tb_i^{m^*}q_i(u), u\in[0,t]$. By Lemma 7, for all n sufficiently large, we have

$$\sum_i b_i^m Q_i(nt+nu)\leq\frac{n\epsilon}{2}+\sum_i b_i^{m*}Q_i(nt+nu)\,,$$

for all $u\in[0,\delta]$. Thus, $m\notin\mathrm{MW}(Q(nt+nu))$ for all $u\in[0,\delta]$, and so $\bar{T}_m(nt+n\delta)-\bar{T}_m(nt)=0$. This also implies $\dot{\bar{T}}_m(t)=0$. Hence, $\dot{\bar{T}}_m(t)=0$ holds for all $m\notin\mathrm{MW}(q(t))$.

Convex combinations (p,θ) of policies $(b^m, m\in\mathrm{MW}(q(t)))$ can be synthesized by choosing probability θ_i^m to become busy or free, and use transition probabilities $(p_{ij}^m, j\in R)$ if free, of any policy $m\in\mathrm{MW}(q(t))$, at each i. Let ϕ_m^i be the probability of following policy (p^m,θ^m) at region i, so $\sum_{m\in\mathrm{MW}(q(t))}\phi_m^i=1$. Then, the stationary distribution $\pi_i^\phi, i\in R$ of the resulting policy will be optimal.

We show that the occupation measure $(\pi_{im}(t), i\in R, m=1,\ldots,M)$ from Lemma 7 in (27), which measures the proportion of time region i is visited and following policy (p^m,θ^m) there, can be expressed as a combination of $(p^m,\theta^m), m\in\mathrm{MW}(q(t))$ for some choice of ϕ. For each $i\in R, m=1,\ldots,M$ define

$$\pi_i=\sum_{m\in\mathrm{MW}(q(t))}\pi_{im}(t)\,,\phi_m^i=\begin{cases}\pi_{im}(t)/\pi_i\,, & \pi_i\neq 0\,,\\ 0\,, & \pi_i=0\,.\end{cases}$$

Since $\dot{\bar{T}}_m(t)=0$ for $m\notin\mathrm{MW}(q(t))$, (27) implies $\pi_{im}(t)=0$. Thus, $\phi_m^i>0$ only for $m\in\mathrm{MW}(q(t))$. Moreover,

$$\sum_i\pi_i=\sum_i\sum_{m\in\mathrm{MW}(q(t))}\pi_{im}(t)=\sum_i\sum_m\pi_{im}(t)=1\,,$$

and $\pi_i=\pi_i^\phi, i\in R$. To see this, note that by substituting $b_i(t), f_{ij}(t)$ in (34) for b_i, f_{ij} in the driver balance equations (15), we get

$$\xi_i\sum_m\theta_i^m\pi_{im}(t)+\xi_i\sum_m(1-\theta_i^m)\pi_{im}(t)=$$
$$\sum_j\xi_j\sum_m\theta_j^m\pi_{jm}(t)q_{ji}+\sum_j\xi_j\sum_m(1-\theta_j^m)\pi_{jm}(t)p_{ji}^m\,,$$

which is equivalent to

$$\pi_i=\sum_j\xi_j\pi_j\sum_m\phi_j^m\left[\theta_j^m q_{ji}+(1-\theta_j^m)p_{ji}^m\right]\,,$$

for all i, i.e., (π_i) is the stationary distribution (π_i^ϕ). Therefore, $b(t)$ is in the convex hull of $\{b^m, m\in\mathrm{MW}(q(t))\}$. □