A Riemannian Statistical Shape Model using Differential Coordinates

Project of Advanced Medical Imaging

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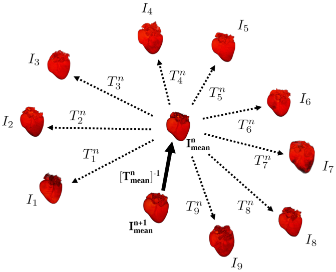
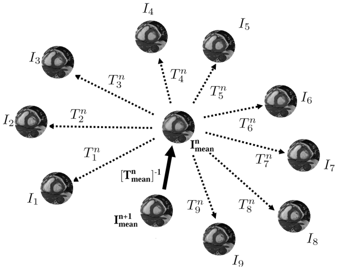
Overview

- Introduction
- Algorithm [9]
- Experiment
- Conclusion

Task

Given a set of shapes, understand the geometric variability of their structures (eg. Mean shape, Variation, Diffeomorphism, Low dimensional representation, etc.).





Fréchet Mean

$$\bar{\phi} = \text{argmin}_{\phi \in \mathcal{F}} \sum_{i=1}^{N} \int_M d_G(\phi, \phi_i)^2 dV$$

where:

- $\mathcal{F}$ is the space containing all shapes
- M is the space where a shape lies in ($\mathbb{R}^2$ or $\mathbb{R}^3$)
- N is the number of shapes in the set
- d_G is a defined metric on $\mathcal{F}$
- $\phi_1, \cdots, \phi_N \in \mathcal{F}$

Related Methods

Embedding Space

Euclidean Space

Non-euclidean Space

Distance Metric

Linear Metric

Non-linear Metric

Related Method

- **Point distribution model** (PDM) [4]: Euclidean space + euclidean distance
- **Kernel-based Method** [8]: Euclidean space + Non-linear distance
- **Connectivity-Independent Method** [3]: Euclidean space + Non-linear distance
- **Lie-Algebra Based Method** [6]: SE(2) / SE(3) + Non-linear distance
- **Deformable Model** [7]: Non-euclidean space + Non-linear distance (Elastic models, viscous flows, etc.)
- **Deep Learning Based Method** [2]: Euclidean / Non-euclidean space + Non-linear distance

Differential Coordinate [1]

Given a d-dimensional simplicial manifold $\mathcal{M}$, there exists a unique gradient tensor field $\nabla\phi$.

Deformation ϕ is represented as piecewise affine coordinate functions by barycentric interpolation:

$$\phi^h(x) = \sum_{j=1}^{n_0} \varphi_j(x) p_j$$

Grident field yields a constant 3×3 matrix on each d-simplex.

$$\nabla\phi^h(x) = \sum_{j=1}^{n_0} \nabla\varphi_j(x) p_j^T$$

Intrinsic Metric

Let $d_G(\cdot, \cdot)$ be a distance for 3×3 matrices with positive determinant, the distance in $\mathcal{F}$ is:

$$d_{\mathcal{F}}(\xi, \xi') = \left(\int_{\mathcal{M}} d_G(\xi(x), \xi'(x))^2 dV \right)^{\frac{1}{2}}$$

How to define d_G?

Intrinsic Metric

Given F the deformation gradient at $x \in \mathcal{M}$, we apply a polar decomposition.

$$F = RU$$

where $R \in SO(d)$ is a unitary matrix and $U \in Sym^{+}(d)$ is a symmetric positive-definite matrix.

- $d_{SO(d)}(R, R') = \| \log(R^T R') \|_F$
- $d_{Sym^{+}(d)}(U, U') = \| \log(U') - \log(U) \|_F$

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$$d_G = d_{SO(d)} + d_{Sym^+(d)}$$

Optimization

Large non-convex optimization $\rightarrow$ Local minimizer

Interior-point method is used in the author's implementation with the help of IPOPT library. Popular deep learning frameworks can also be used to simplify the calculation.

Inverse Map

Given the optimized ξ^*, the corresponding shape $\phi^* \in \mathcal{M}$ is found by solving a poisson equation:

$$\Delta\phi^* = \nabla \cdot \xi^*$$

Principal Geodesic Analysis [5]

At the tangent space of ξ^*, we perform a PGA to estimate the variability of the shapes. The representation of ξ_i is $v_i = \mathrm{Log}_{\xi^*}(\xi_i)$ where Log is the geodesic logarithmic mapping.

Covariance matrix $C = \frac{1}{N}\sum_{i=1}^{N} v_i v_i^T$

An eigendecomposition is performed on C, which gives us the eigenvalues $\lambda_1, \cdots, \lambda_N$ and the eigenvalues $u_1, \cdots, u_N$.

Dataset

OAI Dataset: We choose 58 severely diseased subjects and 58 healthy subjects. After manually preprocessing, they contructed a mesh model with 8988 vertices for each of the examples.

FAUST Dataset: 100 scans of 10 subjects of different shapes. The provided meshes are watertight, with 6890 vertices.

Test 1 - Fréchet Mean

Model:

- Shell PCA[10]
- The proposed model

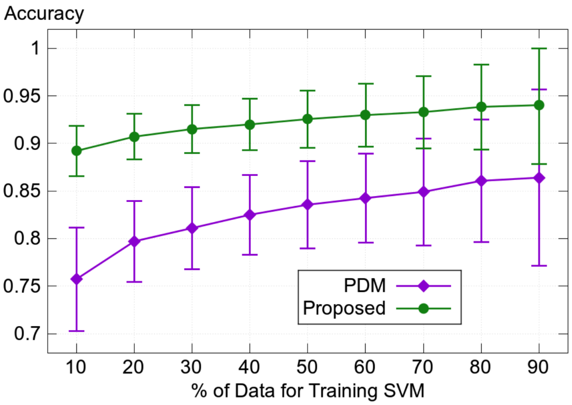
Method	Successful Rate	Lower Discrete Shells Energy	Speed
Shell PCA	74%	60%	9min
Proposed Model	100%	14%	5s

Table: OAI Dataset

Test 2 - Classification

Model:

- Point distribution model with 115-dimensional features
- The proposed model with 115-dimensional features

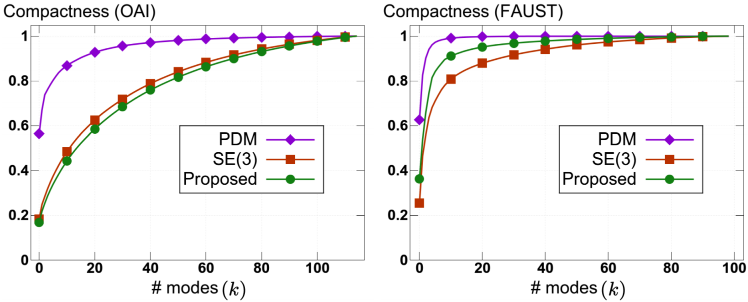


Test 3 - Model Evaluation

Model:

- Point distribution model
- SE model [6]
- The proposed model

Method: Specificity, Generalization ability, Compactness



Perspective

Advantages

- The model accounts for the non-Euclidean nature inherent to shapes
- Numerically robust
- Good performance

Disadvantages

- Need dense correspondence on mesh level
- Need well-localized data

Reference I

[1] Mario Botsch, Robert W. Sumner, Mark Pauly, and Markus H. Gross. Deformation transfer for detail-preserving surface editing. 2006.

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[3] Guillaume Charpiat, Olivier D. Faugeras, and Renaud Keriven. Approximations of shape metrics and application to shape warping and empirical shape statistics. *Foundations of Computational Mathematics*, 5:1–58, 2005.

[4] Timothy F. Cootes, Christopher J. Taylor, David H. Cooper, and Jim Graham. Training models of shape from sets of examples. In *BMVC*, 1992.

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[7] Martin Rumpf and Benedikt Wirth. Variational methods in shape analysis. In *Handbook of Mathematical Methods in Imaging*, 2015.

Reference II

[8] Yogesh Rathi Samuel Dambreville Allen Tannenbaum. Statistical shape analysis using kernel pca. 2005.

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[10] Chao Zhang, Behrend Heeren, Martin Rumpf, and William A. P. Smith. Shell pca: Statistical shape modelling in shell space. *2015 IEEE International Conference on Computer Vision (ICCV)*, pages 1671–1679, 2015.