Robust and Fast Extraction of 3D Symmetric Tensor Field Topology

Topological Data Analysis

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Problem Statement

3D Tensor Field

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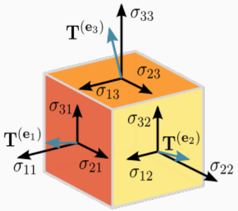

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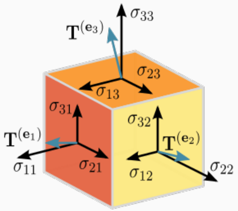

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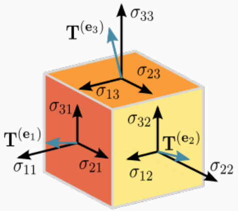

Tensor Field assigns a tensor to each point of a mathematical space

Most mechanical problems can be modelised using **symmetric tensors**. For instance, with stress tensors, this condition comes from the conservation of angular momentum.

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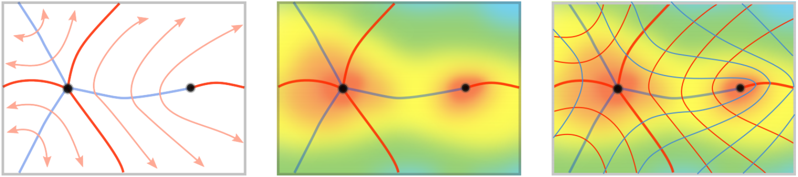
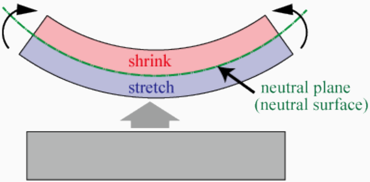
Relationship: Hyperstreamlines can intersect only at degenerate points.

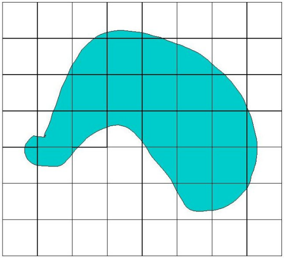


(a) (b) (c)

Neutral Surface

Neutral Surface is the surface within the beam between these zones, where the material of the beam is not under stress, neither compression or tension.





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- Topological features of interest
- Stable even in the presence of noise
- Widely used in topological data analysis

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Neutral surface in tensor field $\Longleftrightarrow$ Object Surface in real coordinate

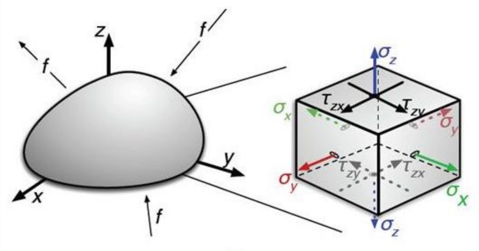
- Topological features of interest
- Widely used in mechanism and topology

Application

1. Mechanical Engineering: Stress Tensor

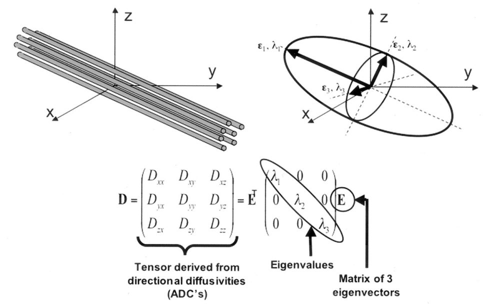
Internal forces or stresses acting within deformable bodies as reaction to external forces.

$$\mathbf{T} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$



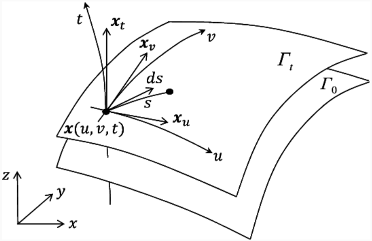
2. Material Science: Diffusion Tensor

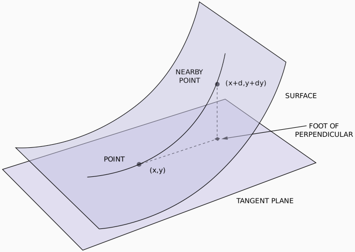
A material property, containing the strength information of the diffusion, according to the direction.



3. Differential Geometry

   - The first fundamental form: gives the infinitesimal distance on the manifold.
   - The second fundamental form: describes the change of the surface normal in any direction.





Existing methods

Degenerate Curve Extraction

- Ridge-and-Valley-Line Based Method (2008) [3]
- A-patches Based Method (2016) [2]: It is slow and it is not stable enough.

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Neutral Surface Reconstruction

- Marching Tetrahedra Method: Fast, but it often has topological errors.
- A-patches Method (2008) [1]: More robust, but rather slow and does not guarantee the convergence.

A more robust and faster algorithm is demanding!

Main Contribution

Contribution

- Provide analysis on degenerate curves and neutral surfaces for 3D linear tensor fields
- Introduce a more robust and faster method to extract degenerate curves and neutral surfaces

(a) (b) (c) (d)

Algorithm Flow

Degenerate Point Parameterization

We consider the space of **traceless symmetric** 3D tensors fields $\mathbb{A}$. Near a point of interest, a linear approximation gives $T(x, y, z) = T_0 + xT_x + yT_y + zT_z$.

The dimension of $\mathbb{A}$ is 5, so there exists a **normal tensor** $\overline{T}$ to the field T.

Theorem: A tensor $t = T(x, y, z)$ of dominant eigenvector $v = (\alpha, \beta, \gamma)$ is degenerate if and only if $v^T \overline{T} v = 0$.

In the coordinate system where $\overline{T} = \mathrm{diag}(a, b, -a-b)$, the previous theorem states that a degenerate tensor satisfies $a\alpha^2 + b\beta^2 = (a+b)\gamma^2$.

Parameterization: degenerate tensors can be found on an **ellipse** of angle parameter $\theta = \mathrm{atan}(\frac{\beta}{\alpha})$.

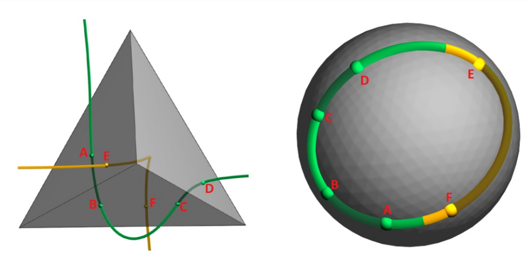
Note that traceless degenerate tensors can be written as $k(v^T v - \frac{I}{3})$.

Degenerate Curve Extraction

Input: a tensor field defined over a tetrahedral mesh with a piece-wise linear interpolation.

Flow:

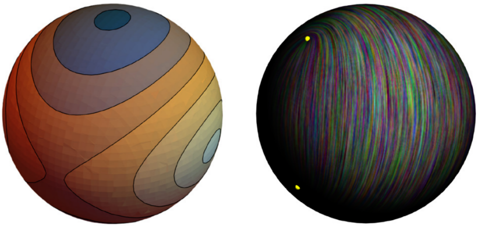
- The mesh is **pre-processed** to eliminate faces without degenerate points.
- **At each face** of a tetrahedron, degenerate points are found by solving a system of equations given by the previous theorem.
- The curve between degenerate points **inside the tetrahedron** is sampled using the curve of the elliptical parameterization.



Neutral Point Parameterization

Given a unit medium eigenvector $v = (\alpha, \beta, \gamma)$, we intend to find a unit major eigenvector and unit minor eigenvector, which reside on the same level set of $v^T \overline{T} v$.

Medium Eigenvector Manifold is a sphere with two pairs of singularities, indicating a map from a neutral tensor t to a tangent line field on the unit sphere.



(a) (b)

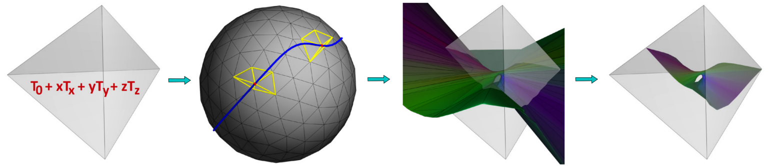
Neutral Surface Extraction

Input A triangulated mesh of a tensor field.

Flow We extract the neutral surface inside each tetrahedron, and then collect them to form the neutral surface of the tensor field.

Surface Extraction

- Triangulate the medium eigenvector manifold
- Modify the mesh so that the each singularity is a vertex
- Map the manifold to the tetrahedron
- Keep the surface inside the tetrahedron as the neutral surface



(a) (b) (c) (d)

Experiments

Speed

Device		
CPU	Intel(R) Xeon(R) E3-1230 CPU	
GPU	NVIDIA Quadro K420 GPU	
RAM	3.40GHz, 64GB of RAM	
Data		
Number of Tets	on the order of 10^6	
Time		
Algorithm	A-patches Based Method	Paper
Degenerate Curve Extraction	$>$10s	0.5s - 1s
Neutral Surface Extraction	$\sim$15s	1s - 10s

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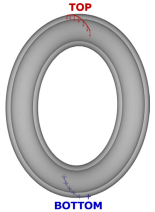
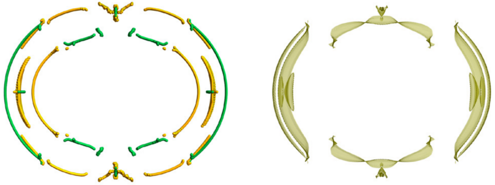
No access to the data they used, thus no further information.

Result: Tire with Symmetric-boundary Condition

We consider a torus that has symmetric boundary conditions at the top and at the bottom.



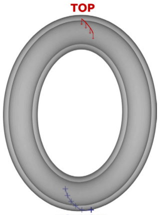
(a)



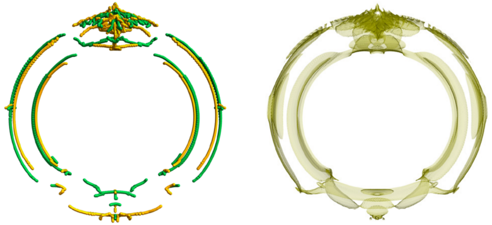
(c) (d)

Result: Tire with Camber-boundary Condition

We consider a torus that has asymmetric boundary conditions at the top and at the bottom.



(b)



(e) (f)

Evaluation

On the method:

- It provides new insights on degenerate curves and neutral surface where there allegedly is a lack of knowledge nowadays.
- It enables to have a analysis of the tensor field instead of the scalar tensor field alone.

Possible improvements:

- Beyond the assumed stable structural condition of the paper: the case of asymmetric tensor fields
- Considering time-dependant tensor fields
- Faster methods that do not rely on the A-patches method for extraction

Other possible use-case:

- Segmentation based on degenerate curves and neutral surfaces

C. Luk.
Tessellating algebraic curves and surfaces using a-patches.
Master's thesis, University of Waterloo, 2008.

J. Palacios, H. Yeh, W. Wang, Y. Zhang, R. S. Laramee, R. Sharma, T. Schultz, and E. Zhang.
Feature surfaces in symmetric tensor fields based on eigenvalue manifold.
IEEE Transactions on Visualization & Computer Graphics, (3):1248–1260, 2016.

X. Tricoche, G. Kindlmann, and C.-F. Westin.
Invariant crease lines for topological and structural analysis of tensor fields.
IEEE Transactions on Visualization and Computer Graphics, 14(6):1627, 2008.