Convex Optimization - DM 3

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1 Lasso Formulation

The primal problem of LASSO is:

$$\min_{w} \frac{1}{2}\|Xw - y\|_2^2 + \lambda\|w\|_1$$

In order to deduce its dual problem, we introduce a new vairable Z to create a constraint. So now we consider the new primal function:

$$\begin{aligned} \min_{Z,w} \quad & \|Z - y\|_2^2 + \lambda\|w\|_1 \\ s.t. \quad & Xw = Z \end{aligned}$$

By introducing a Lagrange multiplier ν, the lagrangian L can be written as:

$$L(Z, w, \nu) = \frac{1}{2}\|Z - y\|_2^2 + \lambda\|w\|_1 + \nu^T(Z - Xw)$$

We now want to minimize the lagrangian with respect to Z and w.

$$\frac{\partial L(Z, w, \nu)}{\partial Z} = Z - y + \nu = 0 \Rightarrow Z = y - \nu$$

So we have:

$$\begin{aligned} L(w, \nu) &= -\frac{1}{2}\nu^T(\nu - 2y) + \lambda\|w\|_1 - \nu^T Xw \\ &= -\frac{1}{2}(\nu - y + y)^T(\nu - y - y) + \lambda\|w\|_1 - \nu^T Xw \\ &= \frac{1}{2}(\|y\|_2^2 - \|y - \nu\|_2^2) + \lambda\|w\|_1 - \nu^T Xw \end{aligned}$$

Then we want

$$\begin{aligned} \min_{Z,w} \quad & \|Z - y\|_2^2 + \lambda\|w\|_1 \\ s.t. \quad & Xw = Z \end{aligned}$$

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with respect to w. Let's take $L'(w, \nu) = \lambda\|w\|_1 - \nu^T Xw$.

$$\begin{aligned} \min_{w} L'(w, \nu) &= \min_{w}\{\lambda\|w\|_1 - \nu^T Xw\} \\ &= -\lambda \max_{w}\{\frac{\nu^T X}{\lambda} w - \|w\|_1\} \end{aligned}$$

This formula corresponds to the definition of the conjugate function. The supper bound exists when $\|\frac{\nu^T X}{\lambda}\|_\infty \leq 1$. So the lagrangian becomes:

$$L(\nu) = \frac{1}{2}(\|y\|_2^2 - \|y-\nu\|_2^2) + \mathbb{1}_{\{\nu^T X \leq \lambda\}}$$

So the dual problem of LASSO is:

$$\begin{aligned} \min_{\nu} \quad & \frac{1}{2}(y-\nu)^T(y-\nu) \\ s.t. \quad & \nu^T X \preceq \lambda \end{aligned}$$